Euclid

Complete Works

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The Collected Works of

EUCLID

(fl. 300 BC)

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Euclid by John Sturgeon Mackay

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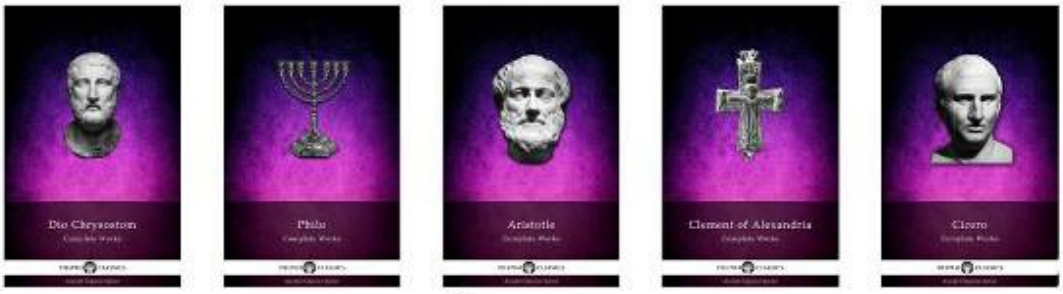

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The Collected Works of

EUCLID

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ISBN: 978 1 78877 946 3

Delphi Classics

is an imprint of

Delphi Publishing Ltd
Hastings, East Sussex
United Kingdom
Contact: sales@delphiclassics.com

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The Translations

Roman remains at Tyre, Lebanon — very little information regarding Euclid's life has survived. A biography by an Arabian author mentions his birth town as Tyre, though this is doubted by many.

A naval action during the siege of Tyre (332 BC) by André Castaigne

Elements

Translated by Thomas Heath, 1908

This famous mathematical treatise consists of 13 books and is attributed to the ancient Greek mathematician Euclid of Alexandria (c. 300 BC). It is composed of a collection of definitions, postulates, propositions and mathematical proofs of the propositions. The oldest extant large-scale deductive treatment of mathematics, *Elements* covers plane and solid Euclidean geometry, elementary number theory and incommensurable lines. The treatise has proven instrumental in the development of logic and modern science and its logical rigor would not be surpassed until the nineteenth century.

It was one of the very earliest mathematical works to be printed after the invention of the printing press and it has been estimated by some to be second only to the Bible in the number of editions published since its first printing in 1482, with the number reaching well over one thousand. For centuries, when the quadrivium was included in the curriculum of all university students, knowledge of at least part of *Elements* was required of all students. Not until the twentieth century, by which time its content was universally taught through other school textbooks, did it cease to be considered something all educated people should read.

Today, scholars believe that *Elements* is largely a compilation of propositions based on works by earlier Greek mathematicians. Proclus (412–485 AD), a Greek mathematician that lived around seven centuries after Euclid, wrote in his commentary on the *Elements*: “Euclid, who put together

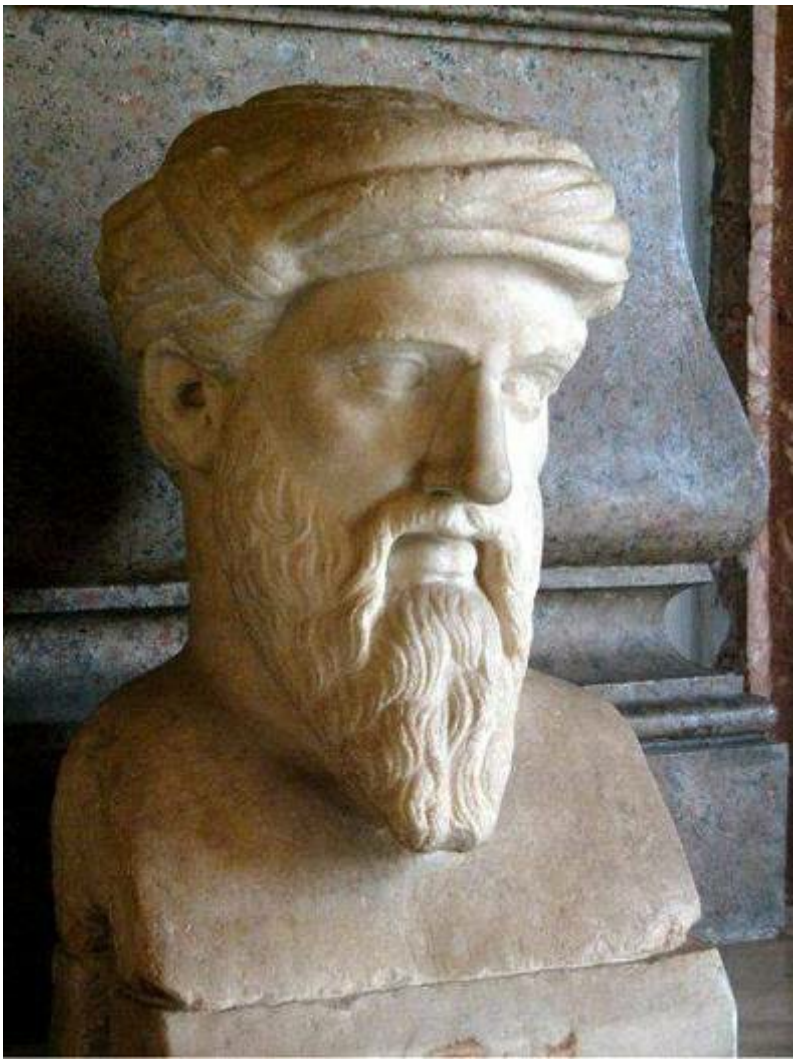
the *Elements*, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors". Pythagoras (c. 570–495 BC) was likely the source for most of the first two books, while Hippocrates of Chios (c. 470–410 BC) was the source of Book III. Eudoxus of Cnidus (c. 408–355 BC) is believed to have inspired Book V, while Books IV, VI, XI and XII probably came from other Pythagorean or Athenian mathematicians.

In the fourth century AD, Theon of Alexandria produced an edition of Euclid which was so widely used that it became the only surviving source until François Peyrard's 1808 discovery at the Vatican of the Heiberg manuscript, which was not derived from Theon's text. This manuscript originates from a Byzantine workshop around 900 and it is the basis of all modern editions. Papyrus Oxyrhynchus 29 is a tiny fragment of an even older manuscript, yet it only contains the statement of one proposition.

Elements is still considered a masterpiece in the application of logic to mathematics. In historical context, it has proven extremely influential in many areas of science. Prominent scientists such as Nicolaus Copernicus, Johannes Kepler, Galileo Galilei and Sir Isaac Newton were all influenced by *Elements*, and applied their knowledge of it to their own work. Mathematicians and philosophers, including Thomas Hobbes, Baruch Spinoza, Alfred North Whitehead and Bertrand Russell have attempted to create their own foundational "Elements" for their respective disciplines, by adopting the axiomatized deductive structures that Euclid's work introduced.

The ascetic beauty of Euclidean geometry has been regarded by many in western culture as a glimpse of an otherworldly system of perfection and certainty. Abraham Lincoln reportedly kept a copy of Euclid in his saddlebag and liked to study it late at night by lamplight; he liked to say to himself,

“You never can make a lawyer if you do not understand what demonstrate means; and I left my situation in Springfield, went home to my father’s house, and stayed there till I could give any proposition in the six books of Euclid at sight”. Einstein recalled a copy of *Elements* and a magnetic compass as two gifts that had a great influence on him as a boy, referring to the Euclid as the “holy little geometry book”.



Bust of Pythagoras of Samos in the Capitoline Museums, Rome — Pythagoras (c. 570 – c. 495 BC) was an ancient Ionian Greek philosopher and the eponymous founder of Pythagoreanism. His ideas survive in the first two books of 'Elements'.

A fragment of Euclid's 'Elements' on part of the Oxyrhynchus papyri

An illumination from a manuscript based on Adelard of Bath's translation of 'Elements', c. 1316; Adelard's is the oldest surviving Latin translation of 'Elements'.

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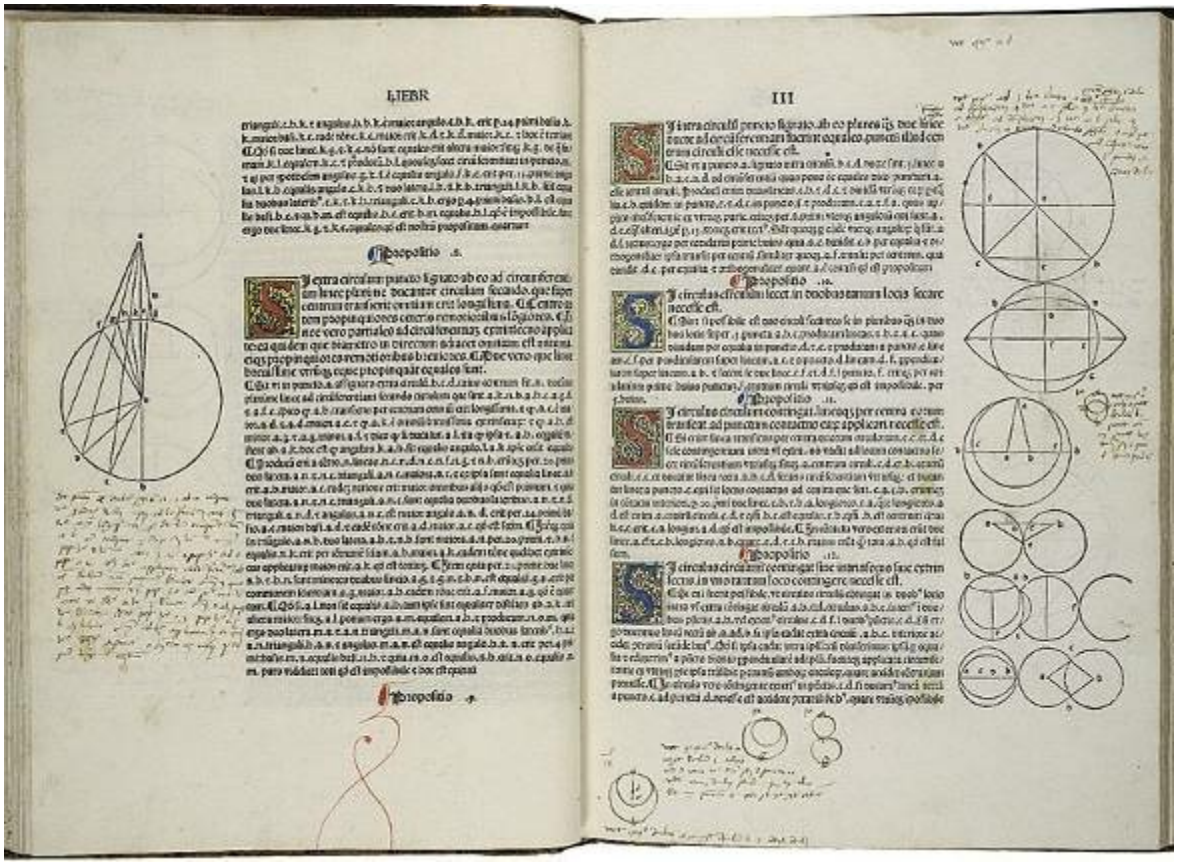
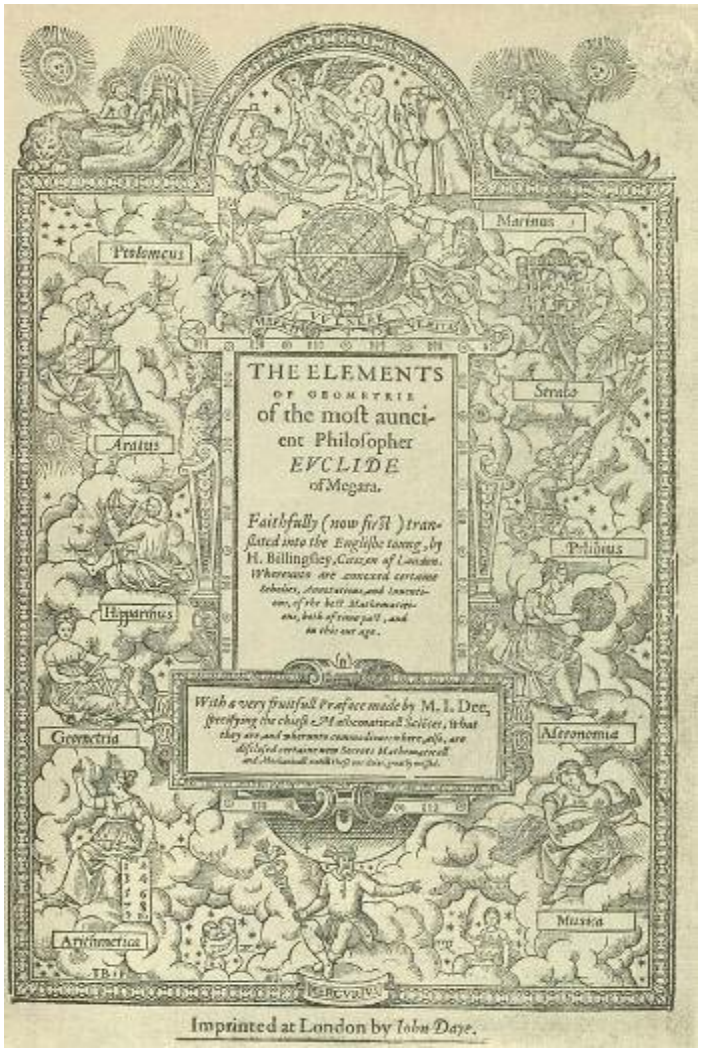
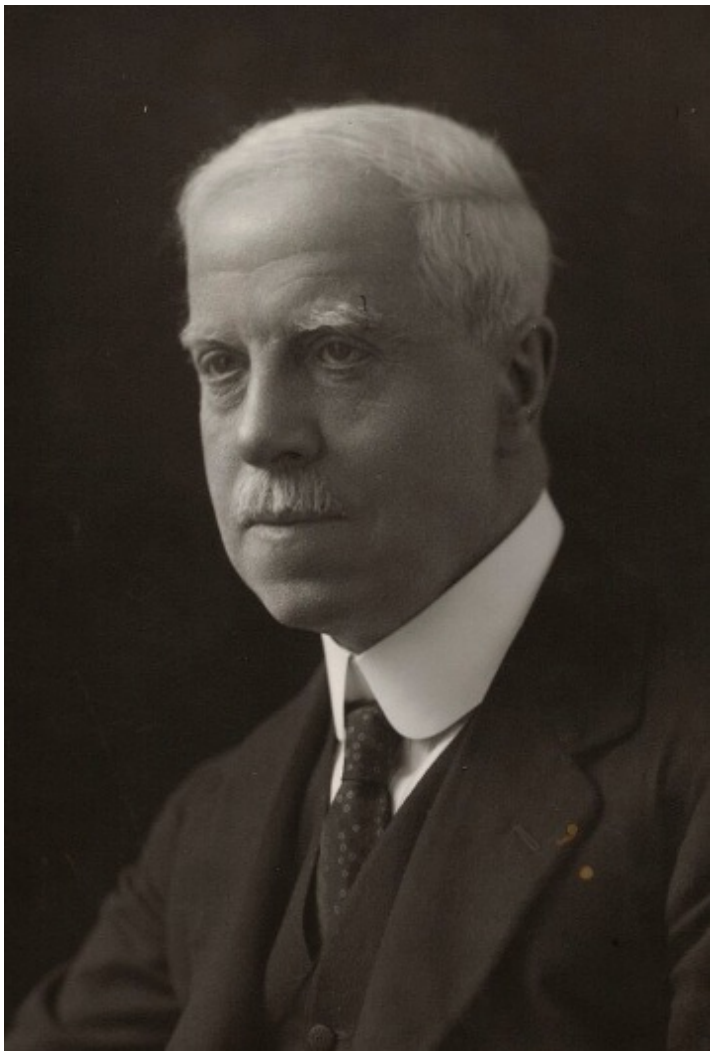


A page with marginalia from the first printed edition of 'Elements', printed by Erhard Ratdolt in 1482



The frontispiece of Sir Henry Billingsley's first English version of Euclid's 'Elements', 1570



Sir Thomas Little Heath (1861-1940) was a British civil servant, mathematician, classical scholar, historian of ancient Greek mathematics, translator, and mountaineer. Heath translated 'The thirteen books of Euclid's Elements' for Cambridge University Press in 1908.

INTRODUCTION

CHAPTER I. EUCLID AND THE TRADITIONS ABOUT HIM.

As in the case of the other great mathematicians of Greece, so in Euclid's case, we have only the most meagre particulars of the life and personality of the man.

Most of what we have is contained in the passage of Proclus' summary relating to him, which is as follows:

"Not much younger than these (sc. Hermotimus of Colophon and Philippus of Medma) is Euclid, who put together the Elements, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors. This man lived in the time of the first Ptolemy. For Archimedes, who came immediately after the first (Ptolemy), makes mention of Euclid: and, further, they say that Ptolemy once asked him if there was in geometry any shorter way than that of the elements, and he answered that there was no royal road to geometry. He is then younger than the pupils of Plato but older than Eratosthenes and Archimedes; for the latter were contemporary with one another, as Eratosthenes somewhere says."

This passage shows that even Proclus had no direct knowledge of Euclid's birthplace or of the date of his birth or death. He proceeds by inference. Since Archimedes lived just after the first Ptolemy, and Archimedes mentions Euclid, while there is an anecdote about some Ptolemy and Euclid, therefore Euclid lived in the time of the first Ptolemy.

We may infer then from Proclus that Euclid was intermediate between the first pupils of Plato and Archimedes. Now Plato died in 347/6, Archimedes lived 287-212, Eratosthenes c. 284-204 B.C. Thus Euclid must have flourished c. 300 B.C., which date agrees well with the fact that Ptolemy reigned from 306 to 283 B.C.

It is most probable that Euclid received his mathematical training in Athens from the pupils of Plato; for most of the geometers who could have taught him were of that school, and it was in Athens that the older writers of elements, and the other mathematicians on whose works Euclid's Elements depend, had lived and taught. He may himself have been a Platonist, but this does not follow from the statements of Proclus on the subject. Proclus says namely that he was of the school of Plato and in close touch with that philosophy. But this was only an attempt of a New Platonist to connect Euclid with his philosophy, as is clear from the next words in the same sentence, "for which reason also he set before himself, as the end of the whole Elements, the construction of the so-called Platonic figures." It is evident that it was only an idea of Proclus' own to infer that Euclid was a Platonist because his Elements end with the investigation of the five regular solids, since a later passage shows him hard put to it to reconcile the view that the construction of the five regular solids was the end and aim of the Elements with the obvious fact that they were intended to supply a foundation for the study of geometry in general, "to make perfect the understanding of the learner in regard to the whole of geometry." To get out of the difficulty he says that, if one should ask him what was the aim (σκοπός) of the treatise, he would reply by making a distinction between Euclid's intentions (1) as regards the subjects with which his investigations are concerned, (2) as regards the learner, and would say as regards (1) that "the whole of the geometer's argument is concerned with the cosmic figures." This latter statement is obviously incorrect. It is true that Euclid's Elements end with the construction of the five regular solids; but the planimetrical portion has no direct relation to them, and the arithmetical no relation at all; the propositions about them are merely the conclusion of the stereometrical division of the work.

One thing is however certain, namely that Euclid taught, and founded a school, at Alexandria. This is clear from the remark of Pappus about Apollonius : “he spent a very long time with the pupils of Euclid at Alexandria, and it was thus that he acquired such a scientific habit of thought.”

It is in the same passage that Pappus makes a remark which might, to an unwary reader, seem to throw some light on the personality of Euclid. He is speaking about Apollonius’ preface to the first book of his Conics, where he says that Euclid had not completely worked out the synthesis of the “three- and four-line locus,” which in fact was not possible without some theorems first discovered by himself. Pappus says on this: “Now Euclid — regarding Aristaeus as deserving credit for the discoveries he had already made in conics, and without anticipating him or wishing to construct anew the same system (such was his scrupulous fairness and his exemplary kindliness towards all who could advance mathematical science to however small an extent), being moreover in no wise contentious and, though exact, yet no braggart like the other [Apollonius] — wrote so much about the locus as was possible by means of the conics of Aristaeus, without claiming completeness for his demonstrations.” It is however evident, when the passage is examined in its context, that Pappus is not following any tradition in giving this account of Euclid: he was offended by the terms of Apollonius’ reference to Euclid, which seemed to him unjust, and he drew a fancy picture of Euclid in order to show Apollonius in a relatively unfavourable light.

Another story is told of Euclid which one would like to believe true. According to Stobaeus, “some one who had begun to read geometry with Euclid, when he had learnt the first theorem, asked Euclid, ‘But what shall I get by-learning these things?’ Euclid called his slave and said ‘Give him threepence, since he must make gain out of what he learns.’”

In the middle ages most translators and editors spoke of Euclid as Euclid of Megara. This description arose out of a confusion between our Euclid and the philosopher Euclid of Megara who lived about 400 B.C. The first trace of this confusion appears in Valerius Maximus (in the time of Tiberius) who says that Plato, on being appealed to for a solution of the problem of doubling the cubical altar, sent the inquirers to "Euclid the geometer." There is no doubt about the reading, although an early commentator on Valerius Maximus wanted to correct "Eucliden" into "Eudoxum," and this correction is clearly right. But, if Valerius Maximus took Euclid the geometer for a contemporary of Plato, it could only be through confusing him with Euclid of Megara. The first specific reference to Euclid as Euclid of Megara belongs to the 14th century, occurring in the ὑπομνηματισμοί of Theodorus Metochita (d. 1332) who speaks of "Euclid of Megara, the Socratic philosopher, contemporary of Plato," as the author of treatises on plane and solid geometry, data, optics etc. : and a Paris MS. of the 14th century has "Euclidis philosophi Socratici liber elementorum." The misunderstanding was general in the period from Campanus' translation (Venice 1482) to those of Tartaglia (Venice 1565) and Candalla (Paris 1566). But one Constantinus Lascaris (d. about 1493) had already made the proper distinction by saying of our Euclid that "he was different from him of Megara of whom Laertius wrote, and who wrote dialogues" ; and to Commandinus belongs the credit of being the first translator to put the matter beyond doubt: "Let us then free a number of people from the error by which they have been induced to believe that our Euclid is the same as the philosopher of Megara" etc.

Another idea, that Euclid was born at Gela in Sicily, is due to the same confusion, being based on Diogenes Laertius' description of the philosopher Euclid as being "of Megara, or, according to some, of Gela, as Alexander says in the Διαδοχαί."

In view of the poverty of Greek tradition on the subject even as early as the time of Proclus (410-485 A.D.), we must necessarily take cum grano the apparently circumstantial accounts of Euclid given by Arabian authors; and indeed the origin of their stories can be explained as the result (1) of the Arabian tendency to romance, and (2) of misunderstandings.

We read that "Euclid, son of Naucrates, grandson of Zenarchus, called the author of geometry, a philosopher of somewhat ancient date, a Greek by nationality domiciled at Damascus, born at Tyre, most learned in the science of geometry, published a most excellent and most useful work entitled the foundation or elements of geometry, a subject in which no more general treatise existed before among the Greeks: nay, there was no one even of later date who did not walk in his footsteps and frankly profess his doctrine. Hence also Greek, Roman and Arabian geometers not a few, who undertook the task of illustrating this work, published commentaries, scholia, and notes upon it, and made an abridgment of the work itself. For this reason the Greek philosophers used to post up on the doors of their schools the well-known notice: 'Let no one come to our school, who has not first learned the elements of Euclid.'" The details at the beginning of this extract cannot be derived from Greek sources, for even Proclus did not know anything about Euclid's father, while it was not the Greek habit to record the names of grandfathers, as the Arabians commonly did. Damascus and Tyre were no doubt brought in to gratify a desire which the Arabians always showed to connect famous Greeks in some way or other with the East. Thus Nas<*>īraddīn, the translator of the Elements, who was of T<*>ūs in Khurāsān, actually makes Euclid out to have been "Thusinus" also. The readiness of the Arabians to run away with an idea is illustrated by the last words of the extract. Everyone knows the story of Plato's inscription over the porch of the Academy: "let no one unversed in geometry enter my doors" ; the Arab turned geometry into

Euclid's geometry, and told the story of Greek philosophers in general and "their Academies."

Equally remarkable are the Arabian accounts of the relation of Euclid and Apollonius. According to them the Elements were originally written, not by Euclid, but by a man whose name was Apollonius, a carpenter, who wrote the work in 15 books or sections. In the course of time some of the work was lost and the rest became disarranged, so that one of the kings at Alexandria who desired to study geometry and to master this treatise in particular first questioned about it certain learned men who visited him and then sent for Euclid who was at that time famous as a geometer, and asked him to revise and complete the work and reduce it to order. Euclid then re-wrote it in 13 books which were thereafter known by his name. (According to another version Euclid composed the 13 books out of commentaries which he had published on two books of Apollonius on conics and out of introductory matter added to the doctrine of the five regular solids.) To the thirteen books were added two more books, the work of others (though some attribute these also to Euclid) which contain several things not mentioned by Apollonius. According to another version Hypsicles, a pupil of Euclid at Alexandria, offered to the king and published Books XIV. and XV., it being also stated that Hypsicles had "discovered" the books, by which it appears to be suggested that Hypsicles had edited them from materials left by Euclid.

We observe here the correct statement that Books XIV. and XV. were not written by Euclid, but along with it the incorrect information that Hypsicles, the author of Book XIV., wrote Book XV. also.

The whole of the fable about Apollonius having preceded Euclid and having written the Elements appears to have been evolved out of the preface to Book XIV. by Hypsicles, and in this way; the Book must in early times have been attributed to Euclid, and the inference based upon this assumption

was left uncorrected afterwards when it was recognised that Hypsicles was the author. The preface is worth quoting:

"Basilides of Tyre, O Protarchus, when he came to Alexandria and met my father, spent the greater part of his sojourn with him on account of their common interest in mathematics. And once, when examining the treatise written by Apollonius about the comparison between the dodecahedron and the icosahedron inscribed in the same sphere, (showing) what ratio they have to one another, they thought that Apollonius had not expounded this matter properly, and accordingly they emended the exposition, as I was able to learn from my father. And I myself, later, fell in with another book published by Apollonius, containing a demonstration relating to the subject, and I was greatly interested in the investigation of the problem. The book published by Apollonius is accessible to all — for it has a large circulation, having apparently been carefully written out later — but I decided to send you the comments which seem to me to be necessary, for you will through your proficiency in mathematics in general and in geometry in particular form an expert judgment on what I am about to say, and you will lend a kindly ear to my disquisition for the sake of your friendship to my father and your goodwill to me."

The idea that Apollonius preceded Euclid must evidently have been derived from the passage just quoted. It explains other things besides. Basilides must have been confused with βασιλεύς, and we have a probable explanation of the "Alexandrian king," and of the "learned men who visited" Alexandria. It is possible also that in the "Tyrian" of Hypsicles' preface we have the origin of the notion that Euclid was born in Tyre. These inferences argue, no doubt, very defective knowledge of Greek: but we could expect no better from those who took the Organon of Aristotle to be "instrumentum musicum pneumaticum," and who explained the name of Euclid, which they

variously pronounced as Uclides or Icludes, to be compounded of Ucli a key, and Dis a measure, or, as some say, geometry, so that Uclides is equivalent to the key of geometry!

Lastly the alternative version, given in brackets above, which says that Euclid made the Elements out of commentaries which he wrote on two books of Apollonius on conics and prolegomena added to the doctrine of the five solids, seems to have arisen, through a like confusion, out of a later passage in Hypsicles' Book XIV.: "And this is expounded by Aristaeus in the book entitled 'Comparison of the five figures,' and by Apollonius in the second edition of his comparison of the dodecahedron with the icosahedron." The "doctrine of the five solids" in the Arabic must be the "Comparison of the five figures" in the passage of Hypsicles, for nowhere else have we any information about a work bearing this title, nor can the Arabians have had. The reference to the two books of Apollonius on conics will then be the result of mixing up the fact that Apollonius wrote a book on conics with the second edition of the other work mentioned by Hypsicles. We do not find elsewhere in Arabian authors any mention of a commentary by Euclid on Apollonius and Aristaeus: so that the story in the passage quoted is really no more than a variation of the fable that the Elements were the work of Apollonius.

CHAPTER II. EUCLID'S OTHER WORKS.

In giving a list of the Euclidean treatises other than the Elements, I shall be brief: for fuller accounts of them, or speculations with regard to them, reference should be made to the standard histories of mathematics.

I will take first the works which are mentioned by Greek authors.

I. The Pseudaria.

I mention this first because Proclus refers to it in the general remarks in praise of the Elements which he gives immediately after the mention of Euclid in his summary. He says: "But, inasmuch as many things, while appearing to rest on truth and to follow from scientific principles, really tend to lead one astray from the principles and deceive the more superficial minds, he has handed down methods for the discriminative understanding of these things as well, by the use of which methods we shall be able to give beginners in this study practice in the discovery of paralogisms, and to avoid being misled. This treatise, by which he puts this machinery in our hands, he entitled (the book) of Pseudaria, enumerating in order their various kinds, exercising our intelligence in each case by theorems of all sorts, setting the true side by side with the false, and combining the refutation of error with practical illustration. This book then is by way of cathartic and exercise, while the Elements contain the irrefragable and complete guide to the actual scientific investigation of the subjects of geometry."

The book is considered to be irreparably lost. We may conclude however from the connexion of it with the Elements and the reference to its usefulness for beginners that it did not go outside the domain of elementary geometry.

2. The Data.

The Data (δεδομένα) are included by Pappus in the Treasury of Analysis (τόπος ἀναλυόμενος), and he describes their contents They are still concerned with elementary geometry, though forming part of the introduction to higher

analysis. Their form is that of propositions proving that, if certain things in a figure are given (in magnitude, in species, etc.), something else is given. The subjectmatter is much the same as that of the planimetrical books of the Elements, to which the Data are often supplementary. We shall see this later when we come to compare the propositions in the Elements which give us the means of solving the general quadratic equation with the corresponding propositions of the Data which give the solution. The Data may in fact be regarded as elementary exercises in analysis.

It is not necessary to go more closely into the contents, as we have the full Greek text and the commentary by Marinus newly edited by Menge and therefore easily accessible.

3. The book On divisions (of figures).

This work (περὶ διαιρέσεων βιβλίον) is mentioned by Proclus. In one place he is speaking of the conception or definition (λόγος) of figure, and of the divisibility of a figure into others differing from it in kind; and he adds: "For the circle is divisible into parts unlike in definition or notion (ἀνόμοια τῷ λόγῳ), and so is each of the rectilineal figures; this is in fact the business of the writer of the Elements in his Divisions, where he divides given figures, in one case into like figures, and in another into unlike." "Like" and "unlike" here mean, not "similar" and "dissimilar" in the technical sense, but "like" or "unlike in definition or notion" (λόγῳ): thus to divide a triangle into triangles would be to divide it into "like" figures, to divide a triangle into a triangle and a quadrilateral would be to divide it into "unlike" figures.

The treatise is lost in Greek but has been discovered in the Arabic. First John Dee discovered a treatise De divisionibus by one Muhammad Bagdadinus and handed over a copy of it (in Latin) in 1563 to Commandinus, who published it, in Dee did not himself translate the tract from the Arabic; he in 1570. Dee did not himself translate the tract from the Arabic; he found

it in Latin in a MS. which was then in his own possession but was about 20 years afterwards stolen or destroyed in an attack by a mob on his house at Mortlake. Dee, in his preface addressed to Commandinus, says nothing of his having translated the book, but only remarks that the very illegible MS. had caused him much trouble and (in a later passage) speaks of "the actual, very ancient, copy from which I wrote out..." (in ipso unde descripsi vetustissimo exemplari). The Latin translation of this tract from the Arabic was probably made by Gherard of Cremona (1114-1187), among the list of whose numerous translations a "liber divisionum" occurs. The Arabic original cannot have been a direct translation from Euclid, and probably was not even a direct adaptation of it; it contains mistakes and unmathematical expressions, and moreover does not contain the propositions about the division of a circle alluded to by Proclus. Hence it can scarcely have contained more than a fragment of Euclid's work.

But Woepcke found in a MS. at Paris a treatise in Arabic on the division of figures, which he translated and published in 1851. It is expressly attributed to Euclid in the MS. and corresponds to the description of it by Proclus. Generally speaking, the divisions are divisions into figures of the same kind as the original figures, e.g. of triangles into triangles; but there are also divisions into "unlike" figures, e.g. that of a triangle by a straight line parallel to the base. The missing propositions about the division of a circle are also here: "to divide into two equal parts a given figure bounded by an arc of a circle and two straight lines including a given angle" and "to draw in a given circle two parallel straight lines cutting off a certain part of the circle." Unfortunately the proofs are given of only four propositions (including the two last mentioned) out of 36, because the Arabic translator found them too easy and omitted them. To illustrate the character of the problems dealt with I need only take one more example: "To cut off a certain fraction from a

(parallel-) trapezium by a straight line which passes through a given point lying inside or outside the trapezium but so that a straight line can be drawn through it cutting both the parallel sides of the trapezium." The genuineness of the treatise edited by Woepcke is attested by the facts that the four proofs which remain are elegant and depend on propositions in the Elements, and that there is a lemma with a true Greek ring: "to apply to a straight line a rectangle equal to the rectangle contained by AB, AC and deficient by a square." Moreover the treatise is no fragment, but finishes with the words "end of the treatise," and is a well-ordered and compact whole. Hence we may safely conclude that Woepcke's is not only Euclid's own work but the whole of it. A restoration of the work, with proofs, was attempted by Ofterdinger, Who however does not give Woepcke's props. 30, 31, 34, 35, 36. We have now a satisfactory restoration, with ample notes and an introduction, by R. C. Archibald, who used for the purpose Woepcke's text and a section of Leonardo of Pisa's Practica geometriae (1220).

4.The Porisms.

It is not possible to give in this place any account of the controversies about the contents and significance of the three lost books of Porisms, or of the important attempts by Robert Simson and Chasles to restore the work. These may be said to form a whole literature, references to which will be found most abundantly given by Heiberg and Loria, the former of whom has treated the subject from the philological point of view, most exhaustively, while the latter, founding himself generally on Heiberg, has added useful details, from the mathematical side, relating to the attempted restorations, etc. It must suffice here to give an extract from the only original source of information about the nature and contents of the Porisms, namely Pappus. In his general preface about the books composing the Treasury of Analysis (τόπος ἀναλυόμενος) he says:

“After the Tangencies (of Apollonius) come, in three books, the Porisms of Euclid, [in the view of many] a collection most ingeniously devised for the analysis of the more weighty problems, [and] although nature presents and unlimited number of such porisms, [they have added nothing to what was written originally by Euclid, except that some before my time have shown their want of taste by adding to a few (of the propositions) second proofs, each (proposition) admitting of a definite number of demonstrations, as we have shown, and Euclid having given one for each, namely that which is the most lucid. These porisms embody a theory subtle, natural, necessary, and of considerable generality, which is fascinating to those who can see and produce results].

“Now all the varieties of porisms belong, neither to theorems nor problems, but to a species occupying a sort of intermediate position [so that their enunciations can be formed like those of either theorems or problems], the result being that, of the great number of geometers, some regarded them as of the class of theorems, and others of problems, looking only to the form of the proposition. But that the ancients knew better the difference between these three things is clear from the definitions. For they said that a theorem is that which is proposed with a view to the demonstration of the very thing proposed, a problem that which is thrown out with a view to the construction of the very thing proposed, and a porism that which is proposed with a view to the producing of the very thing proposed. [But this definition of the porism was changed by the more recent writers who could not produce everything, but used these elements and proved only the fact that that which is sought really exists, but did not produce it and were accordingly confuted by the definition and the whole doctrine. They based their definition on an incidental characteristic, thus: A porism is that which falls short of a locustheorem in respect of its hypothesis. Of this kind of porisms loci are a

species, and they abound in the Treasury of Analysis; but this species has been collected, named and handed down separately from the porisms, because it is more widely diffused than the other species]. But it has further become characteristic of porisms that, owing to their complication, the enunciations are put in a contracted form, much being by usage left to be understood; so that many geometers understand them only in a partial way and are ignorant of the more essential features of their contents.

"[Now to comprehend a number of propositions in one enunciation is by no means easy in these porisms, because Euclid himself has not in fact given many of each species, but chosen, for examples, one or a few out of a great multitude. But at the beginning of the first book he has given some propositions, to the number of ten, of one species, namely that more fruitful species consisting of loci.] Consequently, finding that these admitted of being comprehended in one enunciation, we have set it out thus: " If, in a system of four straight lines which cut each other two and two, three points on one straight line be given while the rest except one lie on different straight lines given in position, the remaining point also will lie on a straight line given in position..

"

"This has only been enunciated of four straight lines, of which not more than two pass through the same point, but it is not known (to most people) that it is true of any assigned number of straight lines if enunciated thus: " If any number of straight lines cut one another, not more than two (passing) through the same point, and all the points (of intersection situated) on one of them be given, and if each of those which are on another (of them) lie on a straight line given in position —

" or still more generally thus: "if any number of straight lines cut one another, not more than two (passing) through the same point, and all the

points (of intersection situated) on one of them be given, while of the other points of intersection in multitude equal to a triangular number a number corresponding to the side of this triangular number lie respectively on straight lines given in position, provided that of these latter points no three are at the angular points of a triangle (sc. having for sides three of the given straight lines) — each of the remaining points will lie on a straight line given in position."

"It is probable that the writer of the Elements was not unaware of this but that he only set out the principle; and he seems, in the case of all the porisms, to have laid down the principles and the seed only [of many important things], the kinds of which should be distinguished according to the differences, not of their hypotheses, but of the results and the things sought. [All the hypotheses are different from one another because they are entirely special, but each of the results and things sought, being one and the same, follow from many different hypotheses.]

"We must then in the first book distinguish the following kinds of things sought:

"At the beginning of the book is this proposition: " I. 'If from two given points straight lines be drawn meeting on a straight line given in position, and one cut off from a straight line given in position (a segment measured) to a given point on it, the other will also cut off from another (straight line a segment) having to the first a given ratio.'

"

"Following on this (we have to prove) " II. that such and such a point lies on a straight line given in position;

III. that the ratio of such and such a pair of straight lines is given;" etc. etc. (up to XXIX.).

"

“The three books of the porisms contain 38 lemmas; of the theorems themselves there are 171.”

Pappus further gives lemmas to the Porisms (p-918, ed. Hultsch).

With Pappus’ account of Porisms must be compared the passages of Proclus on the same subject. Proclus distinguishes two senses in which the word πόρισμα is used. The first is that of corollary where something appears as an incidental result of a proposition, obtained without trouble or special seeking, a sort of bonus which the investigation has presented us with. The other sense is that of Euclid’s Porisms. In this sense “porism is the name given to things which are sought, but need some finding and are neither pure bringing into existence nor simple theoretic argument. For (to prove) that the angles at the base of isosceles triangles are equal is a matter of theoretic argument, and it is with reference to things existing that such knowledge is (obtained). But to bisect an angle, to construct a triangle, to cut off, or to place — all these things demand the making of something; and to find the centre of a given circle, or to find the greatest common measure of two given commensurable magnitudes, or the like, is in some sort between theorems and problems. For in these cases there is no bringing into existence of the things sought, but finding of them, nor is the procedure purely theoretic. For it is necessary to bring that which is sought into view and exhibit it to the eye. Such are the porisms which Euclid wrote, and arranged in three books of Porisms.”

Proclus’ definition thus agrees well enough with the first, “older,” definition of Pappus. A porism occupies a place between a theorem and a problem: it deals with something already existing, as a theorem does, but has to find it (e.g. the centre of a circle), and, as a certain operation is therefore necessary, it partakes to that extent of the nature of a problem, which requires us to construct or produce something not previously existing. Thus, besides

III. I of the Elements and X. 3, 4 mentioned by Proclus, the following propositions are real porisms: III. 25, VI. 11-13, VII. 33, 34, 36, 39, VIII. 2, 4, X. 10, XIII. 18. Similarly in Archimedes On the Sphere and Cylinder I. 2-6 might be called porisms.

The enunciation given by Pappus as comprehending ten of Euclid's propositions may not reproduce the form of Euclid's enunciations; but, comparing the result to be proved, that certain points lie on straight lines given in position, with the class indicated by II. above, where the question is of such and such a point lying on a straight line given in position, and with other classes, e.g. (V.) that such and such a line is given in position, (VI.) that such and such a line verges to a given point, (XXVII.) that there exists a given point such that straight lines drawn from it to such and such (circles) will contain a triangle given in species, we may conclude that a usual form of a porism was "to prove that it is possible to find a point with such and such a property" or "a straight line on which lie all the points satisfying given conditions" etc.

Simson defined a porism thus: "Porisma est propositio in qua proponitur demonstrare rem aliquam, vel plures datas esse, cui, vel quibus, ut et cuilibet ex rebus innumeris, non quidem datis, sed quae ad ea quae data sunt eandem habent relationem, convenire ostendendum est affectionem quandam communem in propositione descriptam."

From the above it is easy to understand Pappus' statement that loci constitute a large class of porisms. A locus is well defined by Simson thus: "Locus est proposition in qua propositum est datam esse demonstrare, vel invenire lineam aut superficiem cuius quodlibet punctum, vel superficiem in qua quaelibet linea data lege descripta, communem quandam habet proprietatem in propositione descriptam." Heiberg cites an excellent instance

of a locus which is a porism, namely the following proposition quoted by Eutocius from the Plane Loci of Apollonius:

"Given two points in a plane, and a ratio between unequal straight lines, it is possible to draw, in the plane, a circle such that the straight lines drawn from the given points to meet on the circumference of the circle have (to one another) a ratio the same as the given ratio."

A difficult point, however, arises on the passage of Pappus, which says that a porism is "that which, in respect of its hypothesis, falls short of a locus-theorem" (τοπικοῦ θεωρήματος). Heiberg explains it by comparing the porism from Apollonius' Plane Loci just given with Pappus' enunciation of the same thing, to the effect that, if from two given points two straight lines be drawn meeting in a point, and these straight lines have to one another a given ratio, the point will lie on either a straight line or a circumference of a circle given in position. Heiberg observes that in this latter enunciation something is taken into the hypothesis which was not in the hypothesis of the enunciation of the porism, viz. "that the ratio of the straight lines is the same." I confess this does not seem to me satisfactory: for there is no real difference between the enunciations, and the supposed difference in hypothesis is very like playing with words. Chasles says: "Ce qui constitue le porisme est ce qui manque à l' hypothèse d'un théorème local (en d'autres termes, le porisme est inférieur, par l'hypothèse, au théorème local; c'est-à-dire que quand quelques parties d'une proposition locale n'ont pas dans l'énoncé la détermination qui leur est propre, cette proposition cesse d'être regardée comme un theéorème et devient un porisme)." But the subject still seems to require further elucidation.

While there is so much that is obscure, it seems certain (1) that the Porisms were distinctly part of higher geometry and not of elementary geometry, (2) that they contained propositions belonging to the modern

theory of transversals and to projective geometry. It should be remembered too that it was in the course of his researches on this subject that Chasles was led to the idea of anharmonic ratios.

Lastly, allusion should be made to the theory of Zeuthen on the subject of the porisms. He observes that the only porism of which Pappus gives the complete enunciation, "If from two given points straight lines be drawn meeting on a straight line given in position, and one cut off from a straight line given in position (a segment measured) towards a given point on it, the other will also cut off from another (straight line a segment) bearing to the first a given ratio," is also true if there be substituted for the first given straight line a conic regarded as the "locus with respect to four lines," and that this extended porism can be used for completing Apollonius' exposition of that locus. Zeuthen concludes that the Porisms were in part byproducts of the theory of conics and in part auxiliary means for the study of conics, and that Euclid called them by the same name as that applied to corollaries because they were corollaries with respect to conics. But there appears to be no evidence to confirm this conjecture.

5. The Surface-loci (τόποι πρὸς ἐπιφανείᾳ).

The two books on this subject are mentioned by Pappus as part of the Treasury of Analysis. As the other works in the list which were on plane subjects dealt only with straight lines, circles, and conic sections, it is a priori likely that among the loci in this treatise (loci which are surfaces) were included such loci as were cones, cylinders and spheres. Beyond this all is conjecture based on two lemmas given by Pappus in connexion with the treatise.

(1) The first of these lemmas and the figure attached to it are not satisfactory as they stand, but a possible restoration is indicated by Tannery. If the latter is right, it suggests that one of the loci contained all the points on

the elliptical parallel sections of a cylinder and was therefore an oblique circular cylinder. Other assumptions with regard to the conditions to which the lines in the figure may be subject would suggest that other loci dealt with were cones regarded as containing all points on particular elliptical parallel sections of the cones.

(2) In the second lemma Pappus states and gives a complete proof of the focus-and-directrix property of a conic, viz. that the locus of a point whose distance from a given point is in a given ratio to its distance from a fixed line is a conic section, which is an ellipse, a parabola or a hyperbola according as the given ratio is less than, equal to, or greater than unity. Two conjectures are possible as to the application of this theorem in Euclid's Surface-loci. (a) It may have been used to prove that the locus of a point whose distance from a given straight line is in a given ratio to its distance from a given plane is a certain cone. (b) It may have been used to prove that the locus of a point whose distance from a given point is in a given ratio to its distance from a given plane is the surface formed by the revolution of a conic about its major or conjugate axis. Thus Chasles may have been correct in his conjecture that the Surface-loci dealt with surfaces of revolution of the second degree and sections of the same.

6. The Conics.

Pappus says of this lost work: "The four books of Euclid's Conics were completed by Apollonius, who added four more and gave us eight books of Conics ." It is probable that Euclid's work was lost even by Pappus'time, for he goes on to speak of "Aristaeus, who wrote the still extant five books of Solid Loci connected with the conics." Speaking of the relation of Euclid's work to that of Aristaeus on conics regarded as loci, Pappus says in a later passage (bracketed however by Hultsch) that Euclid, regarding Aristaeus as deserving credit for the discoveries he had already made in conics, did not

(try to) anticipate him or construct anew the same system. We may no doubt conclude that the book by Aristaeus on solid loci preceded Euclid's on conics and was, at least in point of originality, more important. Though both treatises dealt with the same subject-matter, the object and the point of view were different; had they been the same, Euclid could scarcely have refrained, as Pappus says he did, from attempting to improve upon the earlier treatise. No doubt Euclid wrote on the general theory of conics as Apollonius did, but confined himself to those properties which were necessary for the analysis of the Solid Loci of Aristaeus. The Conics of Euclid were evidently superseded by the treatise of Apollonius.

As regards the contents of Euclid's Conics, the most important source of our information is Archimedes, who frequently refers to propositions in conics as well known and not needing proof, adding in three cases that they are proved in the "elements of conics" or in "the conics," which expressions must clearly refer to the works of Aristaeus and Euclid

Euclid still used the old names for the conics (sections of a rightangled, acute-angled, or obtuse-angled cone), but he was aware that an ellipse could be obtained by cutting a cone in any manner by a plane not parallel to the base (assuming the section to lie wholly between the apex of the cone and its base) and also by cutting a cylinder. This is expressly stated in a passage from the Phaenomena of Euclid about to be mentioned.

7. The Phaenomena.

This is an astronomical work and is still extant. A much interpolated version appears in Gregory's Euclid. An earlier and better recension is however contained in the MS. Vindobonensis philos. Gr. 103, though the end of the treatise, from the middle of pro to the last (18), is missing. The book, now edited by Menge,consists of propositions in spheric geometry. Euclid based it on Autolycus' work περὶ κινουμένης σφαίρας, but also, evidently, on

an earlier textbook of Sphaerica of exclusively mathematical content. It has been conjectured that the latter textbook may have been due to Eudoxus.

8. The Optics.

This book needs no description, as it has been edited by Heiberg recently,both in its genuine form and in the recension by Theon. The Catoptrica published by Heiberg in the same volume is not genuine, and Heiberg suspects that in its present form it may be Theon's. It is not even certain that Euclid wrote Catoptrica at all, as Proclus may easily have had Theon's work before him and inadvertently assigned it to Euclid.

9. Besides the above-mentioned works, Euclid is said to have written the Elements of Music (αἱ κατὰ μουσικὴν στοιχειώσεις). Two treatises are attributed to Euclid in our MSS. of the Musici, the κατατομὴ κανόνος, Sectio canonis (the theory of the intervals), and the εἰσαγωγὴ ἁρμονική (introduction to harmony).The first, resting on the Pythagorean theory of music, is mathematical, and the style and diction as well as the form of the propositions mostly agree with what we find in the Elements. Jan thought it genuine, especially as almost the whole of the treatise (except the preface) is quoted in extenso, and Euclid is twice mentioned by name, in the commentary on Ptolemy's Harmonica published by Wallis and attributed by him to Porphyry. Tannery was of the opposite opinion.The latest editor, Menge, suggests that it may be a redaction by a less competent hand from the genuine Euclidean Elements of Music. The second treatise is not Euclid's, but was written by Cleonides, a pupil of Aristoxenus.

Lastly, it is worth while to give the Arabians' list of Euclid's works. I take this from Suter's translation of the list of philosophers and mathematicians in the Fihrist, the oldest authority of the kind that we possess."To the writings of Euclid belong further [in addition to the Elements]: the book of Phaenomena; the book of Given Magnitudes [Data]; the book of Tones, known under the

name of Music, not genuine; the book of Division, emended by Thābit; the book of Utilisations or Applications [Porisms], not genuine; the book of the Canon; the book of the Heavy and Light; the book of Synthesis, not genuine; and the book of Analysis, not genuine."

It is to be observed that the Arabs already regarded the book of Tones (by which must be meant the εἰσαγωγὴ ἁρμονική) as spurious. The book of Division is evidently the book on Divisions (of figures). The next book is described by Casiri as "liber de utilitate suppositus." Suter gives reason for believing the Porisms to be meant,but does not apparently offer any explanation of why the work is supposed to be spurious. The book of the Canon is clearly the κατατομὴ κανόνος. The book on "the Heavy and Light" is apparently the tract De levi et ponderoso, included in the Basel Latin translation of 1537, and in Gregory's edition. The fragment, however, cannot safely be attributed to Euclid, for (1) we have nowhere any mention of his having written on mechanics, (2) it contains the notion of specific gravity in a form so clear that it could hardly be attributed to anyone earlier than Archimedes.Suter thinksthat the works on Analysis and Synthesis (said to be spurious in the extract) may be further developments of the Data or Porisms, or may be the interpolated proofs of Eucl. XIII. 1-5, divided into analysis and synthesis, as to which see the notes on those propositions.

CHAPTER III. GREEK COMMENTATORS ON THE ELEMENTS OTHER THAN PROCLUS.

That there was no lack of commentaries on the Elements before the time of Proclus is evident from the terms in which Proclus refers to them; and he leaves us in equally little doubt as to the value which, in his opinion, the generality of them possessed. Thus he says in one place (at the end of his second prologue):

"Before making a beginning with the investigation of details, I warn those who may read me not to expect from me the things which have been dinned into our ears ad nauseam (διατεθρύληται) by those who have preceded me, viz. lemmas, cases, and so forth. For I am surfeited with these things and shall give little attention to them. But I shall direct my remarks principally to the points which require deeper study and contribute to the sum of philosophy, therein emulating the Pythagoreans who even had this common phrase for what I mean 'a figure and a platform, but not a figure and sixpence.'"

In another place he says: "Let us now turn to the elucidation of the things proved by the writer of the Elements, selecting the more subtle of the comments made on them by the ancient writers, while cutting down their interminable diffuseness, giving the things which are more systematic and follow scientific methods, attaching more importance to the working-out of the real subject-matter than to the variety of cases and lemmas to which we see recent writers devoting themselves for the most part."

At the end of his commentary on Eucl. I. Prochis remarks that the commentaries then in vogue were full of all sorts of confusion, and contained no account of causes, no dialectical discrimination, and no philosophic thought.

These passages and two others in which Proclus refers to "the commentators" suggest that these commentators were numerous. He does not however give many names; and no doubt the only important commentaries were those of Heron, Porphyry, and Pappus.

1. Heron.

Proclus alludes to Heron twice as Heron mechanicus, in another place he associates him with Ctesibius, and in the three other passages where Heron is mentioned there is no reason to doubt that the same person is meant, namely Heron of Alexandria. The date of Heron is still a vexed question. In the early stages of the controversy much was made of the supposed relation of Heron to Ctesibius. The best MS. of Heron's Belopoeica has the heading Ηρωνος ΚτησιΒίου βελοποιϊκά, and an anonymous Byzantine writer of the tenth century, evidently basing himself on this title, speaks of Ctesibius as Heron's καθηγητής, "master" or "teacher." We know of two men of the name of Ctesibius. One was a barber who lived in the time of Ptolemy Euergetes II, i.e. Ptolemy VII, called Physcon (died 117 B.C.), and who is said to have made an improved water-organ. The other was a mechanician mentioned by Athenaeus as having made an elegant drinking-horn in the time of Ptolemy Philadelphus (285-247 B.C.). Martin took the Ctesibius in question to be the former and accordingly placed Heron at the beginning of the first century B.C., say 126-50 B.C. But Philo of Byzantium, who repeatedly mentions Ctesibius by name, says that the first mechanicians had the advantage of being under kings who loved fame and supported the arts. Hence our Ctesibius is more likely to have been the earlier Ctesibius who was contemporary with Ptolemy II Philadelphus.

But, whatever be the date of Ctesibius, we cannot safely conclude that Heron was his immediate pupil. The title "Heron's (edition of) Ctesibius's

Belopoeica" does not, in fact, justify any inferenee as to the interval of time between the two works.

We now have better evidence for a terminus post quem. The Metrica of Heron, besides quoting Archimedes and Apollonius, twice refers to "the books about straight lines (chords) in a circle" (ἐν τοῖς περὶ τῶν ἐν κύκλῳ εὐθειῶν). Now we know of no work giving a Table of Chords earlier than that of Hipparchus. We get, therefore, at once, 150 B.C. or thereabouts as the terminus post quem. But, again, Heron's Mechanica quotes a definition of "centre of gravity" as given by "Posidonius, a Stoic" : and, even if this Posidonius lived before Archimedes, as the context seems to imply, it is certain that another work of Heron's, the Definitions, owes something to Posidonius of Apamea or Rhodes, Cicero's teacher (135-51 B.C.). This brings Heron's date down to the end of the first century B.C., at least.

We have next to consider the relation, if any, between Heron and Vitruvius. In his De Architectura, brought out apparently in 14 B.C., Vitruvius quotes twelve authorities on machinationes including Archytas (second), Archimedes (third), Ctesibius (fourth) and Philo of Byzantium (sixth), but does not mention Heron. Nor is it possible to establish inter-dependence between Vitruvius and Heron; the differences between them seem on the whole more numerous and important than the resemblances (e.g. Vitruvius uses 3 as the value of π, while Heron always uses the Archimedean value 3 1/7). The inference is that Heron can hardly have written earlier than the first century A.D.

The most recent theory of Heron's date makes him later than Claudius Ptolemy the astronomer (100-178 A.D.). The arguments are mainly these. (1) Ptolemy claims as a discovery of his own a method of measuring the distance between two places (as an arc of a great circle on the earth's surface) in the case where the places are neither on the same meridian nor on the same

parallel circle. Heron, in his Dioptra, speaks of this method as of a thing generally known to experts. (2) The dioptra described in Heron's work is a fine and accurate instrument, much better than anything Ptolemy had at his disposal. (3) Ptolemy, in his work Περὶ ῥοπῶν, asserted that water with water round it has no weight and that the diver, however deep he dives, does not feel the weight of the water above him. Heron, strangely enough, accepts as true what Ptolemy says of the diver, but is dissatisfied with the explanation given by "some," namely that it is because water is uniformly heavy — this seems to be equivalent to Ptolemy's dictum that water in water has no weight — and he essays a different explanation based on Archimedes. (4) It is suggested that the Dionysius to whom Heron dedicated his Definitions is a certain Dionysius who was praefectus urbi in 301 A.D.

On the other hand Heron was earlier than Pappus, who was writing under Diocletian (284-305 A.D.), for Pappus alludes to and draws upon the works of Heron. The net result, then, of the most recent research is to place Heron in the third century A.D. and perhaps little earlier than Pappus. Heiberg accepts this conclusion, which may therefore, perhaps, be said to hold the field for the present..

That Heron wrote a systematic commentary on the Elements might be inferred from Proclus, but it is rendered quite certain by references to the commentary in Arabian writers, and particularly in an-Nairīzī's commentary on the first ten Books of the Elements. The Fihrist says, under Euclid, that "Heron wrote a commentary on this book [the Elements], endeavouring to solve its difficulties" ; and under Heron, "He wrote: the book of explanation of the obscurities in Euclid...." An-Nairīzī's commentary quotes Heron by name very frequently, and often in such a way as to leave no doubt that the author had Heron's work actually before him. Thus the extracts are given in the first person, introduced by "Heron says" ("Dixit Yrinus" or "Heron");

and in other places we are told that Heron "says nothing," or "is not found to have said anything," on such and such a proposition. The commentary of an-Nairīzī is in part edited by Besthorn and Heiberg from a Leiden MS. of the translation of the Elements by al-Hajjāj with the commentary attached. But this MS. only contains six Books, and several pages in the first Book, which contain the comments of Simplicius on the first twenty-two definitions of the first Book, are missing. Fortunately the commentary of an-Nairīzī has been discovered in a more complete form, in a Latin translation by Gherardus Cremonensis of the twelfth century, which contains the missing comments by Simplicius and an-Nairīzī's comments on the first ten Books. This valuable work has recently been edited by Curtze.

Thus from the three sources, Proclus, and the two versions of an-Nairīzī, which supplement one another, we are able to form a very good idea of the character of Heron's commentary. In some cases observations given by Proclus without the name of their author are seen from an-Nairīzī to be Heron's; in a few cases notes attributed by Proclus to Heron are found in an-Nairīzī without Heron's name; and, curiously enough, one alternative proof (of I. 25) given as Heron's by Proclus is introduced by the Arab with the remark that he has not been able to discover who is the author.

Speaking generally, the comments of Heron do not seem to have contained much that can be called important. We find

(1) A few general notes, e.g. that Heron would not admit more than three axioms.

(2) Distinctions of a number of particular cases of Euclid's propositions according as the figure is drawn in one way or in another.

Of this class are the different cases of I. 35, 36, III. 7, 8 (where the chords to be compared are drawn on different sides of the diameter instead of on the same side), III. 12 (which is not Euclid's, but Heron's own, adding the case

of external contact to that of internal contact in III. 11), VI. 19 (where the triangle in which an additional line is drawn is taken to be the smaller of the two), VII. 19 (where he gives the particular case of three numbers in continued proportion, instead of four proportionals).

(3) Alternative proofs. Of these there should be mentioned (a) the proofs of II. 1-10 "without a figure," being simply the algebraic forms of proof, easy but uninstructive, which are so popular nowadays, the proof of III. 25 (placed after III. 30 and starting from the arc instead of the chord), III. 10 (proved by III. 9), III. 13 (a proof preceded by a lemma to the effect that a straight line cannot meet a circle in more than two points). Another class of alternative proof is （b) that which is intended to meet a particular objection (ἔνστασις) which had been or might be raised to Euclid's construction. Thus in certain cases he avoids producing a particular straight line, where Euclid produces it, in order to meet the objection of any one who should deny our right to assume that there is any space available. Of this class are Heron's proofs of I. II, 1.20, and his note on 1. 16. Similarly on 1. 48 he supposes the right-angled triangle which is constructed to be constructed on the same side of the common side as the given triangle is. A third class (c) is that which avoids reductio ad absurdum. Thus, instead of indirect proofs, Heron gives direct proofs of I. 19 (for which he requires, and gives, a preliminary lemma), and of I. 25.

(4) Heron supplies certain converses of Euclid's propositions, e.g. converses of II. 12, 13, VIII. 27.

(5) A few additions to, and extensions of, Euclid's propositions are also found. Some are unimportant, e.g. the construction of isosceles and scalene triangles in a note on I. 1, the construction of two tangents in III. 17, the remark that VII. 3 about finding the greatest common measure of three numbers can be applied to as many numbers as we please (as Euclid tacitly

assumes in VII. 31). The most important extension is that of III. 20 to the case where the angle at the circumference is greater than a right angle, and the direct deduction from this extension of the result of III. 22. Interesting also are the notes on I. 37 (on I. 24 in Proclus), where Heron proves that two triangles with two sides of one equal to two sides of the other and with the included angles supplementary are equal, and compares the areas where the sum of the two included angles (one being supposed greater than the other) is less or greater than two right angles, and on I. 47, where there is a proof (depending on preliminary lemmas) of the fact that, in the figure of the proposition, the straight lines AL, BK, CF meet in a point. After iv. 16 there is a proof that, in a regular polygon with an even number of sides, the bisector of one angle also bisects its opposite, and an enunciation of the corresponding proposition for a regular polygon with an odd number of sides.

Van Pesch gives reason for attributing to Heron certain other notes found in Proclus, viz. that they are designed to meet the same sort of points as Heron had in view in other notes undoubtedly written by him. These are (a) alternative proofs of I. 5, I. 17, and I. 32, which avoid the producing of certain straight lines, (b) an alternative proof of 1.9 avoiding the construction of the equilateral triangle on the side of BC opposite to A; (c) partial converses of I. 35-38, starting from the equality of the areas and the fact of the parallelograms or triangles being in the same parallels, and proving that the bases are the same or equal, may also be Heron's. Van Pesch further supposes that it was in Heron's commentary that the proof by Menelaus of I. 25 and the proof by Philo of I. 8 were given.

The last reference to Heron made by an-Nairīzī occurs in the note on VIII. 27, so that the commentary of the former must at least have reached that point.

II. Porphyry.

The Porphyry here mentioned is of course the Neo-Platonist who lived about 232-304 A.D. Whether he really wrote a systematic commentary on the Elements is uncertain. The passages in Proclus which seem to make this probable are two in which he mentions him (1) as having demonstrated the necessity of the words "not on the same side" in the enunciation of I. 14, and (2) as having pointed out the necessity of understanding correctly the enunciation of I. 26, since, if the particular injunctions as to the sides of the triangles to be taken as equal are not regarded, the student may easily fall into error. These passages, showing that Porphyry carefully anaiysed Euclid's enunciations in these cases, certainly suggest that his remarks were part of a systematic commentary. Further, the list of mathematicians in the Fihrist gives Porphyry as having written "a book on the Elements." It is true that Wenrich takes this book to have been a work by Porphyry mentioned by Suidas and Proclus (Theolog. Platon.), περὶ ἀρχῶν libri II..

There is nothing of importance in the notes attributed to Porphyry by Proclus.

(1) Three alternative proofs of I. 20, which avoid producing a side of the triangle, are assigned to Heron and Porphyry without saying which belonged to which. If the first of the three was Heron's, I agree with van Pesch that it is more probable that the two others were both Porphyry's than that the second was Heron's and only the third Porphyry's. For they are similar in character, and the third uses a result obtained in the second.

(2) Porphyry gave an alternative proof of I. 18 to meet a childish objection which is supposed to require the part of AC equal to AB to be cut off from CA and not from AC.

Proclus gives a precisely similar alternative proof of I. 6 to meet a similar supposed objection; and it may well be that, though Proclus mentions no name, this proof was also Porphyry's, as van Pesch suggests.

Two other references to Porphyry found in Proclus cannot have anything to do with commentaries on the Elements. In the first a work called the Συμμικτά is quoted, while in the second a philosophical question is raised.

III. Pappus.

The references to Pappus in Proclus are not numerous; but we have other evidence that he wrote a commentary on the Elements. Thus a scholiast on the definitions of the Data uses the phrase "as Pappus says at the beginning of his (commentary) on the 10th (book) of Euclid." Again in the Fihrist we are told that Pappus wrote a commentary to the tenth book of Euclid in two parts. Fragments of this still survive in a MS. described by Woepcke, Paris. No. 952. 2 (supplément arabe de la Bibliothèque impériale), which contains a translation by Abū `Uthmān (beginning of 10th century) of a Greek commentary on Book X. It is in two books, and there can now be no doubt that the author of the Greek commentary was Pappus. Again Eutocius, in his note on Archimedes, On the Sphere and Cylinder I. 13, says that Pappus explained in his commentary on the Elements how to inscribe in a circle a polygon similar to a polygon inscribed in another circle; and this would presumably come in his commentary on Book XII., just as the problem is solved in the second scholium on Eucl. XII. I. Thus Pappus' commentary on the Elements must have been pretty complete, an additional confirmation of this supposition being forthcoming in the reference of Marinus (a pupil and follower of Proclus) in his preface to the Data to "the commentaries of Pappus on the book."

The actual references to Pappus in Proclus are as follows:

(1) On the Postulate (4) that all right angles are equal, Pappus is quoted as saying that the converse, viz. that all angles equal to a right angle are right, is not true, since the angle included between the arcs of two semicircles which

are equal, and have their diameters at right angles and terminating at one point, is equal to a right angle, but is not a right angle.

(2) On the axioms Pappus is quoted as saying that, in addition to Euclid's axioms, others are on record as well (συναναγράφεσθαι) about unequals added to equals and equals added to unequals; these, says Proclus, follow from the Euclidean axioms, while others given by Pappus are involved by the definitions, namely those which assert that "all parts of the plane and of the straight line coincide with one another," that "a point divides a straight line, a line a surface, and a surface a solid," and that "the infinite is (obtained) in magnitudes both by addition and diminution."

(3) Pappus gave a pretty proof of 1. 5. This proof has, I think, been wrongly understood; on this point see my note on the proposition.

(4) On 1. 47 Proclus says: "As the proof of the writer of the Elements is manifest, I think that it is not necessary to add anything further, but that what has been said is sufficient, since indeed those who have added more, like Heron and Pappus, were obliged to make use of what is proved in the sixth book, without attaining any important result." We shall see what Heron's addition consisted of; what Pappus may have added we do not know, unless it was something on the lines of his extension of 1. 47 found in the Synagoge (IV. , ed. Hultsch).

We may fairly conclude, with van Pesch, that Pappus is drawn upon in various other passages of Proclus where he quotes no authority, but where the subject-matter reminds us of other notes expressly assigned to Pappus or of what we otherwise know to have been favourite questions with him. Thus:

1. We are reminded of the curvilineal angle which is equal to but not a right angle by the note on 1. 32 to the effect that the converse (that a figure with its interior angles together equal to two right angles is a triangle) is not true unless we confine ourselves to rectilineal figures. This statement is

supported by reference to a figure formed by four semicircles whose diameters form a square, and one of which is turned inwards while the others are turned outwards. The figure forms two angles "equal to" right angles in the sense described by Pappus on Post. 4, while the other curvilineal angles are not considered to be angles at all, and are left out in summing the internal angles. Similarly the allusions in the notes on 1. 4, 23 to curvilineal angles of which certain moon-shaped angles (μηνοειδεῖς) are shown to be "equal to" rectilineal angles savour of Pappus.

2. On 1. 9 Proclus says that "Others, starting from the Archimedean spirals, divided any given rectilineal angle in any given ratio." We cannot but compare this with Pappus iv. , where the spiral is so used; hence this note, including remarks immediately preceding about the conchoid and the quadratrix, which were used for the same purpose, may very well be due to Pappus.

3. The subject of isoperimetric figures was a favourite one with Pappus, who wrote a recension of Zenodorus' treatise on the subject. Now on I. 35 Proclus speaks about the paradox of parallelograms having equal area (between the same parallels) though the two sides between the parallels may be of any length, adding that of parallelograms with equal perimeter the rectangle is greatest if the base be given, and the square greatest if the base be not given etc. He returns to the subject on 1. 37 about triangles. Compare also his note on 1. 4. These notes may have been taken from Pappus.

4. Again, on 1. 21, Proclus remarks on the paradox that straight lines may be drawn from the base to a point within a triangle which are (1) together greater than the two sides, and (2) include a less angle, provided that the straight lines may be drawn from points in the base other than its extremities. The subject of straight lines satisfying condition (1) was treated at length, with reference to a variety of cases, by Pappus, after a collection of

"paradoxes" by Erycinus, of whom nothing more is known. Proclus gives Pappus' first case, and adds a rather useless proof of the possibility of drawing straight lines satisfying condition (2) alone, adding that "the proposition stated has been proved by me without using the parallels of the commentators." By "the commentators" Pappus is doubtless meant.

5. Lastly, the "four-sided triangle," called by Zenodorus the "hollow-angled," is mentioned in the notes on 1. Def. 24-29 and I. 21. As Pappus wrote on Zenodorus' work in which the term occurred, Pappus may be responsible for these notes.

IV. Simplicius.

According to the Fihrist, Simplicius the Greek wrote "a commentary to the beginning of Euclid's book, which forms an introduction to geometry." And in fact this commentary on the definitions, postulates and axioms (including the postulate known as the ParallelAxiom) is preserved in the Arabic commentary of an-Nairĩzĩ. On two subjects this commentary of Simplicius quotes a certain ".Aganis," the first subject being the definition of an angle, and the second the definition of parallels and the parallel-postulate. Simplicius gives word for word, in a long passage placed by an-Nairīzī after 1. 29, an attempt by "Aganis" to prove the parallel-postulate. It starts from a definition of parallels which agrees with Geminus' view of them as given by Proclus, and is closely connected with the definition given by Posidonius. Hence it has been assumed that "Aganis" is none other than Geminus, and the historical importance of the commentary of Simplicius has been judged accordingly. But it has been recently shown by Tannery that the identification of "Aganis" with Geminus is practically impossible In the translation of Besthorn-Heiberg Aganis is called by Simplicius in one place "philosophus Aganis," in another "magister noster Aganis," in Gherard's version he is "socius Aganis" and "socius noster Aganis." These expressions seem to leave

no doubt that Aganis was a contemporary and friend, if not master, of Simplicius; and it is impossible to suppose that Simplicius (fl. about 500 A.D.) could have used them of a man who lived four and a half centuries before his time. A phrase in Simplicius' word-forword quotation from Aganis leads to the same conclusion. He speaks of people who objected "even in ancient times" (iam antiquitus) to the use by geometers of this postulate. This would not have been an appropriate phrase had Geminus been the writer. I do not think that this difficulty can he got over by Suter's suggestion that the passages in question may have been taken out of Heron's commentary, and that an-Nairīzī may have forgotten to name the author; it seems clear that Simplicius is the person who described "Aganis." Hence we are driven to suppose that Aganis was not Geminus, but some unknown contemporary of Simplicius Considerable interest will however continue to attach to the comments of Simplicius so fortunately preserved.

Proclus tells us that one Aegaeas (? Aenaeas) of Hierapolis wrote an epitome of the Elements; but we know nothing more of him or of it.

CHAPTER IV. PROCLUS AND HIS SOURCES.

It is well known that the commentary of Proclus on Eucl. Book I. is one of the two main sources of information as to the history of Greek geometry which we possess, the other being the Collection of Pappus. They are the more precious because the original works of the forerunners of Euclid, Archimedes and Apollonius are lost, having probably been discarded and forgotten almost immediately after the appearance of the masterpieces of that great trio.

Proclus himself lived 410-485 A.D., so that there had already passed a sufficient amount of time for the tradition relating to the pre-Euclidean geometers to become obscure and defective. In this connexion a passage is quoted from Simplicius who, in his account of the quadrature of certain lunes by Hippocrates of Chios, while mentioning two authorities for his statements, Alexander Aphrodisiensis (about 220 A.D.) and Eudemus, says in one place, "As regards Hippocrates of Chios we must pay more attention to Eudemus, since he was nearer the times, being a pupil of Aristotle."

The importance therefore of a critical examination of Proclus' commentary with a view to determining from what original sources he drew need not be further emphasised.

Proclus received his early training in Alexandria, where Olympiodorus was his instructor in the works of Aristotle, and mathematics was taught him by one Heron (of course a different Heron from the "mechanicus Hero" of whom we have already spoken). He afterwards went to Athens where he was imbued by Plutarch, and by Syrianus, with the Neo-Platonic philosophy, to which he then devoted heart and soul, becoming one of its most prominent exponents. He speaks everywhere with the highest respect of his masters, and was in turn regarded with extravagant veneration by his contemporaries, as we learn from Marinus his pupil and biographer. On the death of Syrianus he

was put at the head of the Neo-Platonic school. He was a man of untiring industry, as is shown by the number of books which he wrote, including a large number of commentaries, mostly on the dialogues of Plato. He was an acute dialectician, and pre-eminent among his contemporaries in the range of his learning; he was a competent mathematician; he was even a poet. At the same time he was a believer in all sorts of myths and mysteries and a devout worshipper of divinities both Greek and Oriental.

Though he was a competent mathematician, he was evidently much more a philosopher than a mathematician. This is shown even in his commentary on Eucl. I., where, not only in the Prologues (especially the first), but also in the notes themselves, he seizes any opportunity for a philosophical digression. He says himself that he attaches most importance to "the things which require deeper study and contribute to the sum of philosophy" ; alternative proofs, cases, and the like (though he gives many) have no attraction for him; and, in particular, he attaches no value to the addition of Heron to 1. 47, which is of considerable mathematical interest. Though he esteemed mathematics highly, it was only as a handmaid to philosophy. He quotes Plato's opinion to the effect that "mathematics, as making use of hypotheses, falls short of the non-hypothetical and perfect science" ... "Let us then not say that Plato excludes mathematics from the sciences, but that he declares it to be secondary to the one supreme science." And again, while "mathematical science must be considered desirable in itself, though not with reference to the needs of daily life," "if it is necessary to refer the benefit arising from it to something else, we must connect that benefit with intellectual knowledge (νοερὰν γνῶσιν), to which it leads the way and is a propaedeutic, clearing the eye of the soul and taking away the impediments which the senses place in the way of the knowledge of universals (τῶν ὅλων）."

We know that in the Neo-Platonic school the younger pupils learnt mathematics; and it is clear that Proclus taught this subject, and that this was the origin of the commentary. Many passages show him as a master speaking to scholars. Thus "we have illustrated and made plain all these things in the case of the first problem, but it is necessary that my hearers should make the same inquiry as regards the others as well," and "I do not indicate these things as a merely incidental matter but as preparing us beforehand for the doctrine of the Timaeus." Further, the pupils whom he was addressing were beginners in mathematics; for in one place he says that he omits "for the present" to speak of the discoveries of those who employed the curves of Nicomedes and Hippias for trisecting an angle, and of those who used the Archimedean spiral for dividing an angle in any given ratio, because these things would be too difficult for beginners (δυσθεωρήτους τοῖς εἰσαγομένοις）. Again, if his pupils had not been beginners, it would not have been necessary for Proclus to explain what is meant by saying that sides subtend certain angles, the difference between adjacent and uertical angles etc., or to exhort them, as he often does, to work out other particular cases for themselves, for practice (γυμνασίας ἕνεκα）.

The commentary seems then to have been founded on Proclus' lectures to beginners in mathematics. But there are signs that it was revised and re-edited for a larger public; thus he gives notice in one place "to those who shall come upon" his work (τοῖς ἐντευξομένοις). There are also passages which could not have been understood by the beginners to whom he lectured, e.g. passages about the cylindrical helix, conchoids and cissoids. These passages may have been added in the revised edition, or, as van Pesch conjectures, the explanations given in the lectures may have been much fuller and more comprehensible to beginners, and they may have been shortened on revision.

In his comments on the propositions of Euclid, Proclus generally proceeds in this way: first he gives explanations regarding Euclid's proofs, secondly he gives a few different cases, mainly for the sake of practice, and thirdly he addresses himself to refuting objections raised by cavillers to particular propositions. The latter class of note he deems necessary because of "sophistical cavils" and the attitude of the people who rejoiced in finding paralogisms and in causing annoyance to scientific men. His commentary does not seem to have been written for the purpose of correcting or improving Euclid. For there are very few passages of mathematical content in which Proclus can be supposed to be propounding anything of his own; nearly all are taken from the works of others, mostly earlier commentators, so that, for the purpose of improving on or correcting Euclid, there was no need for his commentary at all. Indeed only in one place does he definitely bring forward anything of his own to get over a difficulty which he finds in Euclid; this is where he tries to prove the parallel-postulate, after first giving Ptolemy's attempt and then pointing out objections to it. On the other hand, there are a number of passages in which he extols Euclid; thrice also he supports Euclid against Apollonius where the latter had given proofs which he considered better than Euclid's (1. 10, 11, and 23).

Allusion must be made to the debated question whether Proclus continued his commentaries beyond Book 1. His intention to do so is clear from the following passages. Just after the words above quoted about the trisection etc. of an angle by means of certain curves he says, "For we may perhaps more appropriately examine these things on the third book, where the writer of the Elements bisects a given circumference." Again, after saying that of all parallelograms which have the same perimeter the square is the greatest "and the rhomboid least of all," he adds: "But this we will prove in another place; for it is more appropriate to the (discussion of the) hypotheses of the second

book." Lastly, when alluding (on I. 45) to the squaring of the circle, and to Archimedes' proposition that any circle is equal to the right-angled triangle in which the perpendicular is equal to the radius of the circle and the base to its perimeter, he adds, "But of this elsewhere" ; this may imply an intention to treat of the subject on Eucl. XII., though Heiberg doubts it. But it is clear that, at the time when the commentary on Book 1. was written, Proclus had not yet begun to write on the other Books and was uncertain whether he would be able to do so: for at the end he says, "For my part, if I should be able to discuss the other books in the same manner, I should give thanks to the gods; but, if other cares should draw me away, I beg those who are attracted by this subject to complete the exposition of the other books as well, following the same method, and addressing themselves throughout to the deeper and better defined questions involved" (τὸ πραγματειῶδες πανταχοῦ καὶ εὐδιαίρετον μεταδιώκοντας).

There is in fact no satisfactory evidence that Proclus did actually write any more commentaries than that on Book 1. The contrary view receives support from two facts pointed out by Heiberg, viz. (1) that the scholiast's copy of Proclus was not so much better than our MSS. as to suggest that the scholiast had further commentaries of Proclus which have vanished for us; (2) that there is no trace in the scholia of the notes which Proclus promised in the passages quoted above.

Coming now to the question of the sources of Proclus, we may say that everything goes to show that his commentary is a compilation, though a compilation "in the better sense" of the term. He does not even give us to understand that we shall find in it much of his own; "let us," he says, "now turn to the exposition of the theorems proved by Euclid, selecting the more subtle of the comments made on them by the ancient writers, and cutting down their interminable diffuseness..." : not a word about anything of his

own. At the same time, he seems to imply that he will not necessarily on each occasion quote the source of each extract from an earlier commentary; and, in fact, while he quotes the name of his authority in many places, especially where the subject is important, in many others, where it is equally certain that he is not giving anything of his own, he mentions no authority. Thus he quotes Heron by name six times; but we now know, from the commentary of an-Nairīzī, that a number of other passages, where he mentions no name, are taken from Heron, and among them the not unimportant addition of an alternative proof to 1. 19. Hence we can by no means conclude that, where no authority is mentioned, Proclus is giving notes of his own. The presumption is génerally the other way; and it is often possible to arrive at a conclusion, either that a particular note is not Proclus' own, or that it is definitely attributable to someone else, by applying the ordinary principles of criticism. Thus, where the note shows an unmistakable affinity to another which Proclus definitely attributes to some commentator by name, especially when both contain some peculiar and distinctive idea, we cannot have much doubt in assigning both to the same commentator. Again, van Pesch finds a criterion in the form of a note, where the explanation is so condensed as to be only just intelligible; the note is that in which a converse of 1. 32 is proved the proposition namely that a rectilineal figure which has all its interior angles together equal to two right angles is a triangle.

It is not safe to attribute a passage to Proclus himself because he uses the first person in such expressions as "I say" or "I will prove" — for he was in the habit of putting into his own words the substance of notes borrowed from others — nor because, in speaking of an objection raised to a particular proposition, he uses such expressions as "perhaps someone may object" (ἴσως δ' ἄν τινες ἐνσταῖεν...): for sometimes other words in the same passage indicate that the objection had actually been taken by someone. Speaking

generally, we shall not be justified in concluding that Proclus is stating something new of his own unless he indicates this himself in express terms.

As regards the form of Proclus' references to others by name, van Pesch notes that he very seldom mentions the particular work from which he is borrowing. If we leave out of account the references to Plato's dialogues, there are only the following references to books: the Bacchae of Philolaus, the Symmikta of Porphyry, Archimedes On the Sphere and Cylinder, Apollonius On the cochlias, a book by Eudemus on The Angle, a whole book of Posidonius directed against Zeno of the Epicurean sect, Carpus' Astronomy, Eudemus' History of Geometry, and a tract by Ptolemy on the parallel-postulate.

Again, Proclus does not always indicate that he is quoting something at second-hand. He often does so, e.g. he quotes Heron as the authority for a statement about Philippus, Eudemus as attributing a certain theorem to Oenopides etc.; but he says on 1. 12 that "Oenopides first investigated this problem, thinking it useful for astronomy" when he cannot have had Oenopides' work before him.

It has been said above that Proclus was in the habit of stating in his own words the substance of the things which he borrowed. We are prepared for this when we find him stating that he will select the best things from ancient commentaries and "cut short their interminable diffuseness," that he will "briefly describe" (συντομως ἱστορῆσαι) the other proofs of 1. 20 given by Heron and Porphyry and also the proofs of 1. 25 by Menelaus and Heron. But the best evidence is of course to be found in the passages where he quotes works still extant, e.g. those of Plato, Aristotle and Plotinus. Examination of these passages shows great divergences from the original; even where he purports to quote textually, using the expressions "Plato says," or "Plotinus says," he by no means quotes word for word. In fact, he seems to have had a

positive distaste for quoting textually from other works. He cannot conquer this even when quoting from Euclid; he says in his note on 1. 22, “we will follow the words of the geometer” but fails, nevertheless, to reproduce the text of Euclid unchanged.

We now come to the sources themselves from which Proclus drew in writing his commentary. Three have already been disposed of, viz. Heron, Porphyry and Pappus, who had all written commentaries on the Elements. We go on to

Eudemus, the pupil of Aristotle, who, among other works, wrote a history of arithmetic, a history of astronomy, and a history of geometry. The importance of the last mentioned work is attested by the frequent use made of it by ancient writers. That there was no other history of geometry written after the time of Eudemus seems to be proved by the remark of Proclus in the course of his famous summary: “Those who compiled histories bring the development of this science up to this point. Not much younger than these is Euclid....” The loss of Eudemus’ history is one of the gravest which fate has inflicted upon us, for it cannot be doubted that Eudemus had before him a number of the actual works of earlier geometers, which, as before observed, seem to have vanished completely when they were superseded by the treatises of Euclid, Archimedes and Apollonius. As it is, we have to be thankful for the fragments from Eudemus which such writers as Proclus have preserved to us.

I agree with van Pesch that there is no sufficient reason for doubting that the work of Eudemus was accessible to Proclus at first hand. For the later writers Simplicius and Eutocius refer to it in terms such as leave no room for doubt that they had it before them. I have already quoted a passage from Simplicius’ account of the lunes of Hippocrates to the effect that Eudemus must be considered the best authority since he lived nearer the times. In the

same place Simplicius says, “I will set out what Eudemus says word for word (κατὰ λέξιν λεγόμενα), adding only a little explanation in the shape of reference to Euclid’s Elements owing to the memorandum-like style of Eudemus (διὰ τὸν ὑπομνηματικὸν τρόπον τοῦ Εὐδήμου) who sets out his explanations in the abbreviated form usual with ancient writers. Now in the second book of the history of geometry he writes as follows.” It is not possible to suppose that Simplicius would have written in this way about the style of Eudemus if he had merely been copying certain passages second-hand out of some other author and had not the original work itself to refer to. In like manner, Eutocius speaks of the paralogisms handed down in connexion with the attempts of Hippocrates and Antiphon to square the circle, “with which I imagine that those are accurately acquainted who have examined (ἐπεσκεμμένους) the geometrical history of Eudemus and know the Ceria Aristotelica.” How could the contemporaries of Eutocius have examined the work of Eudemus unless it was still extant in his time?

The passages in which Proclus quotes Eudemus by name as his authority are as follows:

(1) On 1. 26 he says that Eudemus in his history of geometry referred this theorem to Thales, inasmuch as it was necessary to Thales’ method of ascertaining the distance of ships from the shore.

(2) Eudemus attributed to Thales the discovery of Eucl. 1. 15, and

(3) to Oenopides the problem of 1. 23.

(4) Eudemus referred the discovery of the theorem in 1. 32 to the Pythagoreans, and gave their proof of it, which Proclus reproduees.

(5) On 1. 44 Proclus tells us that Eudemus says that “these things are ancient, being discoveries of the Pythagorean muse, the application (παραβολή) of areas, their exceeding (ὑπερβολή) and their falling short (ἔλλειψις).” The next words about the appropriation of these terms (parabola,

hyperbola and ellipse) by later writers (i.e. Apollonius) to denote the conic sections are of course not due to Eudemus.

Coming now to notes where Eudemus is not named by Proclus, we may fairly conjecture, with van Pesch, that Eudemus was really the authority for the statements (1) that Thales first proved that a circle is bisected by its diameter (though the proof by reductio ad absurdum which follows in Proclus cannot be attributed to Thales), (2) that "Plato made over to Leodamas the analytical method, by means of which it is recorded (ἱστόρηται) that the latter too made many discoveries in geometry," (3) that the theorem of 1. 5 was due to Thales, and that for equal angles he used the more archaic expression "similar" angles, (4) that Oenopides first investigated the problem of 1. 12, and that he called the perpendicular the gnomonic line (κατὰ γνώμονα), (5) that the theorem that only three sorts of polygons can fill up the space round a point, viz. the equilateral triangle, the square and the regular hexagon, was Pythagorean. Eudemus may also be the authority for Proclus' description of the two methods, referred to Plato and Pythagoras respectively, of forming right-angled triangles in whole numbers.

We cannot attribute to Eudemus the beginning of the note on 1. 47 where Proclus says that "if we listen to those who like to recount ancient history, we may find some of them referring this theorem to Pythagoras and saying that he sacrificed an ox in honour of his discovery." As such a sacrifice was contrary to the Pythagorean tenets, and Eudemus could not have been unaware of this, the story cannot rest on his authority. Moreover Proclus speaks as though he were not certain of the correctness of the tradition; indeed, so far as the story of the sacrifice is concerned, the same thing is told of Thales in connexion with his discovery that the angle in a semicircle is a right angle, and Plutarch is not certain whether the ox was sacrificed on the discovery of I. 47 or of the problem about application of areas. Plutarch's

doubt suggests that he knew of no evidence for the story beyond the vague allusion in the distich of Apollodorus "Logisticus" (the "calculator") cited by Diogenes Laertius also; and Proclus may have had in mind this couplet with the passages of Plutarch.

We come now to the question of the famous historical summary given by Proclus. No one appears to maintain that Eudemus is the author of even the early part of this summary in the form in which Proclus gives it. It is, as is well known, divided into two distinct parts, between which comes the remark, "Those who compiled histories bring the development of this science up to this point. Not much younger than these is Euclid, who put together the Elements, collecting many of the theorems of Eudoxus, perfecting many others by Theaetetus, and bringing to irrefragable demonstration the things which had only been somewhat loosely proved by his predecessors." Since Euclid was later than Eudemus, it is impossible that Eudemus can have written this. Yet the style of the summary after this point does not show any such change from that of the former portion as to suggest different authorship. The author of the earlier portion recurs frequently to the question of the origin of the elements of geometry in a way in which no one would be likely to do who was not later than Euclid; and it must be the same hand which in the second portion connects Euclid's Elements with the work of Eudoxus and Theaetetus.

If then the summary is the work of one author, and that author not Eudemus, who is it likely to have been? Tannery answers that it is Geminus; but I think, with van Pesch, that he has failed to show why it should be Geminus rather than another. And certainly the extracts which we have from Geminus' work suggest that the sort of topics which it dealt with was quite different; they seem rather to have been general questions of the content of

mathematics, and even Tannery admits that historical details could only have come incidentally into the work.

Could the author have been Proclus himself? Circumstances which seem to suggest this possibility are (1) that, as already stated, the question of the origin of the Elements is kept prominent, (2) that there is no mention of Democritus, whom Eudemus would not be likely to have ignored, while a follower of Plato would be likely enough to do him the injustice, following the example of Plato who was an opponent of Democritus, never once mentions him, and is said to have wished to burn all his writings, and (3) the allusion at the beginning to the "inspired Aristotle" (ὁ δαιμονιος Ἀριστοτέλης), though this may easily have been inserted by Proclus in a quotation made by him from someone else. On the other hand there are considerations which suggest that Proclus himself was not the writer. (1) The style of the whole passage is not such as to point to him as the author. (2) If he wrote it, it is hardly conceivable that he would have passed over in silence the discovery of the analytical method, the invention of Plato to which he attached so much importance.

There is nothing improbable in the conjecture that Proclus quoted the summary from a compendium of Eudemus' history made by some later writer: but as yet the question has not been definitely settled. All that is certain is that the early part of the summary must have been made up from scattered notices found in the great work of Eudemus.

Proclus refers to another work of Eudemus besides the history, viz. a book on The Angle (βιβλίον περὶ γωνίας). Tannery assumes that this must have been part of the history, and uses this assumption to confirm his idea that the history was arranged according to subjects, not according to chronological order. The phraseology of Proclus however unmistakably suggests a separate work; and that the history was chronologically arranged seems to be clearly

indicated by the remark of Simplicius that Eudemus "also counted Hippocrates among the more ancient writers" (ἐν τοῖς παλαιοτέροις）.

The passage of Simplicius about the lunes of Hippocrates throws considerable light on the style of Eudemus' history. Eudemus wrote in a memorandum-like or summary manner (τὸν ὑπομνηματικὸν τρόπον τοῦ Εὐδήμου） when reproducing what he found in the ancient writers; sometimes it is clear that he left out altogether proofs or constructions of things by no means easy.

Geminus.

The discussions about the date and birthplace of Geminus form a whole literature, as to which I must refer the reader to Manitius and Tittel. Though the name looks like a Latin name (Gem[icaron]nus), Manitius concluded that, since it appears as Γεμῖνος in all Greek MSS. and as Γεμεῖνος in some inscriptions, it is Greek and possibly formed from γεμ as Ἐργῖνος is from ἐργ and Ἀλεξῖνος from ἀλεξ (cf. also Ἰκτῖνος, Κρατῖνος). Tittel is equally positive that it is Gem[icaron]nus and suggests that Γεμῖνος is due to a false analogy with Ἀλεξῖνος etc. and Γεμεῖνος wrongly formed on the model of Ἀντωνεῖνος, Ἀγριππεῖνα. Geminus, a Stoic philosopher, born probably in the island of Rhodes, was the author of a comprehensive work on the classification of mathematics, and also wrote, about 73-67 B.C., a not less comprehensive commentary on the meteorological textbook of his teacher Posidonius of Rhodes.

It is the former work in which we are specially interested here. Though Proclus made great use of it, he does not mention its title, unless we may suppose that, in the passage (, 24) where, after quoting from Geminus a classification of lines which never meet, he says, "these remarks I have selected from the φιλοκαλία of Geminus," φιλοκαλία is a title or an alternative title. Pappus however quotes a work of Geminus "on the

classification of the mathematics" (ἐν τῷ περὶ τῆς τῶν μαθημάτων τάξεως), while Eutocius quotes from "the sixth book of the doctrine of the mathematics" (ἐν τῷ ἕκτῳ τῆς τῶν μαθημάτων θεωρίας). Tannery pointed out that the former title corresponds well enough to the long extract which Proclus gives in his first prologue, and also to the fragments contained in the Anonymi variae collectiones published by Hultsch at the end of his edition of Heron; but it does not suit most of the óther passages borrowed by Proclus. The correct title was therefore probably that given by Eutocius, The Doctrine, or Theory, of the Mathematics; and Pappus probably refers to one particular portion of the work, say the first Book. If the sixth Book treated of conics, as we may conclude from Eutocius, there must have been more Books to follow, because Proclus has preserved us details about higher curves, which must have come later. If again Geminus finished his work and wrote with the same fulness about the other branches of mathematics as he did about geometry, there must have been a considerable number of Books altogether. At all events it seems to have been designed to give a complete view of the whole science of mathematics, and in fact to be a sort of encyclopaedia of the subject.

I shall now indicate first the certain, and secondly the probable, obligations of Proclus to Geminus, in which task I have only to follow van Pesch, who has embodied the results of Tittel's similar inquiry also. I shall only omit the passages as regards which a case for attributing them to Geminus does not seem to me to have been made out.

First come the following passages which must be attributed to Geminus, because Proclus mentions his name:

(1) (In the first prologue of Proclus) on the division of mathematical sciences into arithmetic, geometry, mechanics, astronomy, optics, geodesy,

canonic (science of musical harmony), and logistic (apparently arithmetical problems);

(2) (in the note on the definition of a straight line) on the classification of lines (including curves) as simple (straight or circular) and mixed, composite and incomposite, uniform (ὁμοιομερεῖς) and non-uniform (ἀνομοιομερεῖς), lines "about solids" and lines produced by cutting solids, including conic and spiric sections;

(3) (in the note on the definition of a plane surface) on similar distinctions extended to surfaces and solids;

(4) (in the note on the definition of parallels) on lines which do not meet (ἀσύμπτωτοι) but which are not on that account parallel, e.g. a curve and its asymptote, showing that the property of not meeting does not make lines parallel — a favourite observation of Geminus — and, incidentally, on bounded lines or those which enclose a figure and those which do not;

(5) (in the same note) the definition of parallels given by Posidonius;

(6) on the distinction between postulates and axioms, the futility of trying to prove axioms, as Apollonius tried to prove Axiom 1, and the equal incorrectness of assuming what really requires proof, "as Euclid did in the fourth postulate [equality of right angles] and in the fifth postulate [the parallel-postulate]" ;

(7) on Postulates 1, 2, 3, which Geminus makes depend on the idea of a straight line being described by the motion of a point;

(8) (in the note on Postulate 5) on the inadmissibility in geometry of an argument which is merely plausible, and the danger in this particular case owing to the existence of lines which do converge ad infinitum and yet never meet;

(9) (in the note on I. 1) on the subject-matter of geometry, theorems, problems and διορισμοί (conditions of possibility) for problems;

(10) (in the note on 1. 5) on a generalisation of 1. 5 by Geminus through the substitution for the rectilineal base of "one uniform line (curve)," by means of which he proved that the only "uniform lines" (alike in all their parts) are a straight line, a circle, and a cylindrical helix;

(11) (in the note on I. 10) on the question whether a line is made up of indivisible parts (ἀμερῆ), as affecting the problem of bisecting a given straight line;

(12) (in the note on I. 35) on topical, or locus-theorems, where the illustration of the equal parallelograms described between a hyperbola and its asymptotes may also be due to Geminus.

Other passages which may fairly be attributed to Geminus, though his name is not mentioned, are the following:

(1) in the prologue, where there is the same allusion as in the passage (8) above to a remark of Aristotle that it is equally absurd to expect scientific proofs from a rhetorician and to accept mere plausibilities from a geometer;

(2) a passage in the prologue about the subject-matter, methods, and bases of geometry, the latter including axioms and postulates;

(3) another on the definition and nature of elemenis;

(4) a remark on the Stoic use of the term axiom for every simple statement (ἀπόφανσις ἁπλῆ）;

(5) another discussion on theorems and problems in the middle of which however there are some sentences by Proclus himself.

(6) another passage, in connexion with Def. 3, on lines including or not including a figure (with which cf. part of the passage (4) above);

(7) a classification of different sorts of angles according as they are contained by simple or mixed lines (or curves);

(8) a similar classification of figures, and of plane figures;

(9) Posidonius' definition of a figure;

(10) a classification of triangles into seven kinds;

(11) a note distinguishing lines (or curves) producible indefinitely or not so producible, whether forming a figure or not forming a figure (like the "single-turn spiral");

(12) passages distinguishing different sorts of problems, different sorts of theorems, and two sorts of converses (complete and partial);

(13) the definition of the term "porism" as used in the title of Euclid's Porisms, as distinct from the other meaning of "corollary" ;

(14) a note on the Epicurean objection to I. 20 as being obvious even to an ass;

(15) a passage on the properties of parallels, with allusions to Apollonius' Conics, and the curves invented by Nicomedes, Hippias and Perseus;

(16) a passage on the parallel-postulate regarded as the converse of I. 17.

Of the authors to whom Proclus was indebted in a less degree the most important is Apollonius of Perga. Two passages allude to his Conics, one to a work on irrationals, and two to a treatise On the cochlias (apparently the cylindrical helix) by Apollonius. But more important for our purpose are six references to Apollonius in connexion with elementary geometry.

(1) He appears as the author of an attempt to explain the idea of a line (possessing length but no breadth) by reference to daily experience, e.g. when we tell someone to measure, merely, the length of a road or of a wall; and doubtless the similar passage showing how we may in like manner get a notion of a surface (without depth) is his also.

(2) He gave a new general definition of an angle.

(3) He tried to prove certain axioms, and Proclus gives his attempt to prove Axiom I, word for word.

Proclus further quotes:

(4) Apollonius' solution of the problem in Eucl. I. 10, avoiding Euclid's use of I. 9,

(5) his solution of the problem in I. 11, differing only slightly from Euclid's, and

(6) his solution of the problem in I. 23.

Heiberg conjectures that Apollonius departed from Euclid's method in these propositions because he objected to solving problems of a more general, by means of problems of a more particular, character. Proclus however considers all three solutions inferior to Euclid's; and his remarks on Apollonius' handling of these elementary matters generally suggest that he was nettled by criticisms of Euclid in the work containing the things which he quotes from Apollonius, just as we conclude that Pappus was offended by the remarks of Apollonius about Euclid's incomplete treatment of the "three- and four-line locus." If this was the case, Proclus can hardly have got his information about these things at second-hand; and there seems to be no reason to doubt that he had the actual work of Apollonius before him. This work may have been the treatise mentioned by Marinus in the words "Apollonius in his general treatise" (Ἀπολλώνιος ἐν τῇ καθόλου πραγματείᾳ). If the notice in the Fihrist stating, on the authority of Thābit b. Qurra, that Apollonius wrote a tract on the parallel-postulate be correct, it may have been included in the same work. We may conclude generally that, in it, Apollonius tried to remodel the beginnings of geometry, reducing the number of axioms, appealing, in his definitions of lines, surfaces etc., more to experience than to abstract reason, and substituting for certain proofs others of a more general character.

The probabilities are that, in quoting from the tract of Ptolemy in which he tried to prove the parallel-postulate, Proclus had the actual work before him. For, after an allusion to it as "a certain book" he gives two long extracts, and

at the beginning of the second indicates the title of the tract, "in the (book) about the meeting of straight lines produced from (angles) less than two right angles," as he has very rarely done in other cases.

Certain things from Posidonius are evidently quoted at secondhand, the authority being Geminus (e.g. the definitions of figure and parallels); but besides these we have quotations from a separate work which he wrote to controvert Zeno of Sidon, an Epicurean who had sought to destroy the whole of geometry. We are told that Zeno had argued that, even if we admit the fundamental principles (ἀρχαί) of geometry, the deductions from them cannot be proved without the admission of something else as well, which has not been included in the said principles. On I. 1 Proclus gives at some length the arguments of Zeno and the reply of Posidonius as regards this proposition. In this case Zeno's "something else" which he considers to be assumed is the fact that two straight lines cannot have a common segment, and then, as regards the "proof" of it by means of the bisection of a circle by its diameter, he objects that it has been assumed that two circumferences (arcs) of circles cannot have a common part. Lastly, he makes up, for the purpose of attacking it, another supposed "proof" of the fact that two straight lines cannot have a common part. Proclus appears, more than once, to be quoting the actual words of Zeno and Posidonius; in particular, two expressions used by Posidonius about "the acrid Epicurean" (τὸν δριμὺν Ἐπικούρειον） and his "misrepresentations" (Ποσειδώνιός φησι τὸν Ζήνωνα συκοφαντεῖν）. It is not necessary to suppose that Proclus had the original work of Zeno before him, because Zeno's arguments may easily have been got from Posidonius' reply; but he would appear to have quoted direct from the latter at all events.

The work of Carpus mechanicus (a treatise on astronomy) quoted from by Proclus must have been accessible to him at first-hand, because a portion of the extract from it about the relation of theorems and problems is reproduced

word for word. Moreover, if he were not using the book itself, Proclus would hardly be in a position to question whether the introduction of the subject of theorems and problems was opportune in the place where it was found (εἰ μὲν κατὰ καιρὸν ἢ μή, παρείσθω πρὸς τὸ παρόν）.

It is of course evident that Proclus had before him the original works of Plato, Aristotle, Archimedes and Plotinus, as well as the Συμμικτά of Porphyry and the works of his master Syrianus (ὁ ἡμέτερος καθηγεμών）, from whom he quotes in his note on the definition of an angle. Tannery also points out that he must have had before him a group of works representing the Pythagorean tradition on its mystic, as distinct from its mathematical, side, from Philolaus downwards, and comprising the more or less apocryphal ἱερὸς λόγος of Pythagoras, the Oracles (λόγια), and Orphic verses.

Besides quotations from writers whom we can identify with more or less certainty, there are many other passages which are doubtless quoted from other commentators whose names we do not know. A list of such passages is given by van Pesch, and there is no need to cite them here.

Van Pesch also gives at the end of his work a convenient list of the books which, as the result of his investigation, he deems to have been accessible to and directly used by Proclus. The list is worth giving here, on the same ground of convenience. It is as follows: “Eudemus: history of geometry. Geminus: the theory of the mathematical sciences. Heron: commentary on the Elements of Euclid. Porphyry: commentary on the Elements of Euclid. Pappus: commentary on the Elements of Euclid. Apollonius of Perga: a work relating to elementary geometry. Ptolemy: on the parallel-postulate. Posidonius: a book controverting Zeno of Sidon. Carpus: astronomy. Syrianus: a discussion on the angle. Pythagorean philosophical tradition. Plato’s works. Aristotle’s works. Archimedes’ works. Plotinus: Enneades.”

Lastly we come to the question what passages, if any, in the commentary of Proclus represent his own contributions to the subject. As we have seen, the onus probandi must be held to rest upon him who shall maintain that a particular note is original on the part of Proclus. Hence it is not enough that it should be impossible to point to another writer as the probable source of a note; we must have a positive reason for attributing it to Proclus. The criterion must therefore be found either (1) in the general terms in which Proclus points out the deficiencies in previous commentaries and indicates the respects in which his own will differ from them, or (2) in specific expressions used by him in introducing particular notes which may indicate that he is giving his own views. Besides indicating that he paid more attention than his predecessors to questions requiring deeper study (τὸ πραγματειῶδες) and "pursued clear distinctions" (τὸ εὐδιαίρετον μεταδιώκοντας） — by which he appears to imply that his predecessors had confused the different departments of their commentaries, viz. lemmas, cases, and objections (ἐνστάσεις） — Proclus complains that the earlier commentators had failed to indicate the ultimate grounds or causes of propositions. Although it is from Geminus that he borrowed a passage maintaining that it is one of the proper functions of geometry to inquire into causes (τὴν αἰτίαν καὶ τὸ διὰ τί）, yet it is not likely that Geminus dealt with Euclid's propositions one by one; and consequently, when we find Proclus, on I. 8, 16, 17, 18, 32, and 47, endeavouring to explain causes, we have good reason to suppose that the explanations are his own.

Again, his remarks on certain things which he quotes from Pappus can scarcely be due to anyone else, since Pappus is the latest of the commentators whose works he appears to have used. Under this head come

(1) his objections to certain new axioms introduced by Pappus,

(2) his conjecture as to how Pappus came to think of his alternative proof of I. 5,

(3) an addition to Pappus' remarks about the curvilineal angle which is equal to a right angle without being one.

The defence of Geminus against Carpus, who combated his view of theorems and problems, is also probably due to Proclus, as well as an observation on I. 38 to the effect that I. 35-38 are really comprehended in VI. 1 as particular cases.

Lastly, we can have no hesitation in attributing to Proclus himself (1) the criticism of Ptolemy's attempt to prove the parallel-postulate, and (2) the other attempted proof given ín the same note (on I. 29) and assuming as an axiom that "if from one point two straight lines forming an angle be produced ad infinitum the distance between them when so produced ad infinitum exceeds any finite magnitude (i.e. length)," an assumption which purports to be the equivalent of a statement in Aristotle. It is introduced by words in which the writer appears to claim originality for his proof: "To him who desires to see this proved (κατασκευαζόμενον) let it be said by us (λεγέσθω παῤ η<*>μῶν）" etc. Moreover, Philoponus, in a note on Aristotle's Anal. post. I. 10, says that "the geometer (Euclid) assumes this as an axiom, but it wants a great deal of proof, insomuch that both Ptolemy and Proclus wrote a whole book upon it."

CHAPTER V. THE TEXT.

It is well known that the title of Simson's edition of Euclid (first brought out in Latin and English in 1756) claims that, in it, "the errors by which Theon, or others, have long ago vitiated these books are corrected, and some of Euclid's demonstrations are restored" ; and readers of Simson's notes are familiar with the phrases used, where anything in the text does not seem to him satisfactory, to the effect that the demonstration has been spoiled, or things have been interpolated or omitted, by Theon "or some other unskilful editor." Now most of the MSS. of the Greek text prove by their titles that they proceed from the recension of the Elements by Theon; they purport to be either "from the edition of Theon" (ἐκ τῆς Θέωνος ἐκδόσεως) or "from the lectures of Theon" (ἀπὸ συνουσιῶν τοῦ Θέωνος). This was Theon of Alexandria (4th c. A.D.) who also wrote a commentary on Ptolemy, in which there occurs a passage of the greatest importance in this connexion: "But that sectors in equal circles are to one another as the angles on which they stand has been proved by me in my edition of the Elements at the end of the sixth book." Thus Theon himself says that he edited the Elements and also that the second part of VI. 33, found in nearly all the MSS., is his addition.

This passage is the key to the whole question of Theon's changes in the text of Euclid; for, when Peyrard found in the Vatican the MS. 190 which contained neither the words from the titles of the other MSS. quoted above nor the interpolated second part of VI. 33, he was justified in concluding, as he did, that in the Vatican MS. we have an edition more ancient than Theon's. It is also clear that the copyist of P, or rather of its archetype, had before him the two recensions and systematically gave the preference to the earlier one; for at XIII. 6 in P the first hand has added a note in the margin: "This theorem is not given in most copies of the new edition, but is found in those of the old." Thus we are more fortunate than Simson, since our

judgment of Theon's recension can be formed on the basis, not of mere conjecture, but of the documentary evidence afforded by a comparison of the Vatican MS. just mentioned with what we may conveniently call, after Heiberg, the Theonine MSS.

The MSS. used for Heiberg's edition of the Elements are the following:

(1) P = Vatican MS. numbered 190, 4to, in two volumes (doubtless one originally); 10th c.

This is the MS. which Peyrard was able to use; it was sent from Rome to Paris for his use and bears the stamp of the Paris Imperial Library on the last page. It is well and carefully written. There are corrections some of which are by the original hand, but generally in paler ink, others, still pretty old, by several different hands, or by one hand with different ink in different places (P m. 2), and others again by the latest hand (P m. rec.). It contains, first, the Elements I. — XIII. with scholia, then Marinus' commentary on the Data (without the name of the author), followed by the Data itself and scholia, then the Elements XIV., XV. (so called), and lastly three books and a part of a fourth of a commentary by Theon εἰς τοὺς προχείρους κανόνας Πτολεμαίου.

The other MSS. are "Theonine."

(2) F = MS. XXVIII, 3, in the Laurentian Library at Florence, 4to; 10th c.

This MS. is written in a beautiful and scholarly hand and contains the Elements I. — XV., the Optics and the Phaenomena, but is not well preserved. Not only is the original writing renewed in many places, where it had become faint, by a later hand of the 16th c., but the same hand has filled certain smaller lacunae by gumming on to torn pages new pieces of parchment, and has replaced bodily certain portions of the MS., which had doubtless become illegible, by fresh leaves. The larger gaps so made good extend from Eucl. VII. 12 to IX. 15, and from XII. 3 to the end; so that, besides the conclusion of the Elements, the Optics and Phaenomena are also

in the later hand, and we cannot even tell what in addition to the Elements I. — XIII. the original MS. contained. Heiberg denotes the later hand by φ and observes that, while in restoring words which had become faint and filling up minor lacunae the writer used no other MS., yet in the two larger restorations he used the Laurentian MS. XXVIII, 6, belonging to the 13th — 14th c. The latter MS. (which Heiberg denotes by f) was copied from the Viennese MS. (V) to be described below.

(3) B = Bodleian MS., D'Orville X. 1 inf. 2, 30, 4to; A.D. 888.

This MS. contains the Elements I. — XV. with many scholia. Leaves 15-118 contain I. 14 (from about the middle of the proposition) to the end of Book VI., and leaves 123-387 (wrongly numbered 397) Books VII. — XV. in one and the same elegant hand (9th c.). The leaves preceding leaf 15 seem to have been lost at some time, leaves 6 to 14 (containing Elem. I. to the place in I. 14 above referred to) being carelessly written by a later hand on thick and common parchment (13th c.). On leaves 2 to 4 and 122 are certain notes in the hand of Arethas, who also wrote a two-line epigram on leaf 5, the greater part of the scholia in uncial letters, a few notes and corrections, and two sentences on the last leaf, the first of which states that the MS. was written by one Stephen clericus in the year of the world 6397 (= 888 A.D.), while the second records Arethas' own acquisition of it. Arethas lived from, say, 865 to 939 A.D. He was Archbishop of Caesarea and wrote a commentary on the Apocalypse. The portions of his library which survive are of the greatest interest to palaeography on account of his exact notes of dates, names of copyists, prices of parchment etc. It is to him also that we owe the famous Plato MS. from Patmos (Cod. Clarkianus) which was written for him in November 895.

(4) V = Viennese MS. Philos. Gr. No. 103; probably 12th c.

This MS. contains 292 leaves, Eucl. Elements I. — XV. occupying leaves 1 to 254, after which come the Optics (to leaf 271), the Phaenomena (mutilated at the end) from leaf 272 to leaf 282, and lastly scholia, on leaves 283 to 292, also imperfect at the end. The different material used for different parts and the varieties of handwriting make it necessary for Heiberg to discuss this MS. at some length. The handwriting on leaves 1 to 183 (Book I. to the middle of X. 105) and on leaves 203 to 234 (from XI. 31, towards the end of the proposition, to XIII. 7, a few lines down) is the same; between leaves 184 and 202 there are two varieties of handwriting, that of leaves 184 to 189 and that of leaves 200 (verso) to 202 being the same. Leaf 235 begins in the same handwriting, changes first gradually into that of leaves 184 to 189 and then (verso) into a third more rapid cursive writing which is the same as that of the greater part of the scholia, and also as that of leaves 243 and 282, although, as these leaves are of different material, the look of the writing and of the ink seems altered. There are corrections both by the first and a second hand, and scholia by many hands. On the whole, in spite of the apparent diversity of handwriting in the MS., it is probable that the whole of it was written at about the same time, and it may (allowing for changes of material, ink etc.) even have been written by the same man. It is at least certain that, when the Laurentian MS. XXVIII, 6 was copied from it, the whole MS. was in the condition in which it is now, except as regards the later scholia and leaves 283 to 292 which are not in the Laurentian MS., that MS. coming to an end where the Phaenomena breaks off abruptly in V. Hence Heiberg attributes the whole MS. to the 12th c.

But it was apparently in two volumes originally, the first consisting of leaves 1 to 183; and it is certain that it was not all copied at the same time or from one and the same original. For leaves 184 to 202 were evidently copied from two MSS. different both from one another and from that from which the

rest was copied. Leaves 184 to the middle of leaf 189 (recto) must have been copied from a MS. similar to P, as is proved by similarity of readings, though not from P itself. The rest, up to leaf 202, were copied from the Bologna MS. (b) to be mentioned below. It seems clear that the content of leaves 184 to 202 was supplied from other MSS. because there was a lacuna in the original from which the rest of V was copied.

Heiberg sums up his conclusions thus. The copyist of V first copied leaves 1 to 183 from an original in which two quaterniones were missing (covering from the middle of Eucl. X. 105 to near the end of XI. 31). Noticing the lacuna he put aside one quaternio of the parchment used up to that point. Then he copied onwards from the end of the lacuna in the original to the end of the Phaenomena. After this he looked about him for another MS. from which to fill up the lacuna; finding one, he copied from it as far as the middle of leaf 189 (recto). Then, noticing that the MS. from which he was copying was of a different class, he had recourse to yet another MS. from which he copied up to leaf 202. At the same time, finding that the lacuna was longer than he had reckoned for, he had to use twelve more leaves of a different parchment in addition to the quaternio which he had put aside. The whole MS. at first formed two volumes (the first containing leaves 1 to 183 and the second leaves 184 to 282); then, after the last leaf had perished, the two volumes were made into one to which two more quaterniones were also added. A few leaves of the latter of these two have since perished.

(5) b = MS. numbered 18-19 in the Communal Library at Bologna, in two volumes, 4to; 11th c.

This MS. has scholia in the margin written both by the first hand and by two or three later hands; some are written by the latest hand, Theodorus Cabasilas (a descendant apparently of Nicolaus Cabasilas, 14th c.) who owned the MS. at one time. It contains (a) in 14 quaterniones the definitions

and the enunciations (without proofs) of the Elements I. — XIII. and of the Data, (b) in the remainder of the volumes the Proem to Geometry (published among the Variae Collectiones in Hultsch's edition of Heron, p, 24 to 274, 14) followed by the Elements I. — XIII. (part of XIII. 18 to the end being missing), and then by part of the Data (from the last three words of the enunciation of Pro to the end of the penultimate clause in Pro, ed. Menge). From XI. 36 inclusive to the end of XII. this MS. appears to represent an entirely different recension. Heiberg is compelled to give this portion of b separately in an appendix. He conjectures that it is due to a Byzantine mathematician who thought Euclid's proofs too long and tiresome and consequently contented himself with indicating the course followed. At the same time this Byzantine must have had an excellent MS. before him, probably of the ante-Theonine variety of which the Vatican MS. 190 (P) is the sole representative.

(6) p = Paris MS. 2466, 4to; 12th c.

This manuscript is written in two hands, the finer hand occupying leaves 1 to 53 (recto), and a more careless hand leaves 53 (verso) to 64, which are of the same parchment as the earlier leaves, and leaves 65 to 239, which are of a thinner and rougher parchment showing traces of writing of the 8th — 9th c. (a Greek version of the Old Testament). The MS. contains the Elements I. — XIII. and some scholia after Books XI., XII. and XIII.

(7) q = Paris MS. 2344, folio; 12th c.

It is written by one hand but includes scholia by many hands. On leaves 1 to 16 (recto) are scholia with the same title as that found by Wachsmuth in a Vatican MS. and relied upon by him to prove that Proclus continued his commentaries beyond Book I. Leaves 17 to 357 contain the Elements I. — XIII. (except that there is a lacuna from the middle of VIII. 25 to the ἔκθεσις

of IX. 14); before Books VII. and X. there are some leaves filled with scholia only, and leaves 358 to 366 contain nothing but scholia.

(8) Heiberg also used a palimpsest in the British Museum (Add. 17211). Five pages are of the 7th — 8th c. and are contained (leaves 49-53) in the second volume of the Syrian MS. Brit. Mus. 687 of the 9th c.; half of leaf 50 has perished. The leaves contain various fragments from Book X. enumerated by Heiberg, Vol. III., p. v, and nearly the whole of XIII. 14.

Since his edition of the Elements was published, Heiberg has collected further material bearing on the history of the text. Besides giving the results of further or new examination of MSS., he has collected the fresh evidence contained in an-Nairīzī's commentary, and particularly in the quotations from Heron's commentary given in it (often word for word), which enable us in several cases to trace differences between our text and the text as Heron had it, and to identify some interpolations which actually found their way into the text from Heron's commentary itself; and lastly he has dealt with some valuable fragments of ancient papyri which have recently come to light, and which are especially important in that the evidence drawn from them necessitates some modification in the views expressed in the preface to Vol. V. as to the nature of the changes made in Theon's recension, and in the principles laid down for differentiating between Theon's recension and the original text, on the basis of a comparison between P and the Theonine MSS. alone.

The fragments of ancient papyri referred to are the following.

1. Papyrus Herculanensis No. 1061.

This fragment quotes Def. 15 of Book I. in Greek, and omits the words ἣ καλεῖται περιφέρεια, "which is called the circumference," found in all our MSS., and the further addition πρὸς τὴν τοῦ κύκλου περιφέρειαν also found

in practically all the MSS. Thus Heiberg's assumption that both expressions are interpolations is now confirmed by this oldest of all sources.

2. The Oxyrhynchus Papyri I. , No. XXIX. of the 3rd or 4th c. This fragment contains the enunciation of Eucl. II. 5 (with figure, apparently without letters, immediately following, and not, as usual in our MSS., at the end of the proof) and before it the part of a word περιεχομε belonging to II. 4 (with room for -νῳ ὀρθογωνίῳ: ὅπερ ἔδει δεῖξαι and a stroke to mark the end), showing that the fragment had not the Porism which appears in all the Theonine MSS. and (in a later hand) in P, and thereby confirming Heiberg's assumption that the Porism was due to Theon.

3. A fragment in Fayum towns and their papyri, , No. IX. of 2nd or 3rd c.

This contains I. 39 and I. 41 following one another and almost complete, showing that I. 40 was wanting, whereas it is found in all the MSS. and is recognised by Proclus. Moreover the text of the beginning of I. 39 is better than ours, since it has no double διορισμός but omits the first ("I say that they are also in the same parallels") and has "and" instead of "for let AD be joined" in the next sentence. It is clear that I. 40 was interpolated by someone who thought there ought to be a proposition following I. 39 and related to it as I. 38 is related to I. 37 and I. 36 to I. 35, although Euclid nowhere uses I. 40, and therefore was not likely to include it. The same interpolator failed to realise that the words "let AD be joined" were part of the ἔκθεσις or setting-out, and took them for the κατασκευή or "construction" which generally follows the διορισμός or "particular statement" of the conclusion to be proved, and consequently thought it necessary to insert a διορισμός before the words.

The conclusions drawn by Heiberg from a consideration of particular readings in this papyrus along with those of our MSS. will be referred to below.

We now come to the principles which Heiberg followed, when preparing his edition, in differentiating the original text from the Theonine recension by means of a comparison of the readings of P and of the Theonine MSS. The rules which he gives are subject to a certain number of exceptions (mostly in cases where one MS. or the other shows readings due to copyists' errors), but in general they may be relied upon to give conclusive results.

The possible alternatives which the comparison of P with the Theonine MSS. may give in particular passages are as follows:

I. There may be agreement in three different degrees.

(1) P and all the Theonine MSS. may agree.

In this case the reading common to all, even if it is corrupt or interpolated, is more ancient than Theon, i.e. than the 4th c.

(2) P may agree with some (only) of the Theonine MSS.

In this case Heiberg considered that the latter give the true reading of Theon's recension, and the other Theonine MSS. have departed from it.

(3) P and one only of the Theonine MSS may agree.

In this case too Heiberg assumed that the one Theonine MS. which agrees with P gives the true Theonine reading, and that this rule even supplies a sort of measure of the quality and faithfulness of the Theonine MSS. Now none of them agrees alone with P in preserving the true reading so often as F. Hence F must be held to have preserved Theon's recension more faithfully than the other Theonine MSS.; and it would follow that in those portions where F fails us P must carry rather more weight even though it may differ from the Theonine MSS. BVpq. (Heiberg gives many examples in proof of this, as of his main rules generally, for which reference must be made to his Prolegomena in Vol. v.) The specially close relation of F and P is also illustrated by passages in which they have the same errors; the explanation of these common errors (where not due to accident) is found by Heiberg in the

supposition that they existed, but were not noticed by Theon, in the original copy in which he made his changes.

Although however F is by far the best of the Theonine MSS., there are a considerable number of passages where one of the others (B, V, p or q) alone with P gives the genuine reading of Theon's recension.

As the result of the discovery of the papyrus fragment containing I. 39, 41, the principles above enunciated under (2) and (3) are found by Heiberg to require some qualification. For there is in some cases a remarkable agreement between the papyrus and the Theonine MSS. (some or all) as against P. This shows that Theon took more trouble to follow older MSS., and made fewer arbitrary changes of his own, than has hitherto been supposed. Next, when the papyrus agrees with some of the Theonine MSS. against P, it must now be held that these MSS. (and not, as formerly supposed, those which agree with P) give the true reading of Theon. If it were otherwise, the agreement between the papyrus and the Theonine MSS. would be accidental: but it happens too often for this. It is clear also that there must have been contamination between the two recensions; otherwise, whence could the Theonine MSS. which agree with P and not with the papyrus have got their readings? The influence of the P class on the Theonine F is especially marked.

II. There may be disagreement between P and all the Theonine MSS.

The following possibilities arise.

(1) The Theonine MSS. differ also among themselves.

In this case Heiberg considered that P nearly always has the true reading, and the Theonine MSS. have suffered interpolation in different ways after Theon's time.

(2) The Theonine MSS. all combine against P.

In this case the explanation was assumed by Heiberg to be one or other of the following. “ （α) The common reading is due to an error which cannot be imputed to Theon (though it may have escaped him when putting together the archetype of his edition); such error may either have arisen accidentally in all alike, or (more frequently) may be referred to a common archetype of all the MSS.

“ “ （β) There may be an accidental error in P; e.g. something has dropped out of P in a good many places, generally through ὁμοιοτέλευτον

“ “ （γ) There may be words interpolated in P.

“ “ （δ) Lastly, we may have in the Theonine MSS. a change made by Theon himself.

“

(The discovery of the ancient papyrus showing readings agreeing with some, or with all, of the Theonine MSS. against P now makes it necessary to be very cautious in applying these criteria.)

It is of course the last class (δ) of changes which we have to investigate in order to get a proper idea of Theon’s recension.

Heiberg first observes, as regards these, that we shall find that Theon, in editing the Elements, altered hardly anything without some reason, often inadequate according to our ideas, but still some reason which seemed to him sufficient. Hence, in cases of very slight differences where both the Theonine MSS. and P have readings good and probable in themselves, Heiberg is not prepared to put the differences down to Theon. In those passages where we cannot see the least reason why Theon, if he had the reading of P before him, should have altered it, Heiberg would not at once assume the superiority of P unless there was such a consistency in the differences as would indicate that they were due not to accident but to design. In the absence of such indications, he thinks that the ordinary principles of criticism should be

followed and that proper weight should be attached to the antiquity of the sources. And it cannot be denied that the sources of the Theonine version are the more ancient. For not only is the British Museum palimpsest (L), which is intimately connected with the rest of our MSS., át least two centuries older than P, but the other Theonine MSS. are so nearly allied that they must be held to have had a common archetype intermediate between them and the actual edition of Theon; and, since they themselves are as old as, or older than P, their archetype must have been much older. Heiberg gives (pp. xlvi, xlvii) a list of passages where, for this reason, he has followed the Theonine MSS. in preference to P.

It has been mentioned above that the copyist of P or rather of its archetype wished to give an ancient recension. Therefore (apart from clerical errors and interpolations) the first hand in P may be relied upon as giving a genuine reading even where a correction by the first hand has been made at the same time. But in many places the first hand has made corrections afterwards; on these occasions he must have used new sources, e.g. when inserting the scholia to the first Book which P alone has, and in a number of passages he has made additions from Theonine MSS.

We cannot make out any “family tree” for the different Theonine MSS. Although they all proceeded from a common archetype later than the edition of Theon itself, they cannot have been copied one from the other; for, if they had been, how could it have come about that in one place or other each of them agrees alone with P in preserving the genuine reading? Moreover the great variety in their agreements and disagreements indicates that they have all diverged to about the same extent from their archetype. As we have seen that P contains corrections from the Theonine family, so they show corrections from P or other MSS. of the same family. Thus V has part of the lacuna in the MS. from which it was copied filled up from a MS. similar to P,

and has corrections apparently derived from the same; the copyist, however, in correcting V, also used another MS. to which he alludes in the additions to IX. 19 and 30 (and also on X. 23 Por.): “in the book of the Ephesian (this) is not found.” Who this Ephesian of the 12th c. was, we do not know.

We now come to the alterations made by Theon in his edition of the Elements. I shall indicate classes into which these alterations may be divided but without details (except in cases where they affect the mathematical content as distinct from form or language pure and simple)..

I. Alterations made by Theon where he found, or thought he found, mistakes in the original.

1. Real blots in the original which Theon saw and tried to remove.

（a) Euclid has a porism (corollary) to VI. 19, the enunciation of which speaks of similar and similarly described figures though the proposition itself refers only to triangles, and therefore the porism should have come after VI. 20. Theon substitutes triangle for figure and proves the more general porism after VI. 20.

（b) In IX. 19 there is a statement which is obviously incorrect. Theon saw this and altered the proof by reducing four alternatives to two, with the result that it fails to correspond to the enunciation even with Theon’s substitution of “if” for “when” in the enunciation.

（c) Theon omits a porism to IX. II, although it is necessary for the proof of the succeeding proposition, apparently because, owing to an error in the text (κατὰ τὸν corrected by Heiberg into ἐπὶ τὸ), he could not get out of it the right sense.

（d) I should also put into this category a case which Heiberg classifies among those in which Theon merely fancied that he found mistakes, viz. the porism to V. 7 stating that, if four magnitudes are proportional, they are proportional inversely. Theon puts this after V. 4 with a proof, which

however has no necessary connexion with V. 4 but is obvious from the definition of proportion.

（e) I should also put under this head XI. 1, where Euclid's argument to prove that two straight lines cannot have a common segment is altered.

2. Passages which seemed to Theon to contain blots, and which he therefore set himself to correct, though more careful consideration would have shown that Euclid's words are right or at least may be excused and offer no difficulty to an intelligent reader. Under this head come:

（a) an alteration in III. 24.

（b) a perfectly unnecessary alteration, in Vi. 14, of "equiangular parallelograms" into "parallelograms having one angle equal to one angle," where Theon followed the false analogy of VI. 15.

（c) an omission of words in V. 26, owing to his having been misled by a wrong figure.

（d) an alteration of the order of XI. Deff. 27, 28.

（e) the substitution of "parallelepipedal solid" for "cube" in XI. 38, because Theon observed, correctly enough, that it was true of the parallelepipedal solid in general as well as of the cube, but failed to give weight to the fact that Euclid must have given the particular case of the cube for the simple reason that that was all he wanted for use in XIII. 17.

（f) the substitution of the letter Φ for Ω (V for Z in my figure) because he saw that the perpendicular from K to BΦ would fall on Φ itself, so that Φ, Ω coincide. But, if the substitution is made, it should be proved that Φ, Ω coincide. Euclid can hardly have failed to notice the fact, but it may be that he deliberately ignored it as unnecessary for his purpose, because he did not want to lengthen his proposition by giving the proof.

II. Emendations intended to improve the form or diction of Euclid.

Some of these emendations of Theon affect passages of appreciable length. Heiberg notes about ten such passages; the longest is in Eucl. XII. 4 where a whole page of Heiberg's text is affected and Theon's version is put in the Appendix. The kind of alteration may be illustrated by that in IX. 15 where Euclid uses successively the propositions VII. 24, 25, quoting the enunciation of the former but not of the latter; Theon does exactly the reverse. In a few of the cases here quoted by Heiberg, Theon shortened the original somewhat.

But, as a rule, the emendations affect only a few words in each sentence. Sometimes they are considerable enough to alter the conformation of the sentence, sometimes they are trifling alterations "more magistellorum ineptorum" and unworthy of Theon. Generally speaking, they were prompted by a desire to change anything which was out of the common in expression or in form, in order to reduce the language to one and the same standard or norm. Thus Theon changed the order of words, substituted one word for another where the latter was used in a sense unusual with Euclid (e.g. ἐπειδήπερ, "since," for ὅτι in the sense of "because"), or one expression for another in like circumstances (e.g. where, finding "that which was enjoined would be done" in a theorem, VII. 31, and deeming the phrase more appropriate to a problem, he substituted for it "that which is sought would be manifest" ; probably also and for similar reasons he made certain variations between the two expressions usual at the end of propositions ὅπερ ἔδει δεῖξαι and ὅπερ ἔδει ποιῆσαι, quod erat demonstrandum and quod erat faciendum). Sometimes his alterations show carelessness in the use of technical terms, as when he uses ἅπτεσθαι (to meet) for ἐφάπτεσθαι (to touch) although the ancients carefully distinguished the two words. The desire of keeping to a standard phraseology also led Theon to omit or add words in a number of cases, and also, sometimes, to change the lettering of figures.

But Theon seems, in editing the Elements, to have bestowed the most attention upon

III. Additions designed to supplement or explain Euclid.

First, he did not hesitate to interpolate whole propositions where he thought there was room or use for them. We have already mentioned the addition to VI. 33 of the second part relating to sectors, for which Theon himself takes credit in his commentary on Ptolemy. Again, he interpolated the proposition commonly known as VII. 22 (ex aequo in proportione perturbata for numbers, corresponding to V. 23), and perhaps also VII. 20, a particular case of VII. 19 as VI. 17 is of VI. 16. He added a second case to VI. 27, a porism to II. 4, a second porism to III. 16, and a lemma after X. 12; perhaps also the porism to V. 19 and the first porism to VI. 20. He also inserted alternative proofs here and there, e.g. in II. 4 (where the alternative differs little from the original) and in VII. 31; perhaps also in X. 1, 6, and 9.

Secondly, he sometimes repeats an argument where Euclid had said "For the same reason," adds specific references to points, straight lines etc. in the figures in order to exclude the possibility of mistake arising from Euclid's reference to them in general terms, or inserts words to make the meaning of Euclid more plain, e.g. componendo and alternately, where Euclid had left them out. Sometimes he thought to increase by his additions the mathematical precision of Euclid's language in enunciations or elsewhere, sometimes to make smoother and clearer things which Euclid had expressed with unusual brevity and harshness or carelessness, in reliance on the intelligence of his readers.

Thirdly, he supplied intermediate steps where Euclid's argument seemed too rapid and not easy enough to follow. The form of these additions varies; they are sometimes placed as a definite intermediate step with "therefore" or "so that," sometimes they are additions to the statement of premisses,

sometimes phrases introduced by “since,” “for” and the like, after the inference.

Lastly, there is a very large class of additions of a word, or one or two words, for the sake of clearness or consistency. Heiberg gives a number of examples of the addition of such nouns as “triangle,” “square,” “rectangle,” “magnitude,” “number,” “point,” “side,” “circle,” “straight line,” “area” and the like, of adjectives such as “remaining,” “right,” “whole,” “proportional,” and of other parts of speech, even down to words like “is” (ἐστί) which is added 600 times, δή, ἄρα, μέν, γάρ, καί and the like.

IV. Omissions by Theon.

Heiberg remarks that, Theon’s object having been, as above shown, to amplify and explain Euclid, we should not natuially have expected to find him doing much in the contrary process of compression, and it is only owing to the recurrence of a certain sort of omissions so frequently (especially in the first Books) as to exclude the hypothesis of their being all due to chance that we are bound to credit him with alterations making for greater brevity. We have seen, it is true, that he made omissions as well as additions for the purpose of reducing the language to a certain standard form. But there are also a good number of cases where in the enunciation of propositions, and in the exposition (the re-statement of them with reference to the figure), he has left out words because, apparently, he regarded Euclid’s language as being too careful and precise. Again, he is apparently responsible for the frequent omission of the words ὅπερ ἔδει δεῖξαι (or ποιῆσαι), Q.E.D. (or F.), at the end of propositions. This is often the case at the end of porisms, where, in omitting the words, Theon seems to have deliberately departed from Euclid’s practice. The MS. P seems to show clearly that, where Euclid put a porism at the end of a proposition, he omitted the Q.E.D. at the end of the proposition but inserted it at the end of the porism, as if he regarded the latter as being

actually a part of the proposition itself. As in the Theonine MSS. the Q.E.D. is generally omitted, the omission would seem to have been due to Theon. Sometimes in these cases the Q.E.D. is interpolated at the end of the proposition.

Heiberg summed up the discussion of Theon's edition by the remark that Theon evidently took no pains to discover and restore from MSS. the actual words which Euclid had written, but aimed much more at removing difficulties that might be felt by learners in studying the book. His edition is therefore not to be compared with the editions of the Alexandrine grammarians, but rather with the work done by Eutocius in editing Apollonius and with an interpolated recension of some of the works of Archimedes by a certain Byzantine, Theon occupying a position midway between these two editors, being superior to the latter in mathematical knowledge but behind Eutocius in industry (these views now require to be somewhat modified, as above stated). But however little Theon's object may be approved by those of us who would rather know the ipsissima verba of Euclid, there is no doubt that his work was approved by his pupils at Alexandria for whom it was written; and his edition was almost exclusively used by later Greeks, with the result that the more ancient text is only preserved to us in one MS.

As the result of the above investigation, we may feel satisfied that, where P and the Theonine MSS. agree, they give us (except in a few accidental instances) Euclid as he was read by the Greeks of the 4th c. But even at that time the text had been passed from hand to hand through more than six centuries, so that it is certain that it had already suffered changes, due partly to the fault of copyists and partly to the interpolations of mathematicians. Some errors of copyists escaped Theon and were corrected in some MSS. by later hands. Others appear in all our MSS. and, as they cannot have arisen

accidentally in all, we must put them down to a common source more ancient than Theon. A somewhat serious instance is to be found in III. 8; and the use of ἁπτέσθω for ἐφαπτέσθω in the sense of "touch" may also be mentioned, the proper distinction between the words having been ignored as it was by Theon also. But there are a number of imperfections in the ante-Theonine text which it would be unsafe to put down to the errors of copyists, those namely where the good MSS. agree and it is not possible to see any motive that a copyist could have had for altering a correct reading. In these cases it is possible that the imperfections are due to a certain degree of carelessness on the part of Euclid himself; for it is not possible "Euclidem ab omni naevo vindicare," to use the words of Saccheri, and consequently Simson is not right in attributing to Theon and other editors all the things in Euclid to which mathematical objection can be taken. Thus, when Euclid speaks of "the ratio compounded of the sides" for "the ratio compounded of the ratios of the sides," there is no reason for doubting that Euclid himself is responsible for the more slip-shod expression. Again, in the Books XI. — XIII. relating to solid geometry there are blots neither few nor altogether unimportant which can only be attributed to Euclid himself; and there is the less reason for hesitation in so attributing them because solid geometry was then being treated in a thoroughly systematic manner for the first time. Sometimes the conclusion (συμπέρασμα) of a proposition does not correspond exactly to the enunciation, often it is cut short with the words καὶ τὰ ἑξῆς "and the rest" (especially from Book X. onwards), and very often in Books VIII., IX. it is omitted. Where all the MSS. agree, there is no ground for hesitating to attribute the abbreviation or omission to Euclid; though, of course, where one or more MSS. have the longer form, it must be retained because this is one of the cases where a copyist has a temptation to abbreviate.

Where the true reading is preserved in one of the Theonine MSS. alone, Heiberg attributes the wrong reading to a mistake which arose before Theon's time, and the right reading of the single MS. to a successful correction.

We now come to the most important question of the Interpolations introduced before Theon's time.

I. Alternative proofs or additional cases.

It is not in itself probable that Euclid would have given two proofs of the same proposition; and the doubt as to the genuineness of the alternatives is increased when we consider the character of some of them and the way in which they are introduced. First of all, we have those of VI. 20 and XII. 17 introduced by "we shall prove this otherwise more readily (προχειρότερον）" or that of X. 90 "it is possible to prove more shortly (συντομώτερον)." Now it is impossible to suppose that Euclid would have given one proof as that definitely accepted by him and then added another with the express comment that the latter has certain advantages over the former. Had he considered the two proofs and come to this conclusion, he would have inserted the latter in the received text instead of the former. These alternative proofs must therefore have been interpolated. The same argument applies to alternatives introduced with the words "or even thus" (ἢ καὶ οὕτως), "or even otherwise" (ἢ καὶ ἄλλως). Under this head come the alternatives for the last portions of III. 7, 8; and Heiberg also compares the alternatives for parts of III. 31 (that the angle in a semicircle is a right angle) and XIII. 18, and the alternative proof of the lemma after X. 32. The alternatives to X. 105 and 106, again, are condemned by the place in which they occur, namely after an alternative proof to X. 115. The above alternatives being all admitted to be spurious, suspicion must necessarily attach to the few others which are in themselves unobjectionable. Heiberg instances the alternative proofs to III. 9, III. 10, VI. 30, VI. 31 and XI. 22, observing that it is quite comprehensible that any of

these might have occurred to a teacher or editor and seemed to him, rightly or wrongly, to be better than the corresponding proofs in Euclid. Curiously enough, Simson adopted the alternatives to III. 9, 10 in preference to the genuine proofs. Since Heiberg's preface was written, his suspicion has been amply confirmed as regards III. 10 by the commentary of an-Nairīzī (ed. Curtze) which shows not only that this alternative is Heron's, but also that the substantive proposition III. 12 in Euclid is also Heron's, having been given by him to supplement III. II which must originally have been enunciated of circles "touching one another" simply, i.e. so as to include the case of external as well as internal contact, though the proof covered the case of internal contact only. "Euclid, in the 11th proposition," says Heron, "supposed two circles touching one another internally and wrote the proposition on this case, proving what it was required to prove in it. But I will show how it is to be proved if the contact be external.." This additional proposition of Heron's is by way of adding another case, which brings us to that class of interpolation. It was the practice of Euclid and the ancients to give only one case (generally the most difficult one) and to leave the others to be investigated by the reader for himself. One interpolation of a second case (VI. 27) is due, as we have seen, to Theon. The two extra cases of XI. 23 were manifestly interpolated before Theon's time, for the preliminary distinction of three cases, "(the centre) will either be within the triangle LMN, or on one of the sides, or outside. First let it be within," is a spurious addition (B and V only). Similarly an unnecessary case is interpolated in III. 11.

II. Lemmas.

Heiberg has unhesitatingly placed in his Appendix to Vol. III. certain lemmas interpolated either by Theon (on X. 13) or later writers (on X. 27, 29, 31, 32, 33, 34, where V only has the lemmas). But we are here concerned

with the lemmas found in all the MSS., which however are, for different reasons, necessarily suspected. We will deal with the Book X. lemmas last.

(1) There is an a priori ground of objection to those lemmas which come after the propositions to which they relate and prove properties used in those propositions; for, if genuine, they would be a sign of faulty arrangement such as would not be likely in a systematic work so carefully ordered as the Elements. The lemma to VI. 22 is one of this class, and there is the further objection to it that in VI. 28 Euclid makes an assumption which would equally require a lemma though none is found. The lemma after XII. 4 is open to the further objections that certain altitudes are used but are not drawn in the figure (which is not in the manner of Euclid), and that a peculiar expression "parallelepipedal solids described on (ἀναγραφόμενα ἀπό) prisms" betrays a hand other than Euclid's. There is an objection on the score of language to the lemma after XIII. 2. The lemmas on XI. 23, XIII. 13, XIII. 18, besides coming after the propositions to which they relate, are not very necessary in themselves and, as regards the lemma to XIII. 13, it is to be noticed that the writer of a gloss in the proposition could not have had it, and the words "as will be proved afterwards" in the text are rightly suspected owing to differences between the MS. readings. The lemma to XII. 2 also, to which Simson raised objection, comes after the proposition; but, if it is rejected, the words "as was proved before" used in XII. 5 and 18, and referring to this lemma, must be struck out.

(2) Reasons of substance are fatal to the lemma before X. 60, which is really assumed in X. 44 and therefore should have appeared there if anywhere, and to the lemma on X. 20, which tries to prove what is already stated in X. Def. 4.

We now come to the remaining lemmas in Book X., eleven in number, which come before the propositions to which they relate and remove

difficulties in the way of their demonstration. That before X. 42 introduces a set of propositions with the words “that the said irrational straight lines are uniquely divided ... we will prove after premising the following lemma,” and it is not possible to suppose that these words are due to an interpolator; nor are there any objections to the lemmas before X. 14, 17, 22, 33, 54, except perhaps that they are rather easy. The lemma before X. 10 and X. 10 itself should probably be removed from the Elements; for X. 10 really uses the following proposition X. 11, which is moreover numbered 10 by the first hand in P, and the words in X. 10 referring to the lemma “for we learnt (how to do this)” betray the interpolator. Heiberg gives reason also for rejecting the lemmas before X. 19 and 24 with the words “in any of the aforesaid ways” (omitted in the Theonine MSS.) in the enunciations of X. 19, 24 and in the exposition of X. 20. Lastly, the lemmas before X. 29 may be genuine, though there is an addition to the second of them which is spurious.

Heiberg includes under this heading of interpolated lemmas two which purport to be substantive propositions, XI. 38 and XIII. 6. These must be rejected as spurious for reasons which will be found in detail in my notes on XI. 37 and XIII. 6 respectively. The latter proposition is only quoted once (in XIII. 17); probably the words quoting it (with γραμμή instead of εὐθεῖα) are themselves interpolated, and Euclid thought the fact stated a sufficiently obvious inference from XIII. 1.

III. Porisms (or corollaries).

Most of the porisms in the text are both genuine and necessary; but some are shown by differences in the MSS. not to be so, e.g. those to I. 15 (though Proclus has it), III. 31 and VI. 20 (Por. 2). Sometimes parts of porisms are interpolated. Such are the last few lines in the porisms to IV. 5, VI. 8; the latter addition is proved later by means of VI. 4, 8, so that the writer of these proofs could not have had the addition to VI. 8 Por. before him. Lastly,

interpolators have added a sort of proof to some porisms, as though they were not quite obvious enough; but to add a demonstration is inconsistent with the idea of a porism, which, according to Proclus, is a by-product of a proposition appearing without our seeking it.

IV. Scholia.

Several interpolated scholia betray themselves by their wording, e.g. those given by Heiberg in the Appendix to Book X. and containing the words καλεῖ, ἐκάλεσε ("he calls" or "called"); these scholia were apparently written as marginal notes before Theon's time, and, being adopted as such by Theon, found their way into the text in P and some of the Theonine MSS. The same thing no doubt accounts for the interpolated analyses and syntheses to XIII. 1-5, as to which see my note on XIII. 1.

V. Interpolations in Book X.

First comes the proposition "Let it be proposed to us to show that in square figures the diameter is incommensurable in length with the side," which, with a scholium after it, ends the tenth Book. The form of the enunciation is suspicious enough and the proposition, the proof of which is indicated by Aristotle and perhaps was Pythagorean, is perfectly unnecessary when X. 9 has preceded. The scholium ends with remarks about commensurable and incommensurable solids, which are of course out of place before the Books on solids. The scholiast on Book X. alludes to this particular scholium as being due to "Theon and some others." But it is doubtless much more ancient, and may, as Heiberg conjectures have been the beginning of Apollonius' more advanced treatise on incommensurables. Not only is everything in Book X. after X. 115 interpolated, but Heiberg doubts the genuineness even of X. 112-115, on the ground that X. 111 rounds off the theory of incommensurables as we want it in the Books on solid geometry, while X. 112-115 are not really connected with what precedes, nor wanted

for the later Books, but seem to form the starting-point of a new and more elaborate theory of irrationals.

VI. Other minor interpolations are found of the same character as those above attributed to Theon. First there are two places (XI. 35 and XI. 26) where, after "similarly we shall prove" and "for the same reason," an actual proof is nevertheless given. Clearly the proofs are interpolated; and there are other similar interpolations. There are also interpolations of intermediate steps in proofs, unnecessary explanations and so on, as to which I need not enter into details.

Lastly, following Heiberg's order, I come to

VII. Interpolated definitions, axioms etc.

Apart from VI. Def. 5 (which may have been interpolated by Theon although it is found written in the margin of P by the first hand), the definition of a segment of a circle in Book I. is interpolated, as is clear from the fact that it occurs in a more appropriate place in Book III. and Proclus omits it. VI. Def. 2 (reciprocal figures) is rightly condemned by Simson — perhaps it was taken from Heron — and Heiberg would reject VII. Def. 10, as to which see my note on that definition. Lastly the double definition of a solid angle (XI. Def. 11) constitutes a difficulty. The use of the word ἐπιφάνεια suggests that the first definition may have been older than Euclid, and he may have quoted it from older elements, especially as his own definition which follows only includes solid angles contained by planes, whereas the other includes other sorts (cf. the words γραμμῶν, γραμμαῖς) which are also distinguished by Heron (Def. 22). If the first definition had come last, it could have been rejected without hesitation: but it is not so easy to reject the first part up to and including "otherwise" (ἄλλως). No difficulty need be felt about the definitions of "oblong," "rhombus," and "rhomboid," which are not

actually used in the Elements; they were no doubt taken from earlier elements and given for the sake of completeness.

As regards the axioms or, as they are called in the text, common notions (κοιναὶ ἔννοιαι), it is to be observed that Proclus says that Apollonius tried to prove "the axioms," and he gives Apollonius' attempt to prove Axiom I. This shows at all events that Apollonius had some of the axioms now appearing in the text. But how could Apollonius have taken a controversial line against Euclid on the subject of axioms if these axioms had not been Euclid's to his knowledge? And, if they had been interpolated between Euclid's time and his own, how could Apollonius, living so comparatively short a time after Euclid, have been ignorant of the fact? Therefore some of the axioms are Euclid's (whether he called them common notions, or axioms, as is perhaps more likely since Proclus calls them axioms): and we need not hesitate to accept as genuine the first three discussed by Proclus, viz. (1) things equal to the same equal to one another, (2) if equals be added to equals, wholes equal, (3) if equals be subtracted from equals, remainders equal. The other two mentioned by Proclus (whole greater than part, and congruent figures equal) are more doubtful, since they are omitted by Heron, Martianus Capella, and others. The axiom that "two lines cannot enclose a space" is however clearly an interpolation due to the fact that I. 4 appeared to require it. The others about equals added to unequals, doubles of the same thing, and halves of the same thing are also interpolated; they are connected with other interpolations, and Proclus clearly used some source which did not contain them.

Euclid evidently limited his formal axioms to those which seemed to him most essential and of the widest application; for he not unfrequently assumes other things as axiomatic, e.g. in VII. 28 that, if a number measures two numbers, it measures their difference.

The differences of reading appearing in Proclus suggest the question of the comparative purity of the sources used by Proclus, Heron and others, and of our text. The omission of the definition of a segment in Book I. and of the old gloss "which is called the circumference" in I. Def. 15 (also omitted by Heron, Taurus, Sextus Empiricus and others) indicates that Proclus had better sources than we have; and Heiberg gives other cases where Proclus omits words which are in all our MSS. and where Proclus' reading should perhaps be preferred. But, except in these instances (where Proclus may have drawn from some ancient source such as one of the older commentaries), Proclus' MS. does not seem to have been among the best. Often it agrees with our worst MSS., sometimes it agrees with F where F alone has a certain reading in the text, so that (e.g. in I. 15 Por.) the common reading of Proclus and F must be rejected, thrice only does it agree with P alone, sometimes it agrees with P and some Theonine MSS., and once it agrees with the Theonine MSS. against P and other sources.

Of the other external sources, those which are older than Theon generally agree with our best MSS., e.g. Heron, allowing for the difference in the plan of his definitions and the somewhat free adaptation to his purpose of the Euclidean definitions in Books X., XI.

Heiberg concludes that the Elements were most spoiled by interpolations about the 3rd c., for Sextus Empiricus had a correct text, while Iamblichus had an interpolated one; but doubtless the purer text continued for a long time in circulation, as we conclude from the fact that our MSS. are free from interpolations already found in Iamblichus' MS.

CHAPTER VI. THE SCHOLIA.

Heiberg has collected scholia, to the number of about 1500, in Vol. v. of his edition of Euclid, and has also discussed and classified them in a separate short treatise, in which he added a few others.

These scholia cannot be regarded as doing much to facilitate the reading of the Elements. As a rule, they contain only such observations as any intelligent reader could make for himself. Among the few exceptions are XI. Nos. 33, 35 (where XI. 22, 23 are extended to solid angles formed by any number of plane angles), XII. No. 85 (where an assumption tacitly made by Euclid in XII. 17 is proved), IX. Nos. 28, 29 (where the scholiast has pointed out the error in the text of IX. 19).

Nor are they very rich in historical information; they cannot be compared in this respect with Proclus' commentary on Book I. or with those of Eutocius on Archimedes and Apollonius. But even under this head they contain some things of interest, e.g. II. No. 11 explaining that the gnomon was invented by geometers for the sake of brevity, and that its name was suggested by an incidental characteristic, namely that "from it the whole is known (γνωρίζεται), either of the whole area or of the remainder, when it (the γνώμων) is either placed round or taken away" ; II. No. 13, also on the gnomon; IV. No. 2 stating that Book IV. was the discovery of the Pythagoreans; V. No. 1 attributing the content of Book V. to Eudoxus; X. No. 1 with its allusion to the discovery of incommensurability by the Pythagoreans and to Apollonius' work on irrationals; X. No. 62 definitely attributing X. 9 to Theaetetus; XIII. No. 1 about the "Platonic" figures, which attributes the cube, the pyramid, and the dodecahedron to the Pythagoreans, and the octahedron and icosahedron to Theaetetus.

Sometimes the scholia are useful in connexion with the settlement of the text, (1) directly, e.g. III. No. 16 on the interpolation of the word "within"

(ἐντός) in the enunciation of III. 6, and X. No. 1 alluding to the discussion by "Theon and some others" of irrational "surfaces" and "solids," as well as "lines," from which we may conclude that the scholium at the end of Book X. is not genuine; (2) indirectly in that they sometimes throw light on the connexion of certain MSS.

Lastly, they have their historical importance as enabling us to judge of the state of mathematical science at the times when they were written.

Before passing to the classification of the scholia, Heiberg remarks that we must separate from them a number of additions in the nature of scholia which are found in the text of our MSS. but which can, in one way or another, be proved to be spurious. As they are found both in P and in the Theonine MSS., they must have been in the MSS. anterior to Theon (4th c.). But they are, in great part, only found in the margin of P and the Theonine MSS.; in V they are half in the text and half in the margin. This can hardly be explained except on the supposition that these additions were originally (in the MSS. before Theon) in the margin, and that Theon kept them there in his edition, but that they afterwards found their way gradually into the text of P as well as of the Theonine MSS., or were omitted altogether, while particular MSS. have in certain places preserved the old arrangement. Of such spurious additions Heiberg enumerates the following: the axiom about equals subtracted from unequals, the last lines of the porism to VI. 8, second porisms to V. 19 and to VI. 20, the porism to III. 31, VI. Def. 5, various additions in Book X., the analyses and syntheses of XIII. 1-5, and the proposition XIII. 6.

The two first classes of scholia distinguished by Heiberg are denoted by the convenient abbreviations "Schol. Vat." and "Schol. Vind."

I. Schol. Vat.

It is first necessary to set out the letters by which Heiberg denotes certain collections of scholia.

P=Scholia in P written by the first hand.

B=Scholia in B by a hand of the same date as the MS. itself, generally that of Arethas.

F=Scholia in F by the first hand.

Vat.=Scholia of the Vatican MS. 204 of the 10th c., which has these scholia on leaves 198-205 (the end is missing) as an independent collection. It does not contain the text of the Elements.

V^{c}=Scholia found on leaves 283-292 of V and written in the same hand as that part of the MS. itself which begins at leaf 235.

Vat. 192=a Vatican MS. of the 14th c. which contains, after (1) the Elements I. — XIII. (without scholia), (2) the Data with scholia, (3) Marinus on the Data, the Schol. Vat. as an independent collection and in their entirety, beginning with 1. No. 88 and ending with XIII. No. 44.

The Schol. Vat., the most ancient and important collection of scholia, comprise those which are found in PBF Vat. and, from VII. 12 to IX. 15, in PB Vat. only, since in that portion of the Elements F was restored by a later hand without scholia; they also include 1. No. 88 which only happens to be erased in F, and IX. Nos. 28, 29 which may be left out because F. here has a different text. In F and Vat. the collection ends with Book X.; but it must also include Schol. PB of Books XI. — XIII., since these are found along with Schol. Vat. to Books I. — X. in several MSS. (of which Vat. 192 is one) as a separate collection. The Schol. Vat. to Books X. — XIII. are also found in the collection V^{c} (where, curiously enough, XIII. Nos. 43, 44 are at the beginning). The Schol. Vat. accordingly include Schol. PBV^{c} Vat. 192, and doubtless also those which are found in two of these sources. The total number of scholia classified by Heiberg as Schol. Vat. is 138.

As regards the contents of Schol. Vat. Heiberg has the following observations. The thirteen scholia to Book I. are extracts made from Proclus by a writer thoroughly conversant with the subject, and cleverly recast (with some additions). Their author does not seem to have had the two lacunae which our text of Proclus has (at the end of the note on I. 36 and the beginning of the next note, and at the beginning of the note on I. 43), for the scholia I. Nos. 125 and 137 seem to fill the gaps appropriately, at least in part. In some passages he had better readings than our MSS. have. The rest of Schol. Vat. (on Books II. — XIII.) are essentially of the same character as those on Book I., containing prolegomena, remarks on the object of the propositions, critical remarks on the text, converses, lemmas; they are, in general, exact and true to tradition. The reason of the resemblance between them and Proclus appears to be due to the fact that they have their origin in the commentary of Pappus, of which we know that Proclus also made use. In support of the view that Pappus is the source, heiberg places some of the Schol. Vat. to Book X. side by side with passages from the commentary of Pappus in the Arabic translation discovered by Woepcke;; he also refers to the striking confirmation afforded by the fact that XII. No. 2 contains the solution of the problem of inscribing in a given circle a polygon similar to a polygon inscribed in another circle, which problem Eutocius says that Pappus gave in his commentary on the Elements.

But, on the other hand, Schol. Vat. contain some things which cannot have come from Pappus, e.g. the allusion in X. No. 1 to Theon and irrational surfaces and solids, Theon being later than Pappus; III. No. 10 about porisms is more like Proclus' treatment of the subject than Pappus', though one expression recalls that of Pappus about forming (σχηματίζεσθαι) the enunciations of porisms like those of either theorems or problems.

The Schol. Vat. give us important indications as regards the text of the Elements as Pappus had it. In particular, they show that he could not have had in his text certain of the lemmas in Book X. For example, three of these are identical with what we find in Schol. Vat. (the lemma to X. 17=Schol. X. No. 106, and the lemmas to X. 54, 60 come in Schol. X. No. 328); and it is not possible to suppose. that these lemmas, if they were already in the text, would also be given as scholia. Of these three lemmas, that before X. 60 has already been condemned for other reasons; the other two, unobjectionable in themselves, must be rejected on the ground now stated. There were four others against which Heiberg found nothing to urge when writing his prolegomena to Vol. v., viz. the lemmas before X. 42, X. 14, X. 22 and X. 33. Of these, the lemma to X. 22 is not reconcilable with Schol. X. No. 161, which takes up the assumption in the text of Eucl. X. 22 as if no lemma had gone before. The lemma to X. 42, which, on account of the words introducing it (see above), Heiberg at first hesitated to regard as an interpolation, is identical with Schol. X. No. 270. It is true that in Schol. X. No. 269 we find the words "this lemma has been proved before (ἐν τοῖς ἔμπροσθεν), but it shall also be proved now for convenience' sake (τοῦ ἑτοίμου ἕνεκα)," and it is possible to suppose that "before" may mean in Euclid's text before X. 42; but a proof in that place would surely have been as "convenient" as could be desired, and it is therefore more probable that the proof had been given by Pappus in some earlier place. (It may be added that the lemma to X. 14, which is identical with the lemma to XI. 23, condemned on other grounds, is for that reason open to suspicion.)

Heiberg's conclusion is that all the lemmas are spurious, and that most or all of them have found their way into the text from Pappus' commentary, though at a time anterior to Theon's edition, since they are found in all our

MSS. This enables us to fix a date for these interpolations, namely the first half of the 4th c.

Of course Pappus had not in his text the interpolations which, from the fact of their appearing only in some of our MSS., are seen to be later than those above-mentioned. Such are the lemmas which are found in the text of V only after X. 29 and X. 31 respectively and are given in Heiberg's Appendix to Book X. (numbered 10 and 11). On the other hand it appears from Woepcke's tract that Pappus already had X. 115 in his text: though it does not follow from this that the proposition is genuine but only that interpolations began very early.

Theon interpolated a proposition (or lemma) between X. 12 and X. 13 (No. 5 in Heiberg's Appendix). Schol. Vat. has the same thing (X. No. 125). The writer of the scholia therefore did not find this lemma in the text. Schol. Vat. IX. Nos. 28, 29 show that neither did he find in his text the alterations which Theon made in Eucl. IX. 19; the scholia in fact only agree with the text of P, not with Theon's. This suggests that Schol. Vat. were written for use with a MS. of the ante-Theonine recension such as P is. This probability is further confirmed by a certain independence which P shows in several places when compared with the Theonine MSS. Not only has P better readings in some passages, but more substantial divergences; and, in particular, the absence in P of three notes of a historical character which are added, wholly or partly from Proclus, in the Theonine MSS. attests an independent and more primitive point of view in P.

In view of the distinctive character of P, it is possible that some of the scholia found in it in the first hand, but not in the other sources of Schol. Vat., also belong to that collection; and several circumstances confirm this. Schol. XIII. No. 45, found in P only, which relates to a passage in Eucl. XIII. 13, shows that certain words in the text, though older than Theon, are

interpolated; and, as the scholium is itself older than Theon, is headed "third lemma," and follows a "second lemma" relating to a passage in the text immediately preceding, which "second lemma" belongs to Schol. Vat. and is taken from Pappus, the "third" in all probability came from Pappus also. The same is true of Schol. XII. No. 72 and XIII. No. 69, which are respectively identical with the propositions vulgo XI. 38 (Heiberg, App. to Book XI., No. 3) and XIII. 6; for both of these interpolations are older than Theon. Moreover most of the scholia which P in the first hand alone has are of the same character as Schol. Vat. Thus VII. No. 7 and XIII. No. I introducing Books VII. and XIII. respectively are of the same historical character as several of Schol. Vat.; that VII. No. 7 appears in the text of P at the beginning of Book VII. constitutes no difficulty. There are a number of converses, remarks on the relation of propositions to one another, explanations such as XII. No. 89 in which it is remarked that Φ, Ω in Euclid's figure to XII. 17 (Z, V in my figure) are really the same point but that this makes no difference in the proof. Two other Schol. P on XII. 17 are connected by their headings with XII. No. 72 mentioned above. XI. No. 10 (P) is only another form of XI. No. 11 (B); and B often, alone with P, has preserved Schol. Vat. On the whole Heiberg considers some 40 scholia found in P alone to belong to Schol. Vat.

The history of Schol. Vat. appears to have been, in its main outlines, the following. They were put together after 500 A.D., since they contain extracts from Proclus, to which we ought not to assign a date too near to that of Proclus' work itself; and they must at least be earlier than the latter half of the 9th c., in which B was written. As there must evidently have been several intermediate links between the archetype and B, we must assign them rather to the first half of the period between the two dates, and it is not improbable that they were a new product of the great development of mathematical studies at the end of the 6th c. (Isidorus of Miletus). The author extracted

what he found of interest in the commentary of Proclus on Book I. and in that of Pappus on the rest of the work, and put these extracts in the margin of a MS. of the class of P. As there are no scholia to I. 1-22, the first leaves of the archetype or of one of the earliest copies must have been lost at an early date, and it was from that mutilated copy that partly P and partly a MS. of the Theonine class were taken, the scholia being put in the margin in both. Then the collection spread through the Theonine MSS., gradually losing some scholia which could not be read or understood, or which were accidentally or deliberately omitted. Next it was extracted from one of these MSS. and made into a separate work which has been preserved, in part, in its entirety (Vat. 192 etc.) and, in part, divided into sections, so that the scholia to Books X. — XIII. were detached (V^{c}). It had the same fate in the MSS. which kept the original arrangement (in the margin), and in consequence there are some MSS. where the scholia to the stereometric Books are missing, those Books having come to be less read in the period of decadence. It is from one of these MSS. that the collection was extracted as a separate work such as we find it in Vat. (10th c.).

II. The second great division of the scholia is Schol. Vind.

This title is taken from the Viennese MS. (V), and the letters used by Heiberg to indicate the sources here in question are as follows.

V^{a}=scholia in V written by the same hand that copied the MS. itself from fol. 235 onward.

q=scholia of the Paris MS. 2344 (q) written by the first hand.

l=scholia of the Florence MS. Laurent. XXVIII, 2 written in the 13th — 14th c., mostly in the first hand, but partly in two later hands.

V^{b}=scholia in V written by the same hand as the first part (leaves 1-183) of the MS. itself; V^{b} wrote his scholia after V^{a}.

q^=scholia of the Paris MS. (q) found here and there in another hand of early date.

Schol. Vind. include scholia found in V^{a}q. 1 is nearly related to q; and in fact the three MSS. which, so far as Euclid's text is concerned, show no direct interdependence, are, as regards their scholia, derived from one original. Heiberg proves this by reference to the readings of the three in two passages (found in Schol. 1. No. 109 and X. No. 39 respectively). The common source must have contained, besides the scholia found in the three MSS. V^{a}ql, those also which are contained in two of them, for it is more unlikely that two of the three should contain common interpolations than that a particular scholium should drop out of one of them. Besides V^{a} and q, the scholia V^{b} and q^ must equally be referred to Schol. Vind., since the greater part of their scholia are found in 1. There is a lacuna in q from Eucl. VIII. 25 to IX. 14, so that for this portion of the Elements Schol. Vind. are represented by Vl only. Heiberg gives about 450 numbers in all as belonging to this collection.

Schol. Vind. did not all come from one source; this is shown by differences of substance, e.g. between X. Nos. 36 and 39, and by differences of time of writing: e.g. VI. No. 52 refers at the beginning to No. 55 with the words "as the scholium has it" and is therefore later than that scholium; X. No. 247 is also later than X. No. 246.

The scholia to Book I. are here also extracts from Proclus, but more copious and more verbatim than in Schol. Vat. The author has not always understood Proclus; and he had a text as bad as that of our MSS., with the same lacunae. The scholia to the other Books are partly drawn (1) from Schol. Vat., the MSS. representing Schol. Vind. and Schol. Vat. in these cases showing nearly all possible combinations; but there is no certain trace in Schol. Vind. of the scholia peculiar to P. The author used a copy of Schol.

Vat. in the form in which they were attached to the Theonine text; thus Schol. Vind. correspond to BF Vat., where these diverge from P, and especially closely to B. Besides Schol. Vat., the editors of Schol. Vind. used (2) other old collections of scholia of which we find traces in B and F; Schol. Vind. have also some scholia common with b. The scholia which Schol. Vind. have in common with BF come from two different sources, and were apparently afterwards introduced into the other MSS.; one result of this is that several scholia are reproduced twice.

But, besides the scholia derived from these sources, Schol. Vind. contain a large number of others of late date, characterised by incorrect language or by triviality of content (there are many examples in numbers, citations of propositions used, absurd ἀπορίαι, and the like). Unlike Schol. Vat., these scholia often quote words from Euclid as a heading (in one case a heading is inserted in Schol. Vind. where a scholium without the heading is quoted from Schol. Vat., see V. No. 14). The explanations given often presuppose very little knowledge on the part of the reader and frequently contain obscurities and gross errors.

Schol. Vind. were collected for use with a MS. of the Theonine class; this follows from the fact that they contain a note on the proposition vulgo VII. 22 interpolated by Theon (given in Heiberg's App. to Vol. 11.). Since the scholium to VII. 39 given in V and p in the text after the title of Book VIII. quotes the proposition as VII. 39, it follows that this scholium must have been written before the interpolation of the two propositions vulgo VII. 20, 22; Schol. Vind. contain (VII. No. 80) the first sentence of it, but without the heading referring to VII. 39. Schol. VII. No. 97 quotes VII. 33 as VII. 34, so that the proposition vulgo VII. 22 may have stood in the scholiast's text but not the later interpolation vulgo VII. 20 (later because only found in B in the

margin by the first hand). Of course the scholiast had also the interpolations earlier than Theon.

For the date of the collection we have a lower limit in the date (12th c.) of MSS. in which the scholia appear. That it was not much earlier than the 12th c. is indicated (1) by the poverty of its contents, (2) by the quality of the MS. of Proclus which was used in the compilation of it (the Munich MS. used by Friedlein with which the scholiast's excerpts are essentially in agreement belongs to the 11th — 12th c.), (3) by the fact that Schol. Vind. appear only in MSS. of the 12th c. and no trace of them is found in our MSS. belonging to the 9th — 10th c. in which Schol. Vat. are found. The collection may therefore probably be assigned to the 11th c. Perhaps it may be in part due to Psellus who lived towards the end of that century: for in a Florence MS. (Magliabecch. XI, 53 of the 15th c.) containing a mathematical compendium intended for use in the reading of Aristotle the scholia 1. Nos. 40 and 49 appear with the name of Psellus attached.

Schol. Vind. are not found without the admixture of foreign elements in any of our three sources. In 1 there are only very few such in the first hand. In q there are several new scholia in the first hand, for the most part due to the copyist himself. The collection of scholia on Book X. in q (Heiberg's q^{c}) is also in the first hand; it is not original, and it may perhaps be due to Psellus (Maglb. has some definitions of Book X. with a heading "scholia of...Michael Psellus on the definitions of Euclid's 10th Element" and Schol. X. No. 9), whose name must have been attached to it in the common source of Maglb. and q; to a great extent it consists of extracts from Schol. Vind. taken from the same source as Vl. The scholia q^ (in an ancient hand in q), confined to Book II., partly belong to Schol. Vind. and partly correspond to b^ (Bologna MS.). q^{a} and q^{b} are in one hand (Theodorus Antiochita), the nearest to

the first hand of q; they are doubtless due to an early possessor of the MS. of whom we know nothing more.

V^{a} has, besides Schol. Vind., a number of scholia which also appear in other MSS., one in BFb, some others in P, and some in v (Codex Vat. 1038, 13th c.); these scholia were taken from a source in which many abbreviations were used, as they were often misunderstood by V^{a}. Other scholia in V^{a} which are not found in the older sources — some appearing in V^{a} alone — are also not original, as is proved by mistakes or corruptions which they contain; some others may be due to the copyist himself.

V^{b} seldom has scholia common with the other older sources; for the most part they either appear in V^{b} alone or only in the later sources as v or F^ (later scholia in F), some being original, others not. In Book X. V^{b} has three series of numerical examples, (1) with Greek numerals, (2) alternatives added later, also mostly with Greek numerals, (3) with Arabic numerals. The last class were probably the work of the copyist himself. These examples (cf. below) show the facility with which the Byzantines made calculations at the date of the MS. (12th c.). They prove also that the use of the Arabic numerals (in the East-Arabian form) was thoroughly established in the 12th c.; they were actually known to the Byzantines a century earlier, since they appear, in the first hand, in an Escurial MS. of the 11th c.

Of collections in other hands in V distinguished by Heiberg (see preface to Vol. v.), V^ has very few scholia which are found in other sources, the greater part being original; V^, V^ are the work of the copyist himself; V^ are so in part only, and contain several scholia from Schol. Vat. and other sources. V^ and V^ are later than 13th — 14th c., since they are not found in f (cod. Laurent. XXVIII, 6) which was copied from V and contains, besides V^{a} V^{b}, the greater part of V^ and VI. No. 20 of V^ (in the text).

In P there are, besides P^ (a quite late hand, probably one of the old Scriptores Graeci at the Vatican), two late hands (P^), one of which has some new and independent scholia, while the other has added the greater part of Schol. Vind., partly in the margin and partly on pieces of leaves stitched on.

Our sources for Schol. Vat. also contain other elements. In P there were introduced a certain number of extracts from Proclus, to supplement Schol. Vat. to Book I.; they are all written with a different ink from that used for the oldest part of the MS., and the text is inferior. There are additions in the other sources of Schol. Vat. (F and B) which point to a common source for FB and which are nearly all found in other MSS., and, in particular, in Schol. Vind., which also used the same source; that they are not assignable to Schol. Vat. results only from their not being found in Vat. Of other additions in F, some are peculiar to F and some common to it and b; but they are not original. F^ (scholia in a later hand in F) contains three original scholia; the rest come from V. B contains, besides scholia common to it and F, b or other sources, several scholia which seem to have been put together by Arethas, who wrote at least a part of them with his own hand.

Heiberg has satisfied himself, by a closer study of b, that the scholia which he denotes by b, β and b^ are by one hand; they are mostly to be found in other sources as well, though some are original. By the same hand (Theodorus Cabasilas, 15th c.) are also the scholia denoted by b^, B^, b^ and B^. These scholia come in great part from Schol. Vind., and in making these extracts Theodorus probably used one of our sources, l, mistakes in which often correspond to those of Theodorus. To one scholium is attached the name of Demetrius (who must be Demetrius Cydonius, a friend of Nicolaus Cabasilas, 14th c.); but it could not have been written by him, since it appears in B and Schol. Vind. Nor are all the scholia which bear the name of Theodorus due to Theodorus himself, though some are so.

As B^ (a late hand in B) contains several of the original scholia of b^, B^ must have used b itself as his source, and, as all the scholia in B^ are in b, the latter is also the source of the scholia in B^ which are found in other MSS. B and b were therefore, in the 15th c., in the hands of the same person; this explains, too, the fact that b in a late hand has some scholia which can only come from B. We arrive then at the conclusion that Theodorus Cabasilas, in the 15th c., owned both the MSS. B and b, and that he transferred to B scholia which he had before written in b, either independently or after other sources, and inversely transferred some scholia from B to b. Further, B^ are earlier than Theodorus Cabasilas, who certainly himself wrote B^ as well as b^ and b^.

An author's name is also attached to the scholia VI. No. 6 and X. No. 223, which are attributed to Maximus Planudes (end of 13th c.) along with scholia on I. 31, X. 14 and X. 18 found in l in a quite late hand and published on p, 47 of Heibėrg's dissertation. These seem to have been taken from lectures of Planudes on the Elements by a pupil who used l as his copy.

There are also in l two other Byzantine scholia, written by a late hand, and bearing the names Ioannes and Pediasimus respectively; these must in like manner have been written by a pupil after lectures of Ioannes Pediasimus (first half of 14th c.), and this pupil must also have used l.

Before these scholia were edited by Heiberg, very few of them had been published in the original Greek. The Basel editio princeps has a few (V. No. I, VI. Nos. 3, 4 and some in Book X.) which are taken, some from the Paris MS. (Paris. Gr. 2343) used by Grynaeus, others probably from the Venice MS. (Marc. 301) also used by him; one published by Heiberg, not in his edition of Euclid but in his paper on the scholia, may also be from Venet. 301, but appears also in Paris. Gr. 2342. The scholia in the Basel edition

passed into the Oxford edition in the text, and were also given by August in the Appendix to his Vol. II.

Several specimens of the two series of scholia (Vat. and Vind.) were published by C. Wachsmuth (Rhein. Mus. XVIII. sqq.) and by Knoche (Untersuchungen über die neu aufgefundenen Scholien des Proklus, Herford, 1865).

The scholia published in Latin were much more numerous. G. Valla (De expetendis et fugiendis rebus, 1501) reproduced apparently some 200 of the scholia included in Heiberg's edition. Several of these he obtained from two Modena MSS. which at one time were in his possession (Mutin. III B, 4 and II E, 9, both of the 15th c.); but he must have used another source as well, containing extracts from other series of scholia, notably Schol. Vind. with which he has some 87 scholia in common. He has also several that are new.

Commandinus included in his translation under the title "Scholia antiqua" the greater part of the Schol. Vat. which he certainly obtained from a MS. of the class of Vat. 192; on the whole he adhered closely to the Greek text. Besides these scholia Commandinus has the scholia and lemmas which he found in the Basel editio princeps, and also three other scholia not belonging to Schol. Vat., as well as one new scholium (to XII. 13) not included in Heiberg's edition, which are distinguished by different type and were doubtless taken from the Greek MS. used by him along with the Basel edition.

In Conrad Dasypodius' Lexicon mathematicum published in 1573 there is (on fol. 42-44) "Graecum scholion in definitiones Euclidis libri quinti elementorum appendicis loco propter pagellas vacantes annexum." This contains four scholia, and part of two others, published in Heiberg's edition, with some variations of readings, and with some new matter added (for which see p-6 of Heiberg's pamphlet). The source of these scholia is revealed to us

by another work of Dasypodius, Isaaci Monachi Scholia in Euclidis elementorum geometriae sex priores libros per C. Dasypodium in latinum sermonem translata et in lucem edita (1579). This work contains, besides excerpts from Proclus on Book I. (in part closely related to Schol. Vind.), some 30 scholia included in Heiberg's edition, several new scholia, and the above-mentioned scholia to the definitions of Book V. published in Greek in 1573. After the scholia follow "Isaaci Monachi prolegomena in Euclidis Elementorum geometriae libros" (two definitions of geometry) and "Varia miscellanea ad geometriae cognitionem necessaria ab Isaaco Monacho collecta" (mostly the same as p, 24-272, 27 in the Variae Collectiones included in Hultsch's Heron); lastly, a note of Dasypodius to the reader says that these scholia were taken "ex clarissimi viri Joannis Sambuci antiquo codice manu propria Isaaci Monachi scripto." Isaak Monachus is doubtless Isaak Argyrus, 14th c.; and Dasypodius used a MS. in which, besides the passage in Hultsch's Variae Collectiones, there were a number of scholia marked in the margin with the name of Isaak (cf. those in b under the name of Theodorus Cabasilas). Whether the new scholia are original cannot be decided until they are published in Greek; but it is not improbable that they are at all events independent arrangements of older scholia. All but five of the others, and all but one of the Greek scholia to Book v., are taken from Schol. Vat.; three of the excepted ones are from Schol. Vind., and the other three seem to come from F (where some words of them are illegible, but can be supplied by means of Mut. III B, 4, which has these three scholia and generally shows a certain likeness to Isaak's scholia).

Dasypodius also published in 1564 the arithmetical commentary of Barlaam the monk (14th c.) on Eucl. Book II., which finds a place in Appendix IV. to the Scholia in Heiberg's edition.

Hultsch has some remarks on the origin of the scholia. He observes that the scholia to Book I. contain a considerable portion of Geminus' commentary on the definitions and are specially valuable because they contain extracts from Geminus only, whereas Proclus, though drawing mainly upon him, quotes from others as well. On the postulates and axioms the scholia give more than is found in Proclus. Hultsch conjectures that the scholium on Book v., No. 3, attributing the discovery of the theorems to Eudoxus but their arrangement to Euclid, represents the tradition going back to Geminus, and that the scholium XIII., No. 1, has the same origin.

A word should be added about the numerical illustrations of Euclid's propositions in the scholia to Book X. They contain a large number of calculations with sexagesimal fractions; the fractions go as far as fourth-sixtieths (1/60^). Numbers expressed in these fractions are handled with skill and include some results of surprising accuracy

CHAPTER VII. EUCLID IN ARABIA.

We are told by [Hnull]ājī Khalfa that the Caliph al-Mansūr (754-775) sent a mission to the Byzantine Emperor as the result of which he obtained from him a copy of Euclid among other Greek books, and again that the Caliph al-Ma'mūn (813-833) obtained manuscripts of Euclid, among others, from the Byzantines. The version of the Elements by al-[Hnull]ajjāj b. Yūsuf b. Matar is, if not the very first, at least one of the first books translated from the Greek into Arabic. According to the Fihrist it was translated by al-[Hnull]ajjāj twice; the first translation was known as "Hārūni" ("for Hārūn"), the second bore the name "Ma'mūni" ("for al-Ma'mūn") and was the more trustworthy. Six Books of the second of these versions survive in a Leiden MS. (Codex Leidensis 399, 1) now in part published by Besthorn and Heiberg. In the preface to this MS. it is stated that, in the reign of Hārūn ar-Rashīd (786-809), al-[Hnull]ajjāj was commanded by Ya[hnull]yā b. Khālid b. Barmak to translate the book into Arabic. Then, when al-Ma'mūn became Caliph, as he was devoted to learning, al-[Hnull]ajjāj saw that he would secure the favour of al-Ma'mūn "if he illustrated and expounded this book and reduced it to smaller dimensions. He accordingly left out the superfluities, filled up the gaps, corrected or removed the errors, until he had gone through the book and reduced it, when corrected and explained, to smaller dimensions, as in this copy, but without altering the substance, for the use of men endowed with ability and devoted to learning, the earlier edition being left in the hands of readers."

The Fihrist goes on to say that the work was next translated by Ishāq b. Hunain, and that this translation was improved by Thābit b. Qurra. This Abū Ya`qūb Is[hnull ]āq b. [Hnull ]unain b. Is[hnull ]āq al-`Ibādī (d. 910) was the son of the most famous of Arabic translators, Hunain b. Ishāq al-`Ibādī (809-873), a Christian and physician to the Caliph alMutawakkil (847-861). There

seems to be no doubt that Is[hnull]āq, who must have known Greek as well as his father, made his translation direct from the Greek. The revision must apparently have been the subject of an arrangement between Is[hnull]āq and Thābit as the latter died in 901 or nine years before Is[hnull]āq. Thābit undoubtedly consulted Greek MSS. for the purposes of his revision. This is expressly stated in a marginal note to a Hebrew version of the Elements, made from Ishāq's, attributed to one of two scholars belonging to the same family, viz. either to Moses b. Tibbon (about 1244-1274) or to Jakob b. Machir (who died soon after 1306). Moreover Thābit observes, on the proposition which he gives as IX. 31, that he had not found this proposition and the one before it in the Greek but only in the Arabic; from which statement Klamroth draws two conclusions, (1) that the Arabs had already begun to interest themselves in the authenticity of the text and (2) that Thābit did not alter the numbers of the propositions in Ishāq's translation. The Fihrist also says that Yu[hnull]annā al-Qass (i.e. "the Priest") had seen in the Greek copy in his possession the proposition in Book I. which Thābit took credit for, and that this was confirmed by Na[znull]īf, the physician, to whom Yuhannā had shown it. This proposition may have been wanting in Ishāq, and Thābit may have added it, but without claiming it as his own discovery. As a fact, I. 45 is missing in the translation by al-[Hnull]ajjāj.

The original version of Is[hnull]āq without the improvements by Thābit has probably not survived any more than the first of the two versions by al-Hajjāj; the divergences between the MSS. are apparently due to the voluntary or involuntary changes of copyists, the former class varying according to the degree of mathematical knowledge possessed by the copyists and the extent to which they were influenced by considerations of practical utility for teaching purposes. Two MSS. of the Ishāq-Thābit version exist in the Bodleian Library (No. 279 belonging to the year 1238, and No. 280 written in

1260-1); Books I. — XIII. are in the Is[hnull]āq-Thābit version, the non-Euclidean Books XIV., XV. in the translation of Qustā b. L'ūqā al-Ba`labakkī (d. about 912). The first of these MSS. (No. 279) is that (O) used by Klamroth for the purpose of his paper on the Arabian Euclid. The other MS. used by Klamroth is (K) Kjobenhavn LXXXI, undated but probably of the 13th c., containing Books V. — XV., Books V. — X. being in the Is[hnull]āq-Thābit version, Books XI. — XIII. purporting to be in al-Hajjāj's translation, and Books XIV., XV. in the version of Qus[tnull]ā b. Lūqā. In not a few propositions K and O show not the slightest difference, and, even where the proofs show considerable differences, they are generally such that, by a careful comparison, it is possible to reconstruct the common archetype, so that it is fairly clear that we have in these cases, not two recensions of one translation, but arbitrarily altered and shortened copies of one and the same recension. The Bodleian MS. No. 280 contains a preface, translated by Nîcoll, which cannot be by Thābit himself because it mentions Avicenna (980-1037) and other later authors. The MS. was written at Marāġa in the year 1260-1 and has in the margin readings and emendations from the edition of Na[snull]īraddīn a[tnull]-[Tnull]īsī (shortly to be mentioned) who was living at Marāġa at the time. Is it possible that a[tnull]-[Tnull]ūsī himself is the author of the preface? Be this as it may, the preface is interesting because it throws light on the liberties which the Arabians allowed themselves to take with the text. After the observation that the book (in spite of the labours of many editors) is not free from errors, obscurities, redundancies, omissions etc., and is without certain definitions necessary for the proofs, it goes on to say that the man has not yet been found who could make it perfect, and next proceeds to explain (1) that Avicenna "cut out postulates and many definitions" and attempted to clear up difficult and obscure passages, (2) that Abū'l Wafā al-Būzjānī (939-997) "introduced unnecessary additions and left

out many things of great importance and entirely necessary," inasmuch as he was too long in various places in Book VI. and too short in Book X. where he left out entirely the proofs of the apotomae, while he made an unsuccessful attempt to emend XII. 14, (3) that Abū Ja`far al-Khāzin (d. between 961 and 971) arranged the postulates excellently but "disturbed the number and order of the propositions, reduced several propositions to one" etc. Next the preface describes the editor's own claims and then ends with the sentences, "But we have kept to the order of the books and propositions in the work itself (i.e. Euclid's) except in the twelfth and thirteenth books. For we have dealt in Book XIII. with the (solid) bodies and in Book XII. with the surfaces by themselves."

After Thābit the Fihrist mentions Abū 'Uthmān ad-Dimashqī as having translated some Books of the Elements including Book X. (It is Abū 'Uthmān's translation of Pappus' commentary on Book X. which Woepcke discovered at Paris.) The Fihrist adds also that "Na[znull]īf the physician told me that he had seen the tenth Book of Euclid in Greek, that it had 40 propositions more than the version in common circulation which had 109 propositions, and that he had determined to translate it into Arabic."

But the third form of the Arabian Euclid actually accessible to us is the edition of Abū Ja`far Mu[hnull]. b. Mu[hnull]. b. al-[Hnull]asan Na[snull]raddīn aṭ-Ṭūsī (whom we shall call aṭ-Ṭūsī for short), born at Tūs (in Khurāsān) in 1201 (d. 1274). This edition appeared in two forms, a larger and a smaller. The larger is said to survive in Florence only (Pal. 272 and 313, the latter MS. containing only six Books); this was published at Rome in 1594, and, remarkably enough, some copies of this edition are to be found with 12 and some with 13 Books, some with a Latin title and some without. But the book was printed in Arabic, so that Kästner remarks that he will say as much about it as can be said about a book which one cannot read. The shorter form,

which however, in most MSS., is in 15 Books, survives at Berlin, Munich, Oxford, British Museum (974, 1334, 1335), Paris (2465, 2466), India Office, and Constantinople; it was printed at Constantinople in 1801, and the first six Books at Calcutta in 1824.

A[tnull]-[Tnull]ũsĩ's work is however not a translation of Euclid's text, but a re-written Euclid based on the older Arabic translations. In this respect it seems to be like the Latin version of the Elements by Campanus (Campano), which was first published by Erhard Ratdolt at Venice in 1482 (the first printed edition of Euclid). Campanus (13th c.) was a mathematician, and it is likely enough that he allowed himself the same liberty as a[tnull]-[Tnull]ĩsī in reproducing Euclid. Whatever may be the relation between Campanus' version and that of Athelhard of Bath (about 1120), and whether, as Curtze thinks, they both used one and the same Latin version of 10th — 11th c., or whether Campanus used Athelhard's version in the same way as a[tnull]-[Tnull]ĩsī used those of his predecessors,it is certain that both versions came from an Arabian source, as is evident from the occurrence of Arabic words in them. Campanus' version is not of much service for the purpose of forming a judgment on the relative authenticity of the Greek and Arabian tradition ; but it sometimes preserves traces of the purer source, as when it omits Theon's addition to VI. 33. A curious circumstance is that, while Campanus' version agrees with aṭ-Ṭĩsī's in the number of the propositions in all the genuine Euclidean Books except V. and IX., it agrees with Athelhard's in having 34 propositions in Book V. (as against 25 in other versions), which confirms the view that the two are not independent, and also leads, as Klamroth says, to this dilemma: either the additions to Book V. are Athelhard's own, or he used an Arabian Euclid which is not known to us. Heiberg also notes that Campanus' Books XIV., XV. show a certain agreement with the preface to the Thābit-Is[hnull]āq version, in which the

author claims to have (1) given a method of inscribing spheres in the five regular solids, (2) carried further the solution of the problem how to inscribe any one of the solids in any other and (3) noted the cases where this could not be done.

With a view to arriving at what may be called a common measure of the Arabian tradition, it is necessary to compare, in the first place, the numbers of propositions in the various Books. [Hnull]ājī Khalfa says that al-[Hnull]ajjāj's translation contained 468 propositions, and Thābit's 478; this is stated on the authority of a[tnull]-[Tnull]īsī, whose own edition contained 468. The fact that Thābit's version had 478 propositions is confirmed by an index in the Bodleian MS. 279 (called O by Klamroth). A register at the beginning of the Codex Leidensis 399, 1 which gives Is[hnull]aq's numbers (although the translation is that of al-[Hnull]ajjāj) apparently makes the total 479 propositions (the number in Book XIV. being apparently 11, instead of the 10 of O). I subjoin a table of relative numbers taken from Klamroth, to which I have added the corresponding numbers in August's and Heiberg's editions of the Greek text.

The numbers in the case of Heiberg include all propositions which he has printed in the text; they include therefore XIII. 6 and III. 12 now to be regarded as spurious, and X. 112-115 which he brackets as doubtful. He does not number the propositions in Books XIV., XV., but I conclude that the numbers in P reach at least 9 in XIV., and 9 in XV.

The Fihrist confirms the number 109 for Book X., from which Klamroth concludes that Is[hnull]āq's version was considered as by far the most authoritative.

In the text of O, Book IV. consists of 17 propositions and Book XIV. of 12, differing in this respect from its own table of contents; IV. 15, 16 in O are really two proofs of the same proposition.

In al-[Hnull]ajjāj's version Book I. consists of 47 propositions only, I. 45 being omitted. It has also one proposition fewer in Book III., the Heronic proposition III. 12 being no doubt omitted.

In speaking of particular propositions, I shall use Heiberg's numbering, except where otherwise stated.

The difference of 10 propositions between Thābit-Is[hnull]āq and a[tnull]-[Tnull]ĩsī is accounted for thus:

(1) The three propositions VI. 12 and X. 28, 29 which both Is[hnull]āq and the Greek text have are omitted in a[tnull]-[Tnull]ĩsī.

(2) Is[hnull]āq divides each of the propositions XIII. 1-3 into two, making six instead of three in a[tnull]-[Tnull]ĩsī and in the Greek.

(3) Is[hnull]āq has four propositions (numbered by him VIII. 24, 25, IX. 30, 31) which are neither in the Greek Euclid nor in aţ-Ţĩsī.

Apart from the above differences al-[Hnull]ajjāj (so far as we know), Ishāq and a[tnull]-[Tnull]ĩsī agree; but their Euclid shows many differences from our Greek text. These differences we will classify as follows.

1. Propositions.

The Arabian Euclid omits VII. 20, 22 of Gregory's and August's editions (Heiberg, App. to Vol. II. p-32); VIII. 16, 17; X. 7, 8, 13, 16, 24, 112, 113, 114, besides a lemma vulgo X. 13, the proposition X. 117 of Gregory's edition, and the scholium at the end of the Book (see for these Heiberg's Appendix to Vol. III. p, 408-416); XI. 38 in Gregory and August (Heiberg, App. to Vol. IV.); XII. 6, 13, 14; (also all but the first third of Book XV.).

The Arabian Euclid makes III. 11, 12 into one proposition, and divides some propositions (X. 31, 32; XI. 31, 34; XIII. 1-3) into two each.

The order is also changed in the Arabic to the following extent. V. 12, 13 are interchanged and the order in Books VI., VII., IX. — XIII. is:

VI. 1-8, 13, 11, 12, 9, 10, 14-17, 19, 20, 18, 21, 22, 24, 26, 23, 25, 27-30, 32, 31, 33.

VII. 1-20, 22, 21, 23-28, 31, 32, 29, 30, 33-39.

IX. 1-13, 20, 14-19, 21-25, 27, 26, 28-36, with two new propositions coming before pro.

X. 1-6, 9-12, 15, 14, 17-23, 26-28, 25, 29-30, 31, 32, 33 — 111, 115.

XI. 1-30, 31, 32, 34, 33, 35-39.

XII. 1-5, 7, 9, 8, 10, 12, 11, 15, 16-18.

XIII. 1-3, 5, 4, 6, 7, 12, 9, 10, 8, 11, 13, 15, 14, 16-18.

2. Definitions.

The Arabic omits the following definitions : IV. Deff. 3-7, VII. Def. 9 (or 10), XI. Deff. 5-7, 15, 17, 23, 25-28; but it has the spurious definitions VI. Deff. 2, 5, and those of proportion and ordered proportion in Book V. (Deff. 8, 19 August), and wrongly interchanges V. Deff. 11, 12 and also VI. Deff. 3, 4.

The order of the definitions is also different in Book VII. where, after Def. 11, the order is 12, 14, 13, 15, 16, 19, 20, 17, 18, 21, 22, 23, and in Book XI. where the order is 1, 2, 3, 4, 8, 10, 9, 13, 14, 16, 12, 21, 22, 18, 19, 20, 11, 24.

3. Lemmas and porisms.

All are omitted in the Arabic except the porisms to VI. 8, VIII. 2, X. 3; but there are slight additions here and there, not found in the Greek, e.g. in VIII. 14, 15 (in K).

4. Alternative proofs.

These are all omitted in the Arabic, except that in X. 105, 106 they are substituted for the genuine proofs; but one or two alternative proofs are peculiar to the Arabic (VI. 32 and VIII. 4, 6).

The analyses and syntheses to XIII. 1-5 are also omitted in the Arabic.

Klamroth is inclined, on a consideration of all these differences, to give preference to the Arabian tradition over the Greek (1) "on historical grounds," subject to the proviso that no Greek MS. as ancient as the 8th c. is found to contradict his conclusions, which are based generally (2) on the improbability that the Arabs would have omitted so much if they had found it in their Greek MSS., it being clear from the Fihrist that the Arabs had already shown an anxiety for a pure text, and that the old translators were subjected in this matter to the check of public criticism. Against the "historical grounds," Heiberg is able to bring a considerable amount of evidence. First of all there is the British Museum palimpsest (L) of the 7th or the beginning of the 8th c. This has fragments of propositions in Book X. which are omitted in the Arabic; the numbering of one proposition, which agrees with the numbering in other Greek MS., is not comprehensible on the assumption that eight preceding propositions were omitted in it, as they are in the Arabic; and lastly, the readings in L are tolerably like those of our MSS., and surprisingly like those of B. It is also to be noted that, although P dates from the 10th c. only, it contains, according to all appearance, an ante-Theonine recension.

Moreover there is positive evidence against certain omissions by the Arabians. A[tnull]-[Tnull]ĩsī omits VI. 12, but it is scarcely possible that, if Eutocius had not had it, he would have quoted VI. 23 by that number. This quotation of VI. 23 by Eutocius also tells against Is[hnull]āq who has the proposition as VI. 25. Again, Simplicius quotes VI. 10 by that number, whereas it is VI. 13 in Is[hnull]āq; and Pappus quotes, by number, XIII. 2 (Is[hnull]āq 3, 4), XIII. 4 (Is[hnull]āq 8), XIII. 16 (Is[hnull]āq 19). On the other hand the contraction of III. 11, 12 into one proposition in the Arabic tells in favour of the Arabic.

Further, the omission of certain porisms in the Arabic cannot be supported; for Pappus quotes the porism to XIII. 17, Proclus those to II. 4, III.

1, VII. 2, and Simplicius that to IV. 15.

Lastly, some propositions omitted in the Arabic are required in later propositions. Thus X. 13 is used in X. 18, 22, 23, 26 etc.; X. 17 is wanted in X. 18, 26, 36; XII. 6, 13 are required for XII. 11 and XII. 15 respectively.

It must also be remembered that some of the things which were properly omitted by the Arabians are omitted or marked as doubtful in Greek MSS. also, especially in P, and others are rightly suspected for other reasons (e.g. a number of alternative proofs, lemmas, and porisms, as well as the analyses and syntheses of XIII. 1-5). On the other hand, the Arabic has certain interpolations peculiar to our inferior MSS. (cf. the definition VI. Def. 2 and those of proportion and ordered proportion).

Heiberg comes to the general conclusion that, not only is the Arabic tradition not to be preferred offhand to that of the Greek MSS., but it must be regarded as inferior in authority. It is a question how far the differences shown in the Arabic are due to the use of Greek MSS. differing from those which have been most used as the basis of our text, and how far to the arbitrary changes made by the Arabians themselves. Changes of order and arbitrary omissions could not surprise us, in view of the preface above quoted from the Oxford MS. of Thābit-Is[hnull]āq, with its allusion to the many important and necessary things left out by Abĩ'l Wafā and to the author's own rearrangement of Books XII., XIII. But there is evidence of differences due to the use by the Arabs of other Greek MSS. Heiberg is able to show considerable resemblances between the Arabic text and the Bologna MS. b in that part of the MS. where it diverges so remarkably from our other MSS. (see the short description of it above, p.49); in illustration he gives a comparison of the proofs of XII. 7 in b and in the Arabic respectively, and points to the omission in both of the proposition given in Gregory's edition as XI. 38, and to a remarkable agreement between them as regards the order of

the propositions of Book XII. As above stated, the remarkable divergence of b only affects Books XI. (at end) and XII.; and Book XIII. in b shows none of the transpositions and other peculiarities of the Arabic. There are many differences between b and the Arabic, especially in the definitions of Book XI., as well as in Book XIII. It is therefore a question whether the Arabians made arbitrary changes, or the Arabic form is the more ancient, and b has been altered through contact with other MSS. Heiberg points out that the Arabians must be alone responsible for their definition of a prism, which only covers a prism with a triangular base. This could not have been Euclid's own, for the word prism already has the wider meaning in Archimedes, and Euclid himself speaks of prisms with parallelograms and polygons as bases (XI. 39; XII. 10). Moreover, a Greek would not have been likely to leave out the definitions of the "Platonic" regular solids.

Heiberg considers that the Arabian translator had before him a MS. which was related to b, but diverged still further from the rest of our MSS. He does not think that there is evidence of the existence of a redaction of Books I. — X. similar to that of Books XI., XII. in b; for Klamroth observes that it is the Books on solid geometry (XI. — XIII.) which are more remarkable than the others for omissions and shorter proofs, and it is a noteworthy coincidence that it is just in these Books that we have a divergent text in b.

An advantage in the Arabic version is the omission of VII. Def. 10, although, as Iamblichus had it, it may have been deliberately omitted by the Arabic translator. Another advantage is the omission of the analyses and syntheses of XIII. 1-5; but again these may have been omitted purpōsely, as were evidently a number of porisms which are really necessary.

One or two remarks may be added about the Arabic versions as compared with one another. Al-[Hnull]ajjāj's object seems to have been less to give a faithful reflection of the original than to write a useful and convenient

mathematical text-book. One characteristic of it is the careful references to earlier propositions when their results are used. Such specific quotations of earlier propositions are rare in Euclid; but in al-[Hnull]ajjāj we find not only such phrases as “by prop. so and so,” “which was proved” or “which we showed how to do in prop. so and so,” but also still longer phrases. Sometimes he kepeats a construction, as in I. 44 where, instead of constructing “the parallelogram BEFG equal to the triangle C in the angle EBG which is equal to the angle D” and placing it in a certain position, he produces AB to G, making BG equal to half DE (the base of the triangle CDE in his figure), and on GB so constructs the parallelogram BHKG by I. 42 that it is equal to the triangle CDE, and its angle GBH is equal to the given angle.

Secondly, al-[hnull]ajjāj, in the arithmetical books, in the theory of proportion, in the applications of the Pythagorean I. 47, and generally where possible, illustrates the proofs by numerical examples. It is true, observes Klamroth, that these examples are not apparently separated from the commentary of an-Nairīzī, and might not therefore have been due to al-[Hnull]ajjāj himself; but the marginal notes to the Hebrew translation in Munich MS. 36 show that these additions were in the copy of al-[Hnull]ajjāj used by the translator, for they expressly give these proofs in numbers as variants taken from al-[Hnull]ajjāj.

These characteristics, together with al-[Hnull]ájjāj’s freer formulation of the propositions and expansion of the proofs, constitute an intelligible reason why Is[hnull]āq should have undertaken a fresh translation from the Greek. Klamroth calls Is[hnull]āq’s version a model of a good translation of a mathematical text; the introductory and transitional phrases are stereotyped and few in number, the technical terms are simply and consistently rendered, and the less formal expressions connect themselves as closely with the Greek as is consistent with intelligibility and the character of the Arabic language.

Only in isolated cases does the formulation of definitions and enunciations differ to any considerable extent from the original. In general, his object seems to have been to get rid of difficulties and unevennesses in the Greek text by next devices, while at the same time giving a faithful reproduction of it..

There are curious points of contact between the versions of al-[Hnull]ajjāj and T[hnull]amacr;bit-Is[hnull]āq. For example, the definitions and enunciations of propositions are often word for word the same. Presumably this is owing to the fact that Is[hnull]āq found these definitions and enunciations already established in the schools in his time, where they would no doubt be learnt by heart, and refrained from translating them afresh, merely adopting the older version with some changes. Secondly, there is remarkable agreement between the Arabic versions as regards the figures, which show considerable variations from the figures of the Greek text, especially as regards the letters; this is also probably to be explained in the same way, all the later translators having most likely borrowed al-[hnull]ajjāj's adaptation of the Greek figures. Lastly, it is remarkable that the version of Books XI. — XIII. in the Kjφbenhavn MS. (K), purporting to be by al-[Hnull]ajjāj, is almost exactly the same as the Thābit-Is[hnull]āq version of the same Books in O. Klamroth conjectures that Is[hnull]āq may not have translated the Books on solid geometry at all, and that Thābit took them from al-[Hnull]ajjāj, only making some changes in order to fit them to the translation of Is[hnull]āq.

From the facts (1) that a[tnull]-[Tnull]īsī's edition had the same number of propositions (468) as al-[hnull]ajjāj's version, while Thābit-Is[hnull]āq's had 478, and (2) that a[tnull]-[Tnull]īsī has the same careful references to earlier propositions, Klamroth concludes that a[tnull]-[Tnull]īsī deliberately preferred al-[hnull]ajjāj's version to that of Is[hnull]āq. Heiberg, however,

points out (1) that a[tnull]-[Tnull]īsī left out VI. 12 which, if we may judge by Klamroth's silence, al-[hnull]ajjāj had, and (2) al-[hnull]ajjāj's version had one proposition less in Books I. and III. than a[tnull]-[Tnull]īsī has. Besides, in a passage quoted by [hnull]ājī Khalfa from a[tnull]-[Tnull]īsī, the latter says that "he separated the things which, in the approved editions, were taken from the archetype from the things which had been added thereto," indicating that he had compiled his edition from both the earlier translations.

There were a large number of Arabian commentaries on, or reproductions of, the Elements or portions thereof, which will be found fully noticed by Steinschneider. I shall mention here the commentators etc. referred to in the Fihrist, with a few others.

1. Abī 'l `Abbās al-Fa[dnull]l b. [hnull]ātim an-Nairīzī (born at Nairīz, died about 922) has already been mentioned. His commentary survives, as regards Books I. — VI., in the Codex Leidensis 399, I, now edited, as to four Books, by Besthorn and Heiberg, and as regards Books I. — X. in the Latin translation made by Gherard of Cremona in the 12th c. and now published by Curtze from a Cracow MS. Its importance lies mainly in the quotations from Heron and Simplicius.

2. Ahmad b. `Umar al-Karābīsī (date uncertain, probably 9th — 10th c.), "who was among the most distinguished geometers and arithmeticians."

3. Al-'Abbās b. Sa`īd al-Jauharī (fl. 830) was one of the astronomical observers under al-Ma'mīn, but devoted himself mostly to geometry. He wrote a commentary to the whole of the Elements, from the beginning to the end; also the "Book of the propositions which he added to the first book of Euclid."

4. Muh. b. `Īsā Abī `Abdallāh al-Māhānī (d. between 874 and 884) wrote, according to the Fihrist, (I) a commentary on Eucl. Book v., (2) "On

proportion," (3) "On the 26 propositions of the first Book of Euclid which are proved without reductio ad absurdum." The work "On proportion" survives and is probably identical with, or part of, the commentary on Book v. He also wrote, what is not mentioned by the Fihrist, a commentary on Eucl. Book X., a fragment of which survives in a Paris MS.

5. Abĩ Ja`far al-Khāzin (i.e. "the treasurer" or "librarian"), one of the first mathematicians and astronomers of his time, was born in Khurāsān and died between the years 961 and 971. The Fihrist speaks of him as having written a commentary on the whole of the Elements, but only the commentary on the beginning of Book X. survives (in Leiden, Berlin and Paris); therefore either the notes on the rest of the Books have perished, or the Fihrist is in error. The latter would seem more probable, for, at the end of his commentary, al-Khāzin remarks that the rest had already been commented on by Sulaimān b. 'Usma (Leiden MS.) or 'Oqba (Suter), to be mentioned below. Al-Khāzin's method is criticised unfavourably in the preface to the Oxford MS. quoted by Nicoll (see above).

6. Abĩ'l Wafā al-Bĩzjānī (940-997), one of the greatest Arabian mathematicians, wrote a commentary on the Elements, but did not complete it. His method is also unfavourably regarded in the same preface to the Oxford MS. 280. According to Hājī Khalfa, he also wrote a book on geometrical constructions, in thirteen chapters. Apparently a book answering to this description was compiled by a gifted pupil from lectures by Abĩ 'l Wafā, and a Paris MS. (Anc. fonds 169) contains a Persian translation of this work, not that of Abĩ 'l Wafā himself. An analysis of the work was given by Woepcke, and some particulars will be found in Cantor. Abĩ 'l Wafā also wrote a commentary on Diophantus, as well as a separate "book of proofs to the propositions which Diophantus used in his book and to what he (Abĩ 'l Wafā) employed in his commentary."

7. Ibn Rāhawaihi al-Arjānī also commented on Eucl. Book X..

8. `Alī b. Ahmad Abī ‘l-Qāsim al-An[tnull]ākī (d. 987) wrote a commentary on the whole book; part of it seems to survive (from the 5th Book onwards) at Oxford (Catal. MSS. orient. II. 281).

9. Sind b. `Alī Abī ‘[tnull ]-[Tnull ]aiyib was a Jew who went over to Islam in the time of al-Ma’mīn, and was received among his astronomical observers, whose head he became (about 830); he died after 864. He wrote a commentary on the whole of the Elements; “Abī `Alī saw nine books of it, and a part of the tenth.” His book “On the Apotomae and the Medials,” mentioned by the Fihrist, may be the same as, or part of, his commentary on Book X.

10. Abī Yīsuf Ya`qīb b. Muh. ar-Rāzī “wrote a commentary on Book X., and that an excellent one, at the instance of Ibn al`Amīd.”

11. The Fihrist next mentions al-Kindī (Abī Yīsuf Ya’qīb b. Ishāq b. as-Sabbāh al-Kindī, d. about 873), as the author (1) of a work “on the objects of Euclid’s book,” in which occurs the statement that the Elements were originally written by Apollonius, the carpenter (see above, and note), (2) of a book “on the improvement of Euclid’s work,” and (3) of another “on the improvement of the 14th and 15th Books of Euclid.” “He was the most distinguished man of his time, and stood alone in the knowledge of the old sciences collectively; he was called ‘the philosopher of the Arabians’; his writings treat of the most different branches of knowledge, as logic, philosophy, geometry, calculation, arithmetic, music, astronomy and others.” Among the other geometrical works of al-Kindī mentioned by the Fihrist are treatises on the closer investigation of the results of Archimedes concerning the measure of the diameter of a circle in terms of its circumference, on the construction of the figure of the two mean proportionals, on the approximate determination of the chords of the circle, on the approximate determination of

the chord (side) of the nonagon, on the division of triangles and quadrilaterals and constructions for that purpose, on the manner of construction of a circle which is equal to the surface of a given cylinder, on the division of the circle, in three chapters etc.

12. The physician Na[znull]ĩf b. Yumn (or Yaman) al-Qass ("the priest") is mentioned by the Fihrist as having seen a Greek copy of Eucl. Book X. which had 40 more propositions than that which was in general circulation (containing 109), and having determined to translate it into Arabic. Fragments of such a translation exist at Paris, Nos. 18 and 34 of the MS. 2457 (952, 2 Suppl. Arab. in Woepcke's tract); No. 18 contains "additions to some propositions of the 10th Book, existing in the Greek language." Nazĩf must have died about 990.

13. Yũ[hnull]annã b. Yūsuf b. al-[Hnull]ãrith b. al-Bitrĩq al-Qass (d. about 980) lectured on the Elements and other geometrical books, made translations from the Greek, and wrote a tract on the "proof" of the case of two straight lines both meeting a third and making with it, on one side, two angles together less than two right angles. Nothing of his appears to survive, except that a tract "on rational and irrational magnitudes," No. 48 in the Paris MS. just mentioned, is attributed to him.

14. Abũ Mu[hnull]. al-[Hnull]asan b. 'Ubaidallãh b. Sulaimãn b. Wahb (d. 901) was a geometer of distinction, who wrote works under the two distinct titles "A commentary on the difficult parts of the work of Euclid" and "The Book on Proportion." Suter thinks that another reading is possible in the case of the second title, and that it may refer to the Euclidean work "on the divisions (of figures)."

15. Qustã b. Lũqã al-Ba'labakkĩ (d. about 912), a physician, philosopher, astronomer, mathematician and translator, wrote "on the difficult passages of Euclid's book" and "on the solution of arithmetical problems from the third

book of Euclid” ; also an “introduction to geometry,” in the form of question and answer.

16. Thãbit b. Qurra (826-901), besides translating some parts of Archimedes and Books V.mdashVII. of the Conics of Apollonius, and revising Ishãq’s translation of Euclid’s Elements, also revised the translation of the Data by the same Ishãq and the book On divisions of figures translated by an anonymous writer. We are told also that he wrote the following works: (I) On the Premisses (Axioms, Postulates etc.) of Euclid, (2) On the Propositions of Euclid, (3) On the propositions and questions which arise when two straight lines are cut by a third (or on the “proof” of Euclid’s famous postulate). The last tract is extant in the MS. discovered by Woepcke (Paris 2457, 32^{o}). He is also credited with “an excellent work” in the shape of an “Introduction to the Book of Euclid,” a treatise on Geometry dedicated to Ismã’il b. Bulbul, a Compendium of Geometry, and a large number of other works for the titles of which reference may be made to Suter, who also gives particulars as to which are extant.

17. Abũ Sa’ĩd Sinãn b. Thãbit b. Qurra, the son of the translator of Euclid, followed in his father’s footsteps as geometer, astronomer and physician. He wrote an “improvement of the book of ..... on the Elements of Geometry, in which he made various additions to the original.” It is natural to conjecture that Euclid is the name missing in this description (by Ibn abĩ U[snull]aibi’a); Casiri has the name Aqãton. The latest editor of the Ta’rĩkh al-[Hnull]ukamã, however, makes the name to be Iflãton (=Plato), and he refers to the statement by the Fihrist and Ibn al-Qiftĩ attributing to Plato a work on the Elements of Geometry translated by Qust<*>ã. It is just possible, therefore, that at the time of Qus[tnull]ã the Arabs were acquainted with a book on the Elements of Geometry translated from the Greek, which they attributed to Plato. Sinãn died in 943.

18. Abũ Sahl Wĩjan (or Waijan) b. Rustam al-Kũhĩ (fl. 988), born at Kũh in [Tnull]abaristãn, a distinguished geometer and astronomer, wrote, according to the Fihrist, a "Book of the Elements" after that of Euclid; the 1st and 2nd Books survive at Cairo, and a part of the 3rd Book at Berlin (5922). He wrote also a number of other geometrical works: Additions to the 2nd Book of Archimedes on the Sphere and Cylinder (extant at Paris, at Leiden, and in the India Office), On the finding of the side of a heptagon in a circle (India Office and Cairo), On two mean proportionals (India Office), which last may be only a part of the Additions to Archimedes' On the Sphere and Cylinder, etc.

19. Abũ Na[snull]r Mu[hnull]. b. Mu[hnull]. b. [Tnull]arkhãn b. Uzlaġ al-Fãrãbĩ (870-950) wrote a commentary on the difficulties of the introductory matter to Books I. and V. This appears to survive in the Hebrew translation which is, with probability, attributed to Moses b. Tibbon.

20. Abũ 'Alĩ al-Hasan b. al-Hasan b. al-Haitham (about 9651039), known by the name Ibn al-Haitham or Abũ 'Alĩ al-Basrĩ, was a man of great powers and knowledge, and no one of his time approached him in the field of mathematical science. He wrote several works on Euclid the titles of which, as translated by Woepcke from Usaibi'a, are as follows: " 1. Commentary and abridgment of the Elements.

2. Collection of the Elements of Geometry and Arithmetic, drawn from the treatises of Euclid and Apollonius.

3. Collection of the Elements of the Calculus deduced from the principles laid down by Euclid in his Elements.

4. Treatise on "measure" after the manner of Euclid's Elements.

5. Memoir on the solution of the difficulties in Book I.

6. Memoir for the solution of a doubt about Euclid, relative to Book V.

7. Memoir on the solution of a doubt about the stereometric portion.

8. Memoir on the solution of a doubt about Book XII.

9. Memoir on the division of the two magnitudes mentioned in X. I (the theorem of exhaustion).

10. Commentary on the definitions in the work of Euclid (where Steinschneider thinks that some more general expression should be substituted for “definitions”).

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The last-named work (which Suter calls a commentary on the Postulates of Euclid) survives in an Oxford MS. (Catal. MSS. orient. I. 908) and in Algiers (1446, 1^{o}).

A Leiden MS. (966) contains his Commentary “on the difficult places” up to Book V. We do not know whether in this commentary, which the author intended to form, with the commentary on the Musãdarãt, a sort of complete commentary, he had collected the separate memoirs on certain doubts and difficult passages mentioned in the above list.

A commentary on Book V. and following Books found in a Bodleian MS. (Catal. II.) with the title “Commentary on Euclid and solution of his difficulties” is attributed to b. Haitham; this might be a continuation of the Leiden MS.

The memoir on X. 1 appears to survive at St Petersburg, MS. de l’Institut des langues orient. 192, 5^{o} (Rosen, Catal.).

21. Ibn Sĩnã, known as Avicenna (980-1037), wrote a Compendium of Euclid, preserved in a Leiden MS. No. 1445, and forming the geometrical portion of an encyclopaedic work embracing Logic, Mathematics, Physics and Metaphysics.

22. Ahmad b. al-Husain al-Ahwãzĩ al-Kãtib wrote a commentary on Book X., a fragment of which (some 10 pages) is to be found at Leiden (970), Berlin (5923) and Paris (2467, 18^{o}).

23. Na[snull]ĩraddĩn a[tnull]-[Tnull]ũsĩ (1201-1274) who, as we have seen, brought out a Euclid in two forms, wrote: “ 1. A treatise on the postulates of Euclid (Paris, 2467, 5^{o}).

2. A treatise on the 5th postulate, perhaps only a part of the foregoing (Berlin, 5942, Paris, 2467, 6^{o}).

3. Principles of Geometry taken from Euclid, perhaps identical with No. 1 above (Florence, Pal. 298).

4. 105 problems out of the Elements (Cairo). He also edited the Data (Berlin, Florence, Oxford etc.).

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24. Muh. b. Ashraf Shamsaddĩn as-Samarqandĩ (fl. 1276) wrote “Fundamental Propositions, being elucidations of 35 selected propositions of the first Books of Euclid,” which are extant at Gotha (1496 and 1497), Oxford (Catal. I. 967, 2^{o}), and Brit. Mus..

25. Mũsã b. Mu[hnull]. b. Ma[hnull]mũd, known as Qã[dnull]ĩzãde ar-Rũmĩ (i.e. the son of the judge from Asia Minor), who died between 1436 and 1446, wrote a commentary on the “Fundamental Propositions” just mentioned, of which many MSS. are extant. It contained biographical statements about Euclid alluded to above (, note).

26. Abũ Dã’ũd Sulaimãn b. ‘Uqba, a contemporary of al-Khãzin (see above, No. 5), wrote a commentary on the second half of Book X., which is, at least partly, extant at Leiden (974) under the title “On the binomials and apotomae found in the 10th Book of Euclid.”

27. The Codex Leidensis 399, 1 containing al-[Hnull]ajjãj’s translation of Books I. — VI. is said to contain glosses to it by Sa`ĩd b. Mas’ũd b. al-Qass, apparently identical with Abũ Nasr Gars al-Na’ma, son of the physician Mas’ũd b. al-Qass al-Bagdãdĩ, who lived in the time of the last Caliph al-Musta’sim (d. 1258).

28. Abũ Mu[hnull]ammad b. Abdalbãqĩ al-Bag[dnull]ãdĩ al-Fara[dnull]ĩ (d. 1141, at the age of over 70 years) is stated in the Ta'rĩkh al-[Hnull]ukamã to have written an excellent commentary on Book X. of the Elements, in which he gave numerical examples of the propositions. This is published in Curtze's edition of an-Nairĩzĩ where it occupies pages 252-386.

29. Ya[hnull]yã b. Mu[hnull]. b. 'Abdãn b. 'Abdalwã[hnull]id, known by the name of Ibn al-Lubũdĩ (1210-1268), wrote a Compendium of Euclid, and a short presentation of the postulates.

30. Abũ 'Abdallãh Mu[hnull]. b. Mu'ãdh al-Jayyãnĩ wrote a commentary on Eucl. Book V. which survives at Algiers (1446, 3^{o}).

31. Abũ Na[snull]r Mansũr b. 'Aliã b.'Irãq wrote, at the instance of Mu[hnull]. b. A[hnull]mad Abũ 'r-Rai[hnull]ãn al-Bĩrũnĩ (973-1048), a tract "on a doubtful (difficult) passage in Eucl. Book XIII." (Berlin, 5925).

CHAPTER VIII. PRINCIPAL TRANSLATIONS AND EDITIONS OF THE ELEMENTS.

Cicero is the first Latin author to mention Euclid; but it is not likely that in Cicero's time Euclid had been translated into Latin or was studied to any considerable extent by the Romans; for, as Cicero says in another place, while geometry was held in high honour among the Greeks, so that nothing was more brilliant than their mathematicians, the Romans limited its scope by having regard only to its utility for measurements and calculations. How very little theoretical geometry satisfied the Roman agrimensores is evidenced by the work of Balbus de mensuris, where some of the definitions of Eucl. Book I. are given. Again, the extracts from the Elements found in the fragment attributed to Censorinus (fl. 238 A.D) are confined to the definitions, postulates, and common notions. But by degrees the Elements passed even among the Romans into the curriculum of a liberal education; for Martianus Capella speaks of the effect of the enunciation of the proposition "how to construct an equilateral triangle on a given straight line" among a company of philosophers, who, recognising the first proposition of the Elements, straightway break out into encomiums on Euclid. But the Elements were then (c. 470 A.D.) doubtless read in Greek; for what Martianus Capella gives was drawn from a Greek source, as is shown by the occurrence of Greek words and by the wrong translation of I. def. 1 ("punctum vero est cuius pars nihil est"). Martianus may, it is true, have quoted, not from Euclid himself, but from Heron or some other ancient source.

But it is clear from a certain palimpsest at Verona that some scholar had already attempted to translate the Elements into Latin. This palimpsest has part of the "Moral reflections on the Book of Job" by Pope Gregory the Great written in a hand of the 9th c. above certain fragments which in the opinion of the best judges date from the 4th c. Among these are fragments of Vergil

and of Livy, as well as a geometrical fragment which purports to be taken from the 14th and 15th Books of Euclid. As a matter of fact it is from Books XII. and XIII. and is of the nature of a free rendering, or rather a new arrangement, of Euclid with the propositions in different order. The MS. was evidently the translator's own copy, because some words are struck out and replaced by synonyms. We do not know whether the translator completed the translation of the whole, or in what relation his version stood to our other sources.

Magnus Aurelius Cassiodorus (b. about 475 A.D.) in the geometrical part of his encyclopaedia De artibus ac disciplinis liberalium literarum says that geometry was represented among the Greeks by Euclid, Apollonius, Archimedes, and others, "of whom Euclid was given us translated into the Latin language by the same great man Boethius" ; also in his collection of letters is a letter from Theodoric to Boethius containing the words, "for in your translations...Nicomachus the arithmetician, and Euclid the geometer, are heard in the Ausonian tongue." The so-called Geometry of Boethius which has come down to us by no means constitutes a translation of Euclid. The MSS. variously give five, four, three or two Books, but they represent only two distinct compilations, one normally in five Books and the other in two. Even the latter, which was edited by Friedlein, is not genuine,but appears to have been put together in the 11th c., from various sources. It begins with the definitions of Eucl. I., and in these are traces of perfectly correct readings which are not found even in the MSS. of the 10th c., but which can be traced in Proclus and other ancient sources; then come the Postulates (five only), the Axioms (three only), and after these some definitions of Eucl. II., III., IV. Next come the enunciations of Eucl. I., of ten propositions of Book II., and of some from Books III., IV., but always without proofs; there follows an extraordinary passage which indicates that

the author will now give something of his own in elucidation of Euclid, though what follows is a literal translation of the proofs of Eucl. I. 1-3. This latter passage, although it affords a strong argument against the genuineness of this part of the work, shows that the Pseudoboethius had a Latin translation of Euclid from which he extracted the three propositions.

Curtze has reproduced, in the preface to his edition of the translation by Gherard of Cremona of an-Nairīzī's Arabic commentary on Euclid, some interesting fragments of a translation of Euclid taken from a Munich MS. of the 10th c. They are on two leaves used for the cover of the MS. (Bibliothecae Regiae Universitatis Monacensis 2^{o} 757) and consist of portions of Eucl. I. 37, 38 and II. 8, translated literally word for word from the Greek text. The translator seems to have been an Italian (cf. the words "capitolo nono" used for the ninth prop. of Book II.) who knew very little Greek and had moreover little mathematical knowledge. For example, he translates the capital letters denoting points in figures as if they were numerals: thus τὰ ΑΒΓ, ΔΕΖ is translated "que primo secundo et tertio quarto quinto et septimo," T becomes "tricentissimo" and so on. The Greek MS. which he used was evidently written in uncials, for ΔΕΖΘ becomes in one place "quod autem septimo nono," showing that he mistook ΔΕ for the particle δέ, and καὶ ὁ ΣΤΥ is rendered "sicut tricentissimo et quadringentissimo," showing that the letters must have been written KAIOCTU.

The date of the Englishman Athelhard (*Æthelhard*) is approximately fixed by some remarks in his work Perdifficiles Quaestiones Naturales which, on the ground of the personal allusions they contain, must be assigned to the first thirty years of the 12th c. He wrote a number of philosophical works. Little is known about his life. He is said to have studied at Tours and Laon, and to have lectured at the latter school. He travelled to Spain, Greece, Asia Minor

and Egypt, and acquired a knowledge of Arabic, which enabled him to translate from the Arabic into Latin, among other works, the Elements of Euclid. The date of this translation must be put at about 1120. MSS. purporting to contain Athelhard's version are extant in the British Museum (Harleian No. 5404 and others), Oxford (Trin. Coll. 47 and Ball. Coll. 257 of 12th c.), Nürnberg (Johannes Regiomontanus' copy) and Erfurt.

Among the very numerous works of Gherard of Cremona (1114 — 1187) are mentioned translations of "15 Books of Euclid" and of the Data. Till recently this translation of the Elements was supposed to be lost; but Axel Anthon Björnbo has succeeded (1904) in discovering a translation from the Arabic which is different from the two others known to us (those by Athelhard and Campanus respectively), and which he, on grounds apparently convincing, holds to be Gherard's. Already in 1901 Björnbo had found Books X. — XV. of this translation in a MS. at Rome (Codex Reginensis lat. 1268 of 14th c.); but three years later he had traced three MSS. containing the whole of the same translation at Paris (Cod. Paris. 7216, 15th c.), Boulogne-sur-Mer (Cod. Bononiens. 196, 14th c.), and Bruges (Cod. Brugens. 521, 14th c.), and another at Oxford (Cod. Digby 174, end of 12th c.) containing a fragment, XI. 2 to XIV. The occurrence of Greek words in this translation such as rombus, romboides (where Athelhard keeps the Arabic terms), ambligonius, orthogonius, gnomo, pyramis etc., show that the translation is independent of Athelhard's. Gherard appears to have had before him an old translation of Euclid from the Greek which Athelhard also often followed, especially in his terminology, using it however in a very different manner. Again, there are some Arabic terms, e.g. meguar for axis of rotation, which Athelhard did not use, but which is found in almost all the translations that are with certainty attributed to Gherard of Cremona; there occurs also the expression "superficies equidistantium laterum et rectorum angulorum,"

found also in Gherard's translation of an-Nairĩzĩ, where Athelhard says "parallelogrammum rectangulum." The translation is much clearer than Athelhard's: it is neither abbreviated nor "edited" as Athelhard's appears to have been; it is a word-for-word translation of an Arabic MS. containing a revised and critical edition of Thãbit's version. It contains several notes quoted from Thãbit himself (Thebit dixit), e.g. about alternative proofs etc. which Thãbit found "in another Greek MS.," and is therefore a further testimony to Thãbit's critical treatment of the text after Greek MSS. The new editor also added critical remarks of his own, e.g. on other proofs which he found in other Arabic versions, but not in the Greek: whence it is clear that he compared the Thãbit version before him with other versions as carefully as Thãbit collated the Greek MSS. Lastly, the new editor speaks of "Thebit qui transtulit hunc librum in arabicam linguam" and of "translatio Thebit," which may tend to confirm the statement of al-Qiftĩ who credited Thãbit with an independent translation, and not (as the Fihrist does) with a mere improvement of the version of Is[hnull]ãq b. Hunain.

Gherard's translation of the Arabic commentary of an-Nairĩzĩ on the first ten Books of the Elements was discovered by Maximilian Curtze in a MS. at Cracow and published as a supplementary volume to Heiberg and Menge's Euclid: it will often be referred to in this work.

Next in chronological order comes Johannes Campanus (Campano) of Novara. He is mentioned by Roger Bacon (1214-1294) as a prominent mathematician of his time, and this indication of his date is confirmed by the fact that he was chaplain to Pope Urban IV, who was Pope from 1261 to 1281. His most important achievement was his edition of the Elements including the two Books XIV. and XV. which are not Euclid's. The sources of Athelhard's and Campanus' translations, and the relation between them, have been the subject of much discussion, which does not seem to have led as

yet to any definite conclusion. Cantor (II,) gives references and some particulars. It appears that there is a MS. at Munich (Cod. lat. Mon. 13021) written by Sigboto in the 12th c. at Prüfning near Regensburg, and denoted by Curtze by the letter R, which contains the enunciations of part of Euclid. The Munich MSS. of Athelhard and Campanus' translations have many enunciations textually identical with those in R, so that the source of all three must, for these enunciations, have been the same; in others Athelhard and Campanus diverge completely from R, which in these places follows the Greek text and is therefore genuine and authoritative. In the 32nd definition occurs the word "elinuam," the Arabic term for "rhombus," and throughout the translation are a number of Arabic figures. But R was not translated from the Arabic, as is shown by (among other things) its close resemblance to the translation from Euclid given on p sqq. of the Gromatici Veteres and to the so-called geometry of Boethius. The explanation of the Arabic figures and the word "elinuam" in Def. 32 appears to be that R was a late copy of an earlier original with corruptions introduced in many places; thus in Def. 32 a part of the text was completely lost and was supplied by some intelligent copyist who inserted the word "elinuam," which was known to him, and also the Arabic figures. Thus Athelhard certainly was not the first to translate Euclid into Latin; there must have been in existence before the 11th c. a Latin translation which was the common source of R, the passage in the Gromatici, and "Boethius." As in the two latter there occur the proofs as well as the enunciations of I. 1mdash;3, it is possible that this translation originally contained the proofs also. Athelhard must have had before had before him this translation of the enunciations, as well as the Arabic source from which he obtained his proofs. That some sort of translation, or at least fragments of one, were available before Athelhard's time even in England is indicated by some old English verses:

The clerk Euclide on this wyse hit fonde Thys craft of gemetry yn Egypte londe Yn Egypte he tawghte hyt ful wyde, In dyvers londe on every syde. Mony erys afterwarde y understonde Yer that the craft com ynto thys londe. Thys craft com into England, as y yow say, Yn tyme of good kyng Adelstone's day,

which would put the introduction of Euclid into England as far back as 924-940 A.D.

We now come to the relation between Athelhard and Campanus. That their translations were not independent, as Weissenborn would have us believe, is clear from the fact that in all MSS. and editions, apart from orthographical differences and such small differences as are bound to arise when MSS. are copied by persons with some knowledge of the subject-matter, the definitions, postulates, axioms, and the 364 enunciations are word for word identical in Athelhard and Campanus; and this is the case not only where both have the same text as R but where they diverge from it. Hence it would seem that Campanus used Athelhard's translation and only developed the proofs by means of another redaction of the Arabian Euclid. It is true that the difference between the proofs of the propositions in the two translations is considerable; Athelhard's are short and compressed, Campanus' clearer and more complete, following the Greek text more closely, though still at some distance. Further, the arrangement in the two is different; in Athelhard the proofs regularly precede the enunciations, Campanus follows the usual order. It is a question how far the differences in the proofs, and certain additions in each, are due to the two translators themselves or go back to Arabic originals. The latter supposition seems to Curtze and Cantor the more probable one. Curtze's general view of the relation of Campanus to Athelhard is to the effect that Athelhard's translation was gradually altered, from the form in which it appears in the two Erfurt MSS. described by Weissenborn, by

successive copyists and commentators who had Arabic originals before them, until it took the form which Campanus gave it and in which it was published. In support of this view Curtze refers to Regiomontanus' copy of the Athelhard-Campanus translation. In Regiomontanus' own preface the title is given, and this attributes the translation to Athelhard; but, while this copy agrees almost exactly with Athelhard in Book I., yet, in places where Campanus is more lengthy, it has similar additions, and in the later Books, especially from Book III. onwards, agrees absolutely with Campanus; Regiomontanus, too, himself implies that, though the translation was Athelhard's, Campanus had revised it; for he has notes in the margin such as the following, "Campani est hec," "dubito an demonstret hic Campanus" etc.

We come now to the printed editions of the whole or of portions of the Elements. This is not the place for a complete bibliography, such as Riccardi has attempted in his valuable memoir issued in five parts between 1887 and 1893, which makes a large book in itself. I shall confine myself to saying something of the most noteworthy translations and editions. It will be convenient to give first the Latin translations which preceded the publication of the editio princeps of the Greek text in 1533, next the most important editions of the Greek text itself, and after them the most important translations arranged according to date of first appearance and languages, first the Latin translations after 1533, then the Italian, German, French and English translations in order.

It may be added here that the first allusion, in the West, to the Greek text as still extant is found in Boccaccio's commentary on the Divina Commedia of Dante. Next Johannes Regiomontanus, who intended to publish the Elements after the version of Campanus, but with the latter's mistakes corrected, saw in Italy (doubtless when staying with his friend Bessarion)

some Greek MSS. and noticed how far they differed from the Latin version (see a letter of his written in the year 1471 to Christian Roder of Hamburg).

I. Latin translations prior to 1533.

1482. In this year appeared the first printed edition of Euclid, which was also the first printed mathematical book of any importance. This was printed at Venice by Erhard Ratdolt and contained Campanus' translation. Ratdolt belonged to a family of artists at Augsburg, where he was born about 1443. Having learnt the trade of printing at home, he went in 1475 to Venice, and founded there a famous printing house which he managed for II years, after which he returned to Augsburg and continued to print important books until 1516. He is said to have died in 1528. Kästner gives a short description of this first edition of Euclid and quotes the dedication to Prince Mocenigo of Venice which occupies the page opposite to the first page of text. The book has a margin of 2 1/2 inches, and in this margin are placed the figures of the propositions. Ratdolt says in his dedication that at that time, although books by ancient and modern authors were printed every day in Venice, little or nothing mathematical had appeared: a fact which he puts down to the difficulty involved by the figures, which no one had up to that time succeeded in printing. He adds that after much labour he had discovered a method by which figures could be produced as easily as letters. Experts are in doubt as to the nature of Ratdolt's discovery. Was it a method of making figures up out of separate parts of figures, straight or curved lines, put together as letters are put together to make words? In a life of Joh. Gottlob Immanuel Breitkopf, a contemporary of Kästner's own, this member of the great house of Breitkopf is credited with this particular discovery. Experts in that same house expressed the opinion that Ratdolt's figures were woodcuts,

while the letters denoting points in the figures were like the other letters in the text; yet it was with carved wooden blocks that printing began. If Ratdolt was the first to print geometrical figures, it was not long before an emulator arose; for in the very same year Mattheus Cordonis of Windischgrätz employed woodcut mathematical figures in printing Oresme's De latitudinibus. How eagerly the opportunity of spreading geometrical knowledge was seized upon is proved by the number of editions which followed in the next few years. Even the year 1482 saw two forms of the book, though they only differ in the first sheet. Another edition came out in 1486 (Ulmae, apud Io. Regerum) and another in 1491 (Vincentiae per Leonardum de Basilea et Gulielmum de Papia), but without the dedication to Mocenigo who had died in the meantime (1485). If Campanus added anything of his own, his additions are at all events not distinguished by any difference of type or otherwise; the enunciations are in large type, and the rest is printed continuously in smaller type. There are no superscriptions to particular passages such as Euclides ex Campano, Campanus, Campani additio, or Campani annotatio, which are found for the first time in the Paris edition of 1516 giving both Campanus' version and that of Zamberti (presently to be mentioned).

1501. G. Valla included in his encyclopaedic work De expetendis et fugiendis rebus published in this year at Venice (in aedibus Aldi Romani) a number of propositions with proofs and scholia translated from a Greek MS. which was once in his possession (cod. Mutin. III B, 4 of the 15th c.).

1505. In this year Bartolomeo Zamberti (Zambertus) brought out at Venice the first translation, from the Greek text, of the whole of the Elements. From the title, as well as from his prefaces to the Catoptrica and Data, with their allusions to previous translators "who take some things out of authors, omit some, and change some," or "to that most barbarous translator" who filled a

volume purporting to be Euclid's "with extraordinary scarecrows, nightmares and phantasies," one object of Zamberti's translation is clear. His animus against Campanus appears also in a number of notes, e.g. when he condemns the terms "helmuain" and "helmuariphe" used by Campanus as barbarous, un-Latin etc., and when he is roused to wrath by Campanus' unfortunate mistranslation of v. Def. 5. He does not seem to have had the penetration to see that Campanus was translating from an Arabic, and not from a Greek, text. Zamberti tells us that he spent seven years over his translation of the thirteen Books of the Elements. As he seems to have been born in 1473, and the Elements were printed as early as 1500, though the complete work (including the Phaenomena, Optica, Catoptrica, Data etc.) has the date 1505 at the end, he must have translated Euclid before the age of 30. Heiberg has not been able to identify the MS. of the Elements which Zamberti used; but it is clear that it belonged to the worse class of MSS., since it contains most of the interpolations of the Theonine variety. Zamberti, as his title shows, attributed the proofs to Theon.

1509. As a counterblast to Zamberti, Luca Paciuolo brought out an edition of Euclid, apparently at the expense of Ratdolt, at Venice (per Paganinum de Paganinis), in which he set himself to vindicate Campanus. The title-page of this now very rare edition begins thus: "The works of Euclid of Megara, a most acute philosopher and without question the chief of all mathematicians, translated by Campanus their most faithful interpreter." It proceeds to say that the translation had been, through the fault of copyists, so spoiled and deformed that it could scarcely be recognised as Euclid. Luca Paciuolo accordingly has polished and emended it with the most critical judgment, has corrected 129 figures wrongly drawn and added others, besides supplying short explanations of difficult passages. It is added that Scipio Vegius of Milan, distinguished for his knowledge "of both languages" (i.e. of course

Latin and Greek), as well as in medicine and the more sublime studies, had helped to make the edition more perfect. Though Zamberti is not once mentioned, this latter remark must have reference to Zamberti's statement that his translation was from the Greek text; and no doubt Zamberti is aimed at in the wish of Paciuolo's "that others too would seek to acquire knowledge instead of merely showing off, or that they would not try to make a market of the things of which they are ignorant, as it were (selling) smoke ." Weissenborn observes that, while there are many trivialities in Paciuolo's notes, they contain some useful and practical hints and explanations of terms, besides some new proofs which of course are not difficult if one takes the liberty, as Paciuolo does, of divering from Euclid's order and assuming for the proof of a proposition results not arrived at till later. Two not inapt terms are used in this edition to describe the figures of III. 7, 8, the former of which is called the goose's foot (pes anseris), the second the peacock's tail (cauda pavonis) Paciuolo as the castigator of Campanus' translation, as he calls himself, failed to correct the mistranslation of V. Def. 5. Before the fifth Book he inserted a discourse which he gave at Venice on the 15th August, 1508, in S. Bartholomew's Church, before a select audience of 500, as an introduction to his elucidation of that Book.

1516. The first of the editions giving Campanus' and Zamberti's translations in conjunction was brought out at Paris (in officina Henrici Stephani e regione scholae Decretorum). The idea that only the enunciations were Euclid's, and that Campanus was the author of the proofs in his translation, while Theon was the author of the proofs in the Greek text, reappears in the title of this edition; and the enunciations of the added Books XIV., XV. are also attributed to Euclid, Hypsicles being credited with the proofs. The date is not on the title-page nor at the end, but the letter of dedication to François Briconnet by Jacques Lefèvre is dated the day after the

Epiphany, 1516. The figures are in the margin. The arrangement of the propositions is as follows: first the enunciation with the heading Euclides ex Campano, then the proof with the note Campanus, and after that, as Campani additio, any passage found in the edition of Campanus' translation but not in the Greek text; then follows the text of the enunciation translated from the Greek with the heading Euclides ex Zamberto, and lastly the proof headed Theo ex Zamberto. There are separate figures for the two proofs. This edition was reissued with few changes in 1537 and 1546 at Basel (apud Iohannem Hervagium), but with the addition of the Phaenomena, Optica, Catoptrica etc. For the edition of 1537 the Paris edition of 1516 was collated with "a Greek copy" (as the preface says) by Christian Herlin, professor of mathematical studies at Strassburg, who however seems to have done no more than correct one or two passages by the help of the Basel editio princeps (1533), and add the Greek word in cases where Zamberti's translation of it seemed unsuitable or inaccurate.

We now come to

II. Editions of the Greek text.

1533 is the date of the editio princeps, the title-page of which reads as follows:

*e*u*k*l*e*i*d*o*u *s*t*o*i*x*e*i*w*n *b*i*b*l<*> *i*e<*><*> *e*k *t*w*n *q*e*w*n*o*s *s*u*n*o*u*s*i*w*n. *ei)s tou= au)tou= to\ prw=ton, e)chghma/twn *pro/klou bibl. d. Adiecta praefatiuncula in qua de disciplinis Mathematicis nonnihil. BASILEAE APVD IOAN. HERVAGIVM ANNO M.D.XXXIII. MENSE SEPTEMBRI.

The editor was Simon Grynaeus the elder (d. 1541), who, after working at Vienna and Ofen, Heidelberg and Tübingen, taught last of all at Basel, where

theology was his main subject. His “praefatiuncula” is addressed to an Englishman, Cuthbert Tonstall (14741559), who, having studied first at Oxford, then at Cambridge, where he became Doctor of Laws, and afterwards at Padua, where in addition he learnt mathematics — mostly from the works of Regiomontanus and Paciuolo — wrote a book on arithmetic as “a farewell to the sciences,” and then, entering politics, became Bishop of London and member of the Privy Council, and afterwards (1530) Bishop of Durham. Grynaeus tells us that he used two MSS. of the text of the Elements, entrusted to friends of his, one at Venice by “Lazarus Bayfius” (Lazare de Baïf, then the ambassador of the King of France at Venice), the other at Paris by “Ioann. Rvellius” (Jean Ruel, a French doctor and a Greek scholar), while the commentaries of Proclus were put at the disposal of Grynaeus himself by “Ioann. Claymundus” at Oxford. Heiberg has been able to identify the two MSS. used for the text; they are (1) cod. Venetus Marcianus 301 and (2) cod. Paris. gr. 2343 of the 16th c., containing Books I. — XV., with some scholia which are embodied in the text. When Grynaeus notes in the margin the readings from “the other copy,” this “other copy” is as a rule the Paris MS., though sometimes the reading of the Paris MS. is taken into the text and the “other copy” of the margin is the Venice MS. Besides these two MSS. Grynaeus consulted Zamberti, as is shown by a number of marginal notes referring to “Zampertus” or to “latinum exemplar” in certain propositions of Books IX. — XI. When it is considered that the two MSS. used by Grynaeus are among the worst, it is obvious how entirely unauthoritative is the text of the editio princeps. Yet it remained the source and foundation of later editions of the Greek text for a long period, the editions which followed being designed, not for the purpose of giving, from other MSS., a text more nearly representing what Euclid himself wrote, but of supplying a handy compendium to students at a moderate price.

1536. Orontius Finaeus (Oronce Fine) published at Paris (apud Simonem Colinaeum) “demonstrations on the first six books of Euclid’s elements of geometry,” “in which the Greek text of Euclid himself is inserted in its proper places, with the Latin translation of Barth. Zamberti of Venice,” which seems to imply that only the enunciations were given in Greek. The preface, from which Kästner quotes, says that the University of Paris at that time required, from all who aspired to the laurels of philosophy, a most solemn oath that they had attended lectures on the said first six Books. Other editions of Fine’s work followed in 1544 and 1551.

1545. The enunciations of the fifteen Books were published in Greek, with an Italian translation by Angelo Caiani, at Rome (apud Antonium Bladum Asulanum). The translator claims to have corrected the books and “purged them of six hundred things which did not seem to savour of the almost divine genius and the perspicuity of Euclid”

1549. Joachim Camerarius published the enunciations of the first six Books in Greek and Latin (Leipzig). The book, with preface, purports to be brought out by Rhaeticus (1514-1576), a pupil of Copernicus. Another edition with proofs of the propositions of the first three Books was published by Moritz Steinmetz in 1577 (Leipzig); a note by the printer attributes the preface to Camerarius himself.

1550. Ioan. Scheubel published at Basel (also per Ioan. Hervagium) the first six Books in Greek and Latin “together with true and appropriate proofs of the propositions, without the use of letters” (i.e. letters denoting points in the figures), the various straight lines and angles being described in words.

1557 (also 1558). Stephanus Gracilis published another edition (repeated 1573, 1578, 1598) of the enunciations (alone) of Books I. — XV. in Greek and Latin at Paris (apud Gulielmum Cavellat). He remarks in the preface that for want of time he had changed scarcely anything in Books I. — VI., but in

the remaining Books he had emended what seemed obscure or inelegant in the Latin translation, while he had adopted in its entirety the translation of Book X. by Pierre Mondoré (Petrus Montaureus), published separately at Paris in 1551. Gracilis also added a few "scholia."

1564. In this year Conrad Dasypodius (Rauchfuss), the inventor and maker of the clock in Strassburg cathedral, similar to the present one, which did duty from 1571 to 1789, edited (Strassburg, Chr. Mylius) (1) Book I. of the Elements in Greek and Latin with scholia, (2) Book II. in Greek and Latin with Barlaam's arithmetical version of Book II., and (3) the enunciations of the remaining Books III. — XIII. Book I. was reissued with "vocabula quaedam geometrica" of Heron, the enunciations of all the Books of the Elements, and the other works of Euclid, all in Greek and Latin. In the preface to (1) he says that it had been for twenty-six years the rule of his school that all who were promoted from the classes to public lectures should learn the first Book, and that he brought it out, because there were then no longer any copies to be had, and in order to prevent a good and fruitful regulation of his school from falling through. In the preface to the edition of 1571 he says that the first Book was generally taught in all gymnasia and that it was prescribed in his school for the first class. In the preface to (3) he tells us that he published the enunciations of Books III. — XIII. in order not to leave his work unfinished, but that, as it would be irksome to carry about the whole work of Euclid in extenso, he thought it would be more convenient to students of geometry to learn the Elements if they were compressed into a smaller book.

1620. Henry Briggs (of Briggs' logarithms) published the first six Books in Greek with a Latin translation after Commandinus, "corrected in many places" (London, G. Jones).

1703 is the date of the Oxford edition by David Gregory which, until the issue of Heiberg and Menge's edition, was still the only edition of the complete works of Euclid. In the Latin translation attached to the Greek text Gregory says that he followed Commandinus in the main, but corrected numberless passages in it by means of the books in the Bodleian Library which belonged to Edward Bernard (1638-1696), formerly Savilian Professor of Astronomy, who had conceived the plan of publishing the complete works of the ancient mathematicians in fourteen volumes, of which the first was to contain Euclid's Elements I. — XV. As regards the Greek text, Gregory tells us that he consulted, as far as was necessary, not a few MSS. of the better sort, bequeathed by the great Savile to the University, as well as the corrections made by Savile in his own hand in the margin of the Basel edition. He had the help of John Hudson, Bodley's Librarian, who punctuated the Basel text before it went to the printer, compared the Latin version with the Greek throughout, especially in the Elements and Data, and, where they differed or where he suspected the Greek text, consulted the Greek MSS. and put their readings in the margin if they agreed with the Latin and, if they did not agree, affixed an asterisk in order that Gregory might judge which reading was geometrically preferable. Hence it is clear that no Greek MS., but the Basel edition, was the foundation of Gregory's text, and that Greek MSS. were only referred to in the special passages to which Hudson called attention.

1814-1818. A most important step towards a good Greek text was taken by F. Peyrard, who published at Paris, between these years, in three volumes, the Elements and Data in Greek, Latin and French. At the time (1808) when Napoleon was having valuable MSS. selected from Italian libraries and sent to Paris, Peyrard managed to get two ancient Vatican MSS. (190 and 1038) sent to Paris for his use (Vat. 204 was also at Paris at the time, but all three

were restored to their owners in 1814). Peyrard noticed the excellence of Cod. Vat. 190, adopted many of its readings, and gave in an appendix a conspectus of these readings and those of Gregory's edition; he also noted here and there readings from Vat. 1038 and various Paris MSS. He therefore pointed the way towards a better text, but committed the error of correcting the Basel text instead of rejecting it altogether and starting afresh.

1824-1825. A most valuable edition of Books I. — VI. is that of J. G. Camerer (and C. F. Hauber) in two volumes published at Berlin. The Greek text is based on Peyrard, although the Basel and Oxford editions were also used. There is a Latin translation and a collection of notes far more complete than any other I have seen and well nigh inexhaustible. There is no editor or commentator of any mark who is not quoted from; to show the variety of important authorities drawn upon by Camerer, I need only mention the following names: Proclus, Pappus, Tartaglia, Commandinus, Clavius, Peletier, Barrow, Borelli, Wallis, Tacquet, Austin, Simson, Playfair. No words of praise would be too warm for this veritable encyclopaedia of information.

1825. J. G. C. Neide edited, from Peyrard, the text of Books I. — VI., XI. and XII. (Halis Saxoniae).

1826-9. The last edition of the Greek text before Heiberg's is that of E. F. August, who followed the Vatican MS. more closely than Peyrard did, and consulted at all events the Viennese MS. Gr. 103 (Heiberg's V). August's edition (Berlin, 1826-9) contains Books I.-XIII.

III. Latin versions or commentaries after 1533.

1545. Petrus Ramus (Pierre de la Ramée, 1515-1572) is credited with a translation of Euclid which appeared in 1545 and again in 1549 at Paris.

Ramus, who was more rhetorician and logician than geometer, also published in his Scholae mathematicae (1559, Frankfurt; 1569, Basel) what amounts to a series of lectures on Euclid's Elements, in which he criticises Euclid's arrangement of his propositions, the definitions, postulates and axioms, all from the point of view of logic.

1557. Demonstrations to the geometrical Elements of Euclid, six Books, by Peletarius (Jacques Peletier). The second edition (1610) contained the same with the addition of the "Greek text of Euclid" ; but only the enunciations of the propositions, as well as the definitions etc., are given in Greek (with a Latin translation), the rest is in Latin only. He has some acute observations, for instance about the "angle" of contact.

1559. Johannes Buteo, or Borrel (1492-1572), published in an appendix to his book De quadratura circuli some notes "on the errors of Campanus, Zambertus, Orontius, Peletarius, Pena, interpreters of Euclid." Buteo in these notes proved, by reasoned argument based on original authorities, that Euclid himself and not Theon was the author of the proofs of the propositions.

1566. Franciscus Flussates Candalla (François de Foix, Comte de Candale, 1502-1594) "restored" the fifteen Books, following, as he says, the terminology of Zamberti's translation from the Greek, but drawing, for his proofs, on both Campanus and Theon (i.e. Zamberti) except where mistakes in them made emendation necessary. Other editions followed in 1578, 1602, 1695 (in Dutch).

1572. The most important Latin translation is that of Commandinus (1509-1575) of Urbino, since it was the foundation of most translations which followed it up to the time of Peyrard, including that of Simson and therefore of those editions, numerous in England, which give Euclid "chiefly after the text of Simson." Simson's first (Latin) edition (1756) has "ex versione Latina Federici Commandini" on the title-page. Commandinus not only followed the

original Greek more closely than his predecessors but added to his translation some ancient scholia as well as good notes of his own. The title of his work is " Euclidis elementorum libri XV, una cum scholiis antiquis. A Federico Commandino Urbinate nuper in latinum conversi, commentariisque quibusdam illustrati (Pisauri, apud Camillum Francischinum).

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He remarks in his preface that Orontius Finaeus had only edited six Books without reference to any Greek MS., that Peletarius had followed Campanus' version from the Arabic rather than the Greek text, and that Candalla had diverged too far from Euclid, having rejected as inelegant the proofs given in the Greek text and substituted faulty proofs of his own. Commandinus appears to have used, in addition to the Basel editio princeps, some Greek MS., so far not identified; he also extracted his "scholia antiqua" from a MS. of the class of Vat. 192 containing the scholia distinguished by Heiberg as "Schol. Vat." New editions of Commandinus' translation followed in 1575 (in Italian), 1619, 1749 (in English, by Keill and Stone), 1756 (Books I. — VI., XI., XII. in Latin and English, by Simson), 1763 (Keill). Besides these there were many editions of parts of the whole work, e.g. the first six Books.

1574. The first edition of the Latin version by Clavius (Christoph Klau [?]. born at Bamberg 1537, died 1612) appeared in 1574, and new editions of it in 1589, 1591, 1603, 1607, 1612. It is not a translation, as Clavius himself states in the preface, but it contains a vast amount of notes collected from previous commentators and editors, as well as some good criticisms and elucidations of his own. Among other things, Clavius finally disposed of the error by which Euclid had been identified with Euclid of Megara. He speaks of the differences between Campanus who followed the Arabic tradition and the "commentaries of Theon," by which he appears to mean the Euclidean proofs as handed down by Theon; he complains of predecessors who have either

only given the first six Books, or have rejected the ancient proofs and substituted worse proofs of their own, but makes an exception as regards Commandinus, “a geometer not of the common sort, who has lately restored Euclid, in a Latin translation, to his original brilliancy.” Clavius, as already stated, did not give a translation of the Elements but rewrote the proofs, compressing them or adding to them, where he thought that he could make them clearer. Altogether his book is a most useful work.

1621. Henry Savile’s lectures (Praelectiones tresdecim in principium Elementorum Euclidis Oxoniae habitae MDC.XX., Oxonii 1621), though they do not extend beyond I. 8, are valuable because they grapple with the difficulties connected with the preliminary matter, the definitions etc., and the tacit assumptions contained in the first propositions.

1654. André Tacquet’s Elementa geometriae planae et solidae containing apparently the eight geometrical Books arranged for general use in schools. It came out in a large number of editions up to the end of the eighteenth century.

1655. Barrow’s Euclidis Elementorum Libri XV breviter demonstrati is a book of the same kind. In the preface (to the edition of 1659) he says that he would not have written it but for the fact that Tacquet gave only eight Books of Euclid. He compressed the work into a very small compass (less than 400 small pages, in the edition of 1659, for the whole of the fifteen Books and the Data) by abbreviating the proofs and using a large quantity of symbols (which, he says, are generally Oughtred’s). There were several editions up to 1732 (those of 1660 and 1732 and one or two others are in English).

1658. Giovanni Alfonso Borelli (1608-1679) published Euclides restitutus, on apparently similar lines, which went through three more editions (one in Italian, 1663).

1660. Claude François Milliet Dechales' eight geometrical Books of Euclid's Elements made easy. Dechales' versions of the Elements had great vogue, appearing in French, Italian and English as well as Latin. Riccardi enumerates over twenty editions.

1733. Saccheri's Euclides ab omni naevo vindicatus sive conatus geometricus quo stabiliuntur prima ipsa geometriae principia is important for his elaborate attempt to prove the parallel-postulate, forming an important stage in the history of the development of nonEuclidean geometry.

1756. Simson's first edition, in Latin and in English. The Latin title is " Euclidis elementorum libri priores sex, item undecimus et duodecimus, ex versione latina Federici Commandini; sublatis iis quibus olim libri hi a Theone, aliisve, vitiati sunt, et quibusdam Euclidis demonstrationibus restitutis. A Roberto Simson M.D. Glasguae, in aedibus Academicis excudebant Robertus et Andreas Foulis, Academiae typographi.

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1802. Euclidis elementorum libri priores XII ex Commandini et Gregorii versionibus latinis. In usum juventutis Academicae...by Samuel Horsley, Bishop of Rochester. (Oxford, Clarendon Press.)

IV. Italian versions or commentaries.

1543. Tartaglia's version, a second edition of which was published in 1565, and a third in 1585. It does not appear that he used any Greek text, for in the edition of 1565 he mentions as available only "the first translation by Campano," "the second made by Bartolomeo Zamberto Veneto who is still alive," "the editions of Paris or Germany in which they have included both the aforesaid translations," and "our own translation into the vulgar (tongue)."

1575. Commandinus' translation turned into Italian and revised by him.

1613. The first six Books "reduced to practice" by Pietro Antonio Cataldi, re-issued in 1620, and followed by Books VII. — IX. (1621) and Book X. (1625).

1663. Borelli's Latin translation turned into Italian by Domenico Magni.

1680. Euclide restituto by Vitale Giordano.

1690. Vincenzo Viviani's Elementi piani e solidi di Euclide (Book V. in 1674).

1731. Elementi geometrici piani e solidi di Euclide by Guido Grandi. No translation, but an abbreviated version, of which new editions followed one another up to 1806.

1749. Italian translation of Dechales with Ozanam's corrections and additions, re-issued 1785, 1797.

1752. Leonardo Ximenes (the first six Books). Fifth edition, 1819.

1818. Vincenzo Flauti's Corso di geometria elementare e sublime (4 vols.) contains (Vol. I.) the first six Books, with additions and a dissertation on Postulate 5, and (Vol. II.) Books XI., XII. Flauti also published the first six Books in 1827 and the Elements of geometry of Euclid in 1843 and 1854.

V. German.

1558. The arithmetical Books VII.-IX. by Scheubel (cf. the edition of the first six Books, with enunciations in Greek and Latin, mentioned above, under date 1550).

1562. The version of the first six Books by Wilhelm Holtzmann (Xylander). This work has its interest as the first edition in German, but otherwise it is not of importance. Xylander tells us that it was written for practical people such as artists, goldsmiths, builders etc., and that, as the

simple amateur is of course content to know facts, without knowing how to prove them, he has often left out the proofs altogether. He has indeed taken the greatest possible liberties with Euclid, and has not grappled with any of the theoretical difficulties, such as that of the theory of parallels.

1651. Heinrich Hoffmann's Teutscher Euclides (2nd edition 1653), not a translation.

1694. Ant. Ernst Burkh. v. Pirckenstein's Teutsch Redender Euclides (eight geometrical Books), "for generals, engineers etc." "proved in a new and quite easy manner." Other editions 1699, 1744.

1697. Samuel Reyher's In teutscher Sprache vorgestellter Euclides (six Books), "made easy, with symbols algebraical or derived from the newest art of solution."

1714. Euclidis XV Bücher teutsch, "treated in a special and brief manner, yet completely," by Chr. Schessler (another edition in 1729).

1773. The first six Books translated from the Greek for the use of schools by J. F. Lorenz. The first attempt to reproduce Euclid in German word for word.

1781. Books XI., XII. by Lorenz (supplementary to the preceding). Also Euklid's Elemente fünfzehn Bücher translated from the Greek by Lorenz (second edition 1798; editions of 1809, 1818, 1824 by Mollweide, of 1840 by Dippe). The edition of 1824, and I presume those before it, are shortened by the use of symbols and the compression of the enunciation and "setting-out" into one.

1807. Books I. — VI., XI., XII. "newly translated from the Greek," by J. K. F. Hauff.

1828. The same Books by Joh. Jos. Ign. Hoffmann "as guide to instruction in elementary geometry," followed in 1832 by observations on the text by the same editor.

1833. Die Geometrie des Euklid und das Wesen derselben by E. S. Unger; also 1838, 1851.

1901. Max Simon, Euclid und die sechs planimetrischen Bücher.

VI. French.

1564-1566. Nine Books translated by Pierre Forcadel, a pupil and friend of P. de la Ramée.

1604. The first nine Books translated and annotated by Jean Errard de Bar-le-Duc; second edition, 1605.

1615. Denis Henrion's translation of the 15 Books (seven editions up to 1676).

1639. The first six Books "demonstrated by symbols, by a method very brief and intelligible," by Pierre Hérigone, mentioned by Barrow as the only editor who, before him, had used symbols for the exposition of Euclid.

1672. Eight Books "rendus plus faciles" by Claude FranÇis Milliet Dechales, who also brought out Les élémens d'Euclide expliqués d'une manière nouvelle et très facile, which appeared in many editions, 1672, 1677, 1683 etc. (from 1709 onwards revised by Ozanam), and was translated into Italian (1749 etc.) and English (by William Halifax, 1685).

1804. In this year, and therefore before his edition of the Greek text, F. Peyrard published the Elements literally translated into French. A second edition appeared in 1809 with the addition of the fifth Book. As this second edition contains Books I. — VI. XI., XII. and X. I, it would appear that the first edition contained Books I. — IV., VI., XI., XII. Peyrard used for this translation the Oxford Greek text and Simson.

VII. Dutch.

1606. Jan Pieterszoon Dou (six Books). There were many later editions. Kästner, in mentioning one of 1702, says that Dou explains in his preface that he used Xylander's translation, but, having afterwards obtained the French translation of the six Books by Errard de Bar-le-Duc (see above), the proofs in which sometimes pleased him more than those of the German edition, he made his Dutch version by the help of both.

1617. Frans van Schooten, "The Propositions of the Books of Euclid's Elements" ; the fifteen Books in this version "enlarged" by Jakob van Leest in 1662.

1695. C. J. Vooght, fifteen Books complete, with Candalla's "16th."

1702. Hendrik Coets, six Books (also in Latin, 1692); several editions up to 1752. Apparently not a translation. but an edition for school use.

1763. Pybo Steenstra, Books I. — VI., XI., XII., likewise an abberviated version, several times reissued until 1825.

VIII. English.

1570 saw the first and the most important translation, that of Sir Henry Billingsley. The title-page is as follows:

THE ELEMENTS OF GEOMETRIE of the most auncient Philosopher EVCLIDE of Megara

" Faithfully (now first) translated into the Englishe toung, by H. Billingsley, Citizen of London. Whereunto are annexed certaine Scholies, Annotations, and Inuentions, of the best Mathematiciens, both of time past, and in this our age.

With a very fruitfull Preface by M. I. Dee, specifying the chiefe Mathematicall Sciẽces, what they are, and whereunto commodious: where, also, are disclosed certaine new Secrets Mathematicall and Mechanicall, vntill these our daies, greatly missed.

“

Imprinted at London by John Daye.

The Preface by the translator, after a sentence observing that without the diligent study of Euclides Elementes it is impossible to attain unto the perfect knowledge of Geometry, proceeds thus. “Wherefore considering the want and lacke of such good authors hitherto in our Englishe tounge, lamenting also the negligence, and lacke of zeale to their countrey in those of our nation, to whom God hath geuen both knowledge and also abilitie to translate into our tounge, and to publishe abroad such good authors and bookes (the chiefe instrumentes of all learninges): seing moreouer that many good wittes both of gentlemen and of others of all degrees, much desirous and studious of these artes, and seeking for them as much as they can, sparing no paines, and yet frustrate of their intent, by no meanes attaining to that which they seeke: I haue for their sakes, with some charge and great trauaile, faithfully translated into our vulgare toũge, and set abroad in Print, this booke of Euclide. Whereunto I haue added easie and plaine declarations and examples by figures, of the definitions. In which booke also ye shall in due place finde manifolde additions, Scholies, Annotations, and Inuentions: which I haue gathered out of many of the most famous and chiefe Mathematiciẽs, both of old time, and in our age: as by diligent reading it in course, ye shall well perceaue....”

It is truly a monumental work, consisting of 464 leaves, and therefore 928 pages, of folio size, excluding the lengthy preface by Dee. The notes certainly include all the most important that had ever been written, from those of the

Greek commentators, Proclus and the others whom he quotes, down to those of Dee himself on the last books. Besides the fifteen Books, Billingsley included the “sixteenth” added by Candalla. The print and appearance of the book are worthy of its contents; and, in order that it may be understood how no pains were spared to represent everything in the clearest and most perfect form, I need only mention that the figures of the propositions in Book XI. are nearly all duplicated, one being the figure of Euclid, the other an arrangement of pieces of paper (triangular, rectangular etc.) pasted at the edges on to the page of the book so that the pieces can be turned up and made to show the real form of the solid figures represented.

Billingsley was admitted Lady Margaret Scholar of St John’s College, Cambridge, in 1551, and he is also said to have studied at Oxford, but he did not take a degree at either University. He was afterwards apprenticed to a London haberdasher and rapidly became a wealthy merchant. Sheriff of London in 1584, he was elected Lord Mayor on 31st December, 1596, on the death, during his year of office, of Sir Thomas Skinner. From 1589 he was one of the Queen’s four “customers,” or farmers of customs, of the port of London. In 1591 he founded three scholarships at St John’s College for poor students, and gave to the College for their maintenance two messuages and tenements in Tower Street and in Mark Lane, Allhallows, Barking. He died in 1606.

1651. Elements of Geometry. The first VI Boocks: In a compendious form contracted and demonstrated by Captain Thomas Rudd, with the mathematicall preface of John Dee (London).

1660. The first English edition of Barrow’s Euclid (published in Latin in 1655), appeared in London. It contained “the whole fifteen books compendiously demonstrated” ; several editions followed, in 1705, 1722, 1732, 1751.

1661. Euclid's Elements of Geometry, with a supplement of divers Propositions and Corollaries. To which is added a Treatise of regular Solids by Campane and Flussat; likewise Euclid's Data and Marinus his Preface. Also a Treatise of the Divisions of Superficies, ascribed to Machomet Bagdedine, but published by Commandine at the request of J. Dee of London. Published by care and industry of John Leeke and Geo. Serle, students in the Math. (London). According to Potts this was a second edition of Billingsley's translation.

1685. William Halifax's version of Dechales' "Elements of Euclid explained in a new but most easy method" (London and Oxford).

1705. The English Euclide; being the first six Elements of Geometry, translated out of the Greek, with annotations and usefull supplements by Edmund Scarburgh (Oxford). A noteworthy and useful edition.

1708. Books I. — VI., XI., XII., translated from Commandinus' Latin version by Dr John Keill, Savilian Professor of Astronomy at Oxford.

Keill complains in his preface of the omissions by such editors as Tacquet and Dechales of many necessary propositions (e.g. VI. 27-29), and of their substitution of proofs of their own for Euclid's. He praises Barrow's version on the whole, though objecting to the "algebraical" form of proof adopted in Book II., and to the excessive use of notes and symbols, which (he considers) make the proofs too short and thereby obscure; his edition was therefore intended to hit a proper mean between Barrow's excessive brevity and Clavius' prolixity.

Keill's translation was revised by Samuel Cunn and several times reissued. 1749 saw the eighth edition, 1772 the eleventh, and 1782 the twelfth.

1714. W. Whiston's English version (abridged) of The Elements of Euclid with select theorems out of Archimedes by the learned Andr. Tacquet.

1756. Simson's first English edition appeared in the same year as his Latin version under the title: " The Elements of Euclid, viz. the first six Books together with the eleventh and twelfth. In this Edition the Errors by which Theon or others have long ago vitiated these Books are corrected and some of Euclid's Demonstrations are restored. By Robert Simson (Glasgow).

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As above stated, the Latin edition, by its title, purports to be "ex versione latina Federici Commandini," but to the Latin edition, as well as to the English editions, are appended " Notes Critical and Geometrical; containing an Account of those things in which this Edition differs from the Greek text; and the Reasons of the Alterations which have been made. As also Observations on some of the Propositions.

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Simson says in the Preface to some editions (e.g. the tenth, of 1799) that "the translation is much amended by the friendly assistance of a learned gentleman."

Simson's version and his notes are so well known as not to need any further description. The book went through some thirty successive editions. The first five appear to have been dated 1756, 1762, 1767, 1772 and 1775 respectively; the tenth 1799, the thirteenth 1806, the twenty-third 1830, the twenty-fourth 1834, the twenty-sixth 1844 The Data "in like manner corrected" was added for the first time in the edition of 1762 (the first octavo edition).

1781, 1788. In these years respectively appeared the two volumes containing the complete translation of the whole thirteen Books by James Williamson, the last English translation which reproduced Euclid word for word. The title is " The Elements of Euclid, with Disserlations intended to assist and encourage a critical examination of these Elements, as the most

effectual means of establishing a juster taste upon mathematical subjects than that which at present prevails. By James Williamson.

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In the first volume (Oxford, 1781) he is described as “M.A. Fellow of Hertford College,” and in the second (London, printed by T. Spilsbury, 1788) as “B.D.” simply. Books V., VI. with the Conclusion in the first volume are paged separately from the rest.

1781. An examination of the first six Books of Euclid’s Elements, by William Austin (London).

1795. John Playfair’s first edition, containing “the first six Books of Euclid with two Books on the Geometry of Solids.” The book reached a fifth edition in 1819, an eighth in 1831, a ninth in 1836, and a tenth in 1846.

1826. Riccardi notes under this date Euclid’s Elements of Geometry containing the whole twelve Books translated into English, from the edition of Peyrard, by George Phillips. The editor, who was President of Queens’ College, Cambridge, 1857-1892, was born in 1804 and matriculated at Queens’ in 1826, so that he must have published the book as an undergraduate.

1828. A very valuable edition of the first six Books is that of Dionysius Lardner, with commentary and geometrical exercises, to which he added, in place of Books XI., XII., a Treatise on Solid Geometry mostly based on Legendre. Lardner compresses the propositions by combining the enunciation and the setting-out, and he gives a vast number of riders and additional propositions in smaller print. The book had reached a ninth edition by 1846, and an eleventh by 1855. Among other things, Lardner gives an Appendix “on the theory of parallel lines,” in which he gives a short history of the attempts to get over the difficulty of the parallel-postulate, down to that of Legendre.

1833. T. Perronet Thompson's Geometry without axioms, or the first Book of Euclid's Elements with alterations and notes; and an intercalary book in which the straight line and plane are derived from properties of the sphere, with an appendix containing notices of methods proposed for getting over the difficulty in the twelfth axiom of Euclid.

Thompson (1783-1869) was 7th wrangler 1802, midshipman 1803, Fellow of Queens' College, Cambridge, 1804, and afterwards general and politician. The book went through several editions, but, having been well translated into French by Van Tenac, is said to have received more recognition in France than at home.

1845. Robert Potts' first edition (and one of the best) entitled: " Euclid's Elements of Geometry chiefly from the text of Dr Simson with explanatory notes...to which is prefixed an introduction containing a brief outline of the History of Geometry. Designed for the use of the higher forms in Public Schools and students in the Universities (Cambridge University Press, and London, John W. Parker), to which was added (1847) An Appendix to the larger edition of Euclid's Elements of Geometry, containing additional notes on the Elements, a short tract on transversals, and hints for the solution of the problems etc

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1862. Todhunter's edition.

The later English editions I will not attempt to enumerate; their name is legion and their object mostly that of adapting Euclid for school use, with all possible gradations of departure from his text and order.

IX. Spanish.

1576. The first six Books translated into Spanish by Rodrigo Çamorano.

1637. The first six Books translated, with notes, by L. Carduchi.

1689. Books I. — VI., XI., XII., translated and explained by Jacob Knesa.

X. Russian.

1739. Ivan Astaroff (translation from Latin).

1789. Pr. Suvoroff and Yos. Nikitin (translation from Greek).

1880. Vachtchenko-Zakhartchenko.

(1817. A translation into Polish by Jo. Czecha.)

XI. Swedish.

1744. Mårten Strömer, the first six Books; second edition 1748. The third edition (1753) contained Books XI. — XII. as well; new editions continued to appear till 1884.

1836. H. Falk, the first six Books.

1844, 1845, 1859. P. R. Bråkenhjelm, Books I. — VI., XI., XII.

1850. F. A. A. Lundgren.

1850. H. A. Witt and M. E. Areskong, Books I. — VI., XI., XII.

XII. Danish.

1745. Ernest Gottlieb Ziegenbalg.

1803. H. C. Linderup, Books I. — VI.

XIII. Modern Greek.

1820. Benjamin of Lesbos.

I should add a reference to certain editions which have appeared in recent years.

A Danish translation (Euklid's Elementer oversat af Thyra Eibe) was completed in 1912; Books I. — II. were published (with an Introduction by Zeuthen) in 1897, Books III. — IV. in 1900, Books V. — VI. in 1904, Books VII. — XIII. in 1912.

The Italians, whose great services to elementary geometry are more than once emphasised in this work, have lately shown a noteworthy disposition to make the ipsissima verba of Euclid once more the object of study. Giovanni Vacca has edited the text of Book I. (Il primo libro degli Elementi. Testo greco, versione italiana, introduzione e note, Firenze 1916.) Federigo Enriques has begun the publication of a complete Italian translation (Gli Elementi d' Euclide e la critica antica e moderna); Books I. — IV. appeared in 1925 (Alberto Stock, Roma).

An edition of Book I. by the present writer was published in 1918 (Euclid in Greek, Book I., with Introduction and Notes, Camb. Univ. Press).

CHAPTER IX.

§ 1. ON THE NATURE OF ELEMENTS.

It would not be easy to find a more lucid explanation of the terms element and elementary, and of the distinction between them, than is found in Proclus, who is doubtless, here as so often, quoting from Geminus. There are, says Proclus, in the whole of geometry certain leading theorems, bearing to those which follow the relation of a principle, all-pervading, and furnishing proofs of many properties. Such theorems are called by the name of elements; and their function may be compared to that of the letters of the alphabet in relation to language, letters being indeed called by the same name in Greek (στοιχεῖα).

The term elementary, on the other hand, has a wider application: it is applicable to things "which extend to greater multiplicity, and, though possessing simplicity and elegance, have no longer the same dignity as the elements, because their investigation is not of general use in the whole of the science, e.g. the proposition that in triangles the perpendiculars from the angles to the transverse sides meet in a point."

"Again, the term element is used in two senses, as Menaechmus says. For that which is the means of obtaining is an element of that which is obtained, as the first proposition in Euclid is of the second, and the fourth of the fifth. In this sense many things may even be said to be elements of each other, for they are obtained from one another. Thus from the fact that the exterior angles of rectilineal figures are (together) equal to four right angles we deduce the number of right angles equal to the internal angles (taken together), and vice versa. Such an element is like a lemma. But the term element is otherwise used of that into which, being more simple, the composite is divided; and in this sense we can no longer say that everything

is an element of everything, but only that things which are more of the nature of principles are elements of those which stand to them in the relation of results, as postulates are elements of theorems. It is according to this signification of the term element that the elements found in Euclid were compiled, being partly those of plane geometry, and partly those of stereometry. In like manner many writers have drawn up elementary treatises in arithmetic and astronomy.

"Now it is difficult, in each science, both to select and arrange in due order the elements from which all the rest proceeds, and into which all the rest is resolved. And of those who have made the attempt some were able to put together more and some less; some used shorter proofs, some extended their investigation to an indefinite length; some avoided the method of reductio ad absurdum, some avoided proportion; some contrived preliminary steps directed against those who reject the principles; and, in a word, many different methods have been invented by various writers of elements.

"It is essential that such a treatise should be rid of everything superfluous (for this is an obstacle to the acquisition of knowledge); it should select everything that embraces the subject and brings it to a point (for this is of supreme service to science); it must have great regard at once to clearness and conciseness (for their opposites trouble our understanding); it must aim at the embracing of theorems in general terms (for the piecemeal division of instruction into the more partial makes knowledge difficult to grasp). In all these ways Euclid's system of elements will be found to be superior to the rest; for its utility avails towards the investigation of the primordial figures, its clearness and organic perfection are secured by the progression from the more simple to the more complex and by the foundation of the investigation upon common notions, while generality of demonstration is secured by the progression through the theorems which are primary and of the nature of

principles to the things sought. As for the things which seem to be wanting, they are partly to be discovered by the same methods, like the construction of the scalene and isosceles (triangle), partly alien to the character of a selection of elements as introducing hopeless and boundless complexity, like the subject of unordered irrationals which Apollonius worked out at length, and partly developed from things handed down (in the elements) as causes, like the many species of angles and of lines. These things then have been omitted in Euclid, though they have received full discussion in other works; but the knowledge of them is derived from the simple (elements)."

Proclus, speaking apparently on his own behalf, in another place distinguishes two objects aimed at in Euclid's Elements. The first has reference to the matter of the investigation, and here, like a good Platonist, he takes the whole subject of geometry to be concerned with the "cosmic figures," the five regular solids, which in Book XIII. are constructed, inscribed in a sphere and compared with one another. The second object is relative to the learner; and, from this standpoint, the elements may be described as "a means of perfecting the learner's understanding with reference to the whole of geometry. For, starting from these (elements), we shall be able to acquire knowledge of the other parts of this science as well, while without them it is impossible for us to get a grasp of so complex a subject, and knowledge of the rest is unattainable. As it is, the theorems which are most of the nature of principles, most simple, and most akin to the first hypotheses are here collected, in their appropriate order; and the proofs of all other propositions use these theorems as thoroughly well known, and start from them. Thus Archimedes in the books on the sphere and cylinder, Apollonius, and all other geometers, clearly use the theorems proved in this very treatise as constituting admitted principles"

Aristotle too speaks of elements of geometry in the same sense. Thus: “in geometry it is well to be thoroughly versed in the elements “ ; “in general the first of the elements are, given the definitions, e.g. of a straight line and of a circle, most easy to prove, although of course there are not many data that can be used to establish each of them because there are not many middle terms “; “among geometrical propositions we call those ‘elements’ the proofs of which are contained in the proofs of all or most of such propositions .”; “(as in the case of bodies), so in like manner we speak of the elements of geometrical propositions and, generally, of demonstrations; for the demonstrations which come first and are contained in a variety of other demonstrations are called elements of those demonstrations... the term element is applied by analogy to that which, being one and small, is useful for many purposes .”

§ 2. ELEMENTS ANTERIOR TO EUCLID’S.

The early part of the famous summary of Proclus was no doubt drawn, at least indirectly, from the history of geometry by Eudemus; this is generally inferred from the remark, made just after the mention of Philippus of Medma, a disciple of Plato, that “those who have written histories bring the development of this science up to this point.” We have therefore the best authority for the list of writers of elements given in the summary. Hippocrates of Chios (fl. in second half of 5th c.) is the first; then Leon, who also discovered diorismi, put together a more careful collection, the propositions proved in it being more numerous as well as more serviceable. Leon was a little older than Eudoxus (about 408-355 B.C.) and a little younger than Plato (428/7-347/6 B.C.), but did not belong to the latter’s school. The geometrical text-book of the Academy was written by Theudius

of Magnesia, who, with Amyclas of Heraclea, Menaechmus the pupil of Eudoxus, Menaechmus' brother Dinostratus and Athenaeus of Cyzicus consorted together in the Academy and carried on their investigations in common. Theudius "put together the elements admirably, making many partial (or limited) propositions more general." Eudemus mentions no text-book after that of Theudius, only adding that Hermotimus of Colophon "discovered many of the elements." Theudius then must be taken to be the immediate precursor of Euclid, and no doubt Euclid made full use of Theudius as well as of the discoveries of Hermotimus and all other available material. Naturally it is not in Euclid's Elements that we can find much light upon the state of the subject when he took it up; but we have another source of information in Aristotle. Fortunately for the historian of mathematics, Aristotle was fond of mathematical illustrations; he refers to a considerable number of geometrical propositions, definitions etc., in a way which shows that his pupils must have had at hand some textbook where they could find the things he mentions; and this text-book must have been that of Theudius. Heiberg has made a most valuable collection of mathematical extracts from Aristotle, from which much is to be gathered as to the changes which Euclid made in the methods of his predecessors; and these passages, as well as others not included in Heiberg's selection, will often be referred to in the sequel.

§ 3. FIRST PRINCIPLES: DEFINITIONS, POSTULATES, AND AXIOMS.

On no part of the subject does Aristotle give more valuable information than on that of the first principles as, doubtless, generally accepted at the time when he wrote. One long passage in the Posterior Analytics is particularly

full and lucid, and is worth quoting in extenso. After laying it down that every demonstrative science starts from necessary principles, he proceeds:

"By first principles in each genus I mean those the truth of which it is not possible to prove. What is denoted by the first (terms) and those derived from them is assumed; but, as regards their existence, this must be assumed for the principles but proved for the rest. Thus what a unit is, what the straight (line) is, or what a triangle is (must be assumed); and the existence of the unit and of magnitude must also be assumed, but the rest must be proved. Now of the premisses used in demonstrative sciences some are peculiar to each science and others common (to all), the latter being common by analogy, for of course they are actually useful in so far as they are applied to the subject-matter included under the particular science. Instances of first principles peculiar to a science are the assumptions that a line is of such and such a character, and similarly for the straight (line); whereas it is a common principle, for instance, that, if equals be subtracted from equals, the remainders are equal. But it is enough that each of the common principles is true so far as regards the particular genus (subject-matter); for (in geometry) the effect will be the same even if the common principle be assumed to be true, not of everything, but only of magnitudes, and, in arithmetic, of numbers.

"Now the things peculiar to the science, the existence of which must be assumed, are the things with reference to which the science investigates the essential attributes, e.g. arithmetic with reference to units, and geometry with reference to points and lines. With these things it is assumed that they exist and that they are of such and such a nature. But, with regard to their essential properties, what is assumed is only the meaning of each term employed: thus arithmetic assumes the answer to the question what is (meant by) 'odd' or 'even,' 'a square' or 'a cube,' and geometry to the question what is (meant

by) ‘the irrational’ or ‘deflection’ or (the so-called) ‘verging’ (to a point); but that there are such things is proved by means of the common principles and of what has already been demonstrated. Similarly with astronomy. For every demonstrative science has to do with three things, (1) the things which are assumed to exist, namely the genus (subject-matter) in each case, the essential properties of which the science investigates, (2) the common axioms so-called, which are the primary source of demonstration, and (3) the properties with regard to which all that is assumed is the meaning of the respective terms used. There is, however, no reason why some sciences should not omit to speak of one or other of these things. Thus there need not be any supposition as to the existence of the genus, if it is manifest that it exists (for it is not equally clear that number exists and that cold and hot exist); and, with regard to the properties, there need be no assumption as to the meaning of terms if it is clear: just as in the common (axioms) there is no assumption as to what is the meaning of subtracting equals from equals, because it is well known. But none the less is it true that there are three things naturally distinct, the subject-matter of the proof, the things proved, and the (axioms) from which (the proof starts).

“Now that which is per se necessarily true, and must necessarily be thought so, is not a hypothesis nor yet a postulate. For demonstration has not to do with reasoning from outside but with the reason dwelling in the soul, just as is the case with the syllogism. It is always possible to raise objection to reasoning from outside, but to contradict the reason within us is not always possible. Now anything that the teacher assumes, though it is matter of proof, without proving it himself, is a hypothesis if the thing assumed is believed by the learner, and it is moreover a hypothesis, not absolutely, but relatively to the particular pupil; but, if the same thing is assumed when the learner either has no opinion on the subject or is of a contrary opinion, it is a postulate. This

is the difference between a hypothesis and a postulate; for a postulate is that which is rather contrary than otherwise to the opinion of the learner, or whatever is assumed and used without being proved, although matter for demonstration. Now definitions are not hypotheses, for they do not assert the existence or non-existence of anything, while hypotheses are among propositions. Definitions only require to be understood: a definition is therefore not a hypothesis, unless indeed it be asserted that any audible speech is a hypothesis. A hypothesis is that from the truth of which, if assumed, a conclusion can be established. Nor are the geometer's hypotheses false, as some have said: I mean those who say that 'you should not make use of what is false, and yet the geometer falsely calls the line which he has drawn a foot long when it is not, or straight when it is not straight.' The geometer bases no conclusion on the particular line which he has drawn being that which he has described, but (he refers to) what is illustrated by the figures. Further, the postulate and every hypothesis are either universal or particular statements; definitions are neither" (because the subject is of equal extent with what is predicated of it).

Every demonstrative science, says Aristotle, must start from indemonstrable principles: otherwise, the steps of demonstration would be endless. Of these indemonstrable principles some are (a) common to all sciences, others are (b) particular, or peculiar to the particular science; (a) the common principles are the axioms, most commonly illustrated by the axiom that, if equals be subtracted from equals, the remainders are equal. Coming now to (b) the principles peculiar to the particular science which must be assumed, we have first the genus or subject-matter, the existence of which must be assumed, viz. magnitude in the case of geometry, the unit in the case of arithmetic. Under this we must assume definitions of manifestations or attributes of the genus, e.g. straight lines, triangles, deflection etc. The

definition in itself says nothing as to the existence of the thing defined: it only requires to be understood. But in geometry, in addition to the genus and the definitions, we have to assume the existence of a few primary things which are defined, viz. points and lines only: the existence of everything else, e.g. the various figures made up of these, as triangles, squares, tangents, and their properties, e.g. incommensurability etc., has to be proved (as it is proved by construction and demonstration). In arithmetic we assume the existence of the unit: but, as regards the rest, only the definitions, e.g. those of odd, even, square, cube, are assumed, and existence has to be proved. We have then clearly distinguished, among the indemonstrable principles, axioms and definitions. A postulate is also distinguished from a hypothesis, the latter being made with the assent of the learner, the former without such assent or even in opposition to his opinion (though, strangely enough, immediately after saying this, Aristotle gives a wider meaning to "postulate" which would cover "hypothesis" as well, namely whatever is assumed, though it is matter for proof, and used without being proved). Heiberg remarks that there is no trace in Aristotle of Euclid's Postulates, and that "postulate" in Aristotle has a different meaning. He seems to base this on the alternative description of postulate, indistinguishable from a hypothesis; but, if we take the other description in which it is distinguished from a hypothesis as being an assumption of something which is a proper subject of demonstration without the assent or against the opinion of the learner, it seems to fit Euclid's Postulates fairly well, not only the first three (postulating three constructions), but eminently also the other two, that all right angles are equal, and that two straight lines meeting a third and making the internal angles on the same side of it less than two right angles will meet on that side. Aristotle's description also seems to me to suit the "postulates" with which Archimedes begins his book On the equilibrium of planes, namely that equal

weights balance at equal distances, and that equal weights at unequal distances do not balance but that the weight at the longer distance will prevail.

Aristotle's distinction also between hypothesis and definition, and between hypothesis and axiom, is clear from the following passage: "Among immediate syllogistic principles, I call that a thesis which it is neither possible to prove nor essential for any one to hold who is to learn anything; but that which it is necessary for any one to hold who is to learn anything whatever is an axiom: for there are some principles of this kind, and that is the most usual name by which we speak of them. But, of theses, one kind is that which assumes one or other side of a predication, as, for instance, that something exists or does not exist, and this is a hypothesis; the other, which makes no such assumption, is a definition. For a definition is a thesis: thus the arithmetician posits (τίθεται) that a unit is that which is indivisible in respect of quantity; but this is not a hypothesis, since what is meant by a unit and the fact that a unit exists are different things."

Aristotle uses as an alternative term for axioms "common (things)," τὰ κοινά, or "common opinions" (κοιναὶ δόξαι), as in the following passages. "That, when equals are taken from equals, the remainders are equal is (a) common (principle) in the case of all quantities, but mathematics takes a separate department (ἀπολαβοῦσα) and directs its investigation to some portion of its proper subject-matter, as e.g. lines or angles, numbers, or any of the other quantities." "The common (principles), e.g. that one of two contradictories must be true, that equals taken from equals etc., and the like...." "With regard to the principles of demonstration, it is questionable whether they belong to one science or to several. By principles of demonstration I mean the common opinions from which all demonstration proceeds, e.g. that one of two contradictories must be true, and that it is

impossible for the same thing to be and not be." Similarly "every demonstrative (science) investigates, with regard to some subject-matter, the essential attributes, starting from the common opinions." We have then here, as Heiberg says, a sufficient explanation of Euclid's term for axioms, viz. common notions (κοιναὶ ἔννοιαι), and there is no reason to suppose it to be a substitution for the original term due to the Stoics: cf. Proclus' remark that, according to Aristotle and the geometers, axiom and common notion are the same thing.

Aristotle discusses the indemonstrable character of the axioms in the Metaphysics. Since "all the demonstrative sciences use the axioms," the question arises, to what science does their discussion belong? The answer is that, like that of Being (οὐσία), it is the province of the (first) philosopher. It is impossible that there should be demonstration of everything, as there would be an infinite series of demonstrations: if the axioms were the subject of a demonstrative science, there would have to be here too, as in other demonstrative sciences, a subject-genus, its attributes and corresponding axioms; thus there would be axioms behind axioms, and so on continually. The axiom is the most firmly established of all principles. It is ignorance alone that could lead any one to try to prove the axioms; the supposed proof would be a petitio principii. If it is admitted that not everything can be proved, no one can point to any principle more truly indemonstrable. If any one thought he could prove them, he could at once be refuted; if he did not attempt to say anything, it would be ridiculous to argue with him: he would be no better than a vegetable. The first condition of the possibility of any argument whatever is that words should signify something both to the speaker and to the hearer: without this there can be no reasoning with any one. And, if any one admits that words can mean anything to both hearer and

speaker, he admits that something can be true without demonstration. And so on.

It was necessary to give some sketch of Aristotle's view of the first principles, if only in connexion with Proclus' account, which is as follows. As in the case of other sciences, so "the compiler of elements in geometry must give separately the principles of the science, and after that the conclusions from those principles, not giving any account of the principles but only of their consequences. No science proves its own principles, or even discourses about them: they are treated as self-evident....Thus the first essential was to distinguish the principles from their consequences. Euclid carries out this plan practically in every book and, as a preliminary to the whole enquiry, sets out the common principles of this science. Then he divides the common principles themselves into hypotheses, postulates, and axioms. For all these are different from one another: an axiom, a postulate and a hypothesis are not the same thing, as the inspired Aristotle somewhere says. But, whenever that which is assumed and ranked as a principle is both known to the learner and convincing in itself, such a thing is an axiom, e.g. the statement that things which are equal to the same thing are also equal to one another. When, on the other hand, the pupil has not the notion of what is told him which carries conviction in itself, but nevertheless lays it down and assents to its being assumed, such an assumption is a hypothesis. Thus we do not preconceive by virtue of a common notion, and without being taught, that the circle is such and such a figure, but, when we are told so, we assent without demonstration. When again what is asserted is both unknown and assumed even without the assent of the learner, then, he says, we call this a postulate, e.g. that all right angles are equal. This view of a postulate is clearly implied by those who have made a special and systematic attempt to show, with regard to one of the postulates, that it cannot be assented to by

any one straight off. According then to the teaching of Aristotle, an axiom, a postulate and a hypothesis are thus distinguished."

We observe, first, that Proclus in this passage confuses hypotheses and definitions, although Aristotle had made the distinction quite plain. The confusion may be due to his having in his mind a passage of Plato from which he evidently got the phrase about "not giving an account of" the principles. The passage is: "I think you know that those who treat of geometries and calculations (arithmetic) and such things take for granted (ὑποθέμενοι) odd and even, figures, angles of three kinds, and other things akin to these in each subject, implying that they know these things, and, though using them as hypotheses, do not even condescend to give any account of them either to themselves or to others, but begin from these things and then go through everything else in order, arriving ultimately, by recognised methods, at the conclusion which they started in search of." But the hypothesis is here the assumption, e.g. 'that there may be such a thing as length without breadth, henceforward called a line,' and so on, without any attempt to show that there is such a thing; it is mentioned in connexion with the distinction between Plato's 'superior' and 'inferior' intellectual method, the former of which uses successive hypotheses as stepping-stones by which it mounts upwards to the idea of Good.

We pass now to Proclus' account of the difference between postulates and axioms. He begins with the view of Geminus, according to which "they differ from one another in the same way as theorems are also distinguished from problems. For, as in theorems we propose to see and determine what follows on the premisses, while in problems we are told to find and do something, in like manner in the axioms such things are assumed as are manifest of themselves and easily apprehended by our untaught notions, while in the postulates we assume such things as are easy to find and effect (our

understanding suffering no strain in their assumption), and we require no complication of machinery." ..."Both must have the characteristic of being simple and readily grasped, I mean both the postulate and the axiom; but the postulate bids us contrive and find some subject-matter (ὕλη) to exhibit a property simple and easily grasped, while the axiom bids us assert some essential attribute which is self-evident to the learner, just as is the fact that fire is hot, or any of the most obvious things."

Again, says Proclus, "some claim that all these things are alike postulates, in the same way as some maintain that all things that are sought are problems. For Archimedes begins his first book on Inequilibrium with the remark 'I postulate that equal weights at equal distances are in equilibrium,' though one would rather call this an axiom. Others call them all axioms in the same way as some regard as theorems everything that requires demonstration."

"Others again will say that postulates are peculiar to geometrical subject-matter, while axioms are common to all investigation which is concerned with quantity and magnitude. Thus it is the geometer who knows that all right angles are equal and how to produce in a straight line any limited straight line, whereas it is a common notion that things which are equal to the same thing are also equal to one another, and it is employed by the arithmetician and any scientific person who adapts the general statement to his own subject."

The third view of the distinction between a postulate and an axiom is that of Aristotle above described.

The difficulties in the way of reconciling Euclid's classification of postulates and axioms with any one of the three alternative views are next dwelt upon. If we accept the first view according to which an axiom has reference to something known, and a postulate to something done, then the 4th postulate (that all right angles are equal) is not a postulate; neither is the

5th which states that, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines, if produced indefinitely, will meet on that side on which are the angles less than two right angles. On the second view, the assumption that two straight lines cannot enclose a space, "which even now," says Proclus, "some add as an axiom," and which is peculiar to the subject-matter of geometry, like the fact that all right angles are equal, is not an axiom. According to the third (Aristotelian) view, "everything which is confirmed (πιστοῦται) by a sort of demonstration will be a postulate, and what is incapable of proof will be an axiom." This last statement of Proclus is loose, as regards the axiom, because it omits Aristotle's requirement that the axiom should be a selfevident truth, and one that must be admitted by any one who is to learn anything at all, and, as regards the postulate, because Aristotle calls a postulate something assumed without proof though it is "matter of demonstration" (ἀποδεικτὸν ὅν), but says nothing of a quasi-demonstration of the postulates. On the whole I think it is from Aristotle that we get the best idea of what Euclid understood by a postulate and an axiom or common notion. Thus Aristotle's account of an axiom as a principle common to all sciences, which is self-evident, though incapable of proof, agrees sufficiently with the contents of Euclid's common notions as reduced to five in the most recent text (not omitting the fourth, that "things which coincide are equal to one another"). As regards the postulates, it must be borne in mind that Aristotle says elsewhere that, "other things being equal, that proof is the better which proceeds from the fewer postulates or hypotheses or propositions." If then we say that a geometer must lay down as principles, first certain axioms or common notions, and then an irreducible minimum of postulates in the Aristotelian sense concerned only with the subject-matter of geometry, we are not far from describing what Euclid in fact does. As regards the postulates

we may imagine him saying: “Besides the common notions there are a few other things which I must assume without proof, but which differ from the common notions in that they are not self-evident. The learner may or may not be disposed to agree to them; but he must accept them at the outset on the superior authority of his teacher, and must be left to convince himself of their truth in the course of the investigation which follows. In the first place certain simple constructions, the drawing and producing of a straight line, and the drawing of a circle, must be assumed to be possible, and with the constructions the existence of such things as straight lines and circles; and besides this we must lay down some postulate to form the basis of the theory of parallels.” It is true that the admission of the 4th postulate that all right angles are equal still presents a difficulty to which we shall have to recur.

There is of course no foundation for the idea, which has found its way into many text-books, that “the object of the postulates is to declare that the only instruments the use of which is permitted in geometry are the rule and compass.”

§ 4. THEOREMS AND PROBLEMS.

“Again the deductions from the first principles,” says Proclus, “are divided into problems and theorems, the former embracing the generation, division, subtraction or addition of figures, and generally the changes which are brought about in them, the latter exhibiting the essential attributes of each.”

“Now, of the ancients, some, like Speusippus and Amphinomus, thought proper to call them all theorems, regarding the name of theorems as more appropriate than that of problems to theoretic sciences, especially as these deal with eternal objects. For there is no becoming in things eternal, so that neither could the problem have any place with them, since it promises the

generation and making of what has not before existed, e.g. the construction of an equilateral triangle, or the describing of a square on a given straight line, or the placing of a straight line at a given point. Hence they say it is better to assert that all (propositions) are of the same kind, and that we regard the generation that takes place in them as referring not to actual making but to knowledge, when we treat things existing eternally as if they were subject to becoming: in other words, we may say that everything is treated by way of theorem and not by way of problem (πάντα θεωρηματικῶς ἀλλ᾽ οὐ προβληματικῶς λαμβάνεσθαι).

"Others on the contrary, like the mathematicians of the school of Menaechmus, thought it right to call them all problems, describing their purpose as twofold, namely in some cases to furnish (πορίσασθαι) the thing sought, in others to take a determinate object and see either what it is, or of what nature, or what is its property, or in what relations it stands to something else.

"In reality both assertions are correct. Speusippus is right because the problems of geometry are not like those of mechanics, the latter being matters of sense and exhibiting becoming and change of every sort. The school of Menaechmus are right also because the discoveries even of theorems do not arise without an issuing-forth into matter, by which I mean intelligible matter. Thus forms going out into matter and giving it shape may fairly be said to be like processes of becoming. For we say that the motion of our thought and the throwing-out of the forms in it is what produces the figures in the imagination and the conditions subsisting in them. It is in the imagination that constructions, divisions, placings, applications, additions and subtractions (take place), but everything in the mind is fixed and immune from becoming and from every sort of change."

“Now those who distinguish the theorem from the problem say that every problem implies the possibility, not only of that which is predicated of its subject-matter, but also of its opposite, whereas every theorem implies the possibility of the thing predicated but not of its opposite as well. By the subject-matter I mean the genus which is the subject of inquiry, for example, a triangle or a square or a circle, and by the property predicated the essential attribute, as equality, section, position, and the like. When then any one enunciates thus, To inscribe an equilateral triangle in a circle, he states a problem; for it is also possible to inscribe in it a triangle which is not equilateral. Again, if we take the enunciation On a given limited straight line to construct an equilateral triangle, this is a problem; for it is possible also to construct one which is not equilateral. But, when any one enunciates that In isosceles triangles the angles at the base are equal, we must say that he enunciates a theorem; for it is not also possible that the angles at the base of isosceles triangles should be unequal. It follows that, if any one were to use the form of a problem and say In a semicircle to describe a right angle, he would be set down as no geometer. For every angle in a semicircle is right.”

“Zenodotus, who belonged to the succession of Oenopides, but was a disciple of Andron, distinguished the theorem from the problem by the fact that the theorem inquires what is the property predicated of the subject-matter in it, but the problem what is the cause of what effect (τίνος ὄντος τί ἐστιν). Hence too Posidonius defined the one (the problem) as a proposition in which it is inquired whether a thing exists or not (εἰ ἔστιν ἢ μή), the other (the theorem) as a proposition in which it is inquired what (a thing) is or of what nature (τί ἐστιν ἢ ποῖόν τι); and he said that the theoretic proposition must be put in a declaratory form, e.g., Any triangle has two sides (together) greater than the remaining side and In any isosceles triangle the angles at the base are equal, but that we should state the problematic propositi\on as if inquiring

whether it is possible to construct an equilateral triangle upon such and such a straight line. For there is a difference between inquiring absolutely and indeterminately (ἁπλῶς τε καὶ ἀορίστως) whether there exists a straight line from such and such a point at right angles to such and such a straight line and investigating which is the straight line at right angles."

"That there is a certain difference between the problem and the theorem is clear from what has been said; and that the Elements of Euclid contain partly problems and partly theorems will be made manifest by the individual propositions, where Euclid himself adds at the end of what is proved in them, in some cases, 'that which it was required to do,' and in others, 'that which it was required to prove,' the latter expression being regarded as characteristic of theorems, in spite of the fact that, as we have said, demonstration is found in problems also. In problems, however, even the demonstration is for the purpose of (confirming) the construction: for wė bring in the demonstration in order to show that what was enjoined has been done; whereas in theorems the demonstration is worthy of study for its own sake as being capable of putting before us the nature of the thing sought. And you will find that Euclid sometimes interweaves theorems with problems and employs them in turn, as in the first book, while at other times he makes one or other preponderate. For the fourth book consists wholly of problems, and the fifth of theorems."

Again, in his note on Eucl. 1. 4, Proclus says that Carpus, the writer on mechanics, raised the question of theorems and problems in his treatise on astronomy. Carpus, we are told, "says that the class of problems is in order prior to theorems. For the subjects, the properties of which are sought, are discovered by means of problems. Moreover in a problem the enunciation is simple and requires no skilled intelligence; it orders you plainly to do such and such a thing, to construct an equilateral triangle, or, given two straight lines, to cut off from the greater (a straight line) equal to the lesser, and what

is there obscure or elaborate in these things? But the enunciation of a theorem is a matter of labour and requires much exactness and scientific judgment in order that it may not turn out to exceed or fall short of the truth; an example is found even in this proposition (1. 4), the first of the theorems. Again, in the case of problems, one general way has been discovered, that of analysis, by following which we can always hope to succeed; it is this method by which the more obscure problems are investigated. But, in the case of theorems, the method of setting about them is hard to get hold of since 'up to our time,' says Carpus, 'no one has been able to hand down a general method for their discovery. Hence, by reason of their easiness, the class of problems would naturally be more simple.' After these distinctions, he proceeds: 'Hence it is that in the Elements too problems precede theorems, and the Elements begin from them; the first theorem is fourth in order, not because the fifth is proved from the problems, but because, even if it needs for its demonstration none of the propositions which precede it, it was necessary that they should be first because they are problems, while it is a theorem. In fact, in this theorem he uses the common notions exclusively, and in some sort takes the same triangle placed in different positions; the coincidence and the equality proved thereby depend entirely upon sensible and distinct apprehension. Nevertheless, though the demonstration of the first theorem is of this character, the problems properly preceded it, because in general problems are allotted the order of precedence.'"

Proclus himself explains the position of Pro after Props. 1-3 as due to the fact that a theorem about the essential properties of triangles ought not to be introduced before we know that such a thing as a triangle can be constructed, nor a theorem about the equality of sides or straight lines until we have shown, by constructing them, that there can be two straight lines which are equal to one another. It is plausible enough to argue in this way that Props. 2

and 3 at all events should precede Pro. And Pro is used in Pro, and must therefore precede it. But Prop. I showing how to construct an equilateral triangle on a given base is not important, in relation to Pro, as dealing with the "production of triangles" in general: for it is of no use to say, as Proclus does, that the construction of the equilateral triangle is "common to the three species (of triangles)," as we are not in a position to know this at such an early stage. The existence of triangles in general was doubtless assumed as following from the existence of straight lines and points in one plane and from the possibility of drawing a straight line from one point to another.

Proclus does not however seem to reject definitely the view of Carpus, for he goes on: "And perhaps problems are in order before theorems, and especially for those who need to ascend from the arts which are concerned with things of sense to theoretical investigation. But in dignity theorems are prior to problems....It is then foolish to blame Geminus for saying that the theorem is more perfect than the problem. For Carpus himself gave the priority to problems in respect of order, and Geminus to theorems in point of more perfect dignity," so that there was no real inconsistency between the two.

Problems were classified according to the number of their possible solutions. Amphinomus said that those which had a unique solution (μοναχῶς) were called "ordered" (the word has dropped out in Proclus, but it must be τεταγμένα, in contrast to the third kind, ἄτακτα); those which had a definite number of solutions "intermediate" (μέσα); and those with an infinite variety of solutions "unordered" (ἄτακτα）. Proclus gives as an example of the last the problem To divide a given straight line into three parts in continued proportion. This is the same thing as solving the equations . Proclus' remarks upon the problem show that it was solved, like all quadratic equations, by the method of "application of areas." The straight line a was

first divided into any two parts, (x+z) and y, subject to the sole limitation that (x+z) must not be less than 2y, which limitation is the διορισμός, or condition of possibility. Then an area was applied to (x+z), or (a-y), "falling short by a square figure" (ἐλλεῖπον εἴδει τετραγώνῳ) and equal to the square on y. This determines x and z separately in terms of a and y. For, if z be the side of the square by which the area (i.e. rectangle) "falls short," we have whence And y may be chosen arbitrarily, provided that it is not greater than a/3. Hence there are an infinite number of solutions. If then, as Proclus remarks, the three parts are equal.

Other distinctions between different kinds of problems are added by Proclus. The word "problem," he says, is used in several senses. In its widest sense it may mean anything "propounded" (προτεινόμενον), whether for the purpose of instruction (μαθήσεως) or construction (ποιήσεως). (In this sense, therefore, it would include a theorem.) But its special sense in mathematics is that of something "propounded with a view to a theoretic construction."

Again you may apply the term (in this restricted sense) even to something which is impossible, although it is more appropriately used of what is possible and neither asks too much nor contains too little in the shape of data. According as a problem has one or other of these defects respectively, it is called (1) a problem in excess (πλεονάζον) or (2) a deficient problem (ἐλλιπὲς πρόβλημα). The problem in excess (1) is of two kinds, (a) a problem in which the properties of the figure to be found are either inconsistent (ἀσύμβατα) or non-existent (ἀνύπαρκτα), in which case the problem is called impossible, or (b) a problem in which the enunciation is merely redundant: an example of this would be a problem requiring us to construct an equilateral triangle with its vertical angle equal to two-thirds of a right angle; such a problem is possible and is called "more than a problem" (μεῖζον ἢ πρόβλημα). The deficient problem (2) is similarly called "less than a

problem" (ἔλασσον ἢ πρόβλημα), its characteristic being that something has to be added to the enunciation in order to convert it from indeterminateness (ἀοριστία) to order (τάξις) and scientific determinateness (ὅρος ἐπιστημονικός): such would be a problem bidding you "to construct an isosceles triangle," for the varieties of isosceles triangles are unlimited. Such "problems" are not problems in the proper sense (κυρίως λεγόμενα προβλήματα), but only equivocally.

§ 5. THE FORMAL DIVISIONS OF A PROPOSITION.

"Every problem," says Proclus, "and every theorem which is complete with all its parts perfect purports to contain in itself all of the following elements: enunciation (πρότασις), setting-out (ἔκθεσις), definition or specification (διορισμός), construction or machinery (κατασκευή), proof (ἀπόδειξις), conclusion (συμπέρασμα). Now of these the enunciation states what is given and what is that which is sought, the perfect enunciation consisting of both these parts. The setting-out marks off what is given, by itself, and adapts it beforehand for use in the investigation. The definition or specification states separately and makes clear what the particular thing is which is sought. The construction or machinery adds what is wanting to the datum for the purpose of finding what is sought. The proof draws the required inference by reasoning scientifically from acknowledged facts. The conclusion reverts again to the enunciation, confirming what has been demonstrated. These are all the parts of problems and theorems, but the most essential and those which are found in all are enunciation, proof, conclusion. For it is equally necessary to know beforehand what is sought, to prove this by means of the intermediate steps, and to state the proved fact as a conclusion; it is impossible to dispense with any of these three things. The remaining parts are

often brought in, but are often left out as serving no purpose. Thus there is neither setting-out nor definition in the problem of constructing an isosceles triangle having each of the angles at the base double of the remaining angle, and in most theorems there is no construction because the setting-out suffices without any addition for proving the required property from the data. When then do we say that the setting-out is wanting? The answer is, when there is nothing given in the enunciation; for, though the enunciation is in general divided into what is given and what is sought, this is not always the case, but sometimes it states only what is sought, i.e. what must be known or found, as in the case of the problem just mentioned. That problem does not, in fact, state beforehand with what datum we are to construct the isosceles triangle having each of the equal angles double of the remaining angle, but (simply) that we are to find such a triangle.... When, then, the enunciation contains both (what is given and what is sought), in that case we find both definition and setting-out, but, whenever the datum is wanting, they too are wanting. For not only is the setting-out concerned with the datum, but so is the definition also, as, in the absence of the datum, the definition will be identical with the enunciation. In fact, what could you say in defining the object of the aforesaid problem except that it is required to find an isosceles triangle of the kind referred to? But that is what the enunciation stated. If then the enunciation does not include, on the one hand, what is given and, on the other, what is sought, there is no setting-out in virtue of there being no datum, and the definition is left out in order to avoid a mere repetition of the enunciation."

The constituent parts of an Euclidean proposition will be readily identified by means of the above description. As regards the definition or specification (διορισμός) it is to be observed that we have here only one of its uses. Here it means a closer definition or description of the object aimed at, by means of

the concrete lines or figures set out in the ἔκθεσις instead of the general terms used in the enunciation; and its purpose is to rivet the attention better, as Proclus indicates in a later passage (τρόπον τινὰ προσεχείας ἐστὶν αἴτιος ὁ διορισμός）.

The other technical use of the word to signify the limitations to which the possible solutions of a problem are subject is also described by Proclus, who speaks of διορισμοί determining "whether what is sought is impossible or possible, and how far it is practicable and in how many ways" ; and the διορισμός in this sense appears in Euclid as well as in Archimedes and Apollonius. Thus we have in Eucl. 1. 22 the enunciation "From three straight lines which are equal to three given straight lines to construct a triangle," followed immediately by the limiting condition (διορισμός). "Thus two of the straight lines taken together in any manner must be greater than the remaining one." Similarly in VI. 28 the enunciation "To a given straight line to apply a parallelogram equal to a given rectilineal figure and falling short by a parallelogrammic figure similar to a given one" is at once followed by the necessary condition of possibility: "Thus the given rectilineal figure must not be greater than that described on half the line and similar to the defect."

Tannery supposed that, in giving the other description of the διορισμός as quoted above, Proclus, or rather his guide, was using the term incorrectly. The διορισμός in the better known sense of the determination of limits or conditions of possibility was, we are told, invented by Leon. Pappus uses the word in this sense only. The other use of the term might, Tannery thought, be due to a confusion occasioned by the use of the same words (δεῖ δή) in introducing the parts of a proposition corresponding to the two meanings of the word διορισμός. On the other hand it is to be observed that Eutocius distinguishes clearly between the two uses and implies that the difference was well known. The διορισμός in the sense of condition of possibility follows

immediately on the enunciation, is even part of it; the διορισμός in the other sense of course comes immediately after the setting-out.

Proclus has a useful observation respecting the conclusion of a proposition. “The conclusion they are accustomed to make double in a certain way: I mean, by proving it in the given case and then drawing a general inference, passing, that is, from the partial conclusion to the general. For, inasmuch as they do not make use of the individuality of the subjects taken, but only draw an angle or a straight line with a view to placing the datum before our eyes, they consider that this same fact which is established in the case of the particular figure constitutes a conclusion true of every other figure of the same kind. They pass accordingly to the general in order that we may not conceive the conclusion to be partial. And they are justified in so passing, since they use for the demonstration the particular things set out, not quâ particulars, but quâ typical of the rest. For it is not in virtue of such and such a size attaching to the angle which is set out that I effect the bisection of it, but in virtue of its being rectilineal and nothing more. Such and such size is peculiar to the angle set out, but its quality of being rectilineal is common to all rectilineal angles. Suppose, for example, that the given angle is a right angle. If then I had employed in the proof the fact of its being right, I should not have been able to pass to every species of rectilineal angle; but, if I make no use of its being right, and only consider it as rectilineal, the argument will equally apply to rectilineal angles in general.”

§ 6. OTHER TECHNICAL TERMS.

1. Things said to be given.

Proclus attaches to his description of the formal divisions of a proposition an explanation of the different senses in which the word given or datum

(δεδομένον) is used in geometry. “Everything that is given is given in one or other of the following ways, in position, in ratio, in magnitude, or in species. The point is given in position only, but a line and the rest may be given in all the senses.”

The illustrations which Proclus gives of the four senses in which a thing may be given are not altogether happy, and, as regards things which are given in position, in magnitude, and in species, it is best, I think, to follow the definitions given by Euclid himself in his book of Data. Euclid does not mention the fourth class, things given in ratio, nor apparently do any of the great geometers.

(1) Given in position really needs no definition; and, when Euclid says (Data, Def. 4) that “Points, lines and angles are said to be given in position which always occupy the same place,” we are not really the wiser.

(2) Given in magnitude is defined thus (Data, Def. 1): “Areas, lines and angles are called given in magnitude to which we can find equals.” Proclus’ illustration is in this case the following: when, he says, two unequal straight lines are given from the greater of which we have to cut off a straight line equal to the lesser, the straight lines are obviously given in magnitude, “for greater and less, and finite and infinite are predications peculiar to magnitude.” But he does not explain that part of the implication of the term is that a thing is given in magnitude only, and that, for example, its position is not given and is a matter of indifference

(3) Given in species. Euclid’s definition (Data, Def. 3) is: “Rectilineal figures are said to be given in species in which the angles are severally given and the ratios of the sides to one another are given” And this is the recognised use of the term (cf. Pappus, passim) Proclus uses the term in a much wider sense for which I am not aware of any authority. Thus, he says, when we speak of (bisecting) a given rectilineal angle, the angle is given in species by

the word rectilineal, which prevents our attempting, by the same method, to bisect a curvilineal angle! On Eucl. 1. 9, to which he here refers, he says that an angle is given in species when e.g. we say that it is right or acute or obtuse or rectilineal or "mixed," but that the actual angle in the proposition is given in species only. As a matter of fact, we should say that the actual angle in the figure of the proposition is given in magnitude and not in species, part of the implication of given in species being that the actual magnitude of the thing given in species is indifferent; an angle cannot be given in species in this sense at all. The confusion in Proclus' mind is shown when, after saying that a right angle is given in species, he describes a third of a right angle as given in magnitude.

No better example of what is meant by given in species, in its proper sense, as limited to rectilineal figures, can be quoted than the given parallelogram in Eucl. VI. 28, to which the required parallelogram has to be made similar; the former parallelogram is in fact given in species, though its actual size, or scale, is indifferent.

(4) Given in ratio presumably means something which is given by means of its ratio to some other given thing. This we gather from Proclus' remark (in his note on 1. 9) that an angle may be given in ratio "as when we say that it is double and treble of such and such an angle or, generally, greater and less." The term, however, appears to have no authority and to serve no purpose. Proclus may have derived it from such expressions as "in a given ratio" which are common enough.

2. Lemma.

"The term lemma," says Proclus , "is often used of any proposition which is assumed for the construction of something else: thus it is a common remark that a proof has been made out of such and such lemmas. But the special meaning of lemma in geometry is a proposition requiring confirmation. For

when, in either construction or demonstration, we assume anything which has not been proved but requires argument, then, because we regard what has been assumed as doubtful in itsėlf and therefore worthy of investigation, we call it a lemma , differing as it does from the postulate and the axiom in being matter of demonstration, whereas they are immediately taken for granted, without demonstration, for the purpose of confirming other things. Now in the discovery of lemmas the best aid is a mental aptitude for it. For we may see many who are quick at solutions and yet do not work by method; thus Cratistus in our time was able to obtain the required result from first principles, and those the fewest possible, but it was his natural gift which helped him to the discovery. Nevertheless certain methods have been handed down. The finest is the method which by means of analysis carries the thing sought up to an acknowledged principle, a method which Plato, as they say, communicated to Leodamas , and by which the latter, too, is said to have discovered many things in geometry. The second is the method of division , which divides into its parts the genus proposed for consideration and gives a starting-point for the demonstration by means of the elimination of the other elements in the construction of what is proposed, which method also Plato extolled as being of assistance to all sciences. The third is that by means of the reductio ad absurdum, which does not show what is sought directly; but refutes its opposite and discovers the truth incidentally."

3. Case.

"The case (πτῶσις)," Proclus proceeds, "announces different ways of construction and alteration of positions due to the transposition of points or lines or planes or solids. And, in general, all its varieties are seen in the figure, and this is why it is called case, being a transposition in the construction."

4. Porism.

“The term porism is used also of certain problems such as the Porisms written by Euclid. But it is specially used when from what has been demonstrated some other theorem is revealed at the same time without our propounding it, which theorem has on this very account been called a porism (corollary) as being a sort of incidental gain arising from the scientific demonstration.” Cf. the note on I. 15.

5. Objection.

“The objection (ἔνστασις) obstructs the whole course of the argument by appearing as an obstacle (or crying ‘halt,’ ἀπαντῶσα) either to the construction or to the demonstration. There is this difference between the objection and the case, that, whereas he who propounds the case has to prove the proposition to be true of it, he who makes the objection does not need to prove anything: on the contrary it is necessary to destroy the objection and to show that its author is saying what is false.”

That is, in general the objection endeavours to make it appear that the demonstration is not true in every case; and it is then necessary to prove, in refutation of the objection, either that the supposed case is impossible, or that the demonstration is true even for that case. A good instance is afforded by Eucl. 1. 7. The text-books give a second case which is not in the original text of Euclid. Proclus remarks on the proposition as given by Euclid that the objection may conceivably be raised that what Euclid declares to be impossible may after all be possible in the event of one pair of stiaight lines falling completely within the other pair. Proclus then refutes the objection by proving the impossibility in that case also. His proof then came to be given in the text-books as part of Euclid’s proposition.

The objection is one of the technical terms in Aristotle’s logic and its nature is explained in the Prior Analytics. “An objection is a proposition contrary to a proposition.... Objections are of two sorts, general or partial....

For when it is maintained that an attribute belongs to every (member of a class), we object either that it belongs to none (of the class) or that there is some one (member of the class) to which it does not belong."

6. Reduction.

This is again an Aristotelian term, explained in the Prior Analytics. It is well described by Proclus in the following passage:

"Reduction (ἀπαγωγή) is a transition from one problem or theorem to another, the solution or proof of which makes that which is propounded manifest also. For example, after the doubling of the cube had been investigated, they transformed the investigation into another upon which it follows, namely the finding of the two means; and from that time forward they inquired how between two given straight lines two mean proportionals could be discovered. And they say that the first to effect the reduction of difficult constructions was Hippocrates of Chios, who also squared a lune and discovered many other things in geometry, being second to none in ingenuity as regards constructions."

7. Reductio ad absurdum.

This is variously called by Aristotle "reductio ad absurdum" (ἡ εἰς τὸ ἀδύνατον ἀπαγωγή), "proof per impossibile" (ἡ διὰ τοῦ ἀδυνάτου δεῖξις or ἀπόδειξις), "proof leading to the impossible" (ἡ εἰς τὸ ἀδύνατον ἄγουσα ἀπόδειξις). It is part of "proof (starting) from a hypothesis" (ἐξ ὑποθέσεως). "All (syllogisms) which reach the conclusion per impossibile reason out a conclusion which is false, and they prove the original contention (by the method starting) from a hypothesis, when something impossible results from assuming the contradictory of the original contention, as, for example, when it is proved that the diagonal (of a square) is incommensurable because, if it be assumed commensurable, it will follow that odd (numbers) are equal to even (numbers)." Or again, "proof (leading) to the impossible differs from

the direct (δεικτικῆς) in that it assumes what it desires to destroy [namely the hypothesis of the falsity of the conclusion] and then reduces it to something admittedly false, whereas the direct proof starts from premisses admittedly true."

Proclus has the following description of the reductio ad absurdum. "Proofs by reductio as absurdum in every case reach a conclusion manifestly impossible, a conclusion the contradictory of which is admitted. In some cases the conclusions are found to conflict with the common notions, or the postulates, or the hypotheses (from which we started); in others they contradict propositions previously established" ..."Every reductio ad absurdum assumes what conflicts with the desired result, then, using that as a basis, proceeds until it arrives at an admitted absurdity, and, by thus destroying the hypothesis, establishes the result originally desired. For it is necessary to understand generally that all mathematical arguments either proceed from the first principles or lead back to them, as Porphyry somewhere says. And those which proceed from the first principles are again of two kinds, for they start either from common notions and the clearness of the self-evident alone, or from results previously proved; while those which lead back to the principles are either by way of assuming the principles or by way of destroying them. Those which assume the principles are called analyses, and the opposite of these are syntheses — for it is possible to start from the said principles and to proceed in the regular order to the desired conclusion, and this process is synthesis — while the arguments which would destroy the principles are called reductiones ad absurdum. For it is the function of this method to upset something admitted as clear."

8. Analysis and Synthesis.

It will be seen from the note on Eucl. XIII. I that the MSS. of the Elements contain definitions of Analysis and Synthesis followed by alternative proofs

of XIII. 1-5 after that method. The definitions and alternative proofs are interpolated, but they have great historical interest because of the possibility that they represent an ancient method of dealing with these propositions, anterior to Euclid. The propositions give properties of a line cut "in extreme and mean ratio," and they are preliminary to the construction and comparison of the five regular solids. Now Pappus, in the section of his Collection dealing with the latter subject, says that he will give the comparisons between the five figures, the pyramid, cube, octahedron, dodecahedron and icosahedron, which have equal surfaces, "not by means of the so-called analytical inquiry, by which some of the ancients worked out the proofs, but by the synthetical method...." The conjecture of Bretschneider that the matter interpolated in Eucl. XIII. is a survival of investigations due to Eudoxus has at first sight much to commend it. In the first place, we are told by Proclus that Eudoxus "greatly added to the number of the theorems which Plato originated regarding the section, and employed in them the method of analysis." It is obvious that "the section" was some particular section which by the time of Plato had assumed great importance; and the one section of which this can safely be said is that which was called the "golden section," namely, the division of a straight line in extreme and mean ratio which appears in Eucl. II. 11 and is therefore most probably Pythagorean. Secondly, as Cantor points out, Eudoxus was the founder of the theory of proportions in the form in which we find it in Euclid V., VI., and it was no doubt through meeting, in the course of his investigations, with proportions not expressible by whole numbers that he came to realise the necessity for a new theory of proportions which should be applicable to incommensurable as well as commensurable magnitudes. The "golden section" would furnish such a case. And it is even mentioned by Proclus in this connexion. He is explaining that it is only in arithmetic that all quantities bear "rational" ratios (ῥητὸς λόγος)

to one another, while in geometry there are "irrational" ones (ἄρρητος) as well. "Theorems about sections like those in Euclid's second Book are common to both [arithmetic and geometry] except that in which the straight line is cut in extreme and mean ratio."

The definitions of Analysis and Synthesis interpolated in Eucl. XIII. are as follows (I adopt the reading of B and V, the only intelligible one, for the second).

"Analysis is an assumption of that which is sought as if it were admitted <and the passage> through its consequences to something admitted (to be) true.

"Synthesis is an assumption of that which is admitted <and the passage> through its consequences to the finishing or attainment of what is sought."

The language is by no means clear and has, at the best, to be filled out.

Pappus has a fuller account:

"The so-called ἀναλυόμενος ('Treasury of Analysis') is, to put it shortly, a special body of doctrine provided for the use of those who, after finishing the ordinary Elements, are desirous of acquiring the power of solving problems which may be set them involving (the construction of) lines, and it is useful for this alone. It is the work of three men, Euclid the author of the Elements, Apollonius of Perga, and Aristaeus the elder, and proceeds by way of analysis and synthesis.

"Analysis then takes that which is sought as if it were admitted and passes from it through its successive consequences to something which is admitted as the result of synthesis: for in analysis we assume that which is sought as if it were (already) done (γεγονός), and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by so retracing our steps we come upon something already known or

belonging to the class of first principles, and such a method we call analysis as being solution backwards (ἀνάπαλιν λύσιν).

"But in synthesis, reversing the process, we take as already done that which was last arrived at in the analysis and, by arranging in their natural order as consequences what were before antecedents, and successively connecting them one with another, we arrive finally at the construction of what was sought; and this we call synthesis.

"Now analysis is of two kinds, the one directed to searching for the truth and called theoretical, the other directed to finding what we are told to find and called problematical. (1) In the theoretical kind we assume what is sought as if it were existent and true, after which we pass through its successive consequences, as if they too were true and established by virtue of our hypothesis, to something admitted: then (a), if that something admitted is true, that which is sought will also be true and the proof will correspond in the reverse order to the analysis, but (b), if we come upon something admittedly false, that which is sought will also be false. (2) In the problematical kind we assume that which is propounded as if it were known, after which we pass through its successive consequences, taking them as true, up to something admitted: if then (a) what is admitted is possible and obtainable, that is, what mathematicians call given, what was originally proposed will also be possible, and the proof will again correspond in reverse order to the analysis, but if (b) we come upon something admittedly impossible, the problem will also be impossible."

The ancient Analysis has been made the subject of careful studies by several writers during the last half-century, the most complete being those of Hankel, Duhamel and Zeuthen; others by Ofterdinger and Cantor should also be mentioned.

The method is as follows. It is required, let us say, to prove that a certain proposition A is true. We assume as a hypothesis that A is true and, starting from this we find that, if A is true, a certain other proposition B is true; if B is true, then C; and so on until we arrive at a proposition K which is admittedly true. The object of the method is to enable us to infer, in the reverse order, that, since K is true, the proposition A originally assumed is true. Now Aristotle had already made it clear that false hypotheses might lead to a conclusion which is true. There is therefore a possibility of error unless a certain precaution is taken. While, for example, B may be a necessary consequence of A, it may happen that A is not a necessary consequence of B. Thus, in order that the reverse inference from the truth of K that A is true may be logically justified, it is necessary that each step in the chain of inferences should be unconditionally convertible. As a matter of fact, a very large number of theorems in elementary geometry are unconditionally convertible, so that in practice the difficulty in securing that the successive steps shall be convertible is not so great as might be supposed. But care is always necessary. For example, as Hankel says, a proposition may not be unconditionally convertible in the form in which it is generally quoted. Thus the proposition "The vertices of all triangles having a common base and constant vertical angle lie on a circle" cannot be converted into the proposition that "All triangles with common base and vertices lying on a circle have a constant vertical angle" ; for this is only true if the further conditions are satisfied (1) that the circle passes through the extremities of the common base and (2) that only that part of the circle is taken as the locus of the vertices which lies on one side of the base. If these conditions are added, the proposition is unconditionally convertible. Or again, as Zeuthen remarks, K may be obtained by a series of inferences in which A or some other proposition in the series is only apparently used; this would be the case e.g.

when the method of modern algebra is being employed and the expressions

on each side of the sign of equality have been inadvertently multiplied by some composite magnitude which is in reality equal to zero.

Although the above extract from Pappus does not make it clear that each step in the chain of argument must be convertible in the case taken, he almost implies this in the second part of the definition of Analysis where, instead of speaking of the consequences B, C... successively following from A, he suddenly changes the expression and says that we inquire what it is (B) from which A follows (A being thus the consequence of B, instead of the reverse), and then what (viz. C) is the antecedent cause of B; and in practice the Greeks secured what was wanted by always insisting on the analysis being confirmed by subsequent synthesis, that is, they laboriously worked backwards the whole way from K to A, reversing the order of the analysis, which process would undoubtedly bring to light any flaw which had crept into the argument through the accidental neglect of the necessary precautions.

Reductio ad absurdum a variety of analysis.

In the process of analysis starting from the hypothesis that a proposition A is true and passing through B, C... as successive consequences we may arrive at a proposition K which, instead of being admittedly true, is either admittedly false or the contradictory of the original hypothesis A or of some one or more of the propositions B, C... intermediate between A and K. Now correct inference from a true proposition cannot lead to a false proposition; and in this case therefore we may at once conclude, without any inquiry whether the various steps in the argument are convertible or not, that the hypothesis A is false, for, if it were true, all the consequences correctly inferred from it would be true and no incompatibility could arise. This method of proving that a given hypothesis is false furnishes an indirect method of proving that a given hypothesis A is true, since we have only to take the contradictory of A and to prove that it is false. This is the method of

reductio ad absurdum, which is therefore a variety of analysis. The contradictory of A, or not-A, will generally include more than one case and, in order to prove its falsity, each of the cases must be separately disposed of: e.g., if it is desired to prove that a certain part of a figure is equal to some other part, we take separately the hypotheses (1) that it is greater, (2) that it is less, and prove that each of these hypotheses leads to a conclusion either admittedly false or contradictory to the hypothesis itself or to some one of its consequences.

Analysis as applied to problems.

It is in relation to problems that the ancient analysis has the greatest significance, because it was the one general method which the Greeks used for solving all "the more abstruse problems" (τὰ ἀσαφέστερα τῶν προβλημάτων）.

We have, let us suppose, to construct a figure satisfying a certain set of conditions If we are to proceed at all methodically and not by mere guesswork, it is first necessary to "analyse" those conditions. To enable this to be done we must get them clearly in our minds, which is only possible by assuming all the conditions to be actually fulfilled, in other words, by supposing the problem solved. Then we have to transform those conditions, by all the means which practice in such cases has taught us to employ, into other conditions which are necessarily fulfilled if the original conditions are, and to continue this transformation until we at length arrive at conditions which we are in a position to satisfy. In other words, we must arrive at some relation which enables us to construct a particular part of the figure which, it is true, has been hypothetically assumed and even drawn, but which nevertheless really requires to be found in order that the problem may be solved. From that moment the particular part of the figure becomes one of the data, and a fresh relation has to be found which enables a fresh part of the

figure to be determined by means of the original data and the new one together. When this is done, the second new part of the figure also belongs to the data; and we proceed in this way until all the parts of the required figure are found. The first part of the analysis down to the point of discovery of a relation which enables us to say that a certain new part of the figure not belonging to the original data is given, Hankel calls the transformation; the second part, in which it is proved that all the remaining parts of the figure are "given," he calls the resolution. Then follows the synthesis, which also consists of two parts, (1) the construction, in the order in which it has to be actually carried out, and in general following the course of the second part of the analysis, the resolution; (2) the demonstration that the figure obtained does satisfy all the given conditions, which follows the steps of the first part of the analysis, the transformation, but in the reverse order. The second part of the analysis, the resolution, would be much facilitated and shortened by the existence of a systematic collection of Data such as Euclid's book bearing that title, consisting of propositions proving that, if in a figure certain parts or relations are given, other parts or relations are also given. As regards the first part of the analysis, the transformation, the usual rule applies that every step in the chain must be unconditionally convertible; and any failure to observe this condition will be brought to light by the subsequent synthesis. The second part, the resolution, can be directly turned into the construction since that only is given which can be constructed by the means provided in the Elements.

It would be difficult to find a better illustration of the above than the example chosen by Hankel from Pappus.

Given a circle ABC and two points D, E external to it, to draw straight lines DB, EB from D, E to a point B on the circle such that, if DB, EB produced meet the circle again in C, A, AC shall be parallel to DE.

Analysis.

Suppose the problem solved and the tangent at A drawn, meeting ED produced in F.

(Part I. Transformation.)

Then, since AC is parallel to DE, the angle at C is equal to the angle CDE.

But, since FA is a tangent, the angle at C is equal to the angle FAE.

Therefore the angle FAE is equal to the angle CDE, whence A, B, D, F are concyclic.

Therefore the rectangle AE, EB is equal to the rectangle FE, ED.

(Part II. Resolution.)

But the rectangle AE, EB is given, because it is equal to the square on the tangent from E.

Therefore the rectangle FE, ED is given; and, since ED is given, FE is given (in length). [Data, 57.]

But FE is given in position also, so that F is also given. [Data, 27.]

Now FA is the tangent from a given point F to a circle ABC given in position; therefore FA is given in position and magnitude. [Data, 90.]

And F is given; therefore A is given.

But E is also given; therefore the straight line AE is given in position. [Data, 26.]

And the circle ABC is given in position; therefore the point B is also given. [Data, 25.]

But the points D, E are also given; therefore the straight lines DB, BE are also given in position.

Synthesis.

(Part I. Construction.)

Suppose the circle ABC and the points D, E given.

Take a rectangle contained by ED and by a certain straight line EF equal to the square on the tangent to the circle from E.

From F draw FA touching the circle in A; join ABE and then DB, producing DB to meet the circle at C. Join AC.

I say then that AC is parallel to DE.

(Part II. Demonstration.)

Since, by hypothesis, the rectangle FE, ED is equal to the square on the tangent from E, which again is equal to the rectangle AE, EB, the rectangle AE, EB is equal to the rectangle FE, ED.

Therefore A, B, D, F are concyclic, whence the angle FAE is equal to the angle BDE.

But the angle FAE is equal to the angle ACB in the alternate segment; therefore the angle ACB is equal to the angle BDE.

Therefore AC is parallel to DE.

In cases where a διορισμός is necessary, i.e. where a solution is only possible under certain conditions, the analysis will enable those conditions to be ascertained. Sometimes the διορισμός is stated and proved at the end of the analysis, e.g. in Archimedes, On the Sphere and Cylinder, II. 7; sometimes it is stated in that place and the proof postponed till after the end of the synthesis, e.g. in the solution of the problem subsidiary to On the Sphere and Cylinder, II. 4, preserved in Eutocius' commentary on that proposition. The analysis should also enable us to determine the number of solutions of which the problem is susceptible.

§ 7. THE DEFINITIONS.

General. "Real" and "Nominal" Definitions.

It is necessary, says Aristotle, whenever any one treats of any whole subject, to divide the genus into its primary constituents, those which are indivisible in species respectively: e.g. number must be divided into triad and dyad; then an attempt must be made in this way to obtain definitions, e.g. of a straight line, of a circle, and of a right angle.

The word for definition is ὅρος. The original meaning of this word seems to have been "boundary," "landmark." Then we have it in Plato and Aristotle in the sense of standard or determining principle ("id quo alicuius rei natura constituitur vel definitur," Index Aristotelicus）; and closely connected with this is the sense of definition. Aristotle uses both ὅρος and ὁρισμός for definition, the former occurring more frequently in the Topics, the latter in the Metaphysics.

Let us now first be clear as to what a definition does not do. There is nothing in connexion with definitions which Aristotle takes more pains to emphasise than that a definition asserts nothing as to the existence or non-existence of the thing defined. It is an answer to the question what a thing is (τί ἐστι), and does not say that it is (ὅτι ἔστι). The existence of the various things defined has to be proued, except in the case of a few primary things in each science, the existence of which is indemonstrable and must be assumed among the first principles of each science; e.g. points and lines in geometry must be assumed to exist, but the existence of everything else must be proved. This is stated clearly in the long passage quoted above under First Principles. It is reasserted in such passages as the following. "The (answer to the question) what is a man and the fact that a man exists are different things." "It is clear that, even according to the view of definitions now current, those who define things do not prove that they exist." "We say that it is by demonstration that we must show that everything exists, except essence (εἰ μὴ οὐσία εἴη). But the existence of a thing is never essence; for the

existent is not a genus. Therefore there must be demonstration that a thing exists. Thus, what is meant by triangle the geometer assumes, but that it exists he has to prove." "Anterior knowledge of two sorts is necessary: for it is necessary to presuppose, with regard to some things, that they exist; in other cases it is necessary to understand what the thing described is, and in other cases it is necessary to do both. Thus, with the fact that one of two contradictories must be true, we must know that it exists (is true); of the triangle we must know that it means such and such a thing; of the unit we must know both what it means and that it exists." What is here so much insisted on is the very fact which Mill pointed out in his discussion of earlier views of Definitions, where he says that the so-called real definitions or definitions of things do not constitute a different kind of definition from nominal definitions, or definitions of names; the former is simply the latter plus something else, namely a covert assertion that the thing defined exists. "This covert assertion is not a definition but a postulate. The definition is a mere identical proposition which gives information only about the use of language, and from which no conclusion affecting matters of fact can possibly be drawn. The accompanying postulate, on the other hand, affirms a fact which may lead to consequences of every degree of importance. It affirms the actual or possible existence of Things possessing the combination of attributes set forth in the definition: and this, if true, may be foundation sufficient on which to build a whole fabric of scientific truth." This statement really adds nothing to Aristotle's doctrine: it has even the slight disadvantage, due to the use of the word "postulate" to describe "the covert assertion" in all cases, of not definitely pointing out that there are cases where existence has to be proued as distinct from those where it must be assumed. It is true that the existence of a definiend may have to be taken for granted provisionally until the time comes for proving it; but, so far as regards any case where

existence must be proved sooner or later, the provisional assumption would be for Aristotle, not a postulate, but a hypothesis. In modern times, too, Mill's account of the true distinction between real and nominal definitions had been fully anticipated by Saccheri, the editor of Euclides ab omni naevo vindicatus (1733), famous in the history of non-Euclidean geometry. In his Logica Demonstrativa (to which he also refers in his Euclid) Saccheri lays down the clear distinction between what he calls definitiones quid nominis or nominales, and definitiones quid rei or reales, namely that the former are only intended to explain the meaning that is to be attached to a given term, whereas the latter, besides declaring the meaning of a word, affirm at the same time the existence of the thing defined or, in geometry, the possibility of constructing it. The definitio quid nominis becomes a definitio quid rei "by means of a postulate, or when we come to the question whether the thing exists and it is answered affirmatively . " Definitiones quid nominis are in themselves quite arbitrary, and neither require nor are capable of proof; they are merely provisional and are only intended to be turned as quickly as possible into definitiones quid rei, either (1) by means of a postulate in which it is asserted or conceded that what is defined exists or can be constructed, e.g. in the case of straight lines and circles, to which Euclid's first three postulates refer, or (2) by means of a demonstration reducing the construction of the figure defined to the successive carrying-out of a certain number of those elementary constructions, the possibility of which is postulated. Thus definitiones quid rei are in general obtained as the result of a series of demonstrations. Saccheri gives as an instance the construction of a square in Euclid I. 46. Suppose that it is objected that Euclid had no right to define a square, as he does at the beginning of the Book, when it was not certain that such a figure exists in nature; the objection, he says, could only have force if, before proving and making the construction, Euclid had assumed the

aforesaid figure as given. That Euclid is not guilty of this error is clear from the fact that he never presupposes the existence of the square as defined until after I. 46.

Confusion between the nominal and the real definition as thus described, i.e. the use of the former in demonstration before it has been turned into the latter by the necessary proof that the thing defined exists, is according to Saccheri one of the most fruitful sources of illusory demonstration, and the danger is greater in proportion to the "complexity" of the definition, i.e. the number and variety of the attributes belonging to the thing defined. For the greater is the possibility that there may be among the attributes some that are incompatible, i.e. the simultaneous presence of which in a given figure can be proved, by means of other postulates etc. forming part of the basis of the science, to be impossible.

The same thought is expressed by Leibniz also. "If," he says, "we give any definition, and it is not clear from it that the idea, which we ascribe to the thing, is possible, we cannot rely upon the demonstrations which we have derived from that definition, because, if that idea by chance involves a contradiction, it is possible that even contradictories may be true of it at one and the same time, and thus our demonstrations will be useless. Whence it is clear that definitions are not arbitrary. And this is a secret which is hardly sufficiently known." Leibniz' favourite illustration was the "regular polyhedron with ten faces," the impossibility of which is not obvious at first sight.

It need hardly be added that, speaking generally, Euclid's definitions, and his use of them, agree with the doctrine of Aristotle that the definitions themselves say nothing as to the existence of the things defined, but that the existence of each of them must be proved or (in the case of the "principles") assumed. In geometry, says Aristotle, the existence of points and lines only

must be assumed, the existence of the rest being proved. Accordingly Euclid's first three postulates declare the possibility of constructing straight lines and circles (the only "lines" except straight lines used in the Elements). Other things are defined and afterwards constructed and proved to exist: e.g. in Book I., Def. 20, it is explained what is meant by an equilateral triangle; then (I. 1) it is proposed to construct it, and, when constructed, it is proved to agree with the definition. When a square is defined (I. Def. 22), the question whether such a thing really exists is left open until, in I. 46, it is proposed to construct it and, when constructed, it is proved to satisfy the definition. Similarly with the right angle (I. Def. 10, and I. 11) and parallels (I. Def. 23, and I. 27-29). The greatest care is taken to exclude mere presumption and imagination. The transition from the subjective definition of names to the objective definition of things is made, in geometry, by means of constructions (the first principles of which are postulated), as in other sciences it is made by means of experience.

Aristotle's requirements in a definition.

We now come to the positive characteristics by which, according to Aristotle, scientific definitions must be marked.

First, the different attributes in a definition, when taken separately, cover more than the notion defined, but the combination of them does not. Aristotle illustrates this by the "triad," into which enter the several notions of number, odd and prime, and the last "in both its two senses (a) of not being measured by any (other) number (ὡς μὴ μετρεῖσθαι ἀριθμῷ) and (b) of not being obtainable by adding numbers together" (ὡς μὴ συγκεῖσθαι ἐξ ἀριθμῶν), a unit not being a number. Of these attributes some are present in all other odd numbers as well, while the last [primeness in the second sense] belongs also

to the dyad, but in nothing but the triad are they all present." The fact can be equally well illustrated from geometry. Thus, e.g. into the definition of a square (Eucl. I., Def. 22) there enter the several notions of figure, four-sided, equilateral, and right-angled, each of which covers more than the notion into which all enter as attributes.

Secondly, a definition must be expressed in terms of things which are prior to, and better known than, the things defined. This is clear, since the object of a definition is to give us knowledge of the thing defined, and it is by means of things prior and better known that we acquire fresh knowledge, as in the course of demonstrations. But the terms "prior" and "better known" are, as usual susceptible of two meanings; they may mean (1) absolutely or logically prior and better known, or (2) better known relatively to us. In the absolute sense, or from the standpoint of reason, a point is better known than a line, a line than a plane, and a plane than a solid, as also a unit is better known than number (for the unit is prior to, and the first principle of, any number). Similarly, in the absolute sense, a letter is prior to a syllable. But the case is sometimes different relatively to us; for example, a solid is more easily realised by the senses than a plane, a plane than a line, and a line than a point. Hence, while it is more scientific to begin with the absolutely prior, it may, perhaps, be permissible, in case the learner is not capable of following the scientific order, to explain things by means of what is more intelligible to him. "Among the definitions framed on this principle are those of the point, the line and the plane; all these explain what is prior by means of what is posterior, for the point is described as the extremity of a line, the line of a plane, the plane of a solid." But, if it is asserted that such definitions by means of things which are more intelligible relatively only to a particular individual are really definitions, it will follow that there may be many definitions of the same thing, one for each individual for whom a thing is

being defined, and even different definitions for one and the same individual at different times, since at first sensible objects are more intelligible, while to a better trained mind they become less so. It follows therefore that a thing should be defined by means of the absolutely prior and not the relatively prior, in order that there may be one sole and immutable definition. This is further enforced by reference to the requirement that a good definition must state the genus and the differentiae, for these are among the things which are, in the absolute sense, better known than, and prior to, the species (τῶν ἁπλῶς γνωριμωτέρων καὶ προτέρων τοῦ εἴδους ἐστίν). For to destroy the genus and the differentia is to destroy the species, so that the former are prior to the species; they are also better known, for, when the species is known, the genus and the differentia must necessarily be known also, e.g. he who knows "man" must also know "animal" and "land-animal," but it does not follow, when the genus and differentia are known, that the species is known too, and hence the species is less known than they are. It may be frankly admitted that the scientific definition will require superior mental powers for its apprehension; and the extent of its use must be a matter of discretion. So far Aristotle; and we have here the best possible explanation why Euclid supplemented his definition of a point by the statement in I. Def. 3 that the extremities of a line are points and his definition of a surface by I. Def. 6 to the effect that the extremities of a surface are lines. The supplementary explanations do in fact enable us to arrive at a better understanding of the formal definitions of a point and a line respectively, as is well explained by Simson in his note on Def. 1. Simson says, namely, that we must consider a solid, that is, a magnitude which has length, breadth and thickness, in order to understand aright the definitions of a point, a line and a surface. Consider, for instance, the boundary common to two solids which are contiguous or the boundary which divides one solid into two contiguous parts; this boundary is a surface.

We can prove that it has no thickness by taking away either solid, when it remains the boundary of the other; for, if it had thickness, the thickness must either be a part of one solid or of the other, in which case to take away one or other solid would take away the thickness and therefore the boundary itself: which is impossible. Therefore the boundary or the surface has no thickness. In exactly the same way, regarding a line as the boundary of two contiguous surfaces, we prove that the line has no breadth; and, lastly, regarding a point as the common boundary or extremity of two lines, we prove that a point has no length, breadth or thickness.

Aristotle on unscientific definitions.

Aristotle distinguishes three kinds of definition which are unscientific because founded on what is not prior (μὴ ἐκ προτέρων). The first is a definition of a thing by means of its opposite, e.g. of "good" by means of "bad" ; this is wrong because opposites are naturally evolved together, and the knowledge of opposites is not uncommonly regarded as one and the same, so that one of the two opposites cannot be better known than the other. It is true that, in some cases of opposites, it would appear that no other sort of definition is possible: e.g. it would seem impossible to define double apart from the half and, generally, this would be the case with things which in their very nature (καθ' αὐτά) are relative terms (πρός τι λέγεται), since one cannot be known without the other, so that in the notion of either the other must be comprised as well. The second kind of definition which is based on what is not prior is that in which there is a complete circle through the unconscious use in the definition itself of the notion to be defined though not of the name. Trendelenburg illustrates this by two current definitions, (1) that of magnitude as that which can be increased or diminished, which is bad

because the positive and negative comparatives "more" and "less" presuppose the notion of the positive "great," (2) the famous Euclidean definition of a straight line as that which "lies evenly with the points on itself" (ἐξ ἴσου τοῖς ἐφ᾽ ἑαυτῆς σημείοις κεῖται), where "lies evenly" can only be understood with the aid of the very notion of a straight line which is to be defined. The third kind of vicious definition from that which is not prior is the definition of one of two coordinate species by means of its coordinate (ἀντιδιῃρημένον), e.g. a definition of "odd" as that which exceeds the even by a unit (the second alternative in Eucl. VII. Def. 7); for "odd" and "even" are coordinates, being differentiae of number. This third kind is similar to the first. Thus, says Trendelenburg, it would be wrong to define a square as "a rectangle with equal sides."

Aristotle's third requirement.

A third general observation of Aristotle which is specially relevant to geometrical definitions is that "to know what a thing is (τί ἐστιν) is the same as knowing why it is (διὰ τί ἐστιν）." "What is an eclipse? A deprivation of light from the moon through the interposition of the earth. Why does an eclipse take place? Or why is the moon eclipsed? Because the light fails through the earth obstructing it. What is harmony? A ratio of numbers in high or low pitch. Why does the high-pitched harmonise with the low-pitched? Because the high and the low have a numerical ratio to one another." "We seek the cause (τὸ διότι) when we are already in possession of the fact (τὸ ὅτι). Sometimes they both become evident at the same time, but at all events the cause cannot possibly be known [as a cause] before the fact is known." "It is impossible to know what a thing is if we do not know that it is" Trendelenburg paraphrases: "The definition of the notion does not fulfil its

purpose until it is made genetic. It is the producing cause which first reveals the essence of the thing.... . The nominal definitions of geometry have only a provisional significance and are superseded as soon as they are made genetic by means of construction." E.g. the genetic definition of a parallelogram is evolved from Eucl. I. 31 (giving the construction for parallels) and I. 33 about the lines joining corresponding ends of two straight lines parallel and equal in length. Where existence is proved by construction, the cause and the fact appear together.

Again, "it is not enough that the defining statement should set forth the fact, as most definitions do; it should also contain and present the cause; whereas in practice what is stated in the definition is usually no more than a conclusion (συμπέρασμα). For example, what is quadrature? The construction of an equilateral right-angled figure equal to an oblong. But such a definition expresses merely the conclusion. Whereas, if you say that quadrature is the discovery of a mean proportional, then you state the reason." This is better understood if we compare the statement elsewhere that "the cause is the middle term, and this is what is sought in all cases," and the illustration of this by the case of the proposition that the angle in a semicircle is a right angle. Here the middle term which it is sought to establish by means of the figure is that the angle in the semi-circle is equal to the half of two right angles. We have then the syllogism: Whatever is half of two right angles is a right angle; the angle in a semi-circle is the half of two right angles; therefore (conclusion) the angle in a semi-circle is a right angle. As with the demonstration, so it should be with the definition. A definition which is to show the genesis of the thing defined should contain the middle term or cause; otherwise it is a mere statement of a conclusion. Consider, for instance, the definition of "quadrature" as "making a square equal in area to a rectangle with unequal sides." This gives no hint as to whether a solution of

the problem is possible or how it is solved: but, if you add that to find the mean proportional between two given straight lines gives another straight line such that the square on it is equal to the rectangle contained by the first two straight lines, you supply the necessary middle term or cause.

Technical terms not defined by Euclid.

It will be observed that what is here defined, “quadrature” or “squaring” (τετραγωνισμός), is not a geometrical figure, or an attribute of such a figure or a part of a figure, but a technical term used to describe a certain problem. Euclid does not define such things; but the fact that Aristotle alludes to this particular definition as well as to definitions of deflection (κεκλάσθαι) and of verging (νεύειν) seems to show that earlier text-books included among definitions explanations of a number of technical terms, and that Euclid deliberately omitted these explanations from his Elements as surplusage. Later the tendency was again in the opposite direction, as we see from the much expanded Definitions of Heron, which, for example, actually include a definition of a deflected line (κεκλασμένη γραμμή）. Euclid uses the passive of κλᾶν occasionally, but evidently considered it unnecessary to explain such terms, which had come to bear a recognised meaning.

The mention too by Aristotle of a definition of verging (νεύειν) suggests that the problems indicated by this term were not excluded from elementary text-books before Euclid. The type of problem (νεῦσις) was that of placing a straight line across two lines, e.g. two straight lines, or a straight line and a circle, so that it shall verge to a given point (i.e. pass through it if produced) and at the same time the intercept on it made by the two given lines shall be of given length. In general, the use of conics is required for the theoretical solution of these problems, or a mechanical contrivance for their practical

solution. Zeuthen, following Oppermann, gives reasons for supposing, not only that mechanical constructions were practically used by the older Greek geometers for solving these problems, but that they were theoretically recognised as a permissible means of solution when the solution could not be effected by means of the straight line and circle, and that it was only in later times that it was considered necessary to use conics in every case where that was possible. Heiberg suggests that the allusion of Aristotle to νεύσεις perhaps confirms this supposition, as Aristotle nowhere shows the slightest acquaintance with conics. I doubt whether this is a safe inference, since the problems of this type included in the elementary text-books might easily have been limited to those which could be solved by "plane" methods (i.e. by means of the straight line and circle). We know, e.g., from Pappus that Apollonius wrote two Books on plane νεύσεις. But one thing is certain, namely that Euclid deliberately excluded this class of problem, doubtless as not being essential in a book of Elements.

Definitions not afterwards used.

Lastly, Euclid has definitions of some terms which he never afterwards uses, e.g. oblong (ἑτερόμηκες), rhombus, rhomboid. The "oblong" occurs in Aristotle; and it is certain that all these definitions are survivals from earlier books of Elements.

BOOK I. DEFINITIONS.

1

A point is that which has no part.

2

A line is breadthless length.

3

The extremities of a line are points.

4

A straight line is a line which lies evenly with the points on itself.

5

A surface is that which has length and breadth only.

6

The extremities of a surface are lines.

7

A plane surface is a surface which lies evenly with the straight lines on itself.

8

A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

9

And when the lines containing the angle are straight, the angle is called rectilineal.

10

When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

11

An obtuse angle is an angle greater than a right angle.

12

An acute angle is an angle less than a right angle.

13

A boundary is that which is an extremity of anything.

14

A figure is that which is contained by any boundary or boundaries.

15

A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;

16

And the point is called the centre of the circle.

17

A diameter of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

18

A semicircle is the figure contained by the diameter and the circumference cut off by it. And the centre of the semicircle is the same as that of the circle.

19

Rectilineal figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.

20

Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.

21

Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acuteangled triangle that which has its three angles acute.

22

Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.

23

Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

POSTULATES.

1

Let the following be postulated:

To draw a straight line from any point to any point.

2

To produce a finite straight line continuously in a straight line.

3

To describe a circle with any centre and distance.

4

That all right angles are equal to one another.

5

That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

COMMON NOTIONS.

1

Things which are equal to the same thing are also equal to one another.

2

If equals be added to equals, the wholes are equal.

3

If equals be subtracted from equals, the remainders are equal.

4

Things which coincide with one another are equal to one another.

5

The whole is greater than the part.

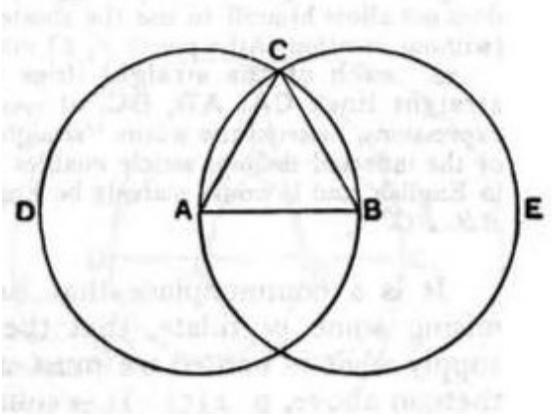
PROPOSITIONS.

Proposition 1.

On a given finite straight line to construct an equilateral triangle.

Let AB be the given finite straight line.

Thus it is required to construct
an equilateral triangle on the straight line AB.



With centre A and distance AB let the circle BCD be described; {Post. 3} again, with centre B and distance BA let the circle ACE be described; {Post. 3} and from the point C, in which the circles cut one another, to the points A, B let the straight lines CA, CB be joined. {Post. 1}

Now, since the point A is the centre of the circle CDB,

AC is equal to AB. {Def. 15}

Again, since the point B is the centre of the circle CAE,

BC is equal to BA. {Def. 15}

But CA was also proved equal to AB;
therefore each of the straight lines CA, CB is equal to AB.

And things which are equal to the same thing are also equal to one another; {C.N. 1}

therefore CA is also equal to CB.

Therefore the three straight lines CA, AB, BC are
equal to one another.

Therefore the triangle ABC is equilateral; and it has been constructed on the given finite straight line AB.

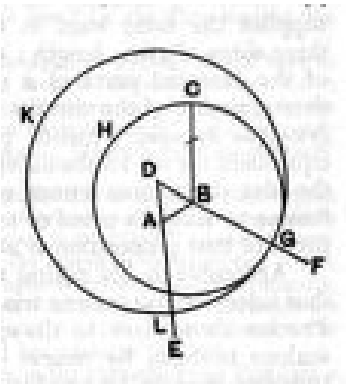
(Being) what it was required to do. 1 2 3 4 5

Proposition 2.

To place at a given point (as an extremity) a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line.

Thus it is required to place at the point A (as an extremity)
a straight line equal to the given straight line BC.



From the point A to the point B let the straight line AB be joined; {Post. 1} and on it let the equilateral triangle
DAB be constructed. {I. 1}

Let the straight lines AE, BF be produced in a straight line with DA, DB; {Post. 2} with centre B and distance BC let the
circle CGH be described; {Post. 3} and again, with centre D and distance DG let the circle GKL be described. {Post. 3}

Then, since the point B is the centre of the circle CGH,

BC is equal to BG.

Again, since the point D is the centre of the circle GKL,

DL is equal to DG.

And in these DA is equal to DB;

therefore the remainder AL is equal to the remainder BG. {C.N. 3}

But BC was also proved equal to BG;

therefore each of the straight lines AL, BC is equal to BG.

And things which are equal to the same thing are also equal to one another; {C.N. 1}

therefore AL is also equal to BC.

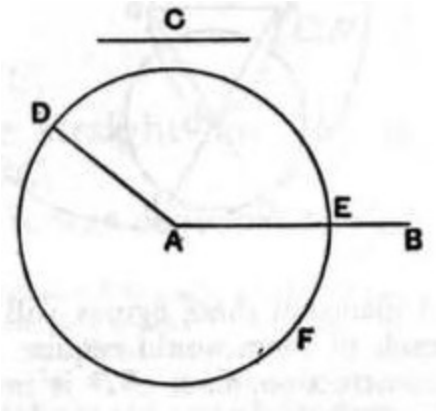
Therefore at the given point A the straight line AL is placed equal to the given straight line BC.

(Being) what it was required to do. [6] [7] [8]

Proposition 3.

Given two unequal straight lines, to cut off from the greater a straight line equal to the less.

Let AB, C be the-two given unequal straight lines, and let AB be the greater of them.



Thus it is required to cut off from AB the greater a straight line equal to C the less.

At the point A let AD be placed equal to the straight line C; {I. 2} and with centre A and distance AD let the circle DEF be described. {Post. 3}

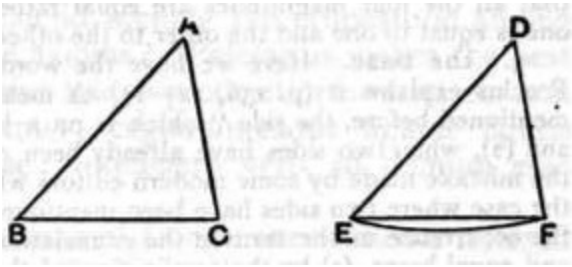
Now, since the point A is the centre of the circle DEF, AE is equal to AD. {Def. 15} But C is also equal to AD. Therefore each of the straight lines AE, C is equal to AD; so that AE is also equal to C. {C.N. 1}

Therefore, given the two straight lines AB, C, from AB the greater AE has been cut off equal to C the less.

(Being) what it was required to do.

Proposition 4.

If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles
will be equal to the remaining angles respectively, namely those which the equal sides subtend.



Let ABC, DEF be two triangles having the two sides AB, AC equal to the two sides DE, DF respectively, namely AB to DE and AC to DF, and the angle BAC equal to the
angle EDF.

I say that the base BC is also equal to the base EF, the triangle ABC will be equal to the triangle DEF, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend,

that

is, the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

For, if the triangle ABC be applied to the triangle DEF,

and if the point A be placed on the point D and the straight line AB on DE,

then the point B will also coincide with E, because AB is equal to DE.

Again, AB coinciding with DE, the straight line AC will also coincide with DF, because the angle BAC is equal to the angle EDF;

hence the point C will also coincide with the point F, because AC is again equal to DF.

But B also coincided with E; hence the base BC will coincide with the base EF.

{For if, when B coincides with E and C with F, the base BC does not coincide with the base EF, two straight lines will enclose a space: which is impossible.

Therefore the base BC will coincide with EF} and will be equal to it.

{C.N. 4}

Thus the whole triangle ABC will coincide with the whole triangle DEF,

and will be equal to it.

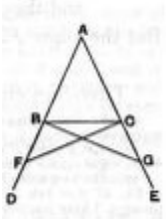
And the remaining angles will also coincide with the remaining angles and will be equal to them, the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

Therefore etc.

(Being) what it was required to prove. 9 10 11 12 13 14 15 16

Proposition 5.

In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.



Let ABC be an isosceles triangle having the side AB
equal to the side AC; and let the straight lines BD, CE be produced further in a straight line with AB, AC. {Post. 2}

I say that the angle ABC is equal to the angle ACB, and the angle CBD to the angle BCE.

Let a point F be taken at random on BD; from AE the greater let AG be cut off equal to AF the less; {I. 3} and let the straight lines FC, GB be joined. {Post. 1}

Then, since AF is equal to AG and AB to AC,

the two sides FA, AC are equal to the two sides GA, AB, respectively;

and they contain a common angle, the angle FAG.

Therefore the base FC is equal to the base GB, and the triangle AFC is equal to the triangle AGB,

and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend,

that is, the angle ACF to the angle ABG, and the angle AFC to the angle AGB. {I. 4}

And, since the whole AF is equal to the whole AG,

and in these AB is equal to AC, the remainder BF is equal to the remainder CG.

But FC was also proved equal to GB;
therefore the two sides BF, FC are equal to the two sides CG, GB respectively; and the angle BFC is equal to the angle CGB,

while the base BC is common to them;

therefore the triangle BFC is also equal to the triangle CGB,
and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend;

therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG.

Accordingly, since the whole angle ABG was proved equal to the angle ACF,

and in these the angle CBG is equal to the angle BCF,

the remaining angle ABC is equal to the remaining angle ACB;

and they are at the base of the triangle ABC.

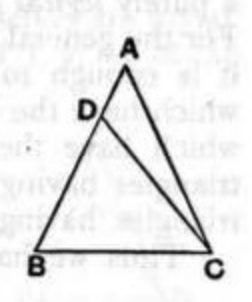
But the angle FBC was also proved equal to the angle GCB;

and they are under the base.

Therefore etc.

Q. E. D. [17] [18] [19] [20] [21]

Proposition 6.



If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB;

I say that the side AB is also equal to the side AC.

For, if AB is unequal to AC, one of them is greater.

Let AB be greater; and from AB the greater let DB be cut off equal to AC the less;

let DC be joined.

Then, since DB is equal to AC, and BC is common,

the two sides DB, BC are equal to the two sides AC, CB respectively;

and the angle DBC is equal to the angle ACB;

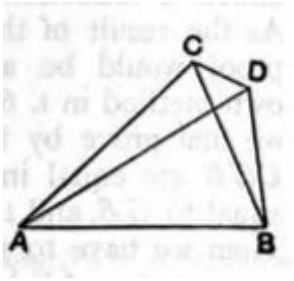
therefore the base DC is equal to the base AB, and the triangle DBC will be equal to the triangle ACB, the less to the greater: which is absurd.

Therefore AB is not unequal to AC; it is therefore equal to it.

Therefore etc.

Q. E. D.

Proposition 7.



Given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there cannot be constructed on the same straight line (from its extremities), and on the same side of it, two other straight lines meeting in
another point and equal to the former two respectively, namely each to that which has the same extremity with it.

For, if possible, given two straight lines AC, CB constructed on the straight line AB and meeting at the point C, let two other straight lines
AD, DB be constructed on the same straight line AB, on the same side of it, meeting in another point D and equal to the former two respectively, namely each to that which has the same extremity with it, so that CA is
equal to DA which has the same extremity A with it, and CB to DB which has the same extremity B with it; and let CD be joined.

Then, since AC is equal to AD,

the angle ACD is also equal to the angle ADC; {I. 5} therefore the angle ADC is greater than the angle DCB;

therefore the angle CDB is much greater than the angle DCB.

Again, since CB is equal to DB,

the angle CDB is also equal to the angle DCB.

But it was also proved much greater than it:

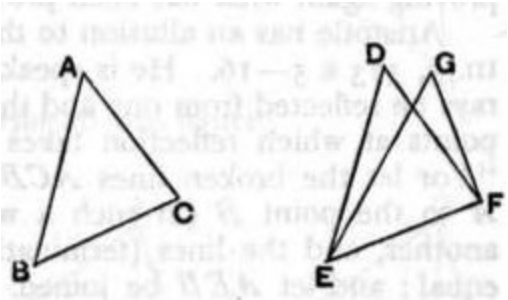
which is impossible.

Therefore etc.

Q. E. D. [22] [23] [24]

Proposition 8.

If two triangles have the two sides equal to two sides respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines.



Let ABC, DEF be two triangles having the two sides AB, AC equal to the two sides DE, DF respectively, namely AB to DE, and AC to DF; and let them have the base BC equal
to the base EF;

I say that the angle BAC is also equal to the angle EDF.

For, if the triangle ABC be applied to the triangle DEF, and if the point B be placed on
the point E and the straight line BC on EF,

the point C will also coincide with F, because BC is equal to EF.

Then, BC coinciding with EF,

BA, AC will also coincide with ED, DF;

for, if the base BC coincides with the base EF, and the sides BA, AC do not coincide with ED, DF but fall beside them as EG, GF,

then, given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there will have been constructed on the same straight line (from its extremities), and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it. But they cannot be so constructed. {I. 7}

Therefore it is not possible that, if the base BC be applied to the base EF, the sides BA, AC should not coincide with ED, DF;

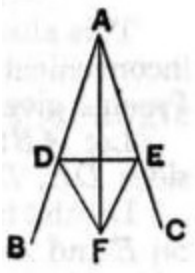
they will therefore coincide,

so that the angle BAC will also coincide with the angle EDF, and will be equal to it.

If therefore etc.

Q. E. D. [25] [26]

Proposition 9.



To bisect a given rectilineal angle.

Let the angle BAC be the given rectilineal angle.

Thus it is required to bisect it.

Let a point D be taken at random on AB; let AE be cut off from AC equal to AD; {I. 3} let DE be joined, and on DE let the equilateral triangle DEF be constructed; let AF be joined.

I say that the angle BAC has been bisected by the straight line AF.

For, since AD is equal to AE, and AF is common,

the two sides DA, AF are equal to the two sides EA, AF respectively.

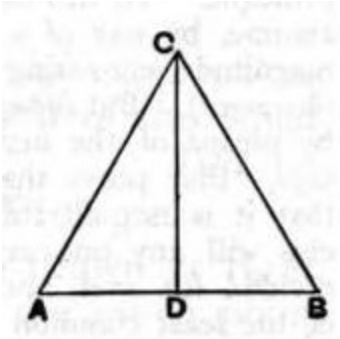
And the base DF is equal to the base EF;

therefore the angle DAF is equal to the angle EAF. {I. 8}

Therefore the given rectilineal angle BAC has been bisected by the straight line AF.

Q. E. F.

Proposition 10.



To bisect a given finite straight line.

Let AB be the given finite straight line.

Thus it is required to bisect the finite straight line AB.

Let the equilateral triangle ABC be constructed on it, {I. 1} and let the angle ACB be bisected by the straight line CD; {I. 9}

I say that the straight line AB has been bisected at the point D.

For, since AC is equal to CB, and CD is common,

the two sides AC, CD are equal to the two sides BC, CD respectively;

and the angle ACD is equal to the angle BCD;

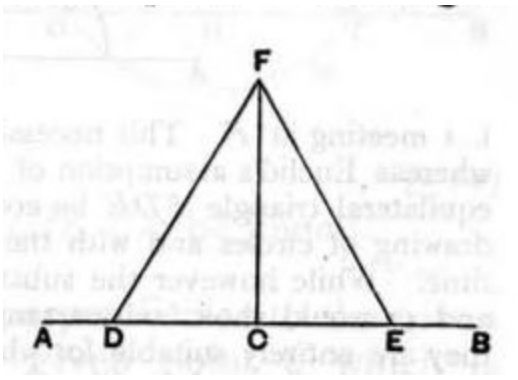
therefore the base AD is equal to the base BD. {I. 4}

Therefore the given finite straight line AB has been bisected at D.

Q. E. F.

Proposition 11.

To draw a straight line at right angles to a given straight line from a given point on it.



Let AB be the given straight line, and C the given point on it.

Thus it is required to draw from the point C a straight line at right angles to the straight line AB.

Let a point D be taken at random on AC;
let CE be made equal to CD; {I. 3} on DE let the equilateral triangle FDE be constructed, {I. 1} and let FC be joined;

I say that the straight line FC has been drawn at right
angles to the given straight line AB from C the given point on it.

For, since DC is equal to CE, and CF is common,

the two sides DC, CF are equal to the two sides EC, CF respectively;

and the base DF is equal to the base FE;

therefore the angle DCF is equal to the angle ECF; {I. 8}

and they are adjacent angles.

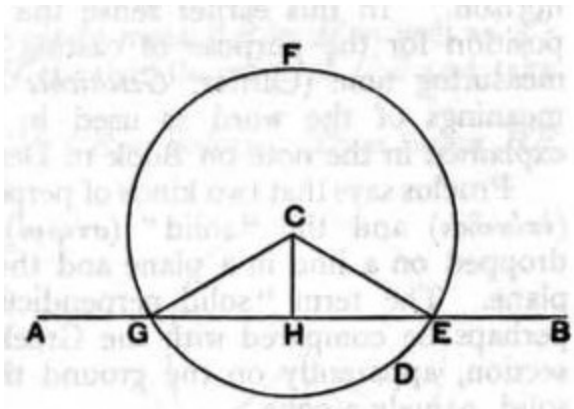
But, when a straight line set up on a straight line makes
the adjacent angles equal to one another, each of the equal angles is right;
{Def. 10}

therefore each of the angles DCF, FCE is right.

Therefore the straight line CF has been drawn at right angles to the given straight line AB from the given point
C on it.

Q. E. F. [27]

Proposition 12.



To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line.

Let AB be the given infinite straight line, and C the given point which is not on it;
thus it is required to draw to the given infinite straight line AB, from the given point C which is not on it, a perpendicular straight line.

For let a point D be taken
at random on the other side of the straight line AB, and with centre C and distance CD let the circle EFG be described; {Post. 3}

let the straight line EG be bisected at H, {I. 10} and let the straight lines CG, CH, CE be joined. {Post. 1}

I say that CH has been drawn perpendicular to the given infinite straight line AB from the given point C which is not on it.

For, since GH is equal to HE, and HC is common,

the two sides GH, HC are equal to the two sides EH, HC respectively;

and the base CG is equal to the base CE;

therefore the angle CHG is equal to the angle EHC. {I. 8} And they are adjacent angles.

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is
called a perpendicular to that on which it stands. {Def. 10}

Therefore CH has been drawn perpendicular to the given infinite straight line AB from the given point C which is not on it.

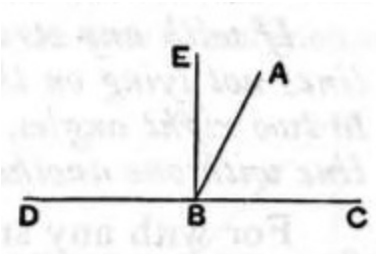
Q. E. F. [28] [29]

Proposition 13.

If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles.

For let any straight line AB set up on the straight line
CD make the angles CBA, ABD;

I say that the angles CBA, ABD are either two right angles or equal to two right angles.



Now, if the angle CBA is equal to
the angle ABD,

they are two right angles. {Def. 10}

But, if not, let BE be drawn from the point B at right angles to CD; {I. 11}

therefore the angles CBE, EBD are two right angles.

Then, since the angle CBE is equal to the two angles CBA, ABE,

let the angle EBD be added to each; therefore the angles CBE, EBD are equal to the three angles CBA, ABE, EBD. {C. N. 2}

Again, since the angle DBA is equal to the two angles DBE, EBA,

let the angle ABC be added to each; therefore the angles DBA. ABC are equal to the three angles DBE, EBA, ABC. {C. N. 2}

But the angles CBE, EBD were also proved equal to the same three angles;

and things which are equal to the same thing are also equal to one another; {C. N. 1} therefore the angles CBE, EBD are also equal to the angles DBA, ABC.

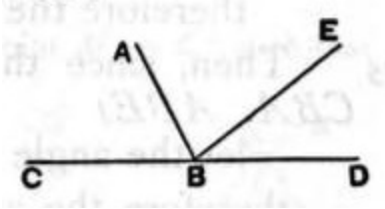
But the angles CBE, EBD are two right angles;

therefore the angles DBA, ABC are also equal to two right angles.

Therefore etc.

Q. E. D. [30]

Proposition 14.



If with any straight line, and at a point on it, two straight lines not lying on the same side make the adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another.

For with any straight line AB, and at the point B on it, let the two straight lines BC, BD not lying on the same side make the adjacent angles ABC, ABD equal to two right angles;

I say that BD is in a straight line with CB.

For, if BD is not in a straight line with BC, let BE be in a straight line with CB.

Then, since the straight line AB stands on the straight line CBE,

the angles ABC, ABE are equal to two right angles. {I. 13}

But the angles ABC, ABD are also equal to two right angles;

therefore the angles CBA, ABE are equal to the angles CBA, ABD. {Post. 4 and C.N. 1}

Let the angle CBA be subtracted from each;
therefore the remaining angle ABE is equal to the remaining angle ABD, {C.N. 3}

the less to the greater: which is impossible. Therefore BE is not in a straight line with CB.

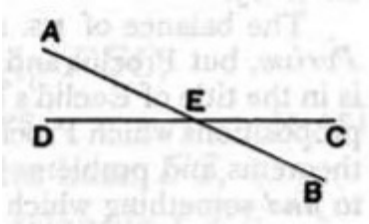
Similarly we can prove that neither is any other straight line except BD.

Therefore CB is in a straight line with BD.

Therefore etc.

Q. E. D. [31] [32] [33]

Proposition 15.



If two straight lines cut one another, they make the vertical angles equal to one another.

For let the straight lines AB, CD cut one another at the point E;
I say that the angle AEC is equal to the angle DEB,

and the angle CEB to the angle AED.

For, since the straight line AE stands
on the straight line CD, making the angles CEA, AED,

the angles CEA, AED are equal to two right angles {I. 13}

Again, since the straight line DE stands on the straight line AB, making the angles AED, DEB,

the angles AED, DEB are equal to two right angles. {I. 13}

But the angles CEA, AED were also proved equal to two right angles;

therefore the angles CEA, AED are equal to the angles AED DEB. {Post. 4 and C. N. 1} Let the angle AED be subtracted from each; therefore the remaining angle CEA is equal to the remaining angle BED. {C. N. 3}

Similarly it can be proved that the angles CEB, DEA are also equal.

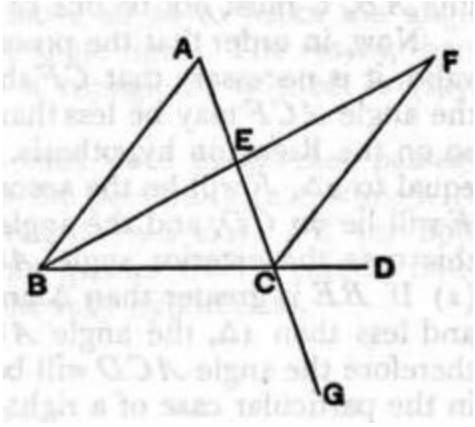
Therefore etc. Q. E. D.

Porism.

{From this it is manifest that, if two straight lines cut one another, they will make the angles at the point of section equal to four right angles. [34] [35]

Proposition 16.

In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.



Let ABC be a triangle, and let one side of it BC be produced to D;

I say that the exterior angle ACD is greater than either of the interior and opposite angles CBA, BAC.

Let AC be bisected at E {I. 10}, and let BE be joined and produced in a straight line to F;

let EF be made equal to BE{I. 3}, let FC be joined {Post. 1}, and let AC be drawn through to G {Post. 2}.

Then, since AE is equal to EC,
and BE to EF,

the two sides AE, EB are equal to the two sides CE, EF respectively; and the angle AEB is equal to the angle FEC, for they are vertical angles. {I. 15}
Therefore the base AB is equal to the base FC, and the triangle ABE is equal to the triangle CFE, and the remaining angles are equal to the remaining angles respectively, namely those which the equal sides subtend; {I. 4}
therefore the angle BAE is equal to the angle ECF.

But the angle ECD is greater than the angle ECF; {C. N. 5}

therefore the angle ACD is greater than the angle BAE.

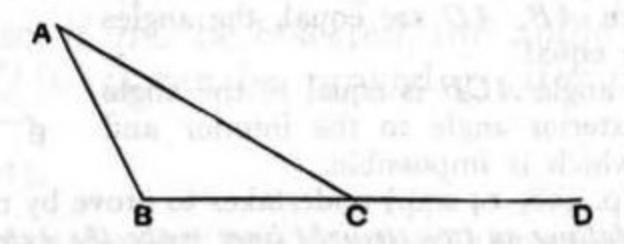
Similarly also, if BC be bisected, the angle BCG, that is, the angle ACD {I. 15}, can be proved greater than the angle ABC as well.

Therefore etc.

Q. E. D. [36] [37] [38] [39]

Proposition 17.

In any triangle two angles taken together in any manner are less than two right angles.



Let ABC be a triangle; I say that two angles of the triangle ABC taken together in any manner are less than two right angles.

For let BC be produced to D. {Post. 2}

Then, since the angle ACD is an exterior angle of the triangle ABC,

it is greater than the interior and opposite angle ABC. {I. 16}

Let the angle ACB be added to each;

therefore the angles ACD, ACB are greater than the angles ABC, BCA. But the angles ACD, ACB are equal to two right angles. {I. 13}

Therefore the angles ABC, BCA are less than two right angles.

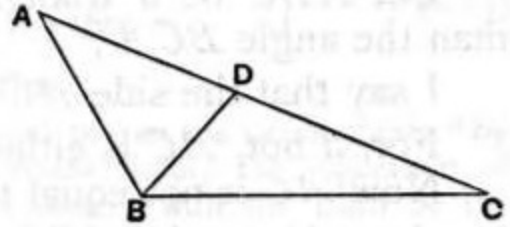
Similarly we can prove that the angles BAC, ACB are also less than two right angles, and so are the angles CAB, ABC as well.

Therefore etc.

Q. E. D. [40]

Proposition 18.

In any triangle the greater side subtends the greater angle.



For let ABC be a triangle having the side AC greater than AB;

I say that the angle ABC is also greater than the angle BCA.

For, since AC is greater than AB, let AD be made equal to AB {I. 3}, and let BD bejoined.

Then, since the angle ADB is an exterior angle of the triangle BCD,
it is greater than the interior and opposite angle DCB. {I. 16}
But the angle ADB is equal to the angle ABD,

since the side AB is equal to AD; therefore the angle ABD is also greater than the angle ACB; therefore the angle ABC is much greater than the angle ACB.

Therefore etc.

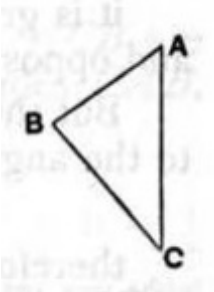
Q. E. D. [41]

Proposition 19.

In any triangle the greater angle is subtended by the greater side.

Let ABC be a triangle having the angle ABC greater than the angle BCA;
I say that the side AC is also greater than the side AB.
For, if not, AC is either equal to AB or less.
Now AC is not equal to AB; for then the angle ABC would also have been equal to the angle ACB; {I. 5} but it is not;

therefore AC is not equal to AB.



Neither is AC less than AB, for then the angle ABC would also have been less than the angle ACB; {I. 18} but it is not;

therefore AC is not less than AB.

And it was proved that it is not equal either.

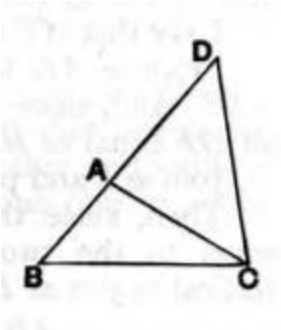
Therefore AC is greater than AB.

Therefore etc.

Q. E. D.

Proposition 20.

In any triangle two sides taken together in any manner are greater than the remaining one.



For let ABC be a triangle; I say that in the triangle ABC two sides taken together in any manner are greater than the remaining one, namely

BA, AC greater than BC, AB, BC greater than AC, BC, CA greater than AB.

For let BA be drawn through to the point D, let DA be made equal to CA, and let DC be joined.

Then, since DA is equal to AC, the angle ADC is also equal to the angle ACD; {I. 5}

therefore the angle BCD is greater than the angle ADC. {C.N. 5}

And, since DCB is a triangle having the angle BCD greater than the angle BDC,

and the greater angle is subtended by the greater side, {I. 19} therefore DB is greater than BC.

But DA is equal to AC;

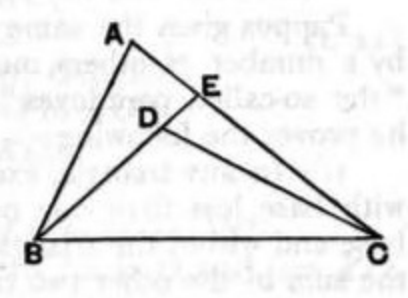
therefore BA, AC are greater than BC.

Similarly we can prove that AB, BC are also greater than CA, and BC, CA than AB.

Therefore etc.

Q. E. D. [42]

Proposition 21.



If on one of the sides of a triangle, from its extremities, there be constructed two straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater
angle.

On BC, one of the sides of the triangle ABC, from its extremities B, C, let the two straight lines BD, DC be constructed meeting within the triangle;

I say that BD, DC are less than the remaining two sides of the triangle BA, AC, but contain an angle BDC greater than the angle BAC.

For let BD be drawn through to E.

Then, since in any triangle two sides are greater than the remaining one, {I. 20}

therefore, in the triangle ABE, the two sides AB, AE are greater than BE.

Let EC be added to each;

therefore BA, AC are greater than BE, EC.

Again, since, in the triangle CED,

the two sides CE, ED are greater than CD, let DB be added to each;

therefore CE, EB are greater than CD, DB.

But BA, AC were proved greater than BE, EC;

therefore BA, AC are much greater than BD, DC.

Again, since in any triangle the exterior angle is greater than the interior and opposite angle, {I. 16} therefore, in the triangle CDE,

the exterior angle BDC is greater than the angle CED.

For the same reason, moreover, in the triangle ABE also,

the exterior angle CEB is greater than the angle BAC.

But the angle BDC was proved greater than the angle CEB;

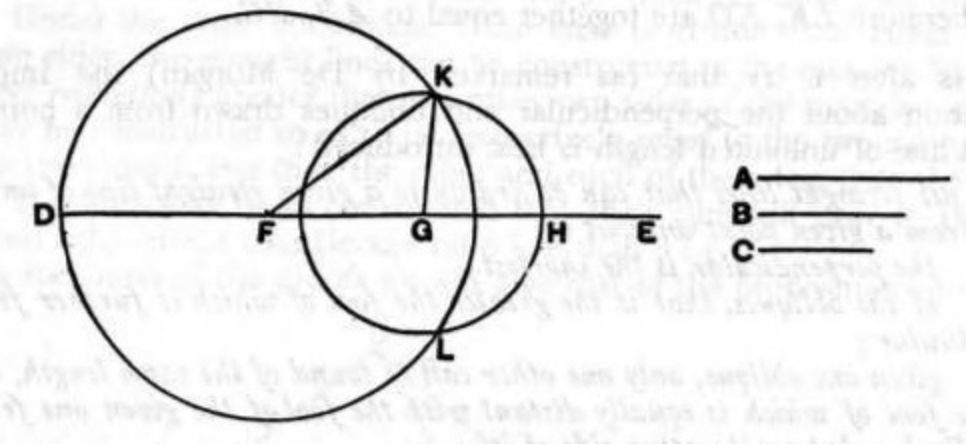
therefore the angle BDC is much greater than the angle BAC.

Therefore etc.

Q. E. D. [43] [44]

Proposition 22.

Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one. {I. 20}



Let the three given straight lines be A, B, C, and of these let two taken together in any manner be greater than the remaining one, namely A, B greater than C,

A, C greater than B,

and B, C greater than A; thus it is required to construct a triangle out of straight lines equal to A, B, C.

Let there be set out a straight line DE, terminated at D but of infinite length in the direction of E, and let DF be made equal to A, FG equal to B, and GH equal to C. {I. 3}

With centre F and distance FD let the circle DKL be described; again, with centre G and distance GH let the circle KLH be described; and let KF, KG be joined;

I say that the triangle KFG has been constructed out of three straight lines equal to A, B, C.

For, since the point F is the centre of the circle DKL,

FD is equal to FK.

But FD is equal to A;

therefore KF is also equal to A.

Again, since the point G is the centre of the circle LKH,

GH is equal to GK.

But GH is equal to C;

therefore KG is also equal to C. And FG is also equal to B;

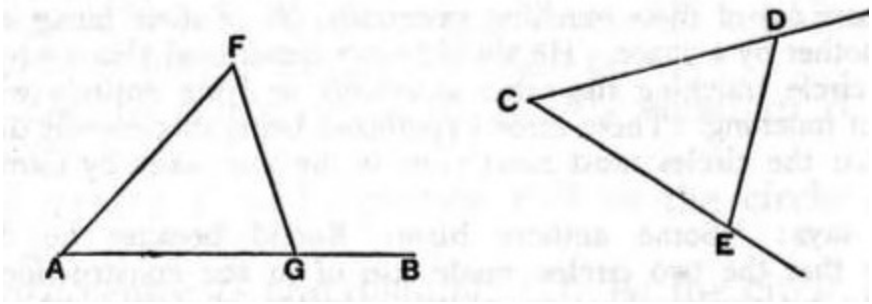
therefore the three straight lines KF, FG, GK are equal to the three straight lines A, B, C.

Therefore out of the three straight lines KF, FG, GK, which are equal to the three given straight lines A, B, C, the triangle KFG has been constructed.

Q. E. F. [45] [46] [47]

Proposition 23.

On a given straight line and at a point on it to construct a rectilineal angle equal to a given rectilineal angle.



Let AB be the given straight line, A the point on it, and the angle DCE the given rectilineal angle;

thus it is required to construct on the given straight line AB, and at the point A on it, a rectilineal angle equal to the given rectilineal angle DCE.

On the straight lines CD, CE respectively let the points D, E be taken at random; let DE be joined, and out of three straight lines which are equal to the three straight lines CD, DE, CE let the triangle AFG be constructed in such a way that CD is equal to AF, CE to AG, and further DE to FG.

Then, since the two sides DC, CE are equal to the two sides FA, AG respectively,

and the base DE is equal to the base FG, the angle DCE is equal to the angle FAG. {I. 8}

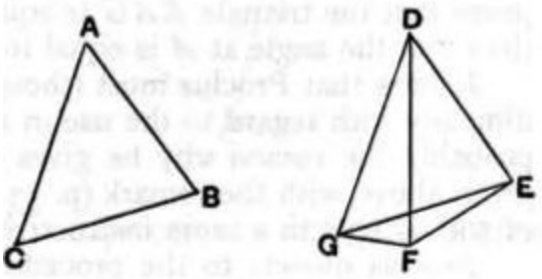
Therefore on the given straight line AB, and at the point A on it, the rectilineal angle FAG has been constructed equal to the given rectilineal angle DCE.

Q. E. F.

Proposition 24.

If two triangles have the two sides equal to two sides respectively, but have the one of the angles contained by the equal straight lines greater than the

other, they will also have the base greater than the base.



Let ABC, DEF be two triangles having the two sides AB, AC equal to the two sides DE, DF respectively, namely AB to DE, and AC to DF, and let the angle at A be greater than the angle at D;

I say that the base BC is also greater than the base EF.

For, since the angle BAC is greater than the angle EDF, let there be constructed, on the straight line DE, and at the point D on it, the angle EDG equal to the angle BAC; {I. 23} let DG be made equal to either of the two straight lines AC, DF, and let EG, FG be joined.

Then, since AB is equal to DE, and AC to DG,
the two sides BA, AC are equal to the two sides ED, DG, respectively;

and the angle BAC is equal to the angle EDG; therefore the base BC is equal to the base EG. {I. 4}

Again, since DF is equal to DG,

the angle DGF is also equal to the angle DFG; {I. 5} therefore the angle DFG is greater than the angle EGF.

Therefore the angle EFG is much greater than the angle EGF.

And, since EFG is a triangle having the angle EFG
greater than the angle EGF,

and the greater angle is subtended by the greater side, {I. 19} the side EG is also greater than EF.

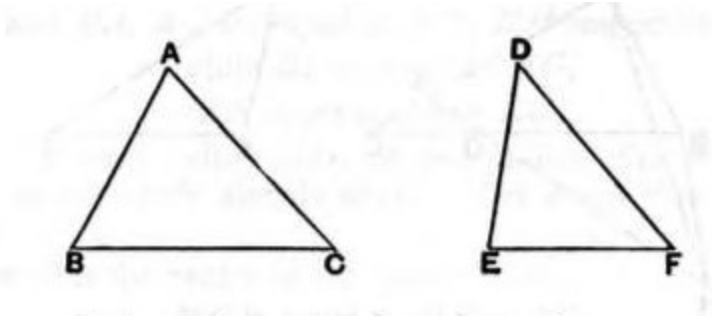
But EG is equal to BC.

Therefore BC is also greater than EF.

Therefore etc.

Q. E. D. [48]

Proposition 25.



If two triangles have the two sides equal to two sides respectively, but have the base greater than the base, they will also have the one of the angles contained by the equal straight lines greater than the other.

Let ABC, DEF be two triangles having the two sides AB, AC equal to the two sides DE, DF respectively, namely AB to DE, and AC to DF; and let the base BC be greater than the base EF;

I say that the angle BAC is also greater than the angle EDF.

For, if not, it is either equal to it or less.

Now the angle BAC is not equal to the angle EDF; for then the base BC would also have been equal to the base EF, {I. 4}

but it is not; therefore the angle BAC is not equal to the angle EDF.

Neither again is the angle BAC less than the angle EDF; for then the base BC would also have been less than the base EF, {I. 24}

but it is not; therefore the angle BAC is not less than the angle EDF.

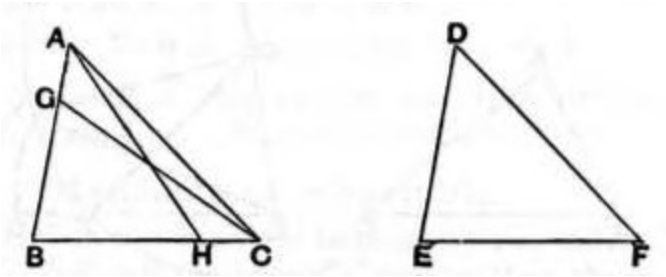
But it was proved that it is not equal either;

therefore the angle BAC is greater than the angle EDF.

Therefore etc.

Q. E. D.

Proposition 26.



If two triangles have the two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that subtending one of the equal angles, they will also have the remaining sides equal to
the remaining sides and the remaining angle to the remaining angle.

Let ABC, DEF be two triangles having the two angles ABC, BCA equal to the two angles DEF, EFD respectively, namely the angle ABC to the angle

DEF, and the angle
BCA to the angle EFD; and let them also have one side equal to one side, first that adjoining the equal angles, namely BC to EF;

I say that they will also have the remaining sides equal to the remaining sides respectively, namely AB to DE and
AC to DF, and the remaining angle to the remaining angle, namely the angle BAC to the angle EDF.

For, if AB is unequal to DE, one of them is greater.
Let AB be greater, and let BG be made equal to DE; and let GC be joined.
Then, since BG is equal to DE, and BC to EF, the two sides GB, BC are equal to the two sides DE, EF respectively; and the angle GBC is equal to the angle DEF;

therefore the base GC is equal to the base DF, and the triangle GBC is equal to the triangle DEF, and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend; {I. 4}
therefore the angle GCB is equal to the angle DFE.

But the angle DFE is by hypothesis equal to the angle BCA;

therefore the angle BCG is equal to the angle BCA, the less to the greater: which is impossible. Therefore AB is not unequal to DE, and is therefore equal to it.

But BC is also equal to EF;

therefore the two sides AB, BC are equal to the two sides DE, EF respectively, and the angle ABC is equal to the angle DEF; therefore the base AC is equal to the base DF, and the remaining angle BAC is equal to the remaining angle EDF. {I. 4}

Again, let sides subtending equal angles be equal, as AB to DE;

I say again that the remaining sides will be equal to the remaining sides, namely AC to DF and BC to EF, and
further the remaining angle BAC is equal to the remaining angle EDF.

For, if BC is unequal to EF, one of them is greater.

Let BC be greater, if possible, and let BH be made equal to EF; let AH be joined.

Then, since BH is equal to EF, and AB to DE, the two sides AB, BH are equal to the two sides DE, EF respectively, and they contain equal angles;

therefore the base AH is equal to the base DF,

and the triangle ABH is equal to the triangle DEF,
and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend; {I. 4}

therefore the angle BHA is equal to the angle EFD.

But the angle EFD is equal to the angle BCA; therefore, in the triangle AHC, the exterior angle BHA is
equal to the interior and opposite angle BCA:

which is impossible. {I. 16}

Therefore BC is not unequal to EF,

and is therefore equal to it.

But AB is also equal to DE;
therefore the two sides AB, BC are equal to the two sides DE, EF
respectively, and they contain equal angles;

therefore the base AC is equal to the base DF, the triangle ABC equal to the triangle DEF,

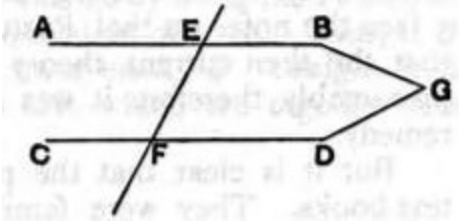
and the remaining angle BAC equal to the remaining angle EDF. {I. 4}

Therefore etc.

Q. E. D. [49] [50]

Proposition 27.

If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another.



For let the straight line EF falling on the two straight lines AB, CD make the alternate angles AEF, EFD equal to one another;

I say that AB is parallel to CD.

For, if not, AB, CD when produced will meet either in the direction of B, D or towards A, C.

Let them be produced and meet, in the direction of B, D, at G.

Then, in the triangle GEF, the exterior angle AEF is equal to the interior and opposite

angle EFG: which is impossible. {I. 16}

Therefore AB, CD when produced will not meet in the direction of B, D.

Similarly it can be proved that neither will they meet
towards A, C.

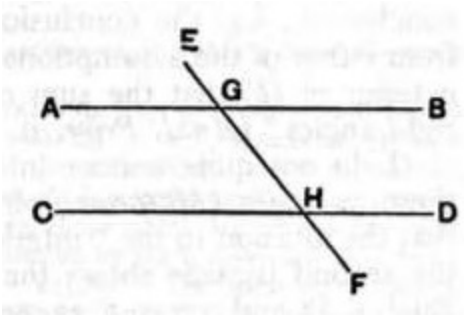
But straight lines which do not meet in either direction are parallel; {Def. 23}

therefore AB is parallel to CD.

Therefore etc.

Q. E. D. [51] [52] [53]

Proposition 28.



If a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side, or the interior angles on the same side equal to two right angles, the straight lines will be parallel to one another.

For let the straight line EF falling on the two straight lines AB, CD make the exterior angle EGB equal to the interior and opposite angle GHD, or the

interior angles on the same side, namely BGH, GHD, equal to two right angles;

I say that AB is parallel to CD.

For, since the angle EGB is equal to the angle GHD, while the angle EGB is equal to the angle AGH, {I. 15} the angle AGH is also equal to the angle GHD; and they are alternate;

therefore AB is parallel to CD. {I. 27}

Again, since the angles BGH, GHD are equal to two right angles, and the angles AGH, BGH are also equal to two right angles, {I. 13} the angles AGH, BGH are equal to the angles BGH, GHD.

Let the angle BGH be subtracted from each; therefore the remaining angle AGH is equal to the remaining angle GHD; and they are alternate;

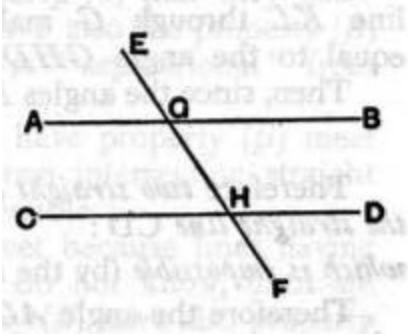
therefore AB is parallel to CD. {I. 27}

Therefore etc.

Q. E. D.

Proposition 29.

A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.



For let the straight line EF fall on the parallel straight lines AB, CD;

I say that it makes the alternate angles AGH, GHD equal, the exterior angle EGB equal to the interior and opposite angle GHD, and the interior angles on the same
side, namely BGH, GHD, equal to two right angles.

For, if the angle AGH is unequal to the angle GHD, one of them is greater.

Let the angle AGH be greater.

Let the angle BGH be added to each; therefore the angles AGH, BGH are greater than the angles BGH, GHD.

But the angles AGH, BGH are equal to two right angles; {I. 13}

therefore the angles BGH, GHD are less than two right angles.

But straight lines produced indefinitely from angles less than two right angles meet; {Post. 5}

therefore AB, CD, if produced indefinitely, will meet; but they do not meet, because they are by hypothesis parallel.

Therefore the angle AGH is not unequal to the angle GHD,

and is therefore equal to it.

Again, the angle AGH is equal to the angle EGB; {I. 15}

therefore the angle EGB is also equal to the angle GHD. {C.N. 1}

Let the angle BGH be added to each;

therefore the angles EGB, BGH are equal to the angles BGH, GHD. {C.N. 2}

But the angles EGB, BGH are equal to two right angles; {I. 13}

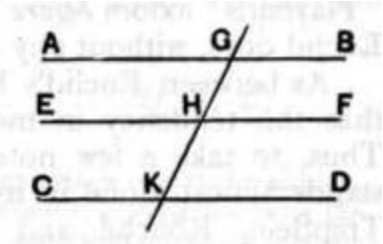
therefore the angles BGH, GHD are also equal to two right angles.

Therefore etc.

Q. E. D. [54] [55]

Proposition 30.

Straight lines parallel to the same straight line are also parallel to one another.



Let each of the straight lines AB, CD be parallel to EF; I say that AB is also parallel to CD.

For let the straight line GK fall upon them;

Then, since the straight line GK has fallen on the parallel straight lines AB, EF,

the angle AGK is equal to the angle GHF. {I. 29}

Again, since the straight line GK has fallen on the parallel straight lines EF, CD,

the angle GHF is equal to the angle GKD. {I. 29}

But the angle AGK was also proved equal to the angle GHF;

therefore the angle AGK is also equal to the angle GKD; {C.N. 1}

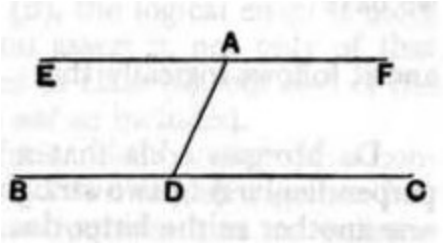
and they are alternate.

Therefore AB is parallel to CD.

Q. E. D. [56]

Proposition 31.

Through a given point to draw a straight line parallel to a given straight line.



Let A be the given point, and BC the given straight line; thus it is required to draw through the point A a straight line parallel to the straight line BC.

Let a point D be taken at random on BC, and let AD be joined; on the straight line DA, and at the point A on it, let the angle DAE be constructed equal to the angle ADC {I. 23}; and let the straight line AF be produced in a straight line with EA.

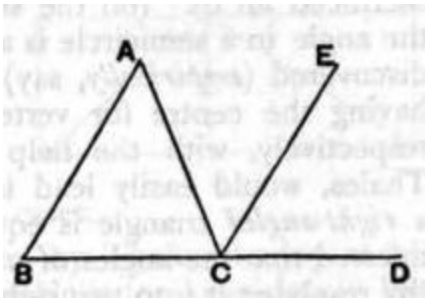
Then, since the straight line AD falling on the two straight lines BC, EF has made the alternate angles EAD, ADC equal to one another,

therefore EAF is parallel to BC. {I. 27}

Therefore through the given point A the straight line EAF has been drawn parallel to the given straight line BC.

Q. E. F.

Proposition 32.



In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.

Let ABC be a triangle, and let one side of it BC be produced to D;

I say that the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC, and the three interior angles of the triangle ABC, BCA, CAB are equal to two right angles.

For let CE be drawn through the point C parallel to the straight line AB. {I. 31}

Then, since AB is parallel to CE,

and AC has fallen upon them, the alternate angles BAC, ACE are equal to one another. {I. 29}

Again, since AB is parallel to CE,

and the straight line BD has fallen upon them,

the exterior angle ECD is equal to the interior and opposite angle ABC. {I. 29}

But the angle ACE was also proved equal to the angle BAC;

therefore the whole angle ACD is equal to the two interior and opposite angles BAC, ABC.

Let the angle ACB be added to each;

therefore the angles ACD, ACB are equal to the three angles ABC, BCA, CAB.

But the angles ACD, ACB are equal to two right angles; {I. 13}

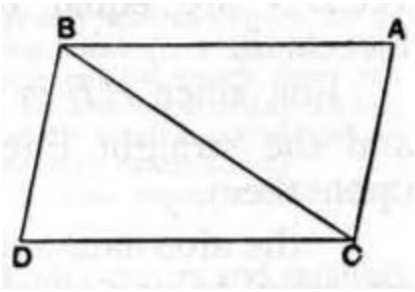
therefore the angles ABC, BCA, CAB are also equal to two right angles.

Therefore etc.

Q. E. D.

Proposition 33.

The straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are themselves also equal and parallel.



Let AB, CD be equal and parallel, and let the straight lines AC, BD join them (at the extremities which are) in the same directions (respectively); I say that AC, BD are also equal and parallel.

Let BC be joined.

Then, since AB is parallel to CD,
and BC has fallen upon them,

the alternate angles ABC, BCD are equal to one another. {I. 29}

And, since AB is equal to CD,

and BC is common, the two sides AB, BC are equal to the two sides DC, CB; and the angle ABC is equal to the angle BCD; therefore the base AC is equal to the base BD, and the griangle ABC is equal to the triangle DCB, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend; {I. 4} therefore the angle ACB is equal to the angle CBD.

And, since the straight line BC falling on the two straight lines AC, BD has made the alternate angles equal to one another,

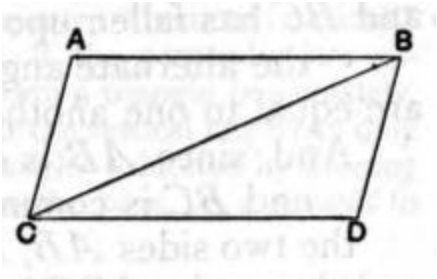
AC is parallel to BD. {I. 27}

And it was also proved equal to it.

Therefore etc.

Q. E. D. [57] [58]

Proposition 34.



In parallelogrammic areas the opposite sides and angles are equal to one another, and the diameter bisects the areas.

Let ACDB be a parallelogrammic area, and BC its diameter;
I say that the opposite sides and angles of the parallelogram ACDB are equal to one another, and the diameter BC bisects it.

For, since AB is parallel to CD, and the straight line BC has fallen upon them,

the alternate angles ABC, BCD are equal to one another. {I. 29}

Again, since AC is parallel to BD, and BChas fallen upon them,

the alternate angles ACB, CBD are equal to one another. {I. 29}

Therefore ABC, DCB are two triangles having the two angles ABC, BCA equal to the two angles DCB, CBD respectively, and one side equal to one side, namely that
adjoining the equal angles and common to both of them, BC;

therefore they will also have the remaining sides equal to the remaining sides respectively, and the remaining angle to the remaining angle; {I. 26}
therefore the side AB is equal to CD, and AC to BD,

and further the angle BAC is equal to the angle CDB.
And, since the angle ABC is equal to the angle BCD,

and the angle CBD to the angle ACB, the whole angle ABD is equal to the whole angle ACD. {C.N. 2}

And the angle BAC was also proved equal to the angle CDB.

Therefore in parallelogrammic areas the opposite sides and angles are equal to one another.

I say, next, that the diameter also bisects the areas.

For, since AB is equal to CD,
and BC is common, the two sides AB, BC are equal to the two sides DC, CB respectively;

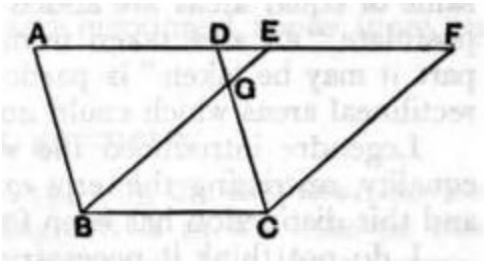
and the angle ABC is equal to the angle BCD; therefore the base AC is also equal to DB, and the triangle ABC is equal to the triangle DCB. {I. 4}

Therefore the diameter BC bisects the parallelogram ACDB.

Q. E. D. [59] [60]

Proposition 35.

Parallelograms which are on the same base and in the same parallels are equal to one another.



Let ABCD, EBCF be parallelograms on the same base BC and in the same parallels AF, BC;
I say that ABCD is equal to the parallelogram EBCF.

For, since ABCD is a parallelogram,

AD is equal to BC. {I. 34}

For the same reason also

EF is equal to BC, so that AD is also equal to EF; {C.N. 1}

and DE is common;

therefore the whole AE is equal to the whole DF. {C.N. 2}

But AB is also equal to DC; {I. 34} therefore the two sides EA, AB are equal to the two sides
FD, DC respectively,

and the angle FDC is equal to the angle EAB, the exterior to the interior; {I. 29} therefore the base EB is equal to the base FC, and the triangle EAB will be equal to the triangle FDC. {I. 4}

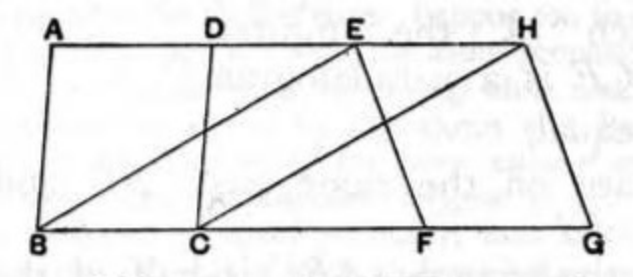
Let DGE be subtracted from each; therefore the trapezium ABGD which remains is equal to the trapezium EGCF which remains. {C.N. 3}

Let the triangle GBC be added to each; therefore the whole parallelogram ABCD is equal to the whole parallelogram EBCF. {C.N. 2}

Therefore etc.

Q. E. D. [61] [62]

Proposition 36.



Parallelograms which are on equal bases and in the same parallels are equal to one another.

Let ABCD, EFGH be parallelograms which are on equal bases BC, FG and in the same parallels AH, BG; I say that the parallelogram ABCD is equal to EFGH.

For let BE, CH be joined.

Then, since BC is equal to FG while FG is equal to EH,

BC is also equal to EH. {C.N. 1}

But they are also parallel.

And EB, HC join them; but straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are equal and parallel. {I. 33}

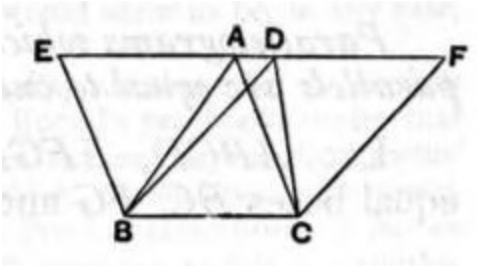
Therefore EBCH is a parallelogram. {I. 34}

And it is equal to ABCD; for it has the same base BC with it, and is in the same parallels BC, AH with it. {I. 35}

For the same reason also EFGH is equal to the same EBCH; {I. 35} so that the parallelogram ABCD is also equal to EFGH. {C.N. 1}

Therefore etc. Q. E. D.

Proposition 37.



Triangles which are on the same base and in the same parallels are equal to one another.

Let ABC, DBC be triangles on the same base BC and in the same parallels AD, BC;
I say that the triangle ABC is equal to the triangle DBC.

Let AD be produced in both directions to E, F; through B let BE be drawn parallel to CA, {I. 31}
and through C let CF be drawn parallel to BD. {I. 31}

Then each of the figures EBCA, DBCF is a parallelogram; and they are equal,

for they are on the same base BC and in the same parallels BC, EF. {I. 35}

Moreover the triangle ABC is half of the parallelogram EBCA; for the diameter AB bisects it. {I. 34}

And the triangle DBC is half of the parallelogram DBCF;
for the diameter DC bisects it. {I. 34}

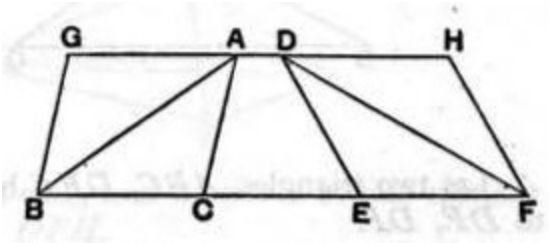
{But the halves of equal things are equal to one another.}

Therefore the triangle ABC is equal to the triangle DBC.

Therefore etc.

Q. E. D. [63]

Proposition 38.



Triangles which are on equal bases and in the same parallels are equal to one another.

Let ABC, DEF be triangles on equal bases BC, EF and in the same parallels BF, AD; I say that the triangle ABC is equal to the triangle DEF.

For let AD be produced in both directions to G, H; through B let BG be drawn parallel to CA, {I. 31} and through F let FH be drawn parallel to DE.

Then each of the figures GBCA, DEFH is a parallelogram; and GBCA is equal to DEFH;

for they are on equal bases BC, EF and in the same parallels BF, GH. {I. 36}

Moreover the triangle ABC is half of the parallelogram GBCA; for the diameter AB bisects it. {I. 34}

And the triangle FED is half of the parallelogram DEFH; for the diameter DF bisects it. {I. 34}

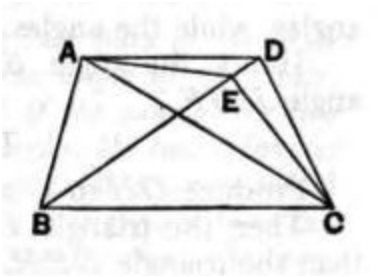
{But the halves of equal things are equal to one another.}

Therefore the triangle ABC is equal to the triangle DEF.

Therefore etc.

Q. E. D.

Proposition 39.



Equal triangles which are on the same base and on the same side are also in the same parallels.

Let ABC, DBC be equal triangles which are on the same base BC and on the same side of it;
{I say that they are also in the same parallels.}

And {For} let AD be joined; I say that AD is parallel to BC.

For, if not, let AE be drawn through the point A parallel to the straight line BC, {I. 31} and let EC be joined.

Therefore the triangle ABC is equal to the triangle EBC; for it is on the same base BC with it and in the same
parallels. {I. 37}

But ABC is equal to DBC;

therefore DBC is also equal to EBC, {C.N. 1} the greater to the less:
which is impossible.

Therefore AE is not parallel to BC.

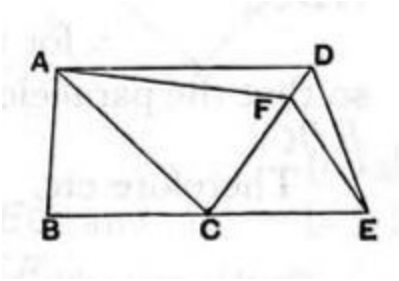
Similarly we can prove that neither is any other straight line except AD;

therefore AD is parallel to BC.

Therefore etc.

Q. E. D. [64]

{Proposition 40.



Equal triangles which are on equal bases and on the same side are also in the same parallels.

Let ABC, CDE be equal triangles on equal bases BC, CE and on the same side.

I say that they are also in the same parallels.

For let AD be joined; I say that AD is parallel to BE.

For, if not, let AF be drawn through A parallel to BE {I. 31}, and let FE be joined.

Therefore the triangle ABC is equal to the triangle FCE; for they are on equal bases BC, CE and in the same parallels BE, AF. {I. 38}

But the triangle ABC is equal to the triangle DCE;

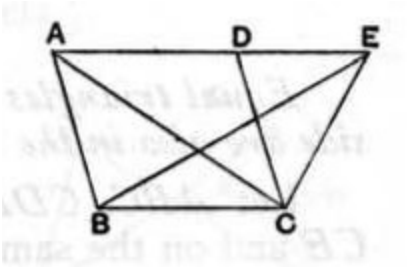
therefore the triangle DCE is also equal to the triangle FCE, {C.N. 1} the greater to the less: which is impossible. Therefore AF is not parallel to BE.

Similarly we can prove that neither is any other straight line except AD;

therefore AD is parallel to BE.

Therefore etc. Q. E. D.}

Proposition 41.



If a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle.

For let the parallelogram ABCD have the same base BC with the triangle EBC, and let it be in the same parallels BC, AE;

I say that the parallelogram ABCD is double of the triangle BEC.

For let AC be joined.

Then the triangle ABC is equal to the triangle EBC; for it is on the same base BC with it and in the same parallels BC, AE. {I. 37}

But the parallelogram ABCD is double of the triangle ABC;

for the diameter AC bisects it; {I. 34}

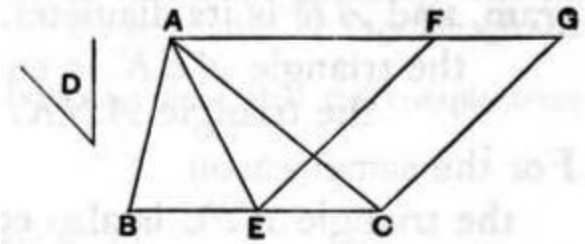
so that the parallelogram ABCD is also double of the triangle EBC.

Therefore etc.

Q. E. D.

Proposition 42.

To construct, in a given rectilineal angle, a parallelogram equal to a given triangle.



Let ABC be the given triangle, and D the given rectilineal angle; thus it is required to construct in the rectilineal angle D a parallelogram equal to the triangle ABC.

Let BC be bisected at E, and let AE be joined; on the straight line EC, and at the point E on it, let the angle CEF be constructed equal to the angle D; {I. 23} through A let AG be drawn parallel to EC, and {I. 31} through C let CG be drawn parallel to EF.

Then FECG is a parallelogram.

And, since BE is equal to EC,

the triangle ABE is also equal to the triangle AEC, for they are on equal bases BE, EC and in the same parallels BC, AG; {I. 38} therefore the triangle ABC is double of the triangle AEC.

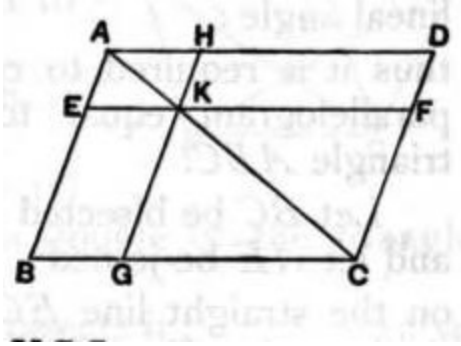
But the parallelogram FECG is also double of the triangle AEC, for it has the same base with it and is in the same parallels with it; {I. 41}

therefore the parallelogram FECG is equal to the triangle ABC.

And it has the angle CEF equal to the given angle D.

Therefore the parallelogram FECG has been constructed equal to the given triangle ABC, in the angle CEF which is equal to D. Q. E. F.

Proposition 43.



In any parallelogram the complements of the parallelograms about the diameter are equal to one another.

Let ABCD be a parallelogram, and AC its diameter; and about AC let EH, FG be parallelograms, and BK, KD
the so-called complements;

I say that the complement BK is equal to the complement KD.

For, since ABCD is a parallelogram, and AC its diameter,

the triangle ABC is equal to the triangle ACD. {I. 34}

Again, since EH is a parallelogram, and AK is its diameter,

the triangle AEK is equal to the triangle AHK.

For the same reason

the triangle KFC is also equal to KGC.

Now, since the triangle AEK is equal to the triangle AHK,

and KFC to KGC,

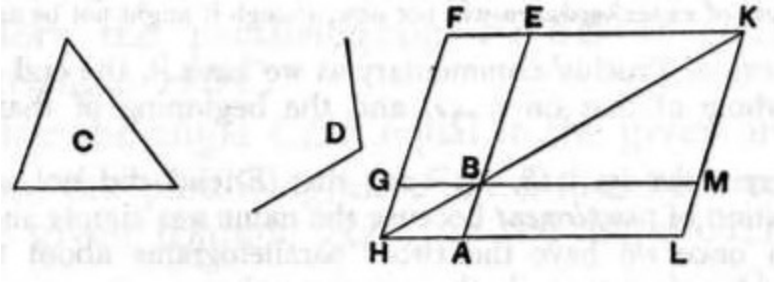
the triangle AEK together with KGC is equal to the triangle AHK together with KFC. {C.N. 2}

And the whole triangle ABC is also equal to the whole ADC; therefore the complement BK which remains is equal to the
complement KD which remains. {C.N. 3}

Therefore etc.

Q. E. D. [65] [66]

Proposition 44.



To a given straight line to apply, in a given rectilineal angle, a parallelogram equal to a given triangle.

Let AB be the given straight line, C the given triangle and D the given rectilineal angle;
thus it is required to apply to the given straight line AB, in an angle equal to the angle D, a parallelogram equal to the given triangle C.

Let the parallelogram BEFG be constructed equal to the triangle C, in the angle EBG which is equal to D {I. 42};
let it be placed so that BE is in a straight line with AB; letFG be drawn through to H, and let AH be drawn through A parallel to either BG or EF. {I. 31}

Let HB be joined.

Then, since the straight line HF falls upon the parallels
AH, EF,

the angles AHF, HFE are equal to two right angles. {I. 29}

Therefore the angles BHG, GFE are less than two right angles; and straight lines produced indefinitely from angles less than
two right angles meet; {Post. 5}

therefore HB, FE, when produced, will meet.

Let them be produced and meet at K; through the point K let KL be drawn parallel to either EA or FH, {I. 31} and let HA, GB be produced to the points L, M.

Then HLKF is a parallelogram, HK is its diameter, and AG, ME are parallelograms. and LB, BF the so-called complements, about HK;

therefore LB is equal to BF. {I. 43}

But BF is equal to the triangle C;

therefore LB is also equal to C. {C.N. 1}

And, since the angle GBE is equal to the angle ABM, {I. 15}

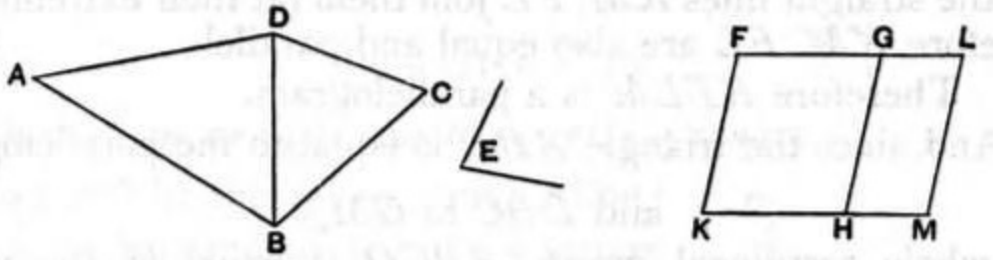
while the angle GBE is equal to D, the angle ABM is also equal to the angle D.

Therefore the parallelogram LB equal to the given triangle
C has been applied to the given straight line AB, in the angle ABM which is equal to D.

Q. E. F. [67]

Proposition 45.

To construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure.



Let ABCD be the given rectilineal figure and E the given rectilineal angle; thus it is required to construct, in the given angle E, a parallelogram equal to the rectilineal figure ABCD.

Let DB be joined, and let the parallelogram FH be constructed equal to the triangle ABD, in the angle HKF which is equal to E; {I. 42}
let the parallelogram GM equal to the triangle DBC be applied to the straight line GH, in the angle GHM which is equal to E. {I. 44}

Then, since the angle E is equal to each of the angles HKF, GHM,

the angle HKF is also equal to the angle GHM. {C.N. 1}

Let the angle KHG be added to each; therefore the angles FKH, KHG are equal to the angles KHG, GHM.

But the angles FKH, KHG are equal to two right angles; {I. 29}
therefore the angles KHG, GHM are also equal to two right angles.

Thus, with a straight line GH, and at the point H on it, two straight lines KH, HM not lying on the same side make the adjacent angles equal to two right angles;

therefore KH is in a straight line with HM. {I. 14}

And, since the straight line HG falls upon the parallels KM, FG, the alternate angles MHG, HGF are equal to one another. {I. 29}

Let the angle HGL be added to each;
therefore the angles MHG, HGL are equal to the angles HGF, HGL. {C.N. 2}

But the angles MHG, HGL are equal to two right angles; {I. 29} therefore the angles HGF, HGL are also equal to two right angles. {C.N. 1}

Therefore FG is in a straight line with GL. {I. 14}

And, since FK is equal and parallel to HG, {I. 34}

and HG to ML also,

KF is also equal and parallel to ML; {C.N. 1; I. 30} and the straight lines KM, FL join them (at their extremities);
therefore KM, FL are also equal and parallel. {I. 33}

Therefore KFLM is a parallelogram.

And, since the triangle ABD is equal to the parallelogram FH,

and DBC to GM,

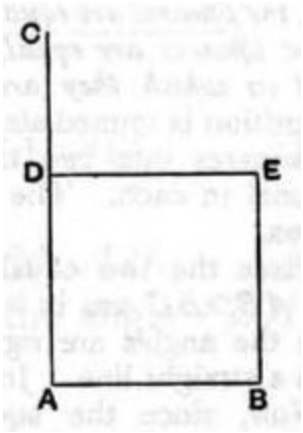
the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM.

Therefore the parallelogram KFLM has been constructed equal to the given rectilineal figure ABCD, in the angle FKM which is equal to the given

angle E.

Q. E. F. [68]

Proposition 46.



On a given straight line to describe a square.

Let AB be the given straight line; thus it is required to describe a square on the straight line AB.

Let AC be drawn at right angles to the straight line AB from the point A on it {I. 11}, and let AD be made equal to AB; through the point D let DE be drawn
parallel to AB, and through the point B let BE be drawn parallel to AD. {I. 31}

Therefore ADEB is a parallelogram;

therefore AB is equal to DE, and AD to BE. {I. 34}

But AB is equal to AD;

therefore the four straight lines BA, AD, DE, EB are equal to one another;

therefore the parallelogram ADEB is equilateral.

I say next that it is also right-angled.

For, since the straight line AD falls upon the parallels
AB, DE,

the angles BAD, ADE are equal to two right angles. {I. 29}

But the angle BAD is right;

therefore the angle ADE is also right.

And in parallelogrammic areas the opposite sides and
angles are equal to one another; {I. 34}

therefore each of the opposite angles ABE, BED is also right. Therefore ADEB is right-angled.

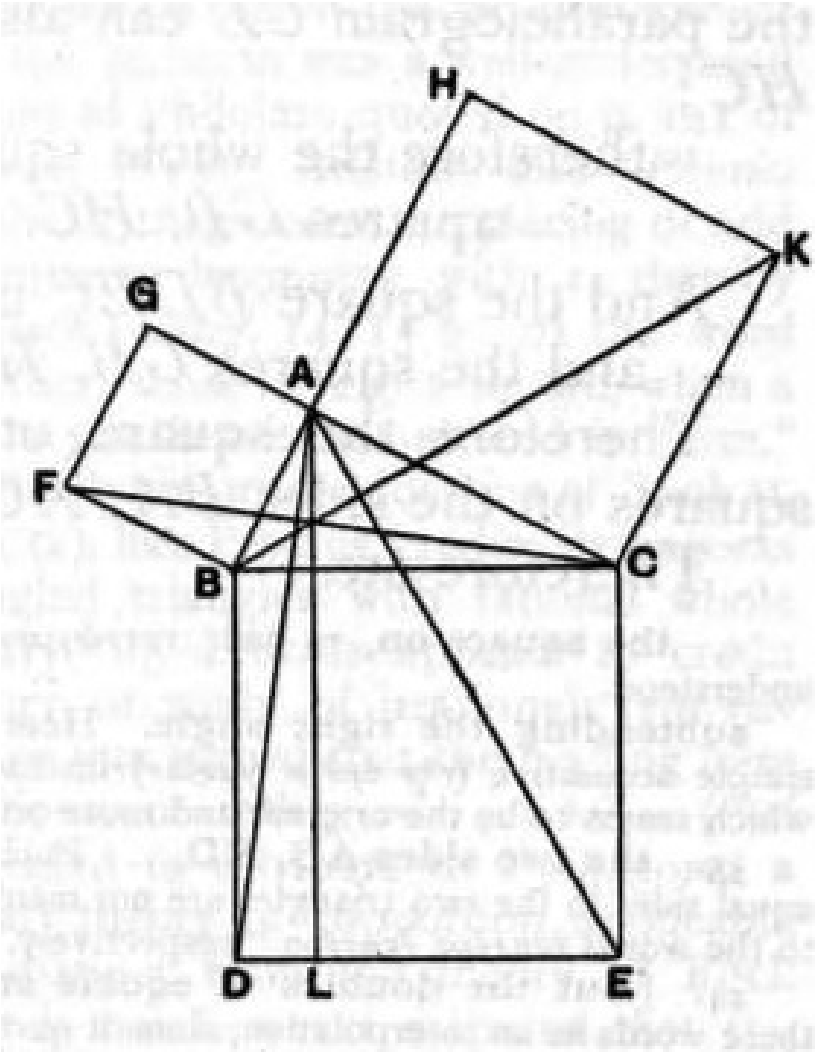
And it was also proved equilateral.

Therefore it is a square; and it is described on the straight line AB.

Q. E. F. [69]

Proposition 47.

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.



Let ABC be a right-angled triangle having the angle
BAC right;

I say that the square on BC is equal to the squares on BA, AC.

For let there be described on BC the square BDEC,
and on BA, AC the squares GB, HC; {I. 46} through A let AL be drawn parallel to either BD or CE, and let AD, FC be joined.

Then, since each of the angles BAC, BAG is right, it follows that with a straight line BA, and at the point A on it, the two straight lines

AC, AG not lying on the same side make the adjacent angles equal to two right angles;

therefore CA is in a straight line with AG. {I. 14}

For the same reason

BA is also in a straight line with AH.

And, since the angle DBC is equal to the angle FBA: for each is right: let the angle ABC be added to each;

therefore the whole angle DBA is equal to the whole angle FBC. {C.N. 2}

And, since DB is equal to BC, and FB to BA, the two sides AB, BD are equal to the two sides FB, BC respectively,

and the angle ABD is equal to the angle FBC; therefore the base AD is equal to the base FC, and the triangle ABD is equal to the triangle FBC. {I. 4}

Now the parallelogram BL is double of the triangle ABD, for they have the same base BD and are in the same parallels
BD, AL. {I. 41}

And the square GB is double of the triangle FBC, for they again have the same base FB and are in the same parallels FB, GC. {I. 41}

{But the doubles of equals are equal to one another.}

Therefore the parallelogram BL is also equal to the square GB.

Similarly, if AE, BK be joined, the parallelogram CL can also be proved equal to the square HC;

therefore the whole square BDEC is equal to the two squares GB, HC. {C.N. 2}

And the square BDEC is described on BC,

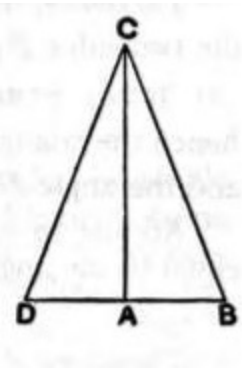
and the squares GB, HC on BA, AC.

Therefore the square on the side BC is equal to the squares on the sides BA, AC.

Therefore etc.

Q. E. D. [70] [71] [72] [73]

Proposition 48.



If in a triangle the square on one of the sides be equal to the squares on the remaining two sides of the triangle, the angle contained by the remaining two sides of the triangle is right.

For in the triangle ABC let the square on one side BC be equal to the squares on the sides BA, AC;

I say that the angle BAC is right.

For let AD be drawn from the point A at right angles to the straight line AC, let AD be made equal to BA, and let DC be joined.

Since DA is equal to AB, the square on DA is also equal to the square on AB.

Let the square on AC be added to each;

therefore the squares on DA, AC are equal to the squares on BA, AC.

But the square on DC is equal to the squares on DA, AC, for the angle DAC is right; {I. 47} and the square on BC is equal to the squares on BA, AC, for this is the hypothesis;

therefore the square on DC is equal to the square on BC, so that the side DC is also equal to BC.

And, since DA is equal to AB, and AC is common,

the two sides DA, AC are equal to the two sides BA, AC;

and the base DC is equal to the base BC;

therefore the angle DAC is equal to the angle BAC. {I. 8}

But the angle DAC is right;

therefore the angle BAC is also right.

Therefore etc.

Q. E. D.

ENDNOTES.

[1] The Greek usage differs from ours in that the definite article is employed in such a phrase as this where we have the indefinite. ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης, "on the given finite straight line," i.e. the finite straight line which we choose to take.

[2] To be strictly literal we should have to translate in the reverse order "let the given finite straight line be the (straight line) AB" ; but this order is inconvenient in other cases where there is more than one datum, e.g. in the setting-out of I. 2, "let the given point be A, and the given straight line BC," the

awkwardness arising from the omission of the verb in the second clause. Hence I have, for clearness' sake, adopted the other order throughout the book.

[3] Two things are here to be noted, (1) the elegant and practically universal use of the perfect passive imperative in constructions, γεγράφθω meaning of course "let it have been described" or "suppose it described," (2) the impossibility of expressing shortly in a translation the force of the words in their original order. κύκλος γεγράφθω ὸ ΒΓΔ means literally "let a circle have been described, the (circle, namely, which I denote by) BCD." Similarly we have lower down "let straight lines, (namely) the (straight lines) CA, CB, be joined," ἐπεζεύχθωσαν εὐθεῖαι αί ΓΑ, ΓΒ. There seems to be no practicable alternative, in English, but to translate as I have done in the text.

[4] Euclid is careful to adhere to the phraseology of Postulate 1 except that he speaks of "joining" (ἐπεζεύχθωσαν) instead of "drawing" (γράφειν). He does not allow himself to use the shortened expression "let the straight line FC be joined" (without mention of the points F, C) until I. 5.

[5] , ἑκατέρα τῶν ΓΑ, ΓΒ and 24. the three straight lines CA, AB, BC, αἱ τρεῖς αἱ ΓΑ, ΑΒ, ΒΓ. I have, here and in all similar expressions, inserted the words "straight lines" which are not in the Greek. The possession of the inflected definite article enables the Greek to omit the words, but this is not possible in English, and it would scarcely be English to write "each of CA, CB" or "the three CA, AB, BC."

[6] I have inserted these words because "to place a straight line at a given point" (πρὸς τῷ δοθέντι σημείῳ) is not quite clear enough, at least in English.

[7] It will be observed that in this first application of Postulate 2, and again in I. 5, Euclid speaks of the continuation of the straight line as that which is produced in such cases, ἐκβεβλήσθωσαν and προσεκβεβλήσθωσαν meaning little more than drawing straight lines "in a straight line with" the given straight lines. The first place in which Euclid uses phraseology exactly corresponding to ours when speaking of a straight line being produced is in I. 16: "let one side of it, BC, be produced to D" (προσεκβεβλήσθω αὐτοῦ μία πλευρὰ ἡ ΒΓ ἐπὶ τὸ Δ).

[8] The Greek expressions are λοιπὴ ἡ ΑΛ and λοιπῇ τῇ ΒΗ, and the literal translation would be "AL (or BG) remaining," but the shade of meaning conveyed by the position of the definite article can hardly be expressed in English.

[9] It is a fact that Euclid's enunciations not infrequently leave something to be desired in point of clearness and precision. Here he speaks of the triangles having "the angle equal to the angle, namely the angle contained by the equal straight lines" (τὴν γωνίαν τῇ γωνίᾳ ἴσην ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην), only one of the two angles being described in the latter expression (in the accusative),

and a similar expression in the dative being left to be understood of the other angle. It is curious too that, after mentioning two "sides," he speaks of the angles contained by the equal "straight lines," not "sides." It may be that he wished to adhere scrupulously, at the outset, to the phraseology of the definitions, where the angle is the inclination to one another of two lines or straight lines. Similarly in the enunciation of I. 5 he speaks of producing the equal "straight lines" as if to keep strictly to the wording of Postulate 2.

[10] I agree with Mr H. M. Taylor (Euclid, p. ix) that it is best to abandon the traditional translation of "each to each," which would naturally seem to imply that all the four magnitudes are equal rather than (as the Greek ἑκατέρα ἑκατέρᾳ does) that one is equal to one and the other to the other.

[11] Here we have the word base used for the first time in the Elements. Proclus explains it (p. 236, 12-15) as meaning (1), when no side of a triangle has been mentioned before, the side "which is on a level with the sight" (τὴν πρὸς τῇ ὄψει κειμένην), and (2), when two sides have already been mentioned, the third side. Proclus thus avoids the mistake made by some modern editors who explain the term exclusively with reference to the case where two sides have been mentioned before. That this is an error is proved (1) by the occurrence of the term in the enunciations of I. 37 etc. about triangles on the same base and equal bases, (2) by the application of the same term to the bases of parallelograms in I. 35 etc. The truth is that the use of the term must have been suggested by the practice of drawing the particular side horizontally, as it were, and the rest of the figure above it. The base of a figure was therefore spoken of, primarily, in the same sense as the base of anything else, e.g. of a pedestal or column; but when, as in I. 5, two triangles were compared occupying other than the normal positions which gave rise to the name, and when two sides had been previously mentioned, the base was, as Proclus says, necessarily the third side.

[12] ὑποτείνειν ὑπό, "to stretch under," with accusative.

[13] The full Greek expression would be ἡ ὑπὸ τῶν ΒΑ, ΑΓ περιεχομένη γωνία, "the angle contained by the (straight lines) BA, AC." But it was a common practice of Greek geometers, e.g. of Archimedes and Apollonius (and Euclid too in Books X. — XIII.), to use the abbreviation αἱ ΒΑΓ for αἱ ΒΑ, ΑΓ, "the (straight lines) BA, AC." Thus, on περιεχομένη being dropped, the expression would become first ἡ ὑπὸ τῶν ΒΑΓ γωνία, then ἡ ὑπὸ ΒΑΓ γωνία, and finally ἡ ὑπὸ ΒΑΓ, without γωνία, as we regularly find it in Euclid.

[14] The difference between the technical use of the passive ἐφαρμόζεσθαι "to be applied (to)," and of the active ἐφαρμόζειν "to coincide (with)" has been noticed above (note on Common Notion 4, pp. 224-5).

[15] Heiberg (Paralipomena su Euklid in Hermes, XXXVIII., 1903, p. 56) has pointed out, as a conclusive reason for regarding these words as an early interpolation, that the text of an-Nairīzī (Codex Leidensis 399, 1, ed. Besthorn-Heiberg, p. 55) does not give the words in this place but after the conclusion Q.E.D., which shows that they constitute a scholium only. They were doubtless added by some commentator who thought it necessary to explain the immediate inference that, since B coincides with E and C with F, the straight line BC coincides with the straight line EF, an inference which really follows from the definition of a straight line and Post. 1; and no doubt the Postulate that "Two straight lines cannot enclose a space" (afterwards placed among the Common Notions) was interpolated at the same time.

[16] Where (as here) Euclid's conclusion merely repeats the enunciation word for word, I shall avoid the repetition and write "Therefore etc." simply.

[17] (meaning the equal sides). Cf. note on the similar expression in Prop. 4, lines 2, 3.

[18] , εἰλήφθω ἐπὶ τῆς ΒΔ τυχὸν σημεῖον τὸ Ζ, where τυχὸν σημεῖονmeans "a chance point."

[19] δύο αἱ ΖΑ, ΑΓ δυσὶ ταῖς ΗΑ, ΑΒ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ. Here, and in numberless later passages, I have inserted the word "sides" for the reason given in the note on I. 1, line 20. It would have been permissible to supply either "straight lines" or "sides" ; but on the whole "sides" seems to be more in accordance with the phraseology of I. 4.

[20] i.e., apparently, common to the angles, as the αὐτῶν in βάσις αὐτῶν κοινὴ can only refer to γωνία and γωνίᾳ preceding. Simson wrote "and the base BC is common to the two triangles BFC, CGB" ; Todhunter left out these words as being of no use and tending to perplex a beginner. But Euclid evidently chose to quote the conclusion of I. 4 exactly; the first phrase of that conclusion is that the bases (of the two triangles) are equal, and, as the equal bases are here the same base, Euclid naturally substitutes the word "common" for "equal."

[21] As "(Being) what it was required to prove" (or "do") is somewhat long, I shall henceforth write the time-honoured "Q. E. D." and "Q. E. F." for ὅπερ ἔδει δεῖξαι and ὅπερ ἔδει ποιῆσαι.

[22] In an English translation of the enunciation of this proposition it is absolutely necessary, in order to make it intelligible, to insert some words which are not in the Greek. The reason is partly that the Greek enunciation is itself very elliptical, and partly that some words used in it conveyed more meaning than the corresponding words in English do. Particularly is this the case with οὐ συσταθήσονται ἐπί "there shall not be constructed upon," since συνίστασθαι is the regular word for constructing a triangle in particular. Thus a Greek would easily understahd συσταθήσονται ἐπί as meaning the construction of

two lines forming a triangle on a given straight line as base; whereas to "construct two straight lines on a straight line" is not in English sufficiently definite unless we explain that they are drawn from the ends of the straight line to meet at a point. I have had the less hesitation in putting in the words "from its extremities" because they are actually used by Euclid in the somewhat similar enunciation of I. 21.

How impossible a literal translation into English is, if it is to convey the meaning of the enunciation intelligibly, will be clear from the following attempt to render literally: "On the same straight line there shall not be constructed two other straight lines equal, each to each, to the same two straight lines, (terminating) at different points on the same side, having the same extremities as the original straight lines" (ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθεῖαι ἴσαι ἑκατέρα ἑκατέρα οὐ συσταθήσονται πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι ταῖς ἐξ ἀρχῆς εὐθείαις).

The reason why Euclid allowed himself to use, in this enunciation, language apparently so obscure is no doubt that the phraseology was traditional and therefore, vague as it was, had a conventional meaning which the contemporary geometer well understood. This is proved, I think, by the occurrence in Aristotle (Meteorologica III. 5, 376 a 2 sqq.) of the very same, evidently technical, expressions. Aristotle is there alluding to the theorem given by Eutocius from Apollonius' Plane Loci to the effect that, if H, K be two fixed points and M such a variable point that the ratio of MH to MK is a given ratio (not one of equality), the locus of M is a circle. (For an account of this theorem see note on VI. 3 below.) Now Aristotle says "The lines drawn up from H, K in this ratio cannot be constructed to two different points of the semicircle A" (αἱ οὖν ἀπὸ τῶν HK ἀναγόμεναι γραμμαὶ ἐν τούτῳ τῷ λόγῳ οὐ συσταθήσουται τοῦ ἐφ᾽ ᾧ Α ἡμικυκλίου πρὸς ἄλλο καὶ ἄλλο σημεῖον).

If a paraphrase is allowed instead of a translation adhering as closely as possible to the original, Simson's is the best that could be found, since the fact that the straight lines form triangles on the same base is really conveyed in the Greek. Simson's enunciation is, Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated at the other extremity. Th. Taylor (the translator of Proclus) attacks Simson's alteration as "indiscreet" and as detracting from the beauty and accuracy of Euclid's enunciation which are enlarged upon by Proclus in his commentary. Yet, when Taylor says "Whatever difficulty learners may find in conceiving this proposition abstractedly is easily removed by its exposition in the figure," he really gives his case away. The fact is that Taylor, always enthusiastic over his author, was nettled by Simson's slighting remarks on Proclus' comments on the proposition. Simson had said, with reference to Proclus' explanation of the bearing of the second part of I. 5 on I. 7, that it was not "worth while to relate his trifles at full length," to which Taylor retorts "But Mr Simson was no philosopher; and therefore the greatest part of these Commentaries must be considered by him as trifles, from the want of a philosophic genius to

comprehend their meaning, and a taste superior to that of a mere mathematician, to discover their beauty and elegance."

[23] It would be natural to insert here the step "but the angle ACD is greater than the angle BCD. {C.N. 5}."

[24] much greater, literally "greater by much" (πολλῷ μειζων). Simson and those who follow him translate: "much more then is the angle BDC greater than the angle BCD," but the Greek for this would have to be πολλῷ (or πολὺ{ρπαρ } μᾶλλόν ἐστι...μείζων. πολλῷ μᾶλλον, however, though used by Apollonius, is not, apparently, found in Euclid or Archimedes.

[25] The text has here "BA, CA."

[26] fall beside them. The Greek has the future, παραλλάξουσι. παραλλάττω means "to pass by without touching," "to miss" or "to deviate."

[27] The verb is κείσθω which, as well as the other parts of κεῖμαι, is constantly used for the passive of τίθημι "to place" ; and the latter word is constantly used in the sense of making, e.g., one straight line equal to another straight line.

[28] , κάθετον εὐθεῖαν γραμμήν. This is the full expression for a perpendicular, κάθετος meaning let fall or let down, so that the expression corresponds to our plumb-line. ἡ κάθετος is however constantly used alone for a perpendicular, γραμμή being understood.

[29] , literally "towards the other parts of the straight line AB," ἐπὶ τὰ ἕτερα μέρη τῆς AB. Cf. "on the same side" (ἐπὶ τὰ αὐτὰ μέρη) in Post. 5 and "in both directions" (ἐφ᾽ ἑκάτερα τὰ μέρη) in Def. 23.

[30] literally "let the angle EBD be added (so as to be) common," κοινὴ προσκείσθω ἡ ὑπὸ ΕΒΔ. Similarly κοινὴ ἀφηρήσθω is used of subtracting a straight line or angle from each of two others. "Let the common angle EBD be added" is clearly an inaccurate translation, for the angle is not common before it is added, i.e. the κοινὴ is proleptic. "Let the common angle be subtracted" as a translation of κοινὴ ἀφηρήσθω would be less unsatisfactory, it is true, but, as it is desirable to use corresponding words when translating the two expressions, it seems hopeless to attempt to keep the word "common," and I have therefore said "to each" and "from each" simply.

[31] There is no greater difficulty in translating the works of the Greek geometers than that of accurately giving the force of prepositions. πρός, for instance, is used in all sorts of expressions with various shades of meaning. The present enunciation begins ἐὰν πρός τινι εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ, and

it is really necessary in this one sentence to translate πρός by three different words, with, at, and on. The first πρός must be translated by with because two straight lines "make" an angle with one another. On the other hand, where the similar expression πρὸς τῇ δοθείσῃ εὐθείᾳ occurs in I. 23, but it is a question of "constructing" an angle (συστήσασθαι), we have to say "to construct on a given straight line." Against would perhaps be the English word coming nearest to expressing all these meanings of πρός, but it would be intolerable as a translation.

[32] Todhunter points out that for the inference in this line Post. 4, that all right angles are equal, is necessary as well as the Common Notion that things which are equal to the same thing (or rather, here, to equal things) are equal. A similar remark applies to steps in the proofs of I. 15 and I. 28.

[33] The Greek expresses this by the future of the verb, δείξομεν, "we shall prove," which however would perhaps be misleading in English.

[34] The difference between adjacent angles (αἱ ἐφεξῆς γωνίαι) and vertical angles (αἱ κατὰ κορυφὴν γωνίαι) is thus explained by Proclus (p. 298, 14-24). The first term describes the angles made by two straight lines when one only is divided by the other, i.e. when one straight line meets another at a point which is not either of its extremities, but is not itself produced beyond the point of meeting. When the first straight line is produced, so that the lines cross at the point, they make two pairs of vertical angles (which are more clearly described as vertically opposite angles), and which are so called because their convergence is from opposite directions to one point (the intersection of the lines) as vertex (κορυφή).

[35] literally "at the section," πρὸς τῇ τομῇ.

[36] literally "the outside angle," ἡ ἐκτὸς γωνία.

[37] τῶν ἐντὸς καὶ ἀπεναντίον γωνιῶν.

[38] The word is διήχθω, a variation on the more usual ἐκβεβλήσθω, "let it be produced."

[39] in the text "FEC."

[40] πάντη μεταλαμβανόμεναι, i.e. any pair added together.

[41] In the enunciation of this number we have ὑποτείνειν ("subtend") used with the simple accusative instead of the more usual ὑπό with accusative. The latter construction is used in the enunciation of I. 19, which otherwise only differs from that of I. 18 in the order of the words. The point to remember in order to distinguish the two is that the datum comes first and the quaesitum second, the datum being in

this proposition the greater side and in the next the greater angle. Thus the enunciations are (I. 18) παντὸς τριγώνου ἡ μείζων πλευρὰ τὴν μείζονα γωνίαν ὑποτείνει and (I. 19) παντὸς τριγώνου ὑπὸ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει. In order to keep the proper order in English we must use the passive of the verb in I. 19. Aristotle quotes the result of I. 19, using the exact wording, ὑπὸ γὰρ τὴν μείζω γωνίαν ὑποτείνει (Meteorologica III. 5, 376 a 12).

[42] It was the habit of the Epicureans, says Proclus (p. 322), to ridicule this theorem as being evident even to an ass and requiring no proof, and their allegation that the theorem was "known" (γνώριμον) even to an ass was based on the fact that, if fodder is placed at one angular point and the ass at another, he does not, in order to get to his food, traverse the two sides of the triangle but only the one side separating them (an argument which makes Savile exclaim that its authors were "digni ipsi, qui cum Asino foenum essent," p. 78). Proclus replies truly that a mere perception of the truth of the theorem is a different thing from a scientific proof of it and a knowledge of the reason why it is true. Moreover, as Simson says, the number of axioms should not be increased without necessity.

[43] The word "meeting" is not in the Greek, where the words are ἐντὸς συσταθῶσιν. συνίστασθαι is the word used of constructing two straight lines to a point (cf. I. 7) or so as to form a triangle; but it is necessary in English to indicate that they meet.

[44] Observe the elegant brevity of the Greek αἱ συσταθεῖσαι.

[45] This is the first case in the Elements of a διορισμός to a problem in the sense of a statement of the conditions or limits of the possibility of a solution. The criterion is of course supplied by the preceding proposition.

[46] This is usually translated (e.g. by Williamson and Simson) "But it is necessary," which is however inaccurate, since the Greek is not δεῖ δέ but δεῖ δή. The words are the same as those used to introduce the διορισμός in the other sense of the "definition" or "particular statement" of a construction to be effected. Hence, as in the latter case we say "thus it is required" (e.g. to bisect the finite straight line AB, I. 10), we should here translate "thus it is necessary."

[47] To this enunciation all the MSS. and Boethius add, after the διορισμός, the words "because in any triangle two sides taken together in any manner are greater than the remaining one." But this explanation has the appearance of a gloss, and it is omitted by Proclus and Campanus. Moreover there is no corresponding addition to the διορισμός of VI. 28.

[48] I have naturally left out the well-known words added by Simson in order to avoid the necessity of considering three cases: "Of the two sides DE, DF let DE be the side which is not greater than the

other." I doubt whether Euclid could have been induced to insert the words himself, even if it had been represented to him that their omission meant leaving two possible cases out of consideration. His habit and that of the great Greek geometers was, not to set out all possible cases, but to give as a rule one case, generally the most difficult, as here, and to leave the others to the reader to work out for himself. We have already seen one instance in I. 7.

[49] πλευρὰν τὴν πρὸς ταῖς ἴσαις γωνίαις.

[50] ὑπόκειται ἴση, according to the elegant Greek idiom. ὑπόκειμαι is used for the passive of ὑποτίθημι, as κεῖμαι is used for the passive of τίθημι, and so with the other compounds. Cf. προσκεῖσθαι, "to be added."

[51] εἰς δύο εὐθείας ἐμπίπτουσα, the phrase being the same as that used in Post. 5, meaning a transversal.

[52] αἱ ἐναλλὰξ γωνίαι. Proclus (p. 357, 9) explains that Euclid uses the word alternate (or, more exactly, alternately, ἐναλλάξ) in two connexions, (1) of a certain transformation of a proportion, as in Book V. and the arithmetical Books, (2) as here, of certain of the angles formed by parallels with a straight line crossing them. Alternate angles are, according to Euclid as interpreted by Proclus, those which are not on the same side of the transversal, and are not adjacent, but are separated by the transversal, both being within the parallels but one "above" and the other "below." The meaning is natural enough if we imagine the four internal angles to be taken in cyclic order and alternate angles to be any two of them not successive but separated by one angle of the four.

[53] literally "towards the parts B, D or towards A, C," ἐπὶ τὰ Β, Δ μέρη ἢ ἐπὶ τὰ Α Γ.

[54] αἱ δὲ ἀπ: ἐλασσόνων ἢ δύο ὀρθῶν ἐκβαλλόμεναι εἰς ἄπειπον συμπίπτουσιν, a variation from the more explicit language of Postulate 5. A good deal is left to be understood, namely that the straight lines begin from points at which they meet a transversal, and make with it internal angles on the same side the sum of which is less than two right angles.

[55] literally "because they are supposed parallel," διὰ τὸ παραλλήλους αὐτὰς ὑποκεῖσθαι.

[56] The usual conclusion in general terms ("Therefore etc.") repeating the enunciation is, curiously enough, wanting at the end of this proposition.

[57] I have for clearness' sake inserted the words in brackets though they are not in the original Greek, which has "joining...in the same directions" or "on the same sides," ἐπὶ τὰ αυτὰ μέρη ἐπιζευγνύουσαι.

The expression "tiwards the same parts," though usage has sanctioned it, is perhaps not quite satisfactory.

58 and 18. DCB. The Greek has " BC, CD" and "BCD" in these places respectively. Euclid is not always careful to write in corresponding order the letters denoting corresponding points in congruent figures. On the contrary, he evidently prefers the alphabetical order, and seems to disdain to alter it for the sake of beginners or others who might be confused by it. In the case of angles alteration is perhaps unnecessary; but in the case of triangles and pairs of corresponding sides I have ventured to alter the order to that which the mathematician of to-day expects.

59 It is to be observed that, when parallelograms have to be mentioned for the first time, Euclid calls them "parallelogrammic areas" or, more exactly, "parallelogram" areas (παραλληλόγραμμα χωρία). The meaning is simply areas bounded by parallel straight lines with the further limitation placed upon the term by Euclid that only four-sided figures are so called, although of course there are certain regular polygons which have opposite sides parallel, and which therefore might be said to be areas bounded by parallel straight lines. We gather from Proclus (p. 393) that the word "parallelogram" was first introduced by Euclid, that its use was suggested by I. 33, and that the formation of the word παραλληλόγραμμος (parallel-lined) was analogous to that of εὐθύγραμμος (straight-lined or rectilineal).

60 and 36. DC, CB. The Greek has in these places "BCD" and "CD, BC" respectively. Cf. note on I. 33, lines 15, 18.

61 The text has "DFC."

62 Euclid speaks of the triangle DGE without any explanation that, in the case which he takes (where AD, EF have no point in common), BE, CD must meet at a point G between the two parallels. He allows this to appear from the figure simply.

63 Here and in the next proposition Heiberg brackets the words "But the halves of equal things are equal to one another" on the ground that, since the Common Notion which asserted this fact was interpolated at a very early date (before the time of Theon), it is probable that the words here were interpolated at the same time. Cf. note above (p. 224) on the interpolated Common Notion.

64 {I say that they are also in the same parallels.} Heiberg has proved (Hermes, XXXVIII., 1903, p. 50) from a recently discovered papyrus-fragment (Fayūm towns and their papyri, p. 96, No. IX.) that these words are an interpolation by some one who did not observe that the words "And let AD be joined" are part of the setting-out (ἔκθεσις), but took them as belonging to the construction (κατασκευή) and

consequently thought that a διορισμός or "definition" (of the thing to be proved) should precede. The interpolator then altered "And" into "For" in the next sentence.

65 complements, παραπληρώματα, the figures put in to fill up (interstices).

66 Euclid's phraseology here and in the next proposition implies that the complements as well as the other parallelograms are "about" the diagonal. The words are here περὶ δὲ τὴν ΑΓ παραλληλόγραμμα μὲν ἔστω τὰ ΕΘ, ΖΗ, τὰ δὲ λεγόμενα παραπληρώματα τὰ ΒΚ, ΚΔ. The expression "the so-called complements" indicates that this technical use of παραπληρώματα was not new, though it might not be universally known.

67 The verb is in the aorist (ἐνέπεσεν) here and in similar expressions in the following propositions.

68 "rectilineal" simply, without "figure," εὐθύγραμμον being here used as a substantive, like the similarly formed παραλληλόγραμμον.

69 Proclus (p. 423, 18 sqq.) notes the difference between the word construct (συστήσασθαι) applied by Euclid to the construction of a triangle (and, he might have added, of an angle) and the words describe on (ἀναγράφειν ἀπό) used of drawing a square on a given straight line as one side. The triangle (or angle) is, so to say, pieced together, while the describing of a square on a given straight line is the making of a figure "from" one side, and corresponds to the multiplication of the number representing the side by itself.

70 , τὸ ἀπὸ...τετράγωνον, the word ἀναγραφέν or ἀναγεγραμμένον being understood.

71 subtending the right angle. Here ὑποτεινούσης, "subtending," is used with the simple accusative (τὴν ὀρθὴν γωνίαν) instead of being followed by ὑπό and the accusative, which seems to be the original and more orthodox construction. Cf. I. 18, note.

72 Euclid actually writes "DB, BA," and therefore the equal sides in the two triangles are not mentioned in corresponding order, though he adheres to the words ἑκατέρα ἑκατέρα "respectively." Here DB is equal to BC and BA to FB.

73 {But the doubles of equals are equal to one another.} Heiberg brackets these words as an interpolation, since it quotes a Common Notion which is itself interpolated. Cf. notes on I. 37, p. 332, and on interpolated Common Notions, pp. 223-4.

BOOK II. DEFINITIONS.

1

Any rectangular parallelogram is said to be contained by the two straight lines containing the right angle.

2

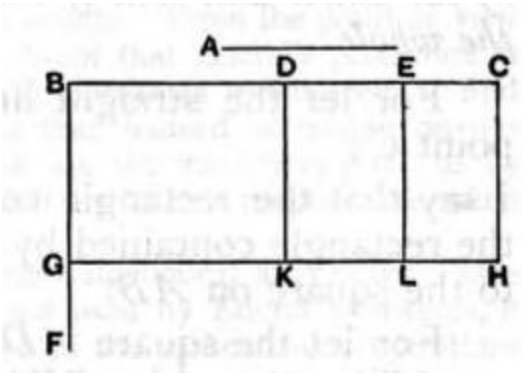
And in any parallelogrammic area let any one whatever of the parallelograms about its diameter with the two complements be called a gnomon.

PROPOSITIONS.

Proposition 1.

If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments.

Let A, BC be two straight lines, and let BC be cut at random at the points D, E; I say that the rectangle contained by A, BC is equal to the rectangle contained by A, BD, that contained by A, DE and
that contained by A, EC.



For let BF be drawn from B at right angles to BC; {I. 11} let BG be made equal to A, {I. 3} through G let GH be drawn
parallel to BC, {I. 31} and through D, E, C let DK, EL, CH be drawn parallel to BG.

Then BH is equal to BK, DL, EH.

Now BH is the rectangle A, BC, for it is contained by GB, BC, and BG is equal to A;

BK is the rectangle A, BD, for it is contained by GB, BD, and BG is equal to A;

and DL is the rectangle A, DE, for DK, that is BG {I. 34}, is equal to A.

Similarly also EH is the rectangle A, EC.

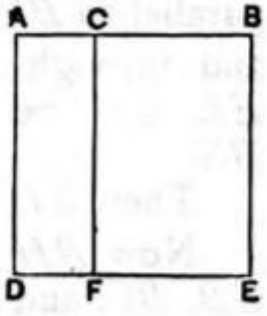
Therefore the rectangle A, BC is equal to the rectangle A, BD, the rectangle A, DE and the rectangle A, EC.

Therefore etc. Q. E. D. [1]

Proposition 2.

If a straight line be cut at random, the rectangle contained by the whole and both of the segments is equal to the square on the whole.

For let the straight line AB be cut at random at the point C; I say that the rectangle contained by AB, BC together with the rectangle contained by BA, AC is equal to the square on AB.



For let the square ADEB be described on AB {I. 46}, and let CF be drawn through C parallel to either AD or BE. {I. 31}

Then AE is equal to AF, CE.

Now AE is the square on AB;

AF is the rectangle contained by BA, AC, for it is contained by DA, AC, and AD is equal to AB;

and CE is the rectangle AB, BC, for BE is equal to AB.

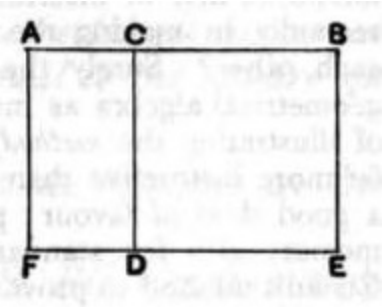
Therefore the rectangle BA, AC together with the rectangle AB, BC is equal to the square on AB.

Therefore etc. Q. E. D.

Proposition 3.

If a straight line be cut at random, the rectangle contained by the whole and one of the segments is equal to the rectangle contained by the segments and the square on the aforesaid segment.

For let the straight line AB be cut at random at C; I say that the rectangle contained by AB, BC is equal to the rectangle contained by AC, CB together with the square on BC.



For let the square CDEB be described on CB; {I. 46} let ED be drawn through to F, and through A let AF be drawn parallel to either CD or BE. {I. 31}

Then AE is equal to AD, CE.

Now AE is the rectangle contained by AB, BC, for it is contained by AB, BE, and BE is equal to BC;

AD is the rectangle AC, CB, for DC is equal to CB;

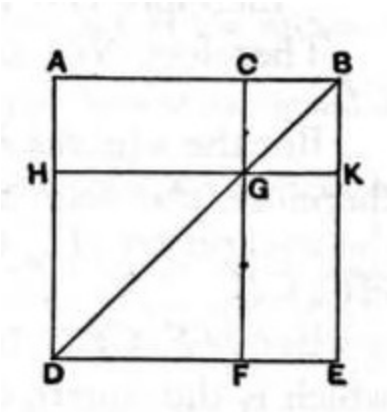
and DB is the square on CB

. Therefore the rectangle contained by AB, BC is equal to the rectangle contained by AC, CB together with the square on BC.

Therefore etc. Q. E. D.

Proposition 4.

If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.



For let the straight line AB be cut at random at C;
I say that the square on AB is equal to the squares on AC, CB and twice the rectangle contained by AC, CB.

For let the square ADEB be described on AB, {I. 46}
let BD be joined; through C let CF be drawn parallel to either AD or EB, and through G let HK be drawn parallel to either AB or DE. {I. 31}

Then, since CF is parallel to AD, and BD has fallen on them, the exterior angle CGB is equal to the interior and opposite angle ADB. {I. 29}

But the angle ADB is equal to the angle ABD,

since the side BA is also equal to AD; {I. 5}

therefore the angle CGB is also equal to the angle GBC, so that the side BC is also equal to the side CG. {I. 6}

But CB is equal to GK, and CG to KB; {I. 34}

therefore GK is also equal to KB; therefore CGKB is equilateral.

I say next that it is also right-angled.

For, since CG is parallel to BK,

the angles KBC, GCB are equal to two right angles. {I. 29}

But the angle KBC is right;

therefore the angle BCG is also right, so that the opposite angles CGK, GKB are also right. {I. 34}

Therefore CGKB is right-angled; and it was also proved equilateral;

therefore it is a square; and it is described on CB.

For the same reason

HF is also a square; and it is described on HG, that is AC. {I. 34}

Therefore the squares HF, KC are the squares on AC, CB.

Now, since AG is equal to GE, and AG is the rectangle AC, CB, for GC is equal to CB,

therefore GE is also equal to the rectangle AC, CB.

Therefore AG, GE are equal to twice the rectangle AC, CB.

But the squares HF, CK are also the squares on AC, CB; therefore the four areas HF, CK, AG, GE are equal to the squares on AC, CB and twice the rectangle contained by AC, CB.

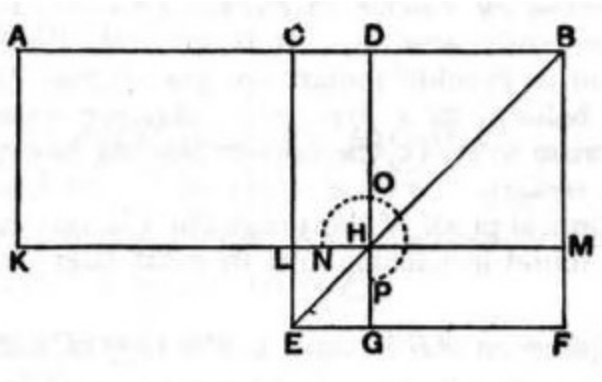
But HF, CK, AG, GE are the whole ADEB,
which is the square on AB.

Therefore the square on AB is equal to the squares on AC, CB and twice the rectangle contained by AC, CB.

Therefore etc. Q. E. D. [2] [3] [4]

Proposition 5.

If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.



For let a straight line AB be cut into equal segments at C and into unequal segments at D; I say that the rectangle contained by AD, DB together with the square on CD is equal to the square on CB.

For let the square CEFB be described on CB, {I. 46} and let BE be joined; through D let DG be drawn parallel to either CE or BF, through H again let KM be drawn parallel to either AB or EF, and again through A let AK be drawn parallel to either CL or BM. {I. 31}

Then, since the complement CH is equal to the complement HF, {I. 43} let DM be added to each;

therefore the whole CM is equal to the whole DF.

But CM is equal to AL,

since AC is also equal to CB; {I. 36} therefore AL is also equal to DF.

Let CH be added to each;

therefore the whole AH is equal to the gnomon NOP.

But AH is the rectangle AD, DB, for DH is equal to DB,

therefore the gnomon NOP is also equal to the rectangle AD, DB.

Let LG, which is equal to the square on CD, be added to each;

therefore the gnomon NOP and LG are equal to the rectangle contained by AD, DB and the square on CD.

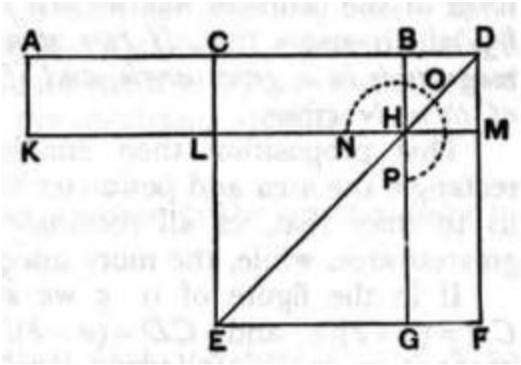
But the gnomon NOP and LG are the whole square CEFB, which is described on CB;

therefore the rectangle contained by AD, DB together with the square on CD is equal to the square on CB.

Therefore etc. Q. E. D. [5]

Proposition 6.

If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.



For let a straight line AB be bisected at the point C, and let a straight line BD be added to it in a straight line;

I say that the rectangle contained by AD, DB together with the square on CB is equal to the square on CD.

For let the square CEFD be described on CD, {I. 46} and let DE be joined; through the point B let BG be drawn parallel to either EC or DF, through the point H let KM be drawn parallel to either AB or EF, and further through A let AK be drawn parallel to either CL or DM. {I. 31}

Then, since AC is equal to CB, AL is also equal to CH. {I. 36}

But CH is equal to HF. {I. 43}

Therefore AL is also equal to HF.

Let CM be added to each;

therefore the whole AM is equal to the gnomon NOP.

But AM is the rectangle AD, DB,

for DM is equal to DB; therefore the gnomon NOP is also equal to the rectangle AD, DB.

Let LG, which is equal to the square on BC, be added to each;

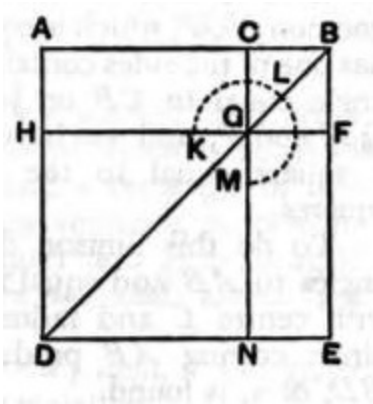
therefore the rectangle contained by AD, DB together with the square on CB is equal to the gnomon NOP and LG.

But the gnomon NOP and LG are the whole square CEFD, which is described on CD;

therefore the rectangle contained by AD, DB together with the square on CB is equal to the square on CD.

Therefore etc. Q. E. D.

Proposition 7.



If a straight line be cut at random, the square on the whole and that on one of the segments both together are equal to twice the rectangle contained by the whole and the said segment and the square on the remaining segment.

For let a straight line AB be cut at random at the point C;

I say that the squares on AB, BC are equal to twice the rectangle contained by AB, BC and the square on CA.

For let the square ADEB be described on AB, {I. 46} and let the figure be drawn.

Then, since AG is equal to GE, {I. 43} let CF be added to each;

therefore the whole AF is equal to the whole CE.

Therefore AF, CE are double of AF.

But AF, CE are the gnomon KLM and the square CF; therefore the gnomon KLM and the square CF are double of AF.

But twice the rectangle AB, BC is also double of AF; for BF is equal to BC; therefore the gnomon KLM and the square CF are equal to twice the rectangle AB, BC.

Let DG, which is the square on AC, be added to each; therefore the gnomon KLM and the squares BG, GD are equal to twice the rectangle contained by AB, BC and the square on AC.

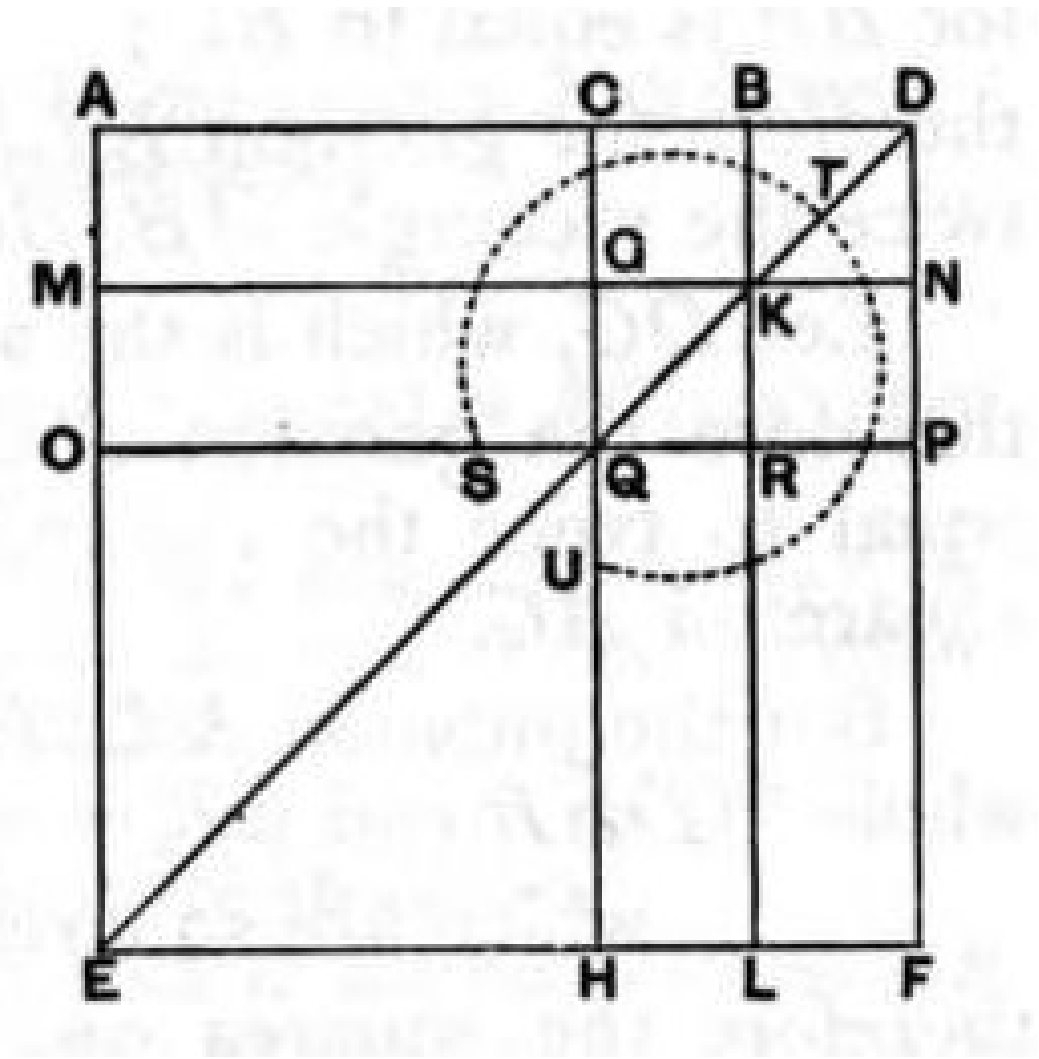
But the gnomon KLM and the squares BG, GD are the whole ADEB and CF,

which are squares described on AB, BC;

therefore the squares on AB, BC are equal to twice the rectangle contained by AB, BC together with the square on AC.

Therefore etc. Q. E. D.

Proposition 8.



If a straight line be cut at random, four times the rectangle contained by the whole and one of the segments together with the square on the remaining segment is equal to the square described on the whole and the aforesaid segment as on one straight line.

For let a straight line AB be cut at random at the point C;

I say that four times the rectangle contained by AB, BC together with the square on AC is equal to the square described on AB, BC as on one straight line.

For let {the straight line} BD be produced in a straight line {with AB}, and let BD be made equal to CB; let the square AEFD be described on AD, and let the figure be drawn double.

Then, since CB is equal to BD, while CB is equal to GK, and BD to KN, therefore GK is also equal to KN.

For the same reason

QR is also equal to RP.

And, since BC is equal to BD, and GK to KN, therefore CK is also equal to KD, and GR to RN. {I. 36}

But CK is equal to RN, for they are complements of the parallelogram CP; {I. 43} therefore KD is also equal to GR; therefore the four areas DK, CK, GR, RN are equal to one another.

Therefore the four are quadruple of CK.

Again, since CB is equal to BD, while BD is equal to BK, that is CG, and CB is equal to GK, that is GQ,

therefore CG is also equal to GQ.

And, since CG is equal to GQ, and QR to RP,

AG is also equal to MQ, and QL to RF. {I. 36}

But MQ is equal to QL, for they are complements of the parallelogram ML; {I. 43}

therefore AG is also equal to RF;

therefore the four areas AG, MQ, QL, RF are equal to one another.

Therefore the four are quadruple of AG.

But the four areas CK, KD, GR, RN were proved to be quadruple of CK;

therefore the eight areas, which contain the gnomon STU, are quadruple of AK.

Now, since AK is the rectangle AB, BD, for BK is equal to BD, therefore four times the rectangle AB, BD is quadruple of AK.

But the gnomon STU was also proved to be quadruple of AK;

therefore four times the rectangle AB, BD is equal to the gnomon STU.

Let OH, which is equal to the square on AC, be added to each;

therefore four times the rectangle AB, BD together with the square on AC is equal to the gnomon STU and OH.

But the gnomon STU and OH are the whole square AEFD,

which is described on AD.

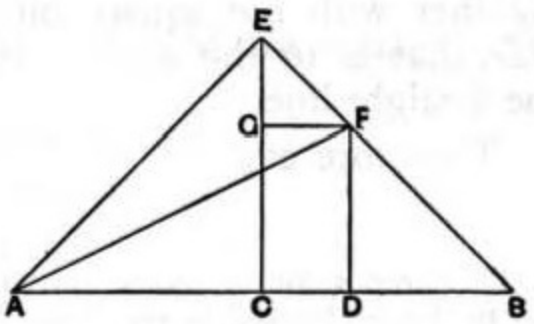
therefore four times the rectangle AB, BD together with the square on AC is equal to the square on AD

But BD is equal to BC; therefore four times the rectangle contained by AB, BC together with the square on AC is equal to the square on AD, that is to the square described on AB and BC as on one straight line.

Therefore etc. Q. E. D.

Proposition 9.

If a straight line be cut into equal and unequal segments, the squares on the unequal segments of the whole are double of the square on the half and of the square on the straight line between the points of section.



For let a straight line AB be cut into equal segments at C, and into unequal segments at D;

I say that the squares on AD, DB are double of the squares on AC, CD.

For let CE be drawn from C at right angles to AB, and let it be made equal to either AC or CB; let EA, EB be joined, let DF be drawn through D parallel to EC, and FG through F parallel to AB, and let AF be joined.

Then, since AC is equal to CE,

the angle EAC is also equal to the angle AEC.

And, since the angle at C is right,

the remaining angles EAC, AEC are equal to one right angle. {I. 32}

And they are equal;

therefore each of the angles CEA, CAE is half a right angle.

For the same reason

each of the angles CEB, EBC is also half a right angle; therefore the whole angle AEB is right.

And, since the angle GEF is half a right angle, and the angle EGF is right, for it is equal to the interior and opposite angle ECB, {I. 29}

the remaining angle EFG is half a right angle; {I. 32} therefore the angle GEF is equal to the angle EFG, so that the side EG is also equal to GF. {I. 6}

Again, since the angle at B is half a right angle, and the angle FDB is right, for it is again equal to the interior and opposite angle ECB, {I. 29}

the remaining angle BFD is half a right angle; {I. 32} therefore the angle at B is equal to the angle DFB, so that the side FD is also equal to the side DB. {I. 6}

Now, since AC is equal to CE,

the square on AC is also equal to the square on CE; therefore the squares on AC, CE are double of the square on AC.

But the square on EA is equal to the squares on AC, CE, for the angle ACE is right; {I. 47}

therefore the square on EA is double of the square on AC.

Again, since EG is equal to GF,

the square on EG is also equal to the square on GF; therefore the squares on EG, GF are double of the square on GF.

But the square on EF is equal to the squares on EG, GF;

therefore the square on EF is double of the square on GF.

But GF is equal to CD; {I. 34} therefore the square on EF is double of the square on CD.

But the square on EA is also double of the square on AC;

therefore the squares on AE, EF are double of the squares on AC, CD.

And the square on AF is equal to the squares on AE, EF, for the angle AEF is right; {I. 47} therefore the square on AF is double of the squares on AC, CD.

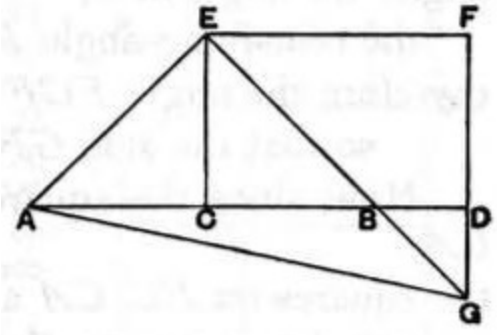
But the squares on AD, DF are equal to the square on AF, for the angle at D is right; {I. 47} therefore the squares on AD, DF are double of the squares on AC, CD.

And DF is equal to DB; therefore the squares on AD, DB are double of the squares on AC, CD.

Therefore etc. Q. E. D.

Proposition 10.

If a straight line be bisected, and a straight line be added to it in a straight line, the square on the whole with the added straight line and the square on the added straight line both together are double of the square on the half and of the square described on the straight line made up of the half and the added straight line as on one straight line.



For let a straight line AB be bisected at C, and let a straight line BD be added to it in a straight line;

I say that the squares on AD, DB are double of the squares on AC, CD.

For let CE be drawn from the point C at right angles to AB {I. 11}, and let it be made equal to either AC or CB {I. 3}; let EA, EB be joined; through E let EF be drawn parallel to AD, and through D let FD be drawn parallel to CE. {I. 31}

Then, since a straight line EF falls on the parallel straight lines EC, FD,

the angles CEF, EFD are equal to two right angles; {I. 29} therefore the angles FEB, EFD are less than two right angles.

But straight lines produced from angles less than two right angles meet; {I. Post. 5}

therefore EB, FD, if produced in the direction B, D, will meet.

Let them be produced and meet at G, and let AG be joined.

Then, since AC is equal to CE, the angle EAC is also equal to the angle AEC; {I. 5} and the angle at C is right;

therefore each of the angles EAC, AEC is half a right angle. {I. 32}

For the same reason

each of the angles CEB, EBC is also half a right angle; therefore the angle AEB is right.

And, since the angle EBC is half a right angle, the angle DBG is also half a right angle. {I. 15}

But the angle BDG is also right,

for it is equal to the angle DCE, they being alternate; {I. 29}

therefore the remaining angle DGB is half a right angle; {I. 32}

therefore the angle DGB is equal to the angle DBG,

so that the side BD is also equal to the side GD. {I. 6}

Again, since the angle EGF is half a right angle, and the angle at F is right, for it is equal to the opposite angle, the angle at C, {I. 34}

the remaining angle FEG is half a right angle; {I. 32} therefore the angle EGF is equal to the angle FEG, so that the side GF is also equal to the side EF. {I. 6}

Now, since the square on EC is equal to the square on CA, the squares on EC, CA are double of the square on CA.

But the square on EA is equal to the squares on EC, CA; {I. 47} therefore the square on EA is double of the square on AC. {C. N. 1}

Again, since FG is equal to EF, the square on FG is also equal to the square on FE; therefore the squares on GF, FE are double of the square on EF.

But the square on EG is equal to the squares on GF, FE; {I. 47} therefore the square on EG is double of the square on EF.

And EF is equal to CD; {I. 34}

therefore the square on EG is double of the square on CD.

But the square on EA was also proved double of the square on AC; therefore the squares on AE, EG are double of the squares on AC, CD.

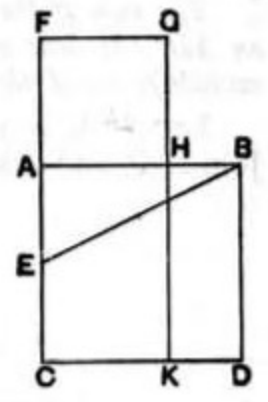
And the square on AG is equal to the squares on AE, EG; {I. 47} therefore the square on AG is double of the squares on AC, CD. But the squares on AD, DG are equal to the square on AG; {I. 47} therefore the squares on AD, DG are double of the squares on AC, CD.

And DG is equal to DB; therefore the squares on AD, DB are double of the squares on AC, CD.

Therefore etc. Q. E. D.

Proposition 11.

To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.



Let AB be the given straight line; thus it is required to cut AB so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

For let the square ABDC be described on AB; {I. 46} let AC be bisected at the point E, and let BE be joined; let CA be drawn through to F, and let EF be made equal to BE; let the square FH be described on AF, and let GH be drawn through to K.

I say that AB has been cut at H so as to make the rectangle contained by AB, BH equal to the square on AH.

For, since the straight line AC has been bisected at E, and FA is added to it,

the rectangle contained by CF, FA together with the square on AE is equal to the square on EF. {II. 6}

But EF is equal to EB;

therefore the rectangle CF, FA together with the square on AE is equal to the square on EB.

But the squares on BA, AE are equal to the square on EB, for the angle at A is right; {I. 47}

therefore the rectangle CF, FA together with the square on AE is equal to the squares on BA, AE.

Let the square on AE be subtracted from each;

therefore the rectangle CF, FA which remains is equal to the square on AB.

Now the rectangle CF, FA is FK, for AF is equal to FG; and the square on AB is AD;

therefore FK is equal to AD.

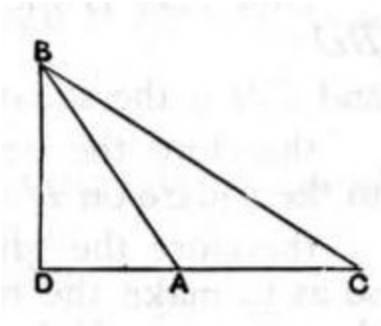
Let AK be subtracted from each;

therefore FH which remains is equal to HD.

And HD is the rectangle AB, BH, for AB is equal to BD; and FH is the square on AH;

therefore the rectangle contained by AB, BH is equal to the square on HA.
therefore the given straight line AB has been cut at H so as to make the rectangle contained by AB, BH equal to the square on HA. Q. E. F.

Proposition 12.



In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle,

namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle.

Let ABC be an obtuse-angled triangle having the angle BAC obtuse, and let BD be drawn from the point B perpendicular to CA produced;

I say that the square on BC is greater than the squares on BA, AC by twice the rectangle contained by CA, AD.

For, since the straight line CD has been cut at random at the point A, the square on DC is equal to the squares on CA, AD and twice the rectangle contained by CA, AD. {II. 4}

Let the square on DB be added to each; therefore the squares on CD, DB are equal to the squares on CA, AD, DB and twice the rectangle CA, AD.

But the square on CB is equal to the squares on CD, DB, for the angle at D is right; {I. 47}

and the square on AB is equal to the squares on AD, DB; {I. 47}

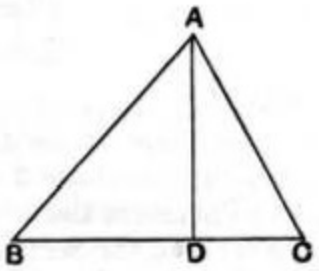
therefore the square on CB is equal to the squares on CA, AB and twice the rectangle contained by CA, AD;

so that the square on CB is greater than the squares on CA, AB by twice the rectangle contained by CA, AD.

Therefore etc. Q. E. D.

Proposition 13.

In acute-angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acutc angle.



Let ABC be an acute-angled triangle having the angle at B acute, and let AD be drawn from the point A perpendicular to BC;

I say that the square on AC is less than the squares on CB, BA by twice the rectangle contained by CB, BD.

For, since the straight line CB has been cut at random at D,

the squares on CB, BD are equal to twice the rectangle contained by CB, BD and the square on DC. {II. 7}

Let the square on DA be added to each; therefore the squares on CB, BD, DA are equal to twice the rectangle contained by CB, BD and the squares on AD, DC.

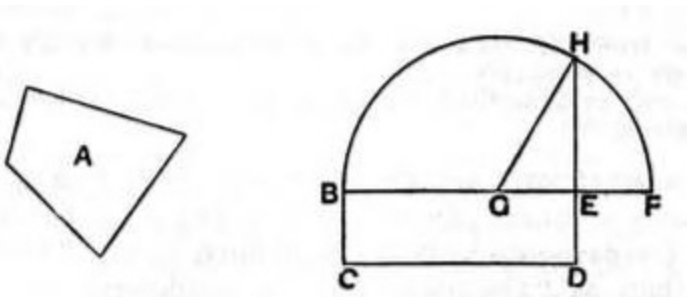
But the square on AB is equal to the squares on BD, DA, for the angle at D is right; {I. 47} and the square on AC is equal to the squares on AD, DC; therefore the squares on CB, BA are equal to the square on AC and twice the rectangle CB, BD,

so that the square on AC alone is less than the squares on CB, BA by twice the rectangle contained by CB, BD.

Therefore etc. Q. E. D.

Proposition 14.

To construct a square equal to a given rectilineal figure.



Let A be the given rectilineal figure; thus it is required to construct a square equal to the rectilineal figure A.

For let there be constructed the rectangular parallelogram BD equal to the rectilineal figure A. {I. 45}

Then, if BE is equal to ED, that which was enjoined will have been done; for a square BD has been constructed equal to the rectilineal figure A.

But, if not, one of the straight lines BE, ED is greater.

Let BE be greater, and let it be produced to F; let EF be made equal to ED, and let BF be bisected at G.

With centre G and distance one of the straight lines GB, GF let the semicircle BHF be described; let DE be produced
to H, and let GH be joined.

Then, since the straight line BF has been cut into equal segments at G, and into unequal segments at E,

the rectangle contained by BE, EF together with the square on EG is equal to the square on GF. {II. 5}

But GF is equal to GH; therefore the rectangle BE, EF together with the square on GE is equal to the square on GH.

But the squares on HE, EG are equal to the square on GH; {I. 47}
therefore the rectangle BE, EF together with the square on GE is equal to the squares on HE, EG.

Let the square on GE be subtracted from each;

therefore the rectangle contained by BE, EF which remains is equal to the square on EH.

But the rectangle BE, EF is BD, for EF is equal to ED;

therefore the parallelogram BD is equal to the square on HE.

And BD is equal to the rectilineal figure A.

Therefore the rectilineal figure A is also equal to the square which can be described on EH.

Therefore a square, namely that which can be described on EH, has been constructed equal to the given rectilineal figure A. Q. E. F. [6] [7]

ENDNOTES.

[1] the rectangle A, BC. From this point onward I shall translate thus in cases where Euclid leaves out the word contained (περιεχόμενον). Though the word "rectangle" is also omitted in the Greek (the neuter article being sufficient to show that the rectangle is meant), it cannot be dispensed with in English. De Morgan advises the use of the expression "the rectangle under two lines." This does not seem to me a very good expression, and, if used in a translation from the Greek, it might suggest that ὑπό in τὸ ὑπό meant under, which it does not.

[2] twice the rectangle contained by the segments. By a curious idiom this is in Greek "the rectangle twice contained by the segments." Similarly "twice the rectangle contained by AC, CB" is expressed as "the rectangle twice contained by AC, CB" (τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ περιεχόμενον ὀρθογ<*>ώνιον).

[3] described. 39, 45. the squares (before "on"). These words are not in the Greek, which simply says that the squares "are on" (εἰσίν ἀπό) their respective sides.

[4] areas. It is necessary to supply some substantive (the Greek leaves it to be understood); and I prefer "areas" to "figures."

[5] between the points of section, literally “between the sections,” the word being the same (τομή) as that used of a conic section.

It will be observed that the gnomon is indicated in the figure by three separate letters and a dotted curve. This is no doubt a clearer way of showing what exactly the gnomon is than the method usual in our text-books. In this particular case the figure of the MSS. has two M’s in it, the gnomon being MNΞ. I have corrected the lettering to avoid confusion.

[6] that which was enjoined will have been done, literally “would have been done,” γεγονὸς ἂν εἴη τὸ ἐπιταχθέν.

[7] which can be described, expressed by the future passive participle, ἀναγραφησομένῳ, ἀναγραφησόμενον.

BOOK III. DEFINITIONS.

1

Equal circles are those the diameters of which are equal, or the radii of which are equal.

2

A straight line is said to touch a circle which, meeting the circle and being produced, does not cut the circle.

3

Circles are said to touch one another which, meeting one another, do not cut one another.

4

In a circle straight lines are said to be equally distant from the centre when the perpendiculars drawn to them from the centre are equal.

5

And that straight line is said to be at a greater distance on which the greater perpendicular falls.

6

A segment of a circle is the figure contained by a straight line and a circumference of a circle.

7

An angle of a segment is that contained by a straight line and a circumference of a circle.

8

An angle in a segment is the angle which, when a point is taken on the circumference of the segment and straight lines are joined from it to the extremities of the straight line which is the base of the segment, is contained by the straight lines so joined.

9

And, when the straight lines containing the angle cut off a circumference, the angle is said to stand upon that circumference.

10

A sector of a circle is the figure which, when an angle is constructed at the centre of the circle, is contained by the straight lines containing the angle and the circumference cut off by them.

11

Similar segments of circles are those which admit equal angles, or in which the angles are equal to one another.

PROPOSITIONS.

PROPOSITION 1.

To find the centre of a given circle.

Let ABC be the given circle; thus it is required to find the centre of the circle ABC.

Let a straight line AB be drawn
through it at random, and let it be bisected at the point D; from D let DC be drawn at right angles to AB and let it be drawn through to E; let CE be bisected at F;
I say that F is the centre of the circle ABC.

For suppose it is not, but, if possible, let G be the centre, and let GA, GD, GB be joined.

Then, since AD is equal to DB, and DG is common,

the two sides AD, DG are equal to the two sides BD, DG respectively;

and the base GA is equal to the base GB, for they are
radii;

therefore the angle ADG is equal to the angle GDB. {I. 8}

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; {I. Def. 10}

therefore the angle GDB is right.

But the angle FDB is also right; therefore the angle FDB is equal to the angle GDB, the greater to the less: which is impossible.

Therefore G is not the centre of the circle ABC.

Similarly we can prove that neither is any other point except F.

Therefore the point F is the centre of the circle ABC.

PORISM.

From this it is manifest that, if in a circle a straight line cut a straight line into two equal parts and at
right angles, the centre of the circle is on the cutting straight line. Q. E. F. [1] [2]

PROPOSITION 2.

If on the circumference of a circle two points be taken at random, the straight line joining the points will fall within the circle.

Let ABC be a circle, and let two points A, B be taken at random on its circumference; I say that the straight line joined from A to B will fall within the circle.

For suppose it does not, but, if possible, let it fall outside, as AEB; let the centre of the circle ABC be taken {III. 1}, and let it be D; let DA, DB be joined, and let DFE be drawn through.

Then, since DA is equal to DB,

the angle DAE is also equal to the angle DBE. {I. 5}

And, since one side AEB of the triangle DAE is produced,

the angle DEB is greater than the angle DAE. {I. 16}

But the angle DAE is equal to the angle DBE;

therefore the angle DEB is greater than the angle DBE.

And the greater angle is subtended by the greater side; {I. 19}

therefore DB is greater than DE. But DB is equal to DF;

therefore DF is greater than DE,

the less than the greater : which is impossible.

Therefore the straight line joined from A to B will not fall outside the circle.

Similarly we can prove that neither will it fall on the circumference itself;

therefore it will fall within.

Therefore etc. Q. E. D.

PROPOSITION 3.

If in a circle a straight line through the centre bisect a straight line not through the centre, it also cuts it at right angles; and if it cut it at right angles, it also bisects it.

Let ABC be a circle, and in it let a straight line CD
through the centre bisect a straight line AB not through the centre at the point F; I say that it also cuts it at right angles.

For let the centre of the circle ABC
be taken, and let it be E; let EA, EB be joined.

Then, since AF is equal to FB, and FE is common,

two sides are equal to two sides;

and the base EA is equal to the base EB; therefore the angle AFE is equal to the angle BFE. {I. 8}

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; {I. Def. 10}

therefore each of the angles AFE, BFE is right.

Therefore CD, which is through the centre, and bisects AB which is not through the centre, also cuts it at right angles.

Again, let CD cut AB at right angles;
I say that it also bisects it. that is, that AF is equal to FB.

For, with the same construction,

since EA is equal to EB,

the angle EAF is also equal to the angle EBF. {I. 5}

But the right angle AFE is equal to the right angle BFE,
therefore EAF, EBF are two triangles having two angles equal to two angles and one side equal to one side, namely EF, which is common to them, and subtends one of the equal angles;

therefore they will also have the remaining sides equal to the remaining sides; {I. 26} therefore AF is equal to FB.

Therefore etc. Q. E. D. [3]

PROPOSITION 4.

If in a circle two straight lines cut one another which are not through the centre, they do not bisect one another.

Let ABCD be a circle, and in it let the two straight lines AC, BD, which are not through the centre, cut one another at E; I say that they do not bisect

one another.

For, if possible, let them bisect one another, so that AE is equal to EC, and BE to ED; let the centre of the circle ABCD be taken {III. 1}, and let it be F; let FE be joined.

Then, since a straight line FE through the centre bisects a straight line AC not through the centre,

it also cuts it at right angles; {III. 3} therefore the angle FEA is right.

Again, since a straight line FE bisects a straight line BD,

it also cuts it at right angles; {III. 3} therefore the angle FEB is right.

But the angle FEA was also proved right;

therefore the angle FEA is equal to the angle FEB, the less to the greater: which is impossible.

Therefore AC, BD do not bisect one another.

Therefore etc. Q. E. D.

PROPOSITION 5.

If two circles cut one another, they will not have the same centre.

For let the circles ABC, CDG cut one another at the points B, C; I say that they will not have the same centre.

For, if possible, let it be E; let EC be joined, and let EFG be drawn through at random.

Then, since the point E is the centre of the circle ABC,

EC is equal to EF. {I. Def. 15}

Again, since the point E is the centre of the circle CDG,

EC is equal to EG.

But EC was proved equal to EF also;

therefore EF is also equal to EG, the less to the greater : which is impossible.

Therefore the point E is not the centre of the circles ABC, CDG.

Therefore etc. Q. E. D.

PROPOSITION 6.

If two circles touch one another, they will not have the same centre.

For let the two circles ABC, CDE touch one another at the point C; I say that they will not have the same centre.

For, if possible, let it be F; let FC be joined, and let FEB be drawn through at random.

Then, since the point F is the centre of the circle ABC,

FC is equal to FB.

Again, since the point F is the centre of the circle CDE,

FC is equal to FE.

But FC was proved equal to FB;

therefore FE is also equal to FB, the less to the greater: which is impossible.

Therefore F is not the centre of the circles ABC, CDE.

Therefore etc. Q. E. D.

PROPOSITION 7.

If on the diameter of a circle a point be taken which is not the centre of the circle, and from the point straight lines fall upon the circle, that will be greatest on which the centre is, the remainder of the same diameter will be least, and of the rest
the nearer to the straight line through the centre is always greater than the more remote, and only two equal straight lines will fall from the point on the circle, one on each side of the least straight line.

Let ABCD be a circle, and let AD be a diameter of it;
on AD let a point F be taken which is not the centre of the circle, let E be the centre of the circle, and from F let straight lines FB, FC, FG fall upon the circle ABCD; I say that FA is greatest, FD is least, and of the rest FB is greater than FC, and FC than FG.

For let BE, CE, GE be joined.

Then, since in any triangle two sides are greater than the remaining one, {I. 20}

EB, EF are greater than BF.

But AE is equal to BE;

therefore AF is greater than BF.

Again, since BE is equal to CE, and FE is common,
the two sides BE, EF are equal to the two sides CE, EF.

But the angle BEF is also greater than the angle CEF; therefore the base BF is greater than the base CF. {I. 24}

For the same reason

CF is also greater than FG.

Again, since GF, FE are greater than EG, and EG is equal to ED,

GF, FE are greater than ED.

Let EF be subtracted from each;

therefore the remainder GF is greater than the remainder FD.

Therefore FA is greatest, FD is least, and FB is greater than FC, and FC than FG.

I say also that from the point F only two equal straight lines will fall on the circle ABCD, one on each side of the
least FD.

For on the straight line EF, and at the point E on it, let the angle FEH be constructed equal to the angle GEF {I. 23}, and let FH be joined.

Then, since GE is equal to EH,
and EF is common,

the two sides GE, EF are equal to the two sides HE, EF; and the angle GEF is equal to the angle HEF;

therefore the base FG is equal to the base FH. {I. 4}

I say again that another straight line equal to FG will not
fall on the circle from the point F.

For, if possible, let FK so fall.
Then, since FK is equal to FG, and FH to FG,

FK is also equal to FH, the nearer to the straight line through the centre being thus equal to the more remote: which is impossible.

Therefore another straight line equal to GF will not fall from the point F upon the circle;

therefore only one straight line will so fall.

Therefore etc. Q. E. D. [4] [5]

PROPOSITION 8.

If a point be taken outside a circle and from the point straight lines be drawn through to the circle, one of which is through the centre and the others are drawn at random, then, of the straight lines which fall on the concave circumference, that through the centre is greatest, while of the rest the nearer to that through the centre is always greater than the more remote, but, of the straight lines falling on the convex circumference, that between the point and the diameter is least, while of the rest the nearer to the least is always less than the more remote, and only two equal straight lines will fall on the circle from the point, one on each side of the least.

Let ABC be a circle, and let a point D be taken outside ABC; let there be drawn through from it straight lines DA, DE, DF, DC, and let DA be through the centre; I say that, of the straight lines falling on the concave circumference AEFC, the straight line DA through the centre is greatest, while DE is greater than DF and DF than DC.; but, of the straight lines falling on the convex circumference HLKG, the straight line DG between the point and the diameter AG is least; and the nearer to the least DG is always less than the more remote, namely DK than DL, and DL than DH.

For let the centre of the circle ABC be taken {III. 1}, and let it be M; let ME, MF, MC, MK, ML, MH be joined.

Then, since AM is equal to EM, let MD be added to each;

therefore AD is equal to EM, MD.

But EM, MD are greater than ED; {I. 20}

therefore AD is also greater than ED.

Again, since ME is equal to MF,

and MD is common,

therefore EM, MD are equal to FM, MD;

and the angle EMD is greater than the angle FMD;

therefore the base ED is greater than the base FD. {I. 24}

Similarly we can prove that FD is greater than CD; therefore DA is greatest, while DE is greater than DF, and DF than DC.

Next, since MK, KD are greater than MD, {I. 20} and MG is equal to MK, therefore the remainder KD is greater than the remainder GD,

so that GD is less than KD.

And, since on MD, one of the sides of the triangle MLD, two straight lines MK, KD were constructed meeting within the triangle, therefore MK, KD are less than ML, LD; {I. 21} and MK is equal to ML;

therefore the remainder DK is less than the remainder DL.

Similarly we can prove that DL is also less than DH;

therefore DG is least, while DK is less than DL, and DL than DH.

I say also that only two equal straight lines will fall from the point D on the circle, one on each side of the least DG.

On the straight line MD, and at the point M on it, let the angle DMB be constructed equal to the angle KMD, and let DB be joined.

Then, since MK is equal to MB, and MD is common,

the two sides KM, MD are equal to the two sides BM, MD respectively;

and the angle KMD is equal to the angle BMD;

therefore the base DK is equal to the base DB. {I. 4}

I say that no other straight line equal to the straight line DK will fall on the circle from the point D.

For, if possible, let a straight line so fall, and let it be DN.

Then, since DK is equal to DN,

while DK is equal to DB,

DB is also equal to DN,

that is, the nearer to the least DG equal to the more remote: which was proved impossible.

Therefore no more than two equal straight lines will fall on the circle ABC from the point D, one on each side of DG the least.

Therefore etc.

PROPOSITION 9.

If a point be taken within a circle, and more than two equal straight lines fall from the point on the circle, the point taken is the centre of the circle.

Let ABC be a circle and D a point within it, and from D let more than two equal straight lines, namely DA, DB, DC, fall on the circle ABC; I say that the point D is the centre of the circle ABC.

For let AB, BC be joined and bisected at the points E, F, and let ED, FD be joined and drawn through to the points G, K, H, L.

Then, since AE is equal to EB, and ED is common,

the two sides AE, ED are equal to the two sides BE, ED;

and the base DA is equal to the base DB;

therefore the angle AED is equal to the angle BED. {I. 8}

Therefore each of the angles AED, BED is right; {I. Def. 10} therefore GK cuts AB into two equal parts and at right angles.

And since, if in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line, {III. 1, Por.}

the centre of the circle is on GK.

For the same reason

the centre of the circle ABC is also on HL.

And the straight lines GK, HL have no other point common but the point D;

therefore the point D is the centre of the circle ABC.

Therefore etc. Q. E. D.

PROPOSITION 10.

A circle does not cut a circle at more points than two.

For, if possible, let the circle ABC cut the circle DEF at more points than two, namely B, C, F, H;

let BH, BG be joined and bisected at the points K, L, and from K, L let KC, LM be drawn at right angles to BH, BG and carried through to the points A, E.

Then, since in the circle ABC a straight line AC cuts a straight line BH into two equal parts and at right angles,

the centre of the circle ABC is on AC. {III. 1, Por.}

Again, since in the same circle ABC a straight line NO cuts a straight line BG into two equal parts and at right angles,

the centre of the circle ABC is on NO.

But it was also proved to be on AC, and the straight lines AC, NO meet at no point except at P;

therefore the point P is the centre of the circle ABC.

Similarly we can prove that P is also the centre of the circle DEF;

therefore the two circles ABC, DEF which cut one another have the same centre P: which is impossible. {III. 5}

Therefore etc. Q. E. D. [6]

PROPOSITION 11.

If two circles touch one another internally, and their centres be taken, the straight line joining their centres, if it be also produced, will fall on the point of contact of the circles.

For let the two circles ABC, ADE touch one another internally at the point A, and let the centre F of the circle ABC, and the centre G of ADE, be taken; I say that the straight line joined from G to F and produced will fall on A.

For suppose it does not, but, if possible, let it fall as FGH, and let AF, AG be joined.

Then, since AG, GF are greater than FA, that is, than FH,

let FG be subtracted from each; therefore the remainder AG is greater than the remainder GH.

But AG is equal to GD;

therefore GD is also greater than GH, the less than the greater: which is impossible.

Therefore the straight line joined from F to G will not fall outside;

therefore it will fall at A on the point of contact.

Therefore etc. Q. E. D. [7] [8]

PROPOSITION 12.

If two circles touch one another externally, the straight line joining their centres will pass through the point of contact.

For let the two circles ABC, ADE touch one another externally at the point A, and let the centre F of ABC, and the centre G of ADE, be taken; I say that the straight line joined from F to G will pass through the point of contact at A.

For suppose it does not,

but, if possible, let it pass as FCDG, and let AF, AG be joined.

Then, since the point F is the centre of the circle ABC,

FA is equal to FC.

Again, since the point G is the centre of the circle ADE,

GA is equal to GD.

But FA was also proved equal to FC;

therefore FA, AG are equal to FC, GD, so that the whole FG is greater than FA, AG;

but it is also less {I. 20}: which is impossible.

Therefore the straight line joined from F to G will not fail to pass through the point of contact at A;

therefore it will pass through it.

Therefore etc. Q. E. D.} [9]

PROPOSITION 13.

A circle does not touch a circle at more points than one, whether it touch it internally or externally.

For, if possible, let the circle ABDC touch the circle EBFD, first internally, at more
points than one, namely D, B.

Let the centre G of the circle ABDC, and the centre H of EBFD, be taken.

Therefore the straight line
joined from G to H will fall on B, D. {III. 11}

Let it so fall, as BGHD.

Then, since the point G is the centre of the circle ABCD,

BG is equal to GD;

therefore BG is greater than HD; therefore BH is much greater than HD.

Again, since the point H is the centre of the circle EBFD,

BH is equal to HD;

but it was also proved much greater than it: which is impossible.

Therefore a circle does not touch a circle internally at more points than one.

I say further that neither does it so touch it externally.

For, if possible, let the circle ACK touch the circle ABDC at more points than one, namely A, C, and let AC be joined.

Then, since on the circumference of each of the circles ABDC, ACK two points A, C have been taken at random, the straight line joining the points will fall within each circle; {III. 2}

but it fell within the circle ABCD and outside ACK {III. Def. 3}: which is absurd.

Therefore a circle does not touch a circle externally at more points than one.

And it was proved that neither does it so touch it internally.

Therefore etc. Q. E. D. [10] [11]

PROPOSITION 14.

In a circle equal straight lines are equally distant from the centre, and those which are equally distant from the centre are equal to one another.

Let ABDC be a circle, and let AB, CD be equal straight lines in it; I say that AB, CD are equally distant from the centre.

For let the centre of the circle ABDC be taken {III. 1}, and let it be E; from E let EF, EG be drawn perpendicular to AB, CD, and let AE, EC be joined.

Then, since a straight line EF through the centre cuts a straight line AB not through the centre at right angles, it also bisects it. {III. 3}

Therefore AF is equal to FB; therefore AB is double of AF.

For the same reason

CD is also double of CG;

and AB is equal to CD;

therefore AF is also equal to CG.

And, since AE is equal to EC,

the square on AE is also equal to the square on EC.

But the squares on AF, EF are equal to the square on AE, for the angle at F is right; and the squares on EG, GC are equal to the square on EC, for the angle at G is right; {I. 47}

therefore the squares on AF, FE are equal to the squares on CG, GE,

of which the square on AF is equal to the square on CG, for AF is equal to CG;

therefore the square on FE which remains is equal to the square on EG,
therefore EF is equal to EG.

But in a circle straight lines are said to be equally distant from the centre when the perpendiculars drawn to them from the centre are equal; {III. Def.

4}

therefore AB, CD are equally distant from the centre.

Next, let the straight lines AB, CD be equally distant from the centre; that is, let EF be equal to EG.

I say that AB is also equal to CD.

For, with the same construction, we can prove, similarly, that AB is double of AF, and CD of CG.

And, since AE is equal to CE,

the square on AE is equal to the square on CE.

But the squares on EF, FA are equal to the square on AE, and the squares on EG, GC equal to the square on CE. {I. 47}

Therefore the squares on EF, FA are equal to the squares on EG, GC, of which the square on EF is equal to the square on EG, for EF is equal to EG; therefore the square on AF which remains is equal to the square on CG;

therefore AF is equal to CG.

And AB is double of AF, and CD double of CG;

therefore AB is equal to CD.

Therefore etc. Q. E. D.

PROPOSITION 15.

Of straight lines in a circle the diameter is greatest, and of the rest the nearer to the centre is always greater than the more remote.

Let ABCD be a circle, let AD be its diameter and E the centre; and let BC be nearer to the diameter AD, and FG more remote; I say that AD is greatest

and BC greater than FG.

For from the centre E let EH, EK be drawn perpendicular to BC, FG.

Then, since BC is nearer to the centre and FG more remote, EK is greater than EH. {III. Def. 5}

Let EL be made equal to EH, through L let LM be drawn at right angles to EK and carried through to N, and let ME, EN, FE, EG be joined.

Then, since EH is equal to EL,

BC is also equal to MN. {III. 14}

Again, since AE is equal to EM, and ED to EN,

AD is equal to ME, EN.

But ME, EN are greater than MN, {I. 20} and MN is equal to BC;

therefore AD is greater than BC.

And, since the two sides ME, EN are equal to the two sides FE, EG, and the angle MEN greater than the angle FEG, therefore the base MN is greater than the base FG. {I. 24}

But MN was proved equal to BC.

Therefore the diameter AD is greatest and BC greater than FG.

Therefore etc. Q. E. D. [12] [13]

PROPOSITION 16.

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.

Let ABC be a circle about D as centre and AB as diameter; I say that the straight line drawn from A at right angles to AB from its extremity will fall outside the circle.

For suppose it does not, but, if possible, let it fall within as CA, and let DC be joined.

Since DA is equal to DC,

the angle DAC is also equal to the angle ACD. {I. 5}

But the angle DAC is right;

therefore the angle ACD is also right:

thus, in the triangle ACD, the two angles DAC, ACD are equal to two right angles: which is impossible. {I. 17}

Therefore the straight line drawn from the point A at right angles to BA will not fall within the circle.

Similarly we can prove that neither will it fall on the circumference;

therefore it will fall outside.

Let it fall as AE; I say next that into the space between the straight line AE and the circumference CHA another straight line cannot be interposed.

For, if possible, let another straight line be so interposed, as FA, and let DG be drawn from the point D perpendicular to FA.

Then, since the angle AGD is right,

and the angle DAG is less than a right angle, AD is greater than DG. {I. 19}

But DA is equal to DH;

therefore DH is greater than DG, the less than the greater: which is impossible.

Therefore another straight line cannot be interposed into the space between the straight line and the circumference.

I say further that the angle of the semicircle contained by the straight line BA and the circumference CHA is greater than any acute rectilineal angle, and the remaining angle contained by the circumference CHA and the straight line AE is less than any acute rectilineal angle.

For, if there is any rectilineal angle greater than the angle contained by the straight line BA and the circumference CHA, and any rectilineal angle less than the angle contained by the circumference CHA and the straight line AE, then into the space between the circumference and the straight line AE a straight line will be interposed such as will make an angle contained by straight lines which is greater than the angle contained by the straight line BA and the circumference CHA, and another angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line AE.

But such a straight line cannot be interposed;

therefore there will not be any acute angle contained by straight lines which is greater than the angle contained by the straight line BA and the circumference CHA, nor yet any acute angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line AE. —

PORISM.

From this it is manifest that the straight line drawn at right angles to the diameter of a circle from its extremity touches the circle. Q. E. D. [14]

PROPOSITION 17.

From a given point to draw a straight line touching a given circle.

Let A be the given point, and BCD the given circle; thus it is required to draw from the point A a straight line touching the circle BCD.

For let the centre E of the circle be taken; {III. 1} let AE be joined, and with centre E and distance EA let the circle AFG be described; from D let DF be drawn at right angles to EA, and let EF, AB be joined; I say that AB has been drawn from the point A touching the circle BCD.

For, since E is the centre of the circles BCD, AFG,

EA is equal to EF, and ED to EB;

therefore the two sides AE, EB are equal to the two sides FE, ED: and they contain a common angle, the angle at E;

therefore the base DF is equal to the base AB, and the triangle DEF is equal to the triangle BEA, and the remaining angles to the remaining angles; {I. 4} therefore the angle EDF is equal to the angle EBA.

But the angle EDF is right;

therefore the angle EBA is also right.

Now EB is a radius; and the straight line drawn at right angles to the diameter of a circle, from its extremity, touches the circle; {III. 16, Por.}

therefore AB touches the circle BCD.

Therefore from the given point A the straight line AB has been drawn touching the circle BCD.

PROPOSITION 18.

If a straight line touch a circle, and a straight line be joined from the centre to the point of contact, the straight line so joined will be perpendicular to the tangent.

For let a straight line DE touch the circle ABC at the point C, let the centre F of the circle ABC be taken, and let FC be joined from F to C; I say that FC is perpendicular to DE.

For, if not, let FG be drawn from F perpendicular to DE.

Then, since the angle FGC is right,

the angle FCG is acute; {I. 17}

and the greater angle is subtended by the greater side; {I. 19}

therefore FC is greater than FG.

But FC is equal to FB;

therefore FB is also greater than FG, the less than the greater: which is impossible.

Therefore FG is not perpendicular to DE.

Similarly we can prove that neither is any other straight line except FC;

therefore FC is perpendicular to DE.

Therefore etc. Q. E. D. [15]

PROPOSITION 19.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the tangent, the centre of the circle will be on the straight line so drawn.

For let a straight line DE touch the circle ABC at the point C, and from C let CA be drawn at right angles to DE; I say that the centre of the circle is on AC.

For suppose it is not, but, if possible, let F be the centre, and let CF be joined.

Since a straight line DE touches the circle ABC, and FC has been joined from the centre to the point of contact,

FC is perpendicular to DE; {III. 18} therefore the angle FCE is right.

But the angle ACE is also right;

therefore the angle FCE is equal to the angle ACE, the less to the greater: which is impossible.

Therefore F is not the centre of the circle ABC.

Similarly we can prove that neither is any other point except a point on AC.

Therefore etc. Q. E. D.

PROPOSITION 20.

In a circle the angle at the centre is double of the angle at the circumference, when the angles have the same circumference as base.

Let ABC be a circle, let the angle BEC be an angle
at its centre, and the angle BAC an angle at the circumference, and let them have the same circumference BC as base; I say that the angle BEC is double of
the angle BAC.

For let AE be joined and drawn through to F.

Then, since EA is equal to EB,

the angle EAB is also equal to the angle EBA; {I. 5}

therefore the angles EAB, EBA are double of the angle EAB.

But the angle BEF is equal to the angles EAB, EBA; {I. 32} therefore the angle BEF is also double of the angle
EAB.

For the same reason

the angle FEC is also double of the angle EAC.

Therefore the whole angle BEC is double of the whole angle BAC.

Again let another straight line be inflected, and let there be another angle BDC; let DE be joined and produced to G.

Similarly then we can prove that the angle GEC is double of the angle EDC,

of which the angle GEB is double of the angle EDB; therefore the angle BEC which remains is double of the angle BDC.

Therefore etc. Q. E. D. [16]

PROPOSITION 21.

In a circle the angles in the same segment are equal to one another.

Let ABCD be a circle, and let the angles BAD, BED be angles in the same segment BAED; I say that the angles BAD, BED are equal to one another.

For let the centre of the circle ABCD be taken, and let it be F; let BF, FD be joined.

Now, since the angle BFD is at the centre,

and the angle BAD at the circumference, and they have the same circumference BCD as base, therefore the angle BFD is double of the angle BAD. {III. 20}

For the same reason

the angle BFD is also double of the angle BED; therefore the angle BAD is equal to the angle BED.

Therefore etc. Q. E. D.

PROPOSITION 22.

The opposite angles of quadrilaterals in circles are equal to two right angles.

Let ABCD be a circle, and let ABCD be a quadrilateral in it; I say that the opposite angles are equal to two right angles.

Let AC, BD be joined.

Then, since in any triangle the three angles are equal to two right angles, {I. 32}

the three angles CAB, ABC, BCA of the triangle ABC are equal to two right angles.

But the angle CAB is equal to the angle BDC, for they are in the same segment BADC; {III. 21} and the angle ACB is equal to the angle ADB, for they are in the same segment ADCB; therefore the whole angle ADC is equal to the angles BAC, ACB.

Let the angle ABC be added to each; therefore the angles ABC, BAC, ACB are equal to the angles ABC, ADC. But the angles ABC, BAC, ACB

are equal to two right angles; therefore the angles ABC, ADC are also equal to two right angles.

Similarly we can prove that the angles BAD, DCB are also equal to two right angles.

Therefore etc. Q. E. D.

PROPOSITION 23.

On the same straight line there cannot be constructed two similar and unequal segments of circles on the same side.

For, if possible, on the same straight line AB let two similar and unequal segments of circles ACB, ADB be constructed on the same side; let ACD be drawn through, and let CB, DB be joined.

Then, since the segment ACB is similar to the segment ADB, and similar segments of circles are those which admit equal angles, {III. Def. 11}

the angle ACB is equal to the angle ADB, the exterior to the interior:
which is impossible. {I. 16}

Therefore etc. Q. E. D. [17]

PROPOSITION 24.

Similar segments of circles on equal straight lines are equal to one another.

For let AEB, CFD be similar segments of circles on equal straight lines AB, CD;
I say that the segment AEB is equal to the segment CFD.

For, if the segment AEB be applied to CFD, and if the point A be placed on C and the straight line AB on CD,

the point B will also coincide with the point D, because AB is equal to CD;

and, AB coinciding with CD, the segment AEB will also coincide with CFD.

For, if the straight line AB coincide with CD but the segment AEB do not coincide with CFD, it will either fall with it, or outside it;
or it will fall awry, as CGD, and a circle cuts a circle at more points than two : which is impossible. {III. 10}

Therefore, if the straight line AB be applied to CD, the segment AEB will not fail to coincide with CFD also;

therefore it will coincide with it and will be equal to it.

Therefore etc. Q. E. D. [18]

PROPOSITION 25.

Given a segment of a circle, to describe the complete circle of which it is a segment.

Let ABC be the given segment of a circle; thus it is required to describe the complete circle belonging to the segment ABC, that is, of which it is a segment.

For let AC be bisected at D, let DB be drawn from the point D at right angles to AC, and let AB. be joined;

the angle ABD is then greater than, equal to, or less than the angle BAD.

First let it be greater; and on the straight line BA, and at the point A on it, let the angle BAE be constructed equal to the angle ABD; let DB be drawn through to E, and let EC be joined.

Then, since the angle ABE is equal to the angle BAE,

the straight line EB is also equal to EA. {I. 6}

And, since AD is equal to DC, and DE is common,

the two sides AD, DE are equal to the two sides CD, DE respectively;

and the angle ADE is equal to the angle CDE, for each is right;

therefore the base AE is equal to the base CE.

But AE was proved equal to BE;

therefore BE is also equal to CE;

therefore the three straight lines AE, EB, EC are equal to one another.

Therefore the circle drawn with centre E and distance one of the straight lines AE, EB, EC will also pass through the remaining points and will have been completed. {III. 9}

Therefore, given a segment of a circle, the complete circle has been described.

And it is manifest that the segment ABC is less than a semicircle, because the centre E happens to be outside it.

Similarly, even if the angle ABD be equal to the angle BAD, AD being equal to each of the two BD, DC,

the three straight lines DA, DB, DC will be equal to one another, D will be the centre of the completed circle, and ABC will clearly be a semicircle.

But, if the angle ABD be less than the angle BAD, and if we construct, on the straight line BA and at the point A on it, an angle equal to the angle ABD, the

centre will fall on DB within the segment ABC, and the segment ABC will clearly be greater than a semicircle.

Therefore, given a segment of a circle, the complete circle has been described. Q. E. F. [19]

PROPOSITION 26.

In equal circles equal angles stand on equal circumferences, whether they stand at the centres or at the circumferences.

Let ABC, DEF be equal circles, and in them let there be equal angles, namely at the centres the angles BGC, EHF, and at the circumferences the angles BAC, EDF; I say that the circumference BKC is equal to the circumference ELF.

For let BC, EF be joined.

Now, since the circles ABC, DEF are equal,

the radii are equal.

Thus the two straight lines BG, GC are equal to the two straight lines EH, HF;

and the angle at G is equal to the angle at H; therefore the base BC is equal to the base EF. {I. 4}

And, since the angle at A is equal to the angle at D,

the segment BAC is similar to the segment EDF; {III. Def. 11}

and they are upon equal straight lines.

But similar segments of circles on equal straight lines are equal to one another; {III. 24}

therefore the segment BAC is equal to EDF.

But the whole circle ABC is also equal to the whole circle DEF; therefore the circumference BKC which remains is equal to the circumference ELF.

Therefore etc. Q. E. D.

PROPOSITION 27.

In equal circles angles standing on equal circumferences are equal to one another, whether they stand at the centres or at the circumferences.

For in equal circles ABC, DEF, on equal circumferences BC, EF, let the angles BGC, EHF stand at the centres G, H, and the angles BAC, EDF at the circumferences; I say that the angle BGC is equal to the angle EHF, and the angle BAC is equal to the angle EDF.

For, if the angle BGC is unequal to the angle EHF,

one of them is greater.

Let the angle BGC be greater : and on the straight line BG, and at the point G on it, let the angle BGK be constructed equal to the angle EHF. {I. 23}

Now equal angles stand on equal circumferences, when they are at the centres; {III. 26}

therefore the circumference BK is equal to the circumference EF.

But EF is equal to BC;

therefore BK is also equal to BC, the less to the greater : which is impossible.

Therefore the angle BGC is not unequal to the angle EHF;

therefore it is equal to it.

And the angle at A is half of the angle BGC,

and the angle at D half of the angle EHF; {III. 20}

therefore the angle at A is also equal to the angle at D.

Therefore etc. Q. E. D.

PROPOSITION 28.

In equal circles equal straight lines cut off equal circumferences, the greater equal to the greater and the less to the less.

Let ABC, DEF be equal circles, and in the circles let AB, DE be equal straight lines cutting off ACB, DFE as greater circumferences and AGB, DHE as lesser; I say that the greater circumference ACB is equal to the greater circumference DFE, and the less circumference AGB to DHE.

For let the centres K, L of the circles be taken, and let AK, KB, DL, LE be joined.

Now, since the circles are equal,

the radii are also equal; therefore the two sides AK, KB are equal to the two sides DL, LE;

and the base AB is equal to the base DE;

therefore the angle AKB is equal to the angle DLE. {I. 8}

But equal angles stand on equal circumferences, when they are at the centres; {III. 26}

therefore the circumference AGB is equal to DHE.

And the whole circle ABC is also equal to the whole circle DEF; therefore the circumference ACB which remains is also equal to the circumference DFE which remains.

Therefore etc. Q. E. D.

PROPOSITION 29.

In equal circles equal circumferences are subtended by equal straight lines.

Let ABC, DEF be equal circles, and in them let equal circumferences BGC, EHF be cut off; and let the straight lines BC, EF be joined; I say that BC is equal to EF.

For let the centres of the circles be taken, and let them be K, L; let BK, KC, EL, LF be joined.

Now, since the circumference BGC is equal to the circumference EHF,

the angle BKC is also equal to the angle ELF. {III. 27}

And, since the circles ABC, DEF are equal,

the radii are also equal; therefore the two sides BK, KC are equal to the two sides EL, LF; and they contain equal angles; therefore the base BC is equal to the base EF. {I. 4}

Therefore etc.

PROPOSITION 30.

To bisect a given circumference.

Let ADB be the given circumference; thus it is required to bisect the circumference ADB.

Let AB be joined and bisected at C; from the point C let CD be drawn at right angles to the straight line AB, and let AD, DB be joined.

Then, since AC is equal to CB, and CD is common,

the two sides AC, CD are equal to the two sides BC, CD;

and the angle ACD is equal to the angle BCD, for each is right;

therefore the base AD is equal to the base DB. {I. 4}

But equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less; {III. 28}

and each of the circumferences AD, DB is less than a semicircle; therefore the circumference AD is equal to the circumference DB.

Therefore the given circumference has been bisected at the point D. Q. E. F.

PROPOSITION 31.

In a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle; and further the angle of the greater segment is greater than a right angle, and the angle of the less segment less than a right angle.

Let ABCD be a circle, let BC be its diameter, and E its centre, and let BA, AC, AD, DC be joined; I say that the angle BAC in the semicircle BAC is right, the angle ABC in the segment ABC greater than the semicircle is less than a right angle, and the angle ADC in the segment ADC less than the semicircle is greater than a right angle.

Let AE be joined, and let BA be carried through to F.

Then, since BE is equal to EA,

the angle ABE is also equal to the angle BAE. {I. 5}

Again, since CE is equal to EA,

the angle ACE is also equal to the angle CAE. {I. 5}

Therefore the whole angle BAC is equal to the two angles ABC, ACB.

But the angle FAC exterior to the triangle ABC is also equal to the two angles ABC, ACB; {I. 32}

therefore the angle BAC is also equal to the angle FAC; therefore each is right; {I. Def. 10} therefore the angle BAC in the semicircle BAC is right.

Next, since in the triangle ABC the two angles ABC, BAC are less than two right angles, {I. 17} and the angle BAC is a right angle,

the angle ABC is less than a right angle;

and it is the angle in the segment ABC greater than the semicircle.

Next, since ABCD is a quadrilateral in a circle, and the opposite angles of quadrilaterals in circles are equal to two right angles, {III. 22} while the angle ABC is less than a right angle, therefore the angle ADC which remains is greater than a right angle; and it is the angle in the segment ADC less than the semicircle.

I say further that the angle of the greater segment, namely that contained by the circumference ABC and the straight line AC, is greater than a right angle; and the angle of the less segment, namely that contained by the circumference ADC and the straight line AC, is less than a right angle.

This is at once manifest. For, since the angle contained by the straight lines BA, AC is right,

the angle contained by the circumference ABC and the straight line AC is greater than a right angle.

Again, since the angle contained by the straight lines AC, AF is right,

the angle contained by the straight line CA and the circumference ADC is less than a right angle.

Therefore etc. Q. E. D.

PROPOSITION 32.

If a straight line touch a circle, and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle.

For let a straight line EF touch the circle ABCD at the point B, and from the point B let there be drawn across, in the circle ABCD, a straight line BD cutting it; I say that the angles which BD makes with the tangent EF will be equal to the angles in the alternate segments of the circle, that is, that the angle FBD is equal to the angle constructed in the segment BAD, and the angle EBD is equal to the angle constructed in the segment DCB.

For let BA be drawn from B at right angles to EF, let a point C be taken at random on the circumference BD, and let AD, DC, CB be joined.

Then, since a straight line EF touches the circle ABCD at B, and BA has been drawn from the point of contact at right angles to the tangent, the centre of the circle ABCD is on BA. {III. 19}

Therefore BA is a diameter of the circle ABCD;

therefore the angle ADB, being an angle in a semicircle, is right. {III. 31}

Therefore the remaining angles BAD, ABD are equal to one right angle. {I. 32}

But the angle ABF is also right; therefore the angle ABF is equal to the angles BAD, ABD.

Let the angle ABD be subtracted from each; therefore the angle DBF which remains is equal to the angle BAD in the alternate segment of the circle.

Next, since ABCD is a quadrilateral in a circle, its opposite angles are equal to two right angles. {III. 22}

But the angles DBF, DBE are also equal to two right angles; therefore the angles DBF, DBE are equal to the angles BAD, BCD,

of which the angle BAD was proved equal to the angle DBF; therefore the angle DBE which remains is equal to the angle DCB in the alternate segment DCB of the circle.

Therefore etc. Q. E. D.

PROPOSITION 33.

On a given straight line to describe a segment of a circle admitting an angle equal to a given rectilineal angle.

Let AB be the given straight line, and the angle at C the given rectilineal angle; thus it is required to describe on the given straight line AB a segment of a circle admitting an angle equal to the angle at C.

The angle at C is then acute, or right, or obtuse.

First let it be acute, and, as in the first figure, on the straight line AB, and at the point A, let the angle BAD be constructed equal to the angle at C;

therefore the angle BAD is also acute.

Let AE be drawn at right angles to DA, let AB be bisected at F, let FG be drawn from the point F at right angles to AB, and let GB be joined.

Then, since AF is equal to FB, and FG is common,

the two sides AF, FG are equal to the two sides BF, FG;

and the angle AFG is equal to the angle BFG;

therefore the base AG is equal to the base BG. {I. 4}

Therefore the circle described with centre G and distance GA will pass through B also.

Let it be drawn, and let it be ABE; let EB be joined.

Now, since AD is drawn from A, the extremity of the diameter AE, at right angles to AE,

therefore AD touches the circle ABE. {III. 16, Por.}

Since then a straight line AD touches the circle ABE, and from the point of contact at A a straight line AB is drawn across in the circle ABE,

the angle DAB is equal to the angle AEB in the alternate segment of the circle. {III. 32}

But the angle DAB is equal to the angle at C; therefore the angle at C is also equal to the angle AEB.

Therefore on the given straight line AB the segment AEB of a circle has been described admitting the angle AEB equal to the given angle, the angle at C.

Next let the angle at C be right; and let it be again required to describe on AB a segment of a circle admitting an angle equal to the right angle at C.

Let the angle BAD be constructed equal to the right angle at C, as is the case in the second figure; let AB be bisected at F, and with centre F and distance either FA or FB let the circle AEB be described.

Therefore the straight line AD touches the circle ABE, because the angle at A is right. {III. 16, Por.}

And the angle BAD is equal to the angle in the segment AEB, for the latter too is itself a right angle, being an angle in a semicircle. {III. 31}

But the angle BAD is also equal to the angle at C.

Therefore the angle AEB is also equal to the angle at C.

Therefore again the segment AEB of a circle has been described on AB admitting an angle equal to the angle at C.

Next, let the angle at C be obtuse; and on the straight line AB, and at the point A, let the angle BAD be constructed equal to it, as is the case in the third figure; let AE be drawn at right angles to AD, let AB be again bisected at F, let FG be drawn at right angles to AB, and let GB be joined.

Then, since AF is again equal to FB, and FG is common,

the two sides AF, FG are equal to the two sides BF, FG;

and the angle AFG is equal to the angle BFG;

therefore the base AG is equal to the base BG. {I. 4}

Therefore the circle described with centre G and distance GA will pass through B also; let it so pass, as AEB.

Now, since AD is drawn at right angles to the diameter AE from its extremity,

AD touches the circle AEB. {III. 16, Por.}

And AB has been drawn across from the point of contact at A;

therefore the angle BAD is equal to the angle constructed in the alternate segment AHB of the circle. {III. 32}

But the angle BAD is equal to the angle at C.

Therefore the angle in the segment AHB is also equal to the angle at C.

Therefore on the given straight line AB the segment AHB of a circle has been described admitting an angle equal to the angle at C. Q. E. F.

PROPOSITION 34.

From a given circle to cut off a segment admitting an angle equal to a given rectilineal angle.

Let ABC be the given circle, and the angle at D the given rectilineal angle; thus it is required to cut off from the circle ABC a segment admitting an angle equal to the given rectilineal angle, the angle at D.

Let EF be drawn touching ABC at the point B, and on the straight line FB, and at the point B on it, let the angle FBC be constructed equal to the angle at D. {I. 23}

Then, since a straight line EF touches the circle ABC,

and BC has been drawn across from the point of contact at B,

the angle FBC is equal to the angle constructed in the alternate segment BAC. {III. 32}

But the angle FBC is equal to the angle at D;

therefore the angle in the segment BAC is equal to the angle at D.

Therefore from the given circle ABC the segment BAC. has been cut off admitting an angle equal to the given rectilineal angle, the angle at D. Q. E. F.

PROPOSITION 35.

If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

For in the circle ABCD let the two straight lines AC, BD cut one another at the point E; I say that the rectangle contained by AE, EC is equal to the rectangle contained by DE, EB.

If now AC, BD are through the centre, so that E is the centre of the circle ABCD, it is manifest that, AE, EC, DE, EB being equal, the rectangle

contained by AE, EC is also equal to the rectangle contained by DE, EB.

Next let AC, DB not be through the centre; let the centre of ABCD be taken, and let it be F; from F let FG, FH be drawn perpendicular to the straight lines AC, DB, and let FB, FC, FE be joined.

Then, since a straight line GF through the centre cuts a straight line AC not through the centre at right angles,

it also bisects it; {III. 3} therefore AG is equal to GC.

Since, then, the straight line AC has been cut into equal parts at G and into unequal parts at E, the rectangle contained by AE, EC together with the square on EG is equal to the square on GC; {II. 5}

Let the square on GF be added; therefore the rectangle AE, EC together with the squares on GE, GF is equal to the squares on CG, GF.

But the square on FE is equal to the squares on EG, GF, and the square on FC is equal to the squares on CG, GF; {I. 47}

therefore the rectangle AE, EC together with the square on FE is equal to the square on FC.

And FC is equal to FB; therefore the rectangle AE, EC together with the square on EF is equal to the square on FB.

For the same reason, also, the rectangle DE, EB together with the square on FE is equal to the square on FB.

But the rectangle AE, EC together with the square on FE was also proved equal to the square on FB; therefore the rectangle AE, EC together with the square on FE is equal to the rectangle DE, EB together with the square on FE.

Let the square on FE be subtracted from each; therefore the rectangle contained by AE, EC which remains is equal to the rectangle contained by DE, EB.

Therefore etc.

PROPOSITION 36.

If a point be taken outside a circle and from it there fall on the circle two straight lines, and if one of them cut the circle and the other touch it, the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference will be equal to the square on the tangent.

For let a point D be taken outside the circle ABC, and from D let the two straight lines DCA, DB fall on the circle ABC; let DCA cut the circle ABC and let BD touch it; I say that the rectangle contained by AD, DC is equal to the square on DB.

Then DCA is either through the centre or not through the centre.

First let it be through the centre, and let F be the centre of the circle ABC; let FB be joined;

therefore the angle FBD is right. {III. 18}

And, since AC has been bisected at F, and CD is added to it, the rectangle AD, DC together with the square on FC is equal to the square on FD. {II. 6}

But FC is equal to FB; therefore the rectangle AD, DC together with the square on FB is equal to the square on FD.

And the squares on FB, BD are equal to the square on FD; {I. 47} therefore the rectangle AD, DC together with the square on FB is equal to the squares on FB, BD.

Let the square on FB be subtracted from each; therefore the rectangle AD, DC which remains is equal to the square on the tangent DB.

Again, let DCA not be through the centre of the circle ABC; let the centre E be taken, and from E let EF be drawn perpendicular to AC; let EB, EC, ED

be joined.

Then the angle EBD is right. {III. 18}

And, since a straight line EF through the centre cuts a straight line AC not through the centre at right angles,

it also bisects it; {III. 3} therefore AF is equal to FC.

Now, since the straight line AC has been bisected at the point F, and CD is added to it, the rectangle contained by AD, DC together with the square on FC is equal to the square on FD. {II. 6}

Let the square on FE be added to each; therefore the rectangle AD, DC together with the squares on CF, FE is equal to the squares on FD, FE.

But the square on EC is equal to the squares on CF, FE, for the angle EFC is right; {I. 47} and the square on ED is equal to the squares on DF, FE; therefore the rectangle AD, DC together with the square on EC is equal to the square on ED.

And EC is equal to EB; therefore the rectangle AD, DC together with the square on EB is equal to the square on ED.

But the squares on EB, BD are equal to the square on ED, for the angle EBD is right; {I. 47} therefore the rectangle AD, DC together with the square on EB is equal to the squares on EB, BD.

Let the square on EB be subtracted from each; therefore the rectangle AD, DC which remains is equal to the square on DB.

Therefore etc. Q. E. D.

PROPOSITION 37.

If a point be taken outside a circle and from the point there fall on the circle two straight lines, if one of them cut the circle, and the other fall on it, and if

further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference be equal to the square on the straight line which falls on the circle, the straight line which falls on it will touch the circle.

For let a point D be taken outside the circle ABC; from D let the two straight lines DCA, DB fall on the circle ACB; let DCA cut the circle and DB fall on it; and let the rectangle AD, DC be equal to the square on DB.

I say that DB touches the circle ABC.

For let DE be drawn touching ABC; let the centre of the circle ABC be taken, and let it be F; let FE, FB, FD be joined.

Thus the angle FED is right. {III. 18}

Now, since DE touches the circle ABC, and DCA cuts it, the rectangle AD, DC is equal to the square on DE. {III. 36}

But the rectangle AD, DC was also equal to the square on DB; therefore the square on DE is equal to the square on DB;

therefore DE is equal to DB.

And FE is equal to FB; therefore the two sides DE, EF are equal to the two sides DB, BF; and FD is the common base of the triangles;

therefore the angle DEF is equal to the angle DBF. {I. 8}

But the angle DEF is right;

therefore the angle DBF is also right.

And FB produced is a diameter; and the straight line drawn at right angles to the diameter of a circle, from its extremity, touches the circle; {III. 16, Por.}

therefore DB touches the circle.

Similarly this can be proved to be the case even if the centre be on AC.

Therefore etc. Q. E. D.

ENDNOTES.

[1] 12. For suppose it is not. This is expressed in the Greek by the two words Μὴ γάρ, but such an elliptical phrase is impossible in English.

[2] the two sides AD, DG are equal to the two sides BD, DG respectively. As before observed, Euclid is not always careful to put the equals in corresponding order. The text here has "GD, DB."

[3] with the same construction, τῶν αὐτῶν κατασκευασθέντων.

[4] of the same diameter. I have inserted these words for clearness' sake. The text has simply ἐλαχίστη δὲ ἡ λοιπή, "and the remaining (straight line) least."

[5] one on each side. The word "one" is not in the Greek, but is necessary to give the force of ἐφ' ἑκάτερα τῆς ἐλαχίστης, literally "on both sides," or "on each of the two sides, of the least."

[6] The word circle (κύκλος) is here employed in the unusual sense of the circumference (περιφέρεια) of a circle. Cf. note on I. Def. 15.

[7] the straight line joining their centres, literally "the straight line joined to their centres" (ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη εὐθεῖα).

[8] point of contact is here συναφή, and in the enunciation of the next proposition ἐπαφή.

[9] will not fail to pass. The Greek has the double negative, οὐκ ἄρα ἡ...εὐθεῖα... οὐκ ἐλεύσεται, literally "the straight line...will not not-pass...."

[10] ABDC. Euclid writes ABCD (here and in the next proposition), notwithstanding the order in which the points are placed in the figure.

[11] does it so touch it. It is necessary to supply these words which the Greek (ὅτι οὐδὲ ἐκτός and ὅτι οὐδὲ ἐντός) leaves to be understood.

[12] Of straight lines. The Greek leaves these words to be understood.

[13] Nearer to the diameter AD. As BC, FG are not in general parallel to AD, Euclid should have said "nearer to the centre."

[14] cannot be interposed, literally “will not fall in between” (οὐ παρεμπεσεῖται).

[15] the tangent, ἡ ἐφαπτομένη.

[16] let another straight line be inflected, κεκλάσθω δὴ πάλιν (without εὐθεῖα). The verb κλάω (to break off) was the regular technical term for drawing from a point a (broken) straight line which first meets another straight line or curve and is then bent back from it to another point, or (in other words) for drawing straight lines from two points meeting at a point on a curve or another straight line. κεκλάσθαι is one of the geometrical terms the definition of which must according to Aristotle be assumed (Anal. Post. I. 10, 76 b 9).

[17] cannot be constructed, οὐ συσταθήσεται, the same phrase as in 1. 7.

[18] fall awry, παραλλάξει, the same word as used in the like case in I. 8. The word implies that the applied figure will partly fall short of, and partly overlap, the figure to which it is applied.

[19] to describe the complete circle, προσαναγράψαι τὸν κύκλον, literally “to describe the circle on to it.’

BOOK IV. DEFINITIONS.

1

A rectilineal figure is said to be inscribed in a rectilineal figure when the respective angles of the inscribed figure lie on the respective sides of that in which it is inscribed.

2

Similarly a figure is said to be circumscribed about a figure when the respective sides of the circumscribed figure pass through the respective angles of that about which it is circumscribed.

3

A rectilineal figure is said to be inscribed in a circle when each angle of the inscribed figure lies on the circumference of the circle.

4

A rectilineal figure is said to be circumscribed about a circle, when each side of the circumscribed figure touches the circumference of the circle.

5

Similarly a circle is said to be inscribed in a figure when the circumference of the circle touches each side of the figure in which it is inscribed.

6

A circle is said to be circumscribed about a figure when the circumference of the circle passes through each angle of the figure about which it is

circumscribed.

7

A straight line is said to be fitted into a circle when its extremities are on the circumference of the circle.

PROPOSITIONS

PROPOSITION 1.

Into a given circle to fit a straight line equal to a given straight line which is not greater than the diameter of the circle.

Let ABC be the given circle, and D the given straight line not greater than the diameter of the circle; thus it is required to fit into the circle ABC a straight line equal to the straight line D.

Let a diameter BC of the circle ABC be drawn.

Then, if BC is equal to D, that which was enjoined will have been done; for BC has been fitted into the circle ABC equal to the straight line D.

But, if BC is greater than D, let CE be made equal to D, and with centre C and distance CE let the circle EAF be described; let CA be joined.

Then, since the point C is the centre of the circle EAF,

CA is equal to CE.

But CE is equal to D;

therefore D is also equal to CA.

Therefore into the given circle ABC there has been fitted CA equal to the given straight line D.

PROPOSITION 2.

In a given circle to inscribe a triangle equiangular with a given triangle.

Let ABC be the given circle, and DEF the given triangle; thus it is required to inscribe in the circle ABC a triangle equiangular with the triangle DEF.

Let GH be drawn touching the circle ABC at A {III. 16, Por.}; on the straight line AH, and at the point A on it, let the angle HAC be constructed equal to the angle DEF, and on the straight line AG, and at the point A on it, let the angle GAB be constructed equal to the angle DFE; {I. 23} let BC be joined.

Then, since a straight line AH touches the circle ABC, and from the point of contact at A the straight line AC is drawn across in the circle, therefore the angle HAC is equal to the angle ABC in the alternate segment of the circle. {III. 32}

But the angle HAC is equal to the angle DEF; therefore the angle ABC is also equal to the angle DEF.

For the same reason

the angle ACB is also equal to the angle DFE;

therefore the remaining angle BAC is also equal to the remaining angle EDF. {I. 32}

Therefore in the given circle there has been inscribed a triangle equiangular with the given triangle. Q. E. F.

PROPOSITION 3.

About a given circle to circumscribe a triangle equiangular with a given triangle.

Let ABC be the given circle, and DEF the given triangle;
thus it is required to circumscribe about the circle ABC a triangle equiangular with the triangle DEF.

Let EF be produced in both directions to the points G, H, let the centre K of the circle ABC be taken {III. 1}, and let
the straight line KB be drawn across at random; on the straight line KB, and at the point K on it, let the angle BKA be constructed equal to the angle DEG, and the angle BKC equal to the angle DFH; {I. 23} and through the points A, B, C let LAM, MBN, NCL be
drawn touching the circle ABC. {III. 16, Por.}

Now, since LM, MN, NL touch the circle ABC at the points A, B, C, and KA, KB, KC have been joined from the centre K to the points A, B, C, therefore the angles at the points A, B, C are right. {III. 18}

And, since the four angles of the quadrilateral AMBK are equal to four right angles, inasmuch as AMBK is in fact divisible into two triangles,

and the angles KAM, KBM are right,

therefore the remaining angles AKB, AMB are equal to two right angles.

But the angles DEG, DEF are also equal to two right angles; {I. 13} therefore the angles AKB, AMB are equal to the angles
DEG, DEF, of which the angle AKB is equal to the angle DEG;

therefore the angle AMB which remains is equal to the angle DEF which remains.

Similarly it can be proved that the angle LNB is also
equal to the angle DFE;

therefore the remaining angle MLN is equal to the angle EDF. {I. 32}

Therefore the triangle LMN is equiangular with the triangle DEF; and it has been circumscribed about the

circle ABC.

Therefore about a given circle there has been circumscribed a triangle equiangular with the given triangle. Q. E. F. [1] [2]

PROPOSITION 4.

In a given triangle to inscribe a circle.

Let ABC be the given triangle; thus it is required to inscribe a circle in the triangle ABC.

Let the angles ABC, ACB
be bisected by the straight lines BD, CD {I. 9}, and let these meet one another at the point D; from D let DE, DF, DG be drawn perpendicular to the straight
lines AB, BC, CA.

Now, since the angle ABD is equal to the angle CBD, and the right angle BED is also equal to the right angle BFD,
EBD, FBD are two triangles having two angles equal to two angles and one side equal to one side, namely that subtending one of the equal angles, which is BD common to the triangles;

therefore they will also have the remaining sides equal to the remaining sides; {I. 26}

therefore DE is equal to DF.

For the same reason

DG is also equal to DF.

Therefore the three straight lines DE, DF, DG are equal to one another;

therefore the circle described with centre D and distance one of the straight lines DE, DF, DG will pass also through the remaining points, and will touch the straight lines AB, BC, CA, because the angles at the points E, F, G are right.

For, if it cuts them, the straight line drawn at right angles to the diameter of the circle from its extremity will be found to fall within the circle : which was proved absurd; {III. 16}

therefore the circle described with centre D and distance one of the straight lines DE, DF, DG will not cut the straight lines AB, BC, CA;

therefore it will touch them, and will be the circle inscribed in the triangle ABC. {IV. Def. 5}

Let it be inscribed, as FGE.

Therefore in the given triangle ABC the circle EFG has been inscribed. Q. E. F. [3]

PROPOSITION 5.

About a given triangle to circumscribe a circle.

Let ABC be the given triangle; thus it is required to circumscribe a circle about the given triangle ABC.

Let the straight lines AB, AC be bisected at the points D, E {I. 10}, and from the points D, E let DF, EF be drawn at right angles to AB, AC; they will then meet within the triangle ABC, or on the straight line BC, or outside BC.

First let them meet within at F, and let FB, FC, FA be joined.

Then, since AD is equal to DB, and DF is common and at right angles, therefore the base AF is equal to the base FB. {I. 4}

Similarly we can prove that

CF is also equal to AF;

so that FB is also equal to FC;

therefore the three straight lines FA, FB, FC are equal to one another.

Therefore the circle described with centre F and distance one of the straight lines FA, FB, FC will pass also through the remaining points, and the circle will have been circumscribed about the triangle ABC.

Let it be circumscribed, as ABC.

Next, let DF, EF meet on the straight line BC at F, as is the case in the second figure; and let AF be joined.

Then, similarly, we shall prove that the point F is the centre of the circle circumscribed about the triangle ABC.

Again, let DF, EF meet outside the triangle ABC at F, as is the case in the third figure, and let AF, BF, CF be joined.

Then again, since AD is equal to DB, and DF is common and at right angles, therefore the base AF is equal to the base BF. {I. 4}

Similarly we can prove that

CF is also equal to AF;

so that BF is also equal to FC; therefore the circle described with centre F and distance one of the straight lines FA, FB, FC will pass also through the remaining points, and will have been circumscribed about the triangle ABC.

Therefore about the given triangle a circle has been circumscribed. Q. E. F.

And it is manifest that, when the centre of the circle falls within the triangle, the angle BAC, being in a segment greater than the semicircle, is less than a right angle; when the centre falls on the straight line BC, the angle BAC, being in a semicircle, is right; and when the centre of the circle falls outside the triangle, the angle BAC, being in a segment less than the semicircle, is greater than a right angle. {III. 31}

PROPOSITION 6.

In a given circle to inscribe a square.

Let ABCD be the given circle; thus it is required to inscribe a square in the circle ABCD.

Let two diameters AC, BD of the circle ABCD be drawn at right angles to one another, and let AB, BC, CD, DA be joined.

Then, since BE is equal to ED, for E is the centre, and EA is common and at right angles, therefore the base AB is equal to the base AD. {I. 4}

For the same reason each of the straight lines BC, CD is also equal to each of the straight lines AB, AD;

therefore the quadrilateral ABCD is equilateral.

I say next that it is also right-angled.

For, since the straight line BD is a diameter of the circle ABCD, therefore BAD is a semicircle;

therefore the angle BAD is right. {III. 31}

For the same reason each of the angles ABC, BCD, CDA is also right;

therefore the quadrilateral ABCD is right-angled.

But it was also proved equilateral; therefore it is a square; {I. Def. 22} and it has been inscribed in the circle ABCD.

Therefore in the given circle the square ABCD has been inscribed. Q. E. F.

PROPOSITION 7.

About a given circle to circumscribe a square.

Let ABCD be the given circle; thus it is required to circumscribe a square about the circle ABCD.

Let two diameters AC, BD of the circle ABCD be drawn at right angles to one another, and through the points A, B, C, D let FG, GH, HK, KF be drawn touching the circle ABCD. {III. 16, Por.}

Then, since FG touches the circle ABCD, and EA has been joined from the centre E to the point of contact at A,

therefore the angles at A are right. {III. 18}

For the same reason

the angles at the points B, C, D are also right.

Now, since the angle AEB is right, and the angle EBG is also right,

therefore GH is parailel to AC. {I. 28}

For the same reason

AC is also parallel to FK, so that GH is also parallel to FK. {I. 30}

Similarly we can prove that

each of the straight lines GF, HK is parallel to BED.

Therefore GK, GC, AK, FB, BK are parallelograms; therefore GF is equal to HK, and GH to FK. {I. 34}

And, since AC is equal to BD, and AC is also equal to each of the straight lines GH, FK,

while BD is equal to each of the straight lines GF, HK, {I. 34} therefore the quadrilateral FGHK is equilateral.

I say next that it is also right-angled.

For, since GBEA is a parallelogram, and the angle AEB is right, therefore the angle AGB is also right. {I. 34}

Similarly we can prove that

the angles at H, K, F are also right.

Therefore FGHK is right-angled.

But it was also proved equilateral;

therefore it is a square;

and it has been circumscribed about the circle ABCD.

Therefore about the given circle a square has been circumscribed. Q. E. F.

PROPOSITION 8.

In a given square to inscribe a circle.

Let ABCD be the given square; thus it is required to inscribe a circle in the given square ABCD.

Let the straight lines AD, AB be bisected at the points E, F respectively {I. 10}, through E let EH be drawn parallel to either AB or CD, and through F let FK be drawn parallel to either AD or BC; {I. 31} therefore each of the figures AK, KB, AH, HD, AG, GC, BG, GD is a parallelogram, and their opposite sides are evidently equal. {I. 34}

Now, since AD is equal to AB, and AE is half of AD, and AF half of AB,

therefore AE is equal to AF, so that the opposite sides are also equal;

therefore FG is equal to GE.

Similarly we can prove that each of the straight lines GH, GK is equal to each of the straight lines FG, GE;

therefore the four straight lines GE, GF, GH, GK are equal to one another.

Therefore the circle described with centre G and distance one of the straight lines GE, GF, GH, GK will pass also through the remaining points.

And it will touch the straight lines AB, BC, CD, DA, because the angles at E, F, H, K are right.

For, if the circle cuts AB, BC, CD, DA, the straight line drawn at right angles to the diameter of the circle from its extremity will fall within the circle : which was proved absurd; {III. 16} therefore the circle described with centre G and distance one of the straight lines GE, GF, GH, GK will not cut the straight lines AB, BC, CD, DA.

Therefore it will touch them, and will have been inscribed in the square ABCD.

Therefore in the given square a circle has been inscribed. Q. E. F.

PROPOSITION 9.

About a given square to circumscribe a circle.

Let ABCD be the given square; thus it is required to circumscribe a circle about the square ABCD.

For let AC, BD be joined, and let them cut one another at E.

Then, since DA is equal to AB, and AC is common, therefore the two sides DA, AC are equal to the two sides BA, AC; and the base DC is equal to the base BC;

therefore the angle DAC is equal to the angle BAC. {I. 8}

Therefore the angle DAB is bisected by AC.

Similarly we can prove that each of the angles ABC, BCD, CDA is bisected by the straight lines AC, DB.

Now, since the angle DAB is equal to the angle ABC, and the angle EAB is half the angle DAB, and the angle EBA half the angle ABC, therefore the angle EAB is also equal to the angle EBA; so that the side EA is also equal to EB. {I. 6}

Similarly we can prove that each of the straight lines EA, EB is equal to each of the straight lines EC, ED.

Therefore the four straight lines EA, EB, EC, ED are equal to one another.

Therefore the circle described with centre E and distance one of the straight lines EA, EB, EC, ED will pass also through the remaining points; and it will have been circumscribed about the square ABCD.

Let it be circumscribed, as ABCD.

Therefore about the given square a circle has been circumscribed. Q. E. F.

PROPOSITION 10.

To construct an isosceles triangle having each of the angles at the base double of the remaining one.

Let any straight line AB be set out, and let it be cut at the point C so that the rectangle contained by AB, BC is equal to the square on CA; {II. 11} with centre A and distance AB let the circle BDE be described, and let there be fitted in the circle BDE the straight line BD equal to the straight line AC which is not greater than the diameter of the circle BDE. {IV. 1}

Let AD, DC be joined, and let the circle ACD be circumscribed about the triangle ACD. {IV. 5}

Then, since the rectangle AB, BC is equal to the square on AC, and AC is equal to BD, therefore the rectangle AB, BC is equal to the square on BD.

And, since a point B has been taken outside the circle ACD, and from B the two straight lines BA, BD have fallen on the circle ACD, and one of them cuts it, while the other falls on it, and the rectangle AB, BC is equal to the square on BD,

therefore BD touches the circle ACD. {III. 37}

Since, then, BD touches it, and DC is drawn across from the point of contact at D, therefore the angle BDC is equal to the angle DAC in the alternate segment of the circle. {III. 32}

Since, then, the angle BDC is equal to the angle DAC, let the angle CDA be added to each; therefore the whole angle BDA is equal to the two angles CDA, DAC.

But the exterior angle BCD is equal to the angles CDA, DAC; {I. 32} therefore the angle BDA is also equal to the angle BCD.

But the angle BDA is equal to the angle CBD, since the side AD is also equal to AB; {I. 5} so that the angle DBA is also equal to the angle BCD.

Therefore the three angles BDA, DBA, BCD are equal to one another.

And, since the angle DBC is equal to the angle BCD,

the side BD is also equal to the side DC. {I. 6}

But BD is by hypothesis equal to CA; therefore CA is also equal to CD,

so that the angle CDA is also equal to the angle DAC; {I. 5}

therefore the angles CDA, DAC are double of the angle DAC.

But the angle BCD is equal to the angles CDA, DAC; therefore the angle BCD is also double of the angle CAD.

But the angle BCD is equal to each of the angles BDA, DBA; therefore each of the angles BDA, DBA is also double of the angle DAB.

Therefore the isosceles triangle ABD has been constructed having each of the angles at the base DB double of the remaining one. Q. E. F.

PROPOSITION 11.

In a given circle to inscribe an equilateral and equiangular pentagon.

Let ABCDE be the given circle; thus it is required to inscribe in the circle ABCDE an equilateral and equiangular pentagon.

Let the isosceles triangle FGH be set out having each of the angles at G, H double of the angle at F; {IV. 10} let there be inscribed in the circle ABCDE the triangle ACD equiangular with the triangle FGH, so that the angle CAD is equal to the angle at F and the angles at G, H respectively equal to the angles ACD, CDA; {IV. 2} therefore each of the angles ACD, CDA is also double of the angle CAD.

Now let the angles ACD, CDA be bisected respectively by the straight lines CE, DB {I. 9}, and let AB, BC, DE, EA be joined.

Then, since each of the angles ACD, CDA is double of the angle CAD, and they have been bisected by the straight lines CE, DB, therefore the five angles DAC, ACE, ECD, CDB, BDA are equal to one another.

But equal angles stand on equal circumferences; {III. 26} therefore the five circumferences AB, BC, CD, DE, EA are equal to one another.

But equal circumferences are subtended by equal straight lines; {III. 29} therefore the five straight lines AB, BC, CD, DE, EA are equal to one another;

therefore the pentagon ABCDE is equilateral.

I say next that it is also equiangular.

For, since the circumference AB is equal to the circumference DE, let BCD be added to each;

therefore the whole circumference ABCD is equal to the whole circumference EDCB.

And the angle AED stands on the circumference ABCD, and the angle BAE on the circumference EDCB;

therefore the angle BAE is also equal to the angle AED. {III. 27}

For the same reason each of the angles ABC, BCD, CDE is also equal to each of the angles BAE, AED;

therefore the pentagon ABCDE is equiangular.

But it was also proved equilateral;

therefore in the given circle an equilateral and equiangular pentagon has been inscribed. Q. E. F.

PROPOSITION 12.

About a given circle to circumscribe an equilateral and equiangular pentagon.

Let ABCDE be the given circle; thus it is required to circumscribe an equilateral and equiangular pentagon about the circle ABCDE.

Let A, B, C, D, E be conceived to be the angular points of the inscribed pentagon, so that the circumferences AB, BC, CD, DE, EA are equal; {IV. 11} through A, B, C, D, E let GH, HK, KL, LM, MG be drawn touching the circle; {III. 16, Por.} let the centre F of the circle ABCDE be taken {III. 1}, and let FB, FK, FC, FL, FD be joined.

Then, since the straight line KL touches the circle ABCDE at C, and FC has been joined from the centre F to the point of contact at C,

therefore FC is perpendicular to KL; {III. 18} therefore each of the angles at C is right.

For the same reason

the angles at the points B, D are also right.

And, since the angle FCK is right, therefore the square on FK is equal to the squares on FC, CK.

For the same reason {I. 47}

the square on FK is also equal to the squares on FB, BK; so that the squares on FC, CK are equal to the squares on FB, BK, of which the square on FC is equal to the square on FB;

therefore the square on CK which remains is equal to the square on BK.

Therefore BK is equal to CK.

And, since FB is equal to FC, and FK common,

the two sides BF, FK are equal to the two sides CF, FK; and the base BK equal to the base CK; therefore the angle BFK is equal to the angle KFC, {I. 8} and the angle BKF to the angle FKC.

Therefore the angle BFC is double of the angle KFC,

and the angle BKC of the angle FKC.

For the same reason

the angle CFD is also double of the angle CFL, and the angle DLC of the angle FLC.

Now, since the circumference BC is equal to CD, the angle BFC is also equal to the angle CFD. {III. 27}

And the angle BFC is double of the angle KFC, and the angle DFC of the angle LFC;

therefore the angle KFC is also equal to the angle LFC.

But the angle FCK is also equal to the angle FCL; therefore FKC, FLC are two triangles having two angles equal to two angles and one side equal to one side, namely FC which is common to them; therefore they will also have the remaining sides equal to the remaining sides, and the remaining angle to the remaining angle; {I. 26}

therefore the straight line KC is equal to CL, and the angle FKC to the angle FLC.

And, since KC is equal to CL, therefore KL is double of KC.

For the same reason it can be proved that

HK is also double of BK.

And BK is equal to KC;

therefore HK is also equal to KL.

Similarly each of the straight lines HG, GM, ML can also be proved equal to each of the straight lines HK, KL;

therefore the pentagon GHKLM is equilateral.

I say next that it is also equiangular.

For, since the angle FKC is equal to the angle FLC, and the angle HKL was proved double of the angle FKC,

and the angle KLM double of the angle FLC, therefore the angle HKL is also equal to the angle KLM.

Similarly each of the angles KHG, HGM, GML can also be proved equal to each of the angles HKL, KLM; therefore the five angles GHK, HKL, KLM, LMG, MGH are equal to one another.

Therefore the pentagon GHKLM is equiangular.

And it was also proved equilateral; and it has been circumscribed about the circle ABCDE. Q. E. F.

PROPOSITION 13.

In a given pentagon, which is equilateral and equiangular, to inscribe a circle.

Let ABCDE be the given equilateral and equiangular pentagon; thus it is required to inscribe a circle in the pentagon ABCDE.

For let the angles BCD, CDE be bisected by the straight lines CF, DF respectively; and from the point F, at which the straight lines CF, DF meet one another, let the straight lines FB, FA, FE be joined.

Then, since BC is equal to CD, and CF common, the two sides BC, CF are equal to the two sides DC, CF; and the angle BCF is equal to the angle DCF;

therefore the base BF is equal to the base DF,

and the triangle BCF is equal to the triangle DCF, and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend. {I. 4}

Therefore the angle CBF is equal to the angle CDF.

And, since the angle CDE is double of the angle CDF, and the angle CDE is equal to the angle ABC, while the angle CDF is equal to the angle CBF; therefore the angle CBA is also double of the angle CBF;

therefore the angle ABF is equal to the angle FBC;

therefore the angle ABC has been bisected by the straight line BF.

Similarly it can be proved that the angles BAE, AED have also been bisected by the straight lines FA, FE respectively.

Now let FG, FH, FK, FL, FM be drawn from the point F perpendicular to the straight lines AB, BC, CD, DE, EA.

Then, since the angle HCF is equal to the angle KCF, and the right angle FHC is also equal to the angle FKC, FHC, FKC are two triangles having two angles equal to two angles and one side equal to one side, namely FC which is common to them and subtends one of the equal angles; therefore they will also have the remaining sides equal to the remaining sides; {I. 26} therefore the perpendicular FH is equal to the perpendicular FK.

Similarly it can be proved that each of the straight lines FL, FM, FG is also equal to each of the straight lines FH, FK; therefore the five straight lines FG, FH, FK, FL, FM are equal to one another.

Therefore the circle described with centre F and distance one of the straight lines FG, FH, FK, FL, FM will pass also through the remaining points; and it will touch the straight lines AB, BC, CD, DE, EA, because the angles at the points G, H, K, L, M are right.

For, if it does not touch them. but cuts them, it will result that the straight line drawn at right angles to the diameter of the circle from its extremity falls within the circle: which was proved absurd. {III. 16}

Therefore the circle described with centre F and distance one of the straight lines FG, FH, FK, FL, FM will not cut the straight lines AB, BC, CD, DE, EA;

therefore it will touch them.

Let it be described, as GHKLM.

Therefore in the given pentagon, which is equilateral and equiangular, a circle has been inscribed. Q. E. F.

PROPOSITION 14.

About a given pentagon, which is equilateral and equiangular, to circumscribe a circle.

Let ABCDE be the given pentagon, which is equilateral and equiangular; thus it is required to circumscribe a circle about the pentagon ABCDE.

Let the angles BCD, CDE be bisected by the straight lines CF, DF respectively, and from the point F, at which the straight lines meet, let the straight lines FB, FA, FE be joined to the points B, A, E.

Then in manner similar to the preceding it can be proved that the angles CBA, BAE, AED have also been bisected by the straight lines FB, FA, FE respectively.

Now, since the angle BCD is equal to the angle CDE, and the angle FCD is half of the angle BCD, and the angle CDF half of the angle CDE, therefore the angle FCD is also equal to the angle CDF,

so that the side FC is also equal to the side FD. {I. 6}

Similarly it can be proved that each of the straight lines FB, FA, FE is also equal to each of the straight lines FC, FD; therefore the five straight lines FA, FB, FC, FD, FE are equal to one another.

Therefore the circle described with centre F and distance one of the straight lines FA, FB, FC, FD, FE will pass also through the remaining points, and will have been circumscribed.

Let it be circumscribed, and let it be ABCDE.

Therefore about the given pentagon, which is equilateral and equiangular, a circle has been circumscribed. Q. E. F.

PROPOSITION 15.

In a given circle to inscribe an equilateral and equiangular hexagon.

Let ABCDEF be the given circle; thus it is required to inscribe an equilateral and equiangular hexagon in the circle ABCDEF.

Let the diameter AD of the circle ABCDEF be drawn; let the centre G of the circle be taken, and with centre D and distance DG let the circle EGCH be described; let EG, CG be joined and carried through to the points B, F, and let AB, BC, CD, DE, EF, FA be joined.

I say that the hexagon ABCDEF is equilateral and equiangular.

For, since the point G is the centre of the circle ABCDEF,

GE is equal to GD.

Again, since the point D is the centre of the circle GCH,

DE is equal to DG.

But GE was proved equal to GD;

therefore GE is also equal to ED; therefore the triangle EGD is equilateral;

and therefore its three angles EGD, GDE, DEG are equal to one another, inasmuch as, in isosceles triangles, the angles at the base are equal to one another. {I. 5}

And the three angles of the triangle are equal to two right angles; {I. 32}

therefore the angle EGD is one-third of two right angles.

Similarly, the angle DGC can also be proved to be onethird of two right angles.

And, since the straight line CG standing on EB makes the adjacent angles EGC, CGB equal to two right angles, therefore the remaining angle CGB is also one-third of two right angles.

Therefore the angles EGD, DGC, CGB are equal to one another; so that the angles vertical to them, the angles BGA, AGF, FGE are equal. {I. 15}

Therefore the six angles EGD, DGC, CGB, BGA, AGF, FGE are equal to one another.

But equal angles stand on equal circumferences; {III. 26} therefore the six circumferences AB, BC, CD, DE, EF, FA are equal to one another.

And equal circumferences are subtended by equal straight lines; {III. 29}

therefore the six straight lines are equal to one another; therefore the hexagon ABCDEF is equilateral.

I say next that it is also equiangular.

For, since the circumference FA is equal to the circumference ED,

let the circumference ABCD be added to each; therefore the whole FABCD is equal to the whole EDCBA;

and the angle FED stands on the circumference FABCD, and the angle AFE on the circumference EDCBA;

therefore the angle AFE is equal to the angle DEF. {III. 27}

Similarly it can be proved that the remaining angles of the hexagon ABCDEF are also severally equal to each of the angles AFE, FED;

therefore the hexagon ABCDEF is equiangular.

But it was also proved equilateral; and it has been inscribed in the circle ABCDEF.

Therefore in the given circle an equilateral and equiangular hexagon has been inscribed. Q. E. F.

PORISM.

From this it is manifest that the side of the hexagon is equal to the radius of the circle.

And, in like manner as in the case of the pentagon, if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle an equilateral and equiangular hexagon in conformity with what was explained in the case of the pentagon.

And further by means similar to those explained in the case of the pentagon we can both inscribe a circle in a given hexagon and circumscribe one about it. Q. E. F.

PROPOSITION 16.

In a given circle to inscribe a fifteen-angled figure which shall be both equilateral and equiangular.

Let ABCD be the given circle; thus it is required to inscribe in the circle ABCD a fifteenangled figure which shall be both equilateral and equiangular.

In the circle ABCD let there be inscribed a side AC of the equilateral triangle inscribed in it, and a side AB of an equilateral pentagon; therefore, of the equal segments of which there are fifteen in the circle ABCD, there will be five in the circumference ABC which is one-third of the circle, and there will be three in the circumference AB which is one-fifth of the circle;

therefore in the remainder BC there will be two of the equal segments.

Let BC be bisected at E; {III. 30} therefore each of the circumferences BE, EC is a fifteenth of the circle ABCD.

If therefore we join BE, EC and fit into the circle ABCD straight lines equal to them and in contiguity, a fifteen-angled figure which is both equilateral and equiangular will have been inscribed in it. Q. E. F.

And, in like manner as in the case of the pentagon, if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle a fifteen-angled figure which is equilateral and equiangular.

And further, by proofs similar to those in the case of the pentagon, we can both inscribe a circle in the given fifteenangled figure and circumscribe one about it. Q. E. F.

ENDNOTES.

[1] at random, literally "as it may chance," ὡς ἔτυχεν. The same expression is used in III. 1 and commonly.

[2] is in fact divisible, καὶ διαιρεῖται, literally "is actually divided."

[3] and distance one of the (straight lines D)E, (D)F, (D)G. The words and letters here shown in brackets are put in to fill out the rather careless language of the Greek. Here and in several other places in Book IV. Euclid says literally "and with distance one of the (points) E, F, G" (καὶ διαστήματι ἐνὶ τῶν E, Z, H) and the like. In one case (IV. 13) he actually has "with distance one of the points G, H, K, L, M" (διαστήματι ἐνὶ τῶν H, Θ, K, Λ, M σημείων). Heiberg notes "Graecam locutionem satis miram et negligentem," but, in view of its frequent occurrence in good MSS., does not venture to correct it.

BOOK V. DEFINITIONS.

1

A magnitude is a part of a magnitude, the less of the greater, when it measures the greater.

2

The greater is a multiple of the less when it is measured by the less.

3

A ratio is a sort of relation in respect of size between two magnitudes of the same kind.

4

Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another.

5

Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

6

Let magnitudes which have the same ratio be called proportional.

7

When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to have a greater ratio to the second than the third has to the fourth.

8

A proportion in three terms is the least possible.

9

When three magnitudes are proportional, the first is said to have to the third the duplicate ratio of that which it has to the second.

10

When four magnitudes are <continuously> proportional, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on continually, whatever be the proportion.

11

The term corresponding magnitudes is used of antecedents in relation to antecedents, and of consequents in relation to consequents.

12

Alternate ratio means taking the antecedent in relation to the antecedent and the consequent in relation to the consequent.

13

Inverse ratio means taking the consequent as antecedent in relation to the antecedent as consequent.

14

Composition of a ratio means taking the antecedent together with the consequent as one in relation to the consequent by itself.

15

Separation of a ratio means taking the excess by which the antecedent exceeds the consequent in relation to the consequent by itself.

16

Conversion of a ratio means taking the antecedent in relation to the excess by which the antecedent exceeds the consequent.

17

A ratio ex aequali arises when, there being several magnitudes and another set equal to them in multitude which taken two and two are in the same proportion, as the first is to the last among the first magnitudes, so is the first to the last among the second magnitudes;

Or, in other words, it means taking the extreme terms by virtue of the removal of the intermediate terms.

18

A perturbed proportion arises when, there being three magnitudes and another set equal to them in multitude, as antecedent is to consequent among the first magnitudes, so is antecedent to consequent among the second magnitudes, while, as the consequent is to a third among the first magnitudes, so is a third to the antecedent among the second magnitudes.

PROPOSITIONS.

PROPOSITION I.

If there be any number of magnitudes whatever which are, respectively, equimultiples of any magnitudes equal in multitude, then, whatever multiple one of the magnitudes is of one, that multiple also will all be of all.

Let any number of magnitudes whatever AB, CD be respectively equimultiples of any magnitudes E, F equal in multitude; I say that, whatever multiple AB is of E, that multiple will AB, CD also be of E, F.

For, since AB is the same multiple of E that CD is of F, as many magnitudes as there are in AB equal to E, so many also are there in CD equal to F.

Let AB be divided into the magnitudes AG, GB equal to E, and CD into CH, HD equal to F; then the multitude of the magnitudes AG, GB will be equal to the multitude of the magnitudes CH, HD.

Now, since AG is equal to E, and CH to F, therefore AG is equal to E, and AG, CH to E, F.

For the same reason

GB is equal to E, and GB, HD to E, F; therefore, as many magnitudes as there are in AB equal to E, so many also are there in AB, CD equal to E, F; therefore, whatever multiple AB is of E, that multiple will AB, CD also be of E, F.

Therefore etc. Q. E. D.

PROPOSITION 2.

If a first magnitude be the same multiple of a second that a third is of a fourth, and a fifth also be the same multiple of the second that a sixth is of the fourth, the sum of the first and fifth will also be the same multiple of the second that the sum of the third and sixth is of the fourth.

Let a first magnitude, AB, be the same multiple of a second, C, that a third, DE, is of a fourth, F, and let a fifth, BG, also be the same multiple of the second, C, that a sixth, EH, is of the fourth F; I say that the sum of the first and fifth, AG, will be the same multiple of the second, C, that the sum of the third and sixth, DH, is of the fourth, F.

For, since AB is the same multiple of C that DE is of F, therefore, as many magnitudes as there are in AB equal to C, so many also are there in DE equal to F.

For the same reason also, as many as there are in BG equal to C, so many are there also in EH equal to F; therefore, as many as there are in the whole AG equal to C, so many also are there in the whole DH equal to F.

Therefore, whatever multiple AG is of C, that multiple also is DH of F.

Therefore the sum of the first and fifth, AG, is the same multiple of the second, C, that the sum of the third and sixth, DH, is of the fourth, F.

Therefore etc. Q.E.D.

PROPOSITION 3.

If a first magnitude be the same multiple of a second that a third is of a fourth, and if equimultiples be taken of the first and third, then also ex aequali the magnitudes taken will be equimultiples respectively, the one of the second and the other of the fourth.

Let a first magnitude A be the same multiple of a second B that a third C is of a fourth D, and let equimultiples EF, GH be taken of A, C; I say that EF is

the same multiple of B that GH is of D.

For, since EF is the same multiple of A that GH is of C, therefore, as many magnitudes as there are in EF equal to A, so many also are there in GH equal to C.

Let EF be divided into the magnitudes EK, KF equal to A, and GH into the magnitudes GL, LH equal to C; then the multitude of the magnitudes EK, KF will be equal to the multitude of the magnitudes GL, LH.

And, since A is the same multiple of B that C is of D, while EK is equal to A, and GL to C, therefore EK is the same multiple of B that GL is of D.

For the same reason

KF is the same multiple of B that LH is of D.

Since, then, a first magnitude EK is the same multiple of a second B that a third GL is of a fourth D, and a fifth KF is also the same multiple of the second B that a sixth LH is of the fourth D, therefore the sum of the first and fifth, EF, is also the same multiple of the second B that the sum of the third and sixth, GH, is of the fourth D. {V. 2}

Therefore etc. Q. E. D.

PROPOSITION 4.

If a first magnitude have to a second the same ratio as a third to a fourth, any equimultiples whatever of the first and third will also have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order.

For let a first magnitude A have to a second B the same ratio as a third C to a fourth D; and let equimultiples E, F be taken of A, C, and G, H other, chance, equimultiples of B, D; I say that, as E is to G, so is F to H.

For let equimultiples K, L be taken of E, F, and other, chance, equimultiples M, N of G, H.

Since E is the same multiple of A that F is of C, and equimultiples K, L of E, F have been taken, therefore K is the same multiple of A that L is of C. {V. 3}

For the same reason

M is the same multiple of B that N is of D.

And, since, as A is to B, so is C to D, and of A, C equimultiples K, L have been taken, and of B, D other, chance, equimultiples M, N, therefore, if K is in excess of M, L also is in excess of N, if it is equal, equal, and if less, less. {V. Def. 5}

And K, L are equimultiples of E, F, and M, N other, chance, equimultiples of G, H; therefore, as E is to G, so is F to H. {V. Def. 5}

Therefore etc. Q. E. D.

PROPOSITION 5.

If a magnitude be the same multiple of a magnitude that a part subtracted is of a part subtracted, the remainder will also be the same multiple of the remainder that the whole is of the whole.

For let the magnitude AB be the same multiple of the magnitude CD that the part AE subtracted is of the part CF subtracted; I say that the remainder EB is also the same multiple of the remainder FD that the whole AB is of the whole CD.

For, whatever multiple AE is of CF, let EB be made that multiple of CG.

Then, since AE is the same multiple of CF that EB is of GC, therefore AE is the same multiple of CF that AB is of GF. {V. 1}

But, by the assumption, AE is the same multiple of CF that AB is of CD.

Therefore AB is the same multiple of each of the magnitudes GF, CD;

therefore GF is equal to CD.

Let CF be subtracted from each; therefore the remainder GC is equal to the remainder FD.

And, since AE is the same multiple of CF that EB is of GC, and GC is equal to DF,
therefore AE is the same multiple of CF that EB is of FD.

But, by hypothesis,

AE is the same multiple of CF that AB is of CD; therefore EB is the same multiple of FD that AB is of CD.

That is, the remainder EB will be the same multiple of
the remainder FD that the whole AB is of the whole CD.

Therefore etc. Q. E. D. [1]

PROPOSITION 6.

If two magnitudes be equimultiples of two magnitudes, and any magnitudes subtracted from them be equimultiples of the same, the remainders also are either equal to the same or equimultiples of them.

For let two magnitudes AB, CD be equimultiples of two magnitudes E, F, and let AG, CH subtracted from them be equimultiples of the same two E, F; I say that the remainders also, GB, HD, are either equal to E, F or equimultiples of them.

For, first, let GB be equal to E; I say that HD is also equal to F.

For let CK be made equal to F.

Since AG is the same multiple of E that CH is of F, while GB is equal to E and KC to F, therefore AB is the same multiple of E that KH is of F. {V. 2}

But, by hypothesis, AB is the same multiple of E that CD is of F; therefore KH is the same multiple of F that CD is of F.

Since then each of the magnitudes KH, CD is the same multiple of F,

therefore KH is equal to CD.

Let CH be subtracted from each; therefore the remainder KC is equal to the remainder HD.

But F is equal to KC; therefore HD is also equal to F.

Hence, if GB is equal to E, HD is also equal to F.

Similarly we can prove that, even if GB be a multiple of E, HD is also the same multiple of F.

Therefore etc. Q. E. D.

PROPOSITION 7.

Equal magnitudes have to the same the same ratio, as also has the same to equal magnitudes.

Let A, B be equal magnitudes and C any other, chance, magnitude; I say that each of the magnitudes A, B has the same ratio to C, and C has the same ratio to each of the magnitudes A, B.

For let equimultiples D, E of A, B be taken, and of C another, chance, multiple F.

Then, since D is the same multiple of A that E is of B, while A is equal to B,

therefore D is equal to E.

But F is another, chance, magnitude.

If therefore D is in excess of F, E is also in excess of F, if equal to it, equal; and, if less, less.

And D, E are equimultiples of A, B, while F is another, chance, multiple of C;

therefore, as A is to C, so is B to C. {V. Def. 5}

I say next that C also has the same ratio to each of the magnitudes A, B.

For, with the same construction, we can prove similarly that D is equal to E; and F is some other magnitude.

If therefore F is in excess of D, it is also in excess of E, if equal, equal; and, if less, less.

And F is a multiple of C, while D, E are other, chance, equimultiples of A, B;

therefore, as C is to A, so is C to B. {V. Def. 5}

Therefore etc.

PORISM.

From this it is manifest that, if any magnitudes are proportional, they will also be proportional inversely. Q. E. D.

PROPOSITION 8.

Of unequal magnitudes, the greater has to the same a greater ratio than the less has; and the same has to the less a greater ratio than it has to the greater.

Let AB, C be unequal magnitudes, and let AB be greater; let D be another, chance, magnitude; I say that AB has to D a greater ratio than C has to D, and D has to C a greater ratio than it has to AB.

For, since AB is greater than C, let BE be made equal to C; then the less of the magnitudes AE, EB, if multiplied, will sometime be greater than D. {V. Def. 4}

{Case I.}

First, let AE be less than EB; let AE be multiplied, and let FG be a multiple of it which is greater than D; then, whatever multiple FG is of AE, let GH be made the same multiple of EB and K of C; and let L be taken double of D, M triple of it, and successive multiples increasing by one, until what is taken is a multiple of D and the first that is greater than K. Let it be taken, and let it be N which is quadruple of D and the first multiple of it that is greather than K.

Then, since K is less than N first, therefore K is not less than M.

And, since FG is the same multiple of AE that GH is of EB, therefore FG is the same multiple of AE that FH is of AB. {V. 1}

But FG is the same multiple of AE that K is of C;

therefore FH is the same multiple of AB that K is of C; therefore FH, K are equimultiples of AB, C.

Again, since GH is the same multiple of EB that K is of C, and EB is equal to C,

therefore GH is equal to K.

But K is not less than M;

therefore neither is GH less than M.

And FG is greater than D; therefore the whole FH is greater than D, M together.

But D, M together are equal to N, inasmuch as M is triple of D, and M, D together are quadruple of D, while N is also quadruple of D; whence M, D together are equal to N.

But FH is greater than M, D;

therefore FH is in excess of N,

while K is not in excess of N.

And FH, K are equimultiples of AB, C, while N is another, chance, multiple of D;

therefore AB has to D a greater ratio than C has to D. {V. Def. 7}

I say next, that D also has to C a greater ratio than D has to AB.

For, with the same construction, we can prove similarly that N is in excess of K, while N is not in excess of FH.

And N is a multiple of D, while FH, K are other, chance, equimultiples of AB, C;

therefore D has to C a greater ratio than D has to AB. {V. Def. 7}

{Case 2.}

Again, let AE be greater than EB.

Then the less, EB, if multiplied, will sometime be greater than D. {V. Def. 4}

Let it be multiplied, and let GH be a multiple of EB and greater than D; and, whatever multiple GH is of EB, let FG be made the same multiple of AE, and K of C.

Then we can prove similarly that FH, K are equimultiples of AB, C; and, similarly, let N be taken a multiple of D but the first that is greater than FG,

so that FG is again not less than M.

But GH is greater than D; therefore the whole FH is in excess of D, M, that is, of N.

Now K is not in excess of N, inasmuch as FG also, which is greater than GH, that is, than K, is not in excess of N.

And in the same manner, by following the above argument, we complete the demonstration.

Therefore etc. Q. E. D.

PROPOSITION 9.

Magnitudes which have the same ratio to the same are equal to one another; and magnitudes to which the same has the same ratio are equal.

For let each of the magnitudes A, B have the same ratio to C; I say that A is equal to B.

For, otherwise, each of the magnitudes A, B would not have had the same ratio to C; {V. 8} but it has;

therefore A is equal to B.

Again, let C have the same ratio to each of the magnitudes A, B; I say that A is equal to B.

For, otherwise, C would not have had the same ratio to each of the magnitudes A, B; {V. 8} but it has;

therefore A is equal to B.

Therefore etc. Q. E. D.

PROPOSITION 10.

Of magnitudes which have a ratio to the same, that which has a greater ratio is greater; and that to which the same has a greater ratio is less.

For let A have to C a greater ratio than B has to C; I say that A is greater than B.

For, if not, A is either equal to B or less.

Now A is not equal to B; for in that case each of the magnitudes A, B would have had the same ratio to C; {V. 7} but they have not;

therefore A is not equal to B.

Nor again is A less than B; for in that case A would have had to C a less ratio than B has to C; {V. 8} but it has not;

therefore A is not less than B.

But it was proved not to be equal either;

therefore A is greater than B.

Again, let C have to B a greater ratio than C has to A; I say that B is less than A.

For, if not, it is either equal or greater.

Now B is not equal to A; for in that case C would have had the same ratio to each of the magnitudes A, B; {V. 7} but it has not;

therefore A is not equal to B.

Nor again is B greater than A; for in that case C would have had to B a less ratio than it has to A; {V. 8} but it has not;

therefore B is not greater than A.

But it was proved that it is not equal either;

therefore B is less than A.

Therefore etc. Q. E. D.

PROPOSITION 11.

Ratios which are the same with the same ratio are also the same with one another.

For, as A is to B, so let C be to D, and, as C is to D, so let E be to F; I say that, as A is to B, so is E to F.

For of A, C, E let equimultiples G, H, K be taken, and of B, D, F other, chance, equimultiples L, M, N.

Then since, as A is to B, so is C to D, and of A, C equimultiples G, H have been taken, and of B, D other, chance, equimultiples L, M, therefore, if G is in excess of L, H is also in excess of M, if equal, equal, and if less, less.

Again, since, as C is to D, so is E to F, and of C, E equimultiples H, K have been taken, and of D, F other, chance, equimultiples M, N, therefore, if H is in excess of M, K is also in excess of N, if equal, equal, and if less, less.

But we saw that, if H was in excess of M, G was also in excess of L; if equal, equal; and if less, less; so that, in addition, if G is in excess of L, K is also in excess of N, if equal, equal, and if less, less.

And G, K are equimultiples of A, E, while L, N are other, chance, equimultiples of B, F;

therefore, as A is to B, so is E to F.

Therefore etc. Q. E. D.

PROPOSITION 12.

If any number of magnitudes be proportional, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents.

Let any number of magnitudes A, B, C, D, E, F be proportional, so that, as A is to B, so is C to D and E to F; I say that, as A is to B, so are A, C, E to B, D, F.

For of A, C, E let equimultiples G, H, K be taken, and of B, D, F other, chance, equimultiples L, M, N.

Then since, as A is to B, so is C to D, and E to F, and of A, C, E equimultiples G, H, K have been taken, and of B, D, F other, chance, equimultiples L, M, N, therefore, if G is in excess of L, H is also in excess of M, and K of N, if equal, equal, and if less, less; so that, in addition, if G is in excess of L, then G, H, K are in excess of L, M, N, if equal, equal, and if less, less.

Now G and G, H, K are equimultiples of A and A, C, E, since, if any number of magnitudes whatever are respectively equimultiples of any magnitudes equal in multitude, whatever multiple one of the magnitudes is of one, that multiple also will all be of all. {V. 1}

For the same reason L and L, M, N are also equimultiples of B and B, D, F;

therefore, as A is to B, so are A, C, E to B, D, F. {V. Def. 5}

Therefore etc. Q. E. D.

PROPOSITION 13.

If a first magnitude have to a second the same ratio as a third to a fourth, and the third have to the fourth a greater ratio than a fifth has to a sixth, the first will also have to the second a greater ratio than the fifth to the sixth.

For let a first magnitude A have to a second B the same ratio as a third C has to a fourth D, and let the third C have to the fourth D a greater ratio than a fifth E has to a sixth F; I say that the first A will also have to the second B a greater ratio than the fifth E to the sixth F.

For, since there are some equimultiples of C, E, and of D, F other, chance, equimultiples, such that the multiple of C is in excess of the multiple of D, while the multiple of E is not in excess of the multiple of F, {V. Def. 7} let them be taken, and let G, H be equimultiples of C, E, and K, L other, chance, equimultiples of D, F, so that G is in excess of K, but H is not in excess of L; and, whatever multiple G is of C, let M be also that multiple of A, and, whatever multiple K is of D, let N be also that multiple of B.

Now, since, as A is to B, so is C to D, and of A, C equimultiples M, G have been taken, and of B, D other, chance, equimultiples N, K, therefore, if M is in excess of N, G is also in excess of K, if equal, equal, and if less, less. {V. Def. 5}

But G is in excess of K; therefore M is also in excess of N.

But H is not in excess of L; and M, H are equimultiples of A, E, and N, L other, chance, equimultiples of B, F;

therefore A has to B a greater ratio than E has to F. {V. Def. 7}

Therefore etc. Q. E. D.

PROPOSITION 14.

If a first magnitude have to a second the same ratio as a third has to a fourth, and the first be greater than the third, the second will also be greater than the fourth; if equal, equal; and if less, less.

For let a first magnitude A have the same ratio to a second B as a third C has to a fourth D; and let A be greater than C; I say that B is also greater than D.

For, since A is greater than C, and B is another, chance, magnitude, therefore A has to B a greater ratio than C has to B. {V. 8}

But, as A is to B, so is C to D;

therefore C has also to D a greater ratio than C has to B. {V. 13}

But that to which the same has a greater ratio is less; {V. 10}

therefore D is less than B; so that B is greater than D.

Similarly we can prove that, if A be equal to C, B will also be equal to D; and, if A be less than C, B will also be less than D.

Therefore etc. Q. E. D.

PROPOSITION 15.

Parts have the same ratio as the same multiples of them taken in corresponding order.

For let AB be the same multiple of C that DE is of F; I say that, as C is to F, so is AB to DE.

For, since AB is the same multiple of C that DE is of F, as many magnitudes as there are in AB equal to C, so many are there also in DE equal to F.

Let AB be divided into the magnitudes AG, GH, HB equal to C, and DE into the magnitudes DK, KL, LE equal to F; then the multitude of the magnitudes AG, GH, HB will be equal to the multitude of the magnitudes DK, KL, LE.

And, since AG. GH, HB are equal to one another, and DK, KL, LE are also equal to one another, therefore, as AG is to DK, so is GH to KL, and HB to LE. {V. 7}

Therefore, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents; {V. 12}

therefore, as AG is to DK, so is AB to DE.

But AG is equal to C and DK to F;

therefore, as C is to F, so is AB to DE.

Therefore etc. Q. E. D.

PROPOSITION 16.

If four magnitudes be proportional, they will also be proportional alternately.

Let A, B, C, D be four proportional magnitudes, so that, as A is to B, so is C to D; I say that they will also be so alternately, that is, as A is to C, so is B to D.

For of A, B let equimultiples E, F be taken, and of C, D other, chance, equimultiples G, H.

Then, since E is the same multiple of A that F is of B, and parts have the same ratio as the same multiples of them, {V. 15} therefore, as A is to B, so is E to F.

But as A is to B, so is C to D; therefore also, as C is to D, so is E to F. {V. 11}

Again, since G, H are equimultiples of C, D, therefore, as C is to D, so is G to H. {V. 15}

But, as C is to D, so is E to F; therefore also, as E is to F, so is G to H. {V. 11}

But, if four magnitudes be proportional, and the first be greater than the third,

the second will also be greater than the fourth;

if equal, equal; and if less, less. {V. 14}

Therefore, if E is in excess of G, F is also in excess of H, if equal, equal, and if less, less.

Now E, F are equimultiples of A, B, and G, H other, chance, equimultiples of C, D;

therefore, as A is to C, so is B to D. {V. Def. 5}

Therefore etc. Q. E. D. [2]

PROPOSITION 17.

If magnitudes be proportional componendo, they will also be proportional separando.

Let AB, BE, CD, DF be magnitudes proportional componendo, so that, as AB is to BE, so is CD to DF; I say that they will also be proportional separando, that is, as AE is to EB, so is CF to DF.

For of AE, EB, CF, FD let equimultiples GH, HK, LM, MN be taken, and of EB, FD other, chance, equimultiples, KO, NP.

Then, since GH is the same multiple of AE that HK is of EB, therefore GH is the same multiple of AE that GK is of AB. {V. 1}

But GH is the same multiple of AE that LM is of CF; therefore GK is the same multiple of AB that LM is of CF.

Again, since LM is the same multiple of CF that MN is of FD, therefore LM is the same multiple of CF that LN is of CD. {V. 1}

But LM was the same multiple of CF that GK is of AB; therefore GK is the same multiple of AB that LN is of CD.

Therefore GK, LN are equimultiples of AB, CD.

Again, since HK is the same multiple of EB that MN is of FD,

and KO is also the same multiple of EB that NP is of FD, therefore the sum HO is also the same multiple of EB that MP is of FD. {V. 2}

And, since, as AB is to BE, so is CD to DF, and of AB, CD equimultiples GK, LN have been taken, and of EB, FD equimultiples HO, MP, therefore, if GK is in excess of HO, LN is also in excess of MP, if equal, equal, and if less, less.

Let GK be in excess of HO; then, if HK be subtracted from each,

GH is also in excess of KO.

But we saw that, if GK was in excess of HO, LN was also in excess of MP;

therefore LN is also in excess of MP,

and, if MN be subtracted from each,

LM is also in excess of NP;

so that, if GH is in excess of KO, LM is also in excess of NP.

Similarly we can prove that, if GH be equal to KO, LM will also be equal to NP, and if less, less.

And GH, LM are equimultiples of AE, CF, while KO, NP are other, chance, equimultiples of EB, FD;

therefore, as AE is to EB, so is CF to FD.

Therefore etc. Q. E. D.

PROPOSITION 18.

If magnitudes be proportional separando, they will also be proportional componendo.

Let AE, EB, CF, FD be magnitudes proportional separando, so that, as AE is to EB, so is CF to FD; I say that they will also be proportional componendo, that is, as AB is to BE, so is CD to FD.

For, if CD be not to DF as AB to BE, then, as AB is to BE, so will CD be either to some magnitude less than DF or to a greater.

First, let it be in that ratio to a less magnitude DG.

Then, since, as AB is to BE, so is CD to DG, they are magnitudes proportional componendo;

so that they will also be proportional separando. {V. 17}

Therefore, as AE is to EB, so is CG to GD.

But also, by hypothesis,

as AE is to EB, so is CF to FD.

Therefore also, as CG is to GD, so is CF to FD. {V. 11}

But the first CG is greater than the third CF;

therefore the second GD is also greater than the fourth FD. {V. 14}

But it is also less: which is impossible.

Therefore, as AB is to BE, so is not CD to a less magnitude than FD.

Similarly we can prove that neither is it in that ratio to a greater;

it is therefore in that ratio to FD itself.

Therefore etc. Q. E. D.

PROPOSITION 19

If, as a whole is to a whole, so is a part subtracted to a part subtracted, the remainder will also be to the remainder as whole to whole.

For, as the whole AB is to the whole CD, so let the part AE subtracted be to the part CF subtracted; I say that the remainder EB will also be to the remainder FD as the whole AB to the whole CD.

For since, as AB is to CD, so is AE to CF, alternately also, as BA is to AE, so is DC to CF. {V. 16}

And, since the magnitudes are proportional componendo, they will also be proportional separando, {V. 17} that is, as BE is to EA, so is DF to CF, and, alternately,

as BE is to DF, so is EA to FC. {V. 16}

But, as AE is to CF, so by hypothesis is the whole AB to the whole CD.

Therefore also the remainder EB will be to the remainder FD as the whole AB is to the whole CD. {V. 11}

Therefore etc. {

PORISM.

From this it is manifest that, if magnitudes be proportional componendo, they will also be proportional convertendo.

} Q. E. D.

PROPOSITION 20.

If there be three magnitudes, and others equal to them in multitude, which taken two and two are in the same ratio, and if ex aequali the first be greater than the third, the fourth will also be greater than the sixth; if equal, equal; and, if less, less.

Let there be three magnitudes A, B, C, and others D, E, F equal to them in multitude, which taken two and two are in the same ratio, so that,

as A is to B, so is D to E,

and as B is to C, so is E to F; and let A be greater than C ex aequali; I say that D will also be greater than F; if A is equal to C, equal; and, if less, less.

For, since A is greater than C, and B is some other magnitude, and the greater has to the same a greater ratio than the less has, {V. 8} therefore A has to B a greater ratio than C has to B.

But, as A is to B, so is D to E, and, as C is to B, inversely, so is F to E; therefore D has also to E a greater ratio than F has to E. {V. 13}

But, of magnitudes which have a ratio to the same, that which has a greater ratio is greater; {V. 10}

therefore D is greater than F.

Similarly we can prove that, if A be equal to C, D will also be equal to F; and if less, less.

Therefore etc. Q. E. D.

PROPOSITION 21.

If there be three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, then, if ex aequali the first magnitude is greater than the third, the fourth will also be greater than the sixth; if equal, equal; and if less, less.

Let there be three magnitudes A, B, C, and others D, E, F equal to them in multitude, which taken two and two are in the same ratio, and let the proportion of them be perturbed, so that,

as A is to B, so is E to F,

and, as B is to C, so is D to E, and let A be greater than C ex aequali; I say that D will also be greater than F; if A is equal to C, equal; and if less, less.

For, since A is greater than C, and B is some other magnitude, therefore A has to B a greater ratio than C has to B. {V. 8}

But, as A is to B, so is E to F, and, as C is to B, inversely, so is E to D. Therefore also E has to F a greater ratio than E has to D. {V. 13}

But that to which the same has a greater ratio is less; {V. 10}

therefore F is less than D; therefore D is greater than F.

Similarly we can prove that,

if A be equal to C, D will also be equal to F;

and if less, less.

Therefore etc. Q. E. D.

PROPOSITION 22.

If there be any number of magnitudes whatever, and others equal to them in multitude, which taken two and two together are in the same ratio, they will also be in the same ratio ex aequali.

Let there be any number of magnitudes A, B, C, and others D, E, F equal to them in multitude, which taken two and two together are in the same ratio, so that,

as A is to B, so is D to E,

and, as B is to C, so is E to F; I say that they will also be in the same ratio ex aequali,

<that is, as A is to C, so is D to F>.

For of A, D let equimultiples G, H be taken, and of B, E other, chance, equimultiples K, L; and, further, of C, F other, chance, equimultiples M, N.

Then, since, as A is to B, so is D to E, and of A, D equimultiples G, H have been taken, and of B, E other, chance, equimultiples K, L,

therefore, as G is to K, so is H to L. {V. 4}

For the same reason also,

as K is to M, so is L to N.

Since, then, there are three magnitudes G, K, M, and others H, L, N equal to them in multitude, which taken two and two together are in the same ratio, therefore, ex aequali, if G is in excess of M, H is also in excess of N; if equal, equal; and if less, less. {V. 20}

And G, H are equimultiples of A, D,

and M, N other, chance, equimultiples of C, F.

Therefore, as A is to C, so is D to F. {V. Def. 5}

Therefore etc. Q. E. D.

PROPOSITION 23.

If there be three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, they will also be in the same ratio ex aequali.

Let there be three magnitudes A, B, C, and others equal to them in multitude, which, taken two and two together, are in the same proportion, namely D, E, F; and let the proportion of them be perturbed, so that,

as A is to B, so is E to F,

and, as B is to C, so is D to E; I say that, as A is to C, so is D to F.

Of A, B, D let equimultiples G, H, K be taken, and of C, E, F other, chance, equimultiples L, M, N.

Then, since G, H are equimultiples of A, B, and parts have the same ratio as the same multiples of them, {V. 15}

therefore, as A is to B, so is G to H.

For the same reason also,

as E is to F, so is M to N.

And, as A is to B, so is E to F;

therefore also, as G is to H, so is M to N. {V. 11}

Next, since, as B is to C, so is D to E, alternately, also, as B is to D, so is C to E. {V. 16}

And, since H, K are equimultiples of B, D, and parts have the same ratio as their equimultiples,

therefore, as B is to D, so is H to K. {V. 15}

But, as B is to D, so is C to E;

therefore also, as H is to K, so is C to E. {V. 11}

Again, since L, M are equimultiples of C, E,

therefore, as C is to E, so is L to M. {V. 15}

But, as C is to E, so is H to K;

therefore also, as H is to K, so is L to M, {V. 11}

and, alternately, as H is to L, so is K to M. {V. 16}

But it was also proved that,

as G is to H, so is M to N.

Since, then, there are three magnitudes G, H, L, and others equal to them in multitude K, M, N, which taken two and two together are in the same ratio, and the proportion of them is perturbed, therefore, ex aequali, if G is in excess of L, K is also in excess of N; if equal, equal; and if less, less. {V. 21}

And G, K are equimultiples of A, D, and L, N of C, F.

Therefore, as A is to C, so is D to F.

Therefore etc. Q. E. D.

PROPOSITION 24.

If a first magnitude have to a second the same ratio as a third has to a fourth, and also a fifth have to the second the same ratio as a sixth to the fourth, the first and fifth added together will have to the second the same ratio as the third and sixth have to the fourth.

Let a first magnitude AB have to a second C the same ratio as a third DE has to a fourth F; and let also a fifth BG have to the second C the same ratio as a sixth EH has to the fourth F; I say that the first and fifth added together, AG, will have to the second C the same ratio as the third and sixth, DH, has to the fourth F.

For since, as BG is to C, so is EH to F, inversely, as C is to BG, so is F to EH.

Since, then, as AB is to C, so is DE to F,

and, as C is to BG, so is F to EH,

therefore, ex aequali, as AB is to BG, so is DE to EH. {V. 22}

And, since the magnitudes are proportional separando, they will also be proportional componendo; {V. 18}

therefore, as AG is to GB, so is DH to HE.

But also, as BG is to C, so is EH to F; therefore, ex aequali, as AG is to C, so is DH to F. {V. 22}

Therefore etc. Q. E. D.

PROPOSITION 25.

If four magnitudes be proportional, the greatest and the least are greater than the remaining two.

Let the four magnitudes AB, CD, E, F be proportional so that, as AB is to CD, so is E to F, and let AB be the greatest of them and F the least; I say that AB, F are greater than CD, E.

For let AG be made equal to E, and CH equal to F.

Since, as AB is to CD, so is E to F, and E is equal to AG, and F to CH,

therefore, as AB is to CD, so is AG to CH.

And since, as the whole AB is to the whole CD, so is the part AG subtracted to the part CH subtracted,

the remainder GB will also be to the remainder HD as the whole AB is to the whole CD. {V. 19}

But AB is greater than CD;

therefore GB is also greater than HD.

And, since AG is equal to E, and CH to F, therefore AG, F are equal to CH, E.

And if, GB, HD being unequal, and GB greater, AG, F be added to GB and CH, E be added to HD,

it follows that AB, F are greater than CD, E.

Therefore etc. Q. E. D.

ENDNOTES.

[1] let EB be made that multiple of CG, τοσαυταπλάσιον γεγονέτω καὶ τὸ ΕΒ τοῦ ΓΗ. From this way of stating the construction one might suppose that CG was given and EB had to be found equal to a certain multiple of it. But in fact EB is what is given and CG has to be found, i.e. CG has to be constructed as a certain submultiple of EB.

[2] "Let A, B, C, D be four proportional magnitudes, so that, as A is to B, so is C to D." In a number of expressions like this it is absolutely necessary, when translating into English, to interpolate words which are not in the Greek. Thus the Greek here is: Ἔστω τέσσαρα μεγέθη ἀνάλογον τὰ Α, Β, Γ, Δ, ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, literally "Let A, B, C, D be four proportional magnitudes, as A to B, so C to D." The same remark applies to the corresponding expressions in the next propositions, V. 17, 18, and to other forms of expression in V. 20-23 and later propositions: e.g. in V. 20 we have a phrase meaning literally "Let there be magnitudes...which taken two and two are in the same ratio, as A to B, so D to E," etc.: in V. 21 "(magnitudes)...which taken two and two are in the same ratio, and let the proportion of them be perturbed, as A to B, so E to F," etc. In all such cases (where the Greek is so terse as to be almost ungrammatical) I shall insert the words necessary in English, without further remark.

BOOK VI. DEFINITIONS.

1

Similar rectilineal figures are such as have their angles severally equal and the sides about the equal angles proportional.

2

{Reciprocally related figures. See note.}

3

A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less.

4

The height of any figure is the perpendicular drawn from the vertex to the base.

5

PROPOSITIONS.

PROPOSITION 1.

Triangles and parallelograms which are under the same height are to one another as their bases.

Let ABC, ACD be triangles and EC, CF parallelograms under the same height;
I say that, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD, and the parallelogram EC to the parallelogram CF.

For let BD be produced in both directions to the points H, L and let {any number of straight lines} BG, GH be
made equal to the base BC, and any number of straight lines DK, KL equal to the base CD; let AG, AH, AK, AL be joined.

Then, since CB, BG, GH are equal to one another,

the triangles ABC, AGB, AHG are also equal to one another. {I. 38}

Therefore, whatever multiple the base HC is of the base BC, that multiple also is the triangle AHC of the triangle ABC.

For the same reason,
whatever multiple the base LC is of the base CD, that multiple also is the triangle ALC of the triangle ACD; and, if the base HC is equal to the base CL, the triangle AHC is also equal to the triangle ACL, {I. 38} if the base HC is in excess of the base CL, the triangle AHC
is also in excess of the triangle ACL, and, if less, less.

Thus, there being four magnitudes, two bases BC, CD and two triangles ABC, ACD, equimultiples have been taken of the base BC and the triangle ABC, namely the base HC and the triangle AHC, and of the base CD and the triangle ADC other, chance, equimultiples, namely the base LC and the triangle ALC;

and it has been proved that,

if the base HC is in excess of the base CL, the triangle AHC is also in excess of the triangle ALC; if equal, equal; and, if less, less.

Therefore, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD. {V. Def. 5}

Next, since the parallelogram EC is double of the triangle ABC, {I. 41} and the parallelogram FC is double of the triangle ACD, while parts have the same ratio as the same multiples of them, {V. 15} therefore, as the triangle ABC is to the triangle ACD, so is the parallelogram EC to the parallelogram FC.

Since, then, it was proved that, as the base BC is to CD, so is the triangle ABC to the triangle ACD, and, as the triangle ABC is to the triangle ACD, so is the parallelogram EC to the parallelogram CF, therefore also, as the base BC is to the base CD, so is the parallelogram EC to the parallelogram FC. {V. 11}

Therefore etc. Q. E. D. [1] [2] [3]

PROPOSITION 2.

If a straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally; and, if the sides of the triangle be cut proportionally, the line joining the points of section will be parallel to the remaining side of the triangle.

For let DE be drawn parallel to BC, one of the sides of the triangle ABC; I say that, as BD is to DA, so is CE to EA.

For let BE, CD be joined.

Therefore the triangle BDE is equal to the triangle CDE; for they are on the same base DE and in the same parallels DE, BC. {I. 38}

And the triangle ADE is another area.

But equals have the same ratio to the same; {V. 7} therefore, as the triangle BDE is to the triangle ADE, so is the triangle CDE to the triangle ADE.

But, as the triangle BDE is to ADE, so is BD to DA; for, being under the same height, the perpendicular drawn from E to AB, they are to one another as their bases. {VI. 1}

For the same reason also, as the triangle CDE is to ADE, so is CE to EA.

Therefore also, as BD is to DA, so is CE to EA. {V. 11}

Again, let the sides AB, AC of the triangle ABC be cut proportionally, so that, as BD is to DA, so is CE to EA; and let DE be joined.

I say that DE is parallel to BC.

For, with the same construction, since, as BD is to DA, so is CE to EA, but, as BD is to DA, so is the triangle BDE to the triangle ADE, and, as CE is to EA, so is the triangle CDE to the triangle ADE, {VI. 1} therefore also,

as the triangle BDE is to the triangle ADE, so is the triangle CDE to the triangle ADE. {V. 11}

Therefore each of the triangles BDE, CDE has the same ratio to ADE.

Therefore the triangle BDE is equal to the triangle CDE; {V. 9} and they are on the same base DE.

But equal triangles which are on the same base are also in the same parallels. {I. 39}

Therefore DE is parallel to BC.

Therefore etc. Q. E. D.

PROPOSITION 3.

If an angle of a triangle be bisected and the straight line cutting the angle cut the base also, the segments of the base will have the same ratio as the remaining sides of the triangle; and, if the segments of the base have the same ratio as the remaining sides of the triangle, the straight line joined from the vertex to the point of section will bisect the angle of the triangle.

Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD; I say that, as BD is to CD, so is BA to AC.

For let CE be drawn through C parallel to DA, and let BA be carried through and meet it at E.

Then, since the straight line AC falls upon the parallels AD, EC,

the angle ACE is equal to the angle CAD. {I. 29}

But the angle CAD is by hypothesis equal to the angle BAD; therefore the angle BAD is also equal to the angle ACE.

Again, since the straight line BAE falls upon the parallels AD, EC,

the exterior angle BAD is equal to the interior angle AEC. {I. 29}

But the angle ACE was also proved equal to the angle BAD;

therefore the angle ACE is also equal to the angle AEC, so that the side AE is also equal to the side AC. {I. 6}

And, since AD has been drawn parallel to EC, one of the sides of the triangle BCE, therefore, proportionally, as BD is to DC, so is BA to AE.

But AE is equal to AC; {VI. 2} therefore, as BD is to DC, so is BA to AC.

Again, let BA be to AC as BD to DC, and let AD be joined; I say that the angle BAC has been bisected by the straight line A.D.

For, with the same construction, since, as BD is to DC, so is BA to AC, and also, as BD is to DC, so is BA to AE : for AD has been drawn parallel to EC, one of the sides of the triangle BCE: {VI. 2} therefore also, as BA is to AC, so is BA to AE. {V. 11}

Therefore AC is equal to AE, {V. 9} so that the angle AEC is also equal to the angle ACE. {I. 5}

But the angle AEC is equal to the exterior angle BAD, {I. 29} and the angle ACE is equal to the alternate angle CAD; {id.}

therefore the angle BAD is also equal to the angle CAD.

Therefore the angle BAC has been bisected by the straight line AD.

Therefore etc. Q. E. D.

PROPOSITION 4.

In equiangular triangles the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles.

Let ABC, DCE be equiangular triangles having the angle ABC equal to the angle DCE, the angle BAC to the angle CDE, and further the angle ACB to the angle CED; I say that in the triangles ABC, DCE the sides about the

equal angles are proportional, and those are corresponding sides which subtend the equal angles.

For let BC be placed in a straight line with CE.

Then, since the angles ABC, ACB are less than two right angles, {I. 17} and the angle ACB is equal to the angle DEC, therefore the angles ABC, DEC are less than two right angles; therefore BA, ED, when produced, will meet. {I. Post. 5}

Let them be produced and meet at F.

Now, since the angle DCE is equal to the angle ABC,

BF is parallel to CD. {I. 28}

Again, since the angle ACB is equal to the angle DEC,

AC is parallel to FE. {I. 28}

Therefore FACD is a parallelogram; therefore FA is equal to DC, and AC to FD. {I. 34}

And, since AC has been drawn parallel to FE, one side of the triangle FBE, therefore, as BA is to AF, so is BC to CE. {VI. 2}

But AF is equal to CD;

therefore, as BA is to CD, so is BC to CE,

and alternately, as AB is to BC, so is DC to CE. {V. 16}

Again, since CD is parallel to BF, therefore, as BC is to CE, so is FD to DE. {VI. 2}

But FD is equal to AC;

therefore, as BC is to CE, so is AC to DE,

and alternately, as BC is to CA, so is CE to ED. {V. 16}

Since then it was proved that,

as AB is to BC, so is DC to CE,

and, as BC is to CA, so is CE to ED; therefore, ex aequali, as BA is to AC, so is CD to DE. {V. 22}

Therefore etc. Q. E. D.

PROPOSITION 5.

If two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend.

Let ABC, DEF be two triangles having their sides proportional, so that,

as AB is to BC, so is DE to EF, as BC is to CA, so is EF to FD,

and further, as BA is to AC, so is ED to DF; I say that the triangle ABC is equiangular with the triangle DEF, and they will have those angles equal which the corresponding sides subtend, namely the angle ABC to the angle DEF, the angle BCA to the angle EFD, and further the angle BAC to the angle EDF.

For on the straight line EF, and at the points E, F on it, let there be constructed the angle FEG equal to the angle ABC, and the angle EFG equal to the angle ACB; {I. 23}

therefore the remaining angle at A is equal to the remaining angle at G. {I. 32}

Therefore the triangle ABC is equiangular with the triangle GEF.

Therefore in the triangles ABC, GEF the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles; {VI. 4}

therefore, as AB is to BC, so is GE to EF.

But, as AB is to BC, so by hypothesis is DE to EF;

therefore, as DE is to EF, so is GE to EF. {V. 11}

Therefore each of the straight lines DE, GE has the same ratio to EF;

therefore DE is equal to GE. {V. 9}

For the same reason

DF is also equal to GF.

Since then DE is equal to EG, and EF is common,

the two sides DE, EF are equal to the two sides GE, EF; and the base DF is equal to the base FG;

therefore the angle DEF is equal to the angle GEF, {I. 8} and the triangle DEF is equal to the triangle GEF, and the remaining angles are equal to the remaining angles, namely those which the equal sides subtend. {I. 4}

Therefore the angle DFE is also equal to the angle GFE,

and the angle EDF to the angle EGF.

And, since the angle FED is equal to the angle GEF, while the angle GEF is equal to the angle ABC, therefore the angle ABC is also equal to the angle DEF.

For the same reason

the angle ACB is also equal to the angle DFE,

and further, the angle at A to the angle at D;

therefore the triangle ABC is equiangular with the triangle DEF.

Therefore etc. Q. E. D.

PROPOSITION 6.

If two triangles have one angle equal to one angle and the sides about the equal angles proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend.

Let ABC, DEF be two triangles having one angle BAC equal to one angle EDF and the sides about the equal angles proportional, so that,

as BA is to AC, so is ED to DF;

I say that the triangle ABC is equiangular with the triangle DEF, and will have the angle ABC equal to the angle DEF, and the angle ACB to the angle DFE.

For on the straight line DF, and at the points D, F on it, let there be constructed the angle FDG equal to either of the angles BAC, EDF, and the angle DFG equal to the angle ACB; {I. 23}

therefore the remaining angle at B is equal to the remaining angle at G. {I. 32}

Therefore the triangle ABC is equiangular with the triangle DGF.

Therefore, proportionally, as BA is to AC, so is GD to DF. {VI. 4}

But, by hypothesis, as BA is to AC, so also is ED to DF; therefore also, as ED is to DF, so is GD to DF. {V. 11}

Therefore ED is equal to DG; {V. 9} and DF is common;

therefore the two sides ED, DF are equal to the two sides GD, DF; and the angle EDF is equal to the angle GDF; therefore the base EF is equal to the base GF, and the triangle DEF is equal to the triangle DGF,

and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend. {I. 4}

Therefore the angle DFG is equal to the angle DFE,

and the angle DGF to the angle DEF.

But the angle DFG is equal to the angle ACB; therefore the angle ACB is also equal to the angle DFE.

And, by hypothesis, the angle BAC is also equal to the angle EDF; therefore the remaining angle at B is also equal to the remaining angle at E; {I. 32}

therefore the triangle ABC is equiangular with the triangle DEF.

Therefore etc. Q. E. D.

PROPOSITION 7.

If two triangles have one angle equal to one angle, the sides about other angles proportional, and the remaining angles either both less or both not less than a right angle, the triangles will be equiangular and will have those angles equal, the sides about which are proportional.

Let ABC, DEF be two triangles having one angle equal to one angle, the angle BAC to the angle EDF, the sides about other angles ABC, DEF proportional, so that, as AB is to BC, so is DE to EF, and, first, each of the remaining angles at C, F less than a right angle; I say that the triangle ABC is equiangular with the triangle DEF, the angle ABC will be equal to the angle DEF, and the remaining angle, namely the angle at C, equal to the remaining angle, the angle at F.

For, if the angle ABC is unequal to the angle DEF, one of them is greater.

Let the angle ABC be greater; and on the straight line AB, and at the point B on it, let the angle ABG be constructed equal to the angle DEF. {I. 23}

Then, since the angle A is equal to D, and the angle ABG to the angle DEF, therefore the remaining angle AGB is equal to the remaining angle

DFE. {I. 32}

Therefore the triangle ABG is equiangular with the triangle DEF.

Therefore, as AB is to BG, so is DE to EF {VI. 4}

But, as DE is to EF, so by hypothesis is AB to BC; therefore AB has the same ratio to each of the straight lines BC, BG; {V. 11}

therefore BC is equal to BG, {V. 9}

so that the angle at C is also equal to the angle BGC. {I. 5}

But, by hypothesis, the angle at C is less than a right angle; therefore the angle BGC is also less than a right angle; so that the angle AGB adjacent to it is greater than a right angle. {I. 13}

And it was proved equal to the angle at F; therefore the angle at F is also greater than a right angle.

But it is by hypothesis less than a right angle : which is absurd.

Therefore the angle ABC is not unequal to the angle DEF; therefore it is equal to it.

But the angle at A is also equal to the angle at D; therefore the remaining angle at C is equal to the remaining angle at F. {I. 32}

Therefore the triangle ABC is equiangular with the triangle DEF.

But, again, let each of the angles at C, F be supposed not less than a right angle; I say again that, in this case too, the triangle ABC is equiangular with the triangle DEF.

For, with the same construction, we can prove similarly that

BC is equal to BG;

so that the angle at C is also equal to the angle BGC. {I. 5}

But the angle at C is not less than a right angle; therefore neither is the angle BGC less than a right angle.

Thus in the triangle BGC the two angles are not less than two right angles: which is impossible. {I. 17}

Therefore, once more, the angle ABC is not unequal to the angle DEF; therefore it is equal to it.

But the angle at A is also equal to the angle at D; therefore the remaining angle at C is equal to the remaining angle at F. {I. 32}

Therefore the triangle ABC is equiangular with the triangle DEF.

Therefore etc. Q. E. D.

PROPOSITION 8.

If in a right-angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another.

Let ABC be a right-angled triangle having the angle BAC right, and let AD be drawn from A perpendicular to BC; I say that each of the triangles ABD, ADC is similar to the whole ABC and, further, they are similar to one another.

For, since the angle BAC is equal to the angle ADB, for each is right, and the angle at B is common to the two triangles ABC and ABD, therefore the remaining angle ACB is equal to the remaining angle BAD; {I. 32} therefore the triangle ABC is equiangular with the triangle ABD.

Therefore, as BC which subtends the right angle in the triangle ABC is to BA which subtends the right angle in the triangle ABD, so is AB itself which subtends the angle at C in the triangle ABC to BD which subtends the equal angle BAD in the triangle ABD, and so also is AC to AD which subtends the angle at B common to the two triangles. {VI. 4}

Therefore the triangle ABC is both equiangular to the triangle ABD and has the sides about the equal angles proportional.

Therefore the triangle ABC is similar to the triangle ABD. {VI. Def. 1}

Similarly we can prove that the triangle ABC is also similar to the triangle ADC; therefore each of the triangles ABD, ADC is similar to the whole ABC.

I say next that the triangles ABD, ADC are also similar to one another.

For, since the right angle BDA is equal to the right angle ADC, and moreover the angle BAD was also proved equal to the angle at C, therefore the remaining angle at B is also equal to the remaining angle DAC; {I. 32} therefore the triangle ABD is equiangular with the triangle ADC.

Therefore, as BD which subtends the angle BAD in the triangle ABD is to DA which subtends the angle at C in the triangle ADC equal to the angle BAD, so is AD itself which subtends the angle at B in the triangle ABD to DC which subtends the angle DAC in the triangle ADC equal to the angle at B, and so also is BA to AC, these sides subtending the right angles; {VI. 4} therefore the triangle ABD is similar to the triangle ADC. {VI. Def. 1}

Therefore etc.

PORISM.

From this it is clear that, if in a right-angled triangle a perpendicular be drawn from the right angle to the base, the straight line so drawn is a mean proportional between the segments of the base. Q. E. D.

PROPOSITION 9.

From a given straight line to cut off a prescribed part.

Let AB be the given straight line; thus it is required to cut off from AB a prescribed part.

Let the third part be that prescribed.

Let a straight line AC be drawn through from A containing with AB any angle; let a point D be taken at random on AC, and let DE, EC be made equal to AD. {I. 3}

Let BC be joined, and through D let DF be drawn parallel to it. {I. 31}

Then, since FD has been drawn parallel to BC, one of the sides of the triangle ABC, therefore, proportionally, as CD is to DA, so is BF to FA. {VI. 2}

But CD is double of DA;

therefore BF is also double of FA; therefore BA is triple of AF.

Therefore from the given straight line AB the prescribed third part AF has been cut off. Q. E. F. [4]

PROPOSITION 10.

To cut a given uncut straight line similarly to a given cut straight line.

Let AB be the given uncut straight line, and AC the straight line cut at the points D, E; and let them be so placed as to contain any angle; let CB be joined, and through D, E let DF, EG be drawn parallel to BC, and through D let DHK be drawn parallel to AB. {I. 31}

Therefore each of the figures FH, HB is a parallelogram; therefore DH is equal to FG and HK to GB. {I. 34}

Now, since the straight line HE has been drawn parallel to KC, one of the sides of the triangle DKC, therefore, proportionally, as CE is to ED, so is KH to HD. {VI. 2}

But KH is equal to BG, and HD to GF; therefore, as CE is to ED, so is BG to GF.

Again, since FD has been drawn parallel to GE, one of the sides of the triangle AGE, therefore, proportionally, as ED is to DA, so is GF to FA. {VI. 2}

But it was also proved that,

as CE is to ED, so is BG to GF; therefore, as CE is to ED, so is BG to GF, and, as ED is to DA, so is GF to FA.

Therefore the given uncut straight line AB has been cut similarly to the given cut straight line AC. Q. E. F.

PROPOSITION 11.

To two given straight lines to find a third proportional.

Let BA, AC be the two given straight lines, and let them be placed so as to contain any angle; thus it is required to find a third proportional to BA, AC.

For let them be produced to the points D, E, and let BD be made equal to AC; {I. 3} let BC be joined, and through D let DE be drawn parallel to it. {I. 31}

Since, then, BC has been drawn parallel to DE, one of the sides of the triangle ADE, proportionally, as AB is to BD, so is AC to CE. {VI. 2}

But BD is equal to AC; therefore, as AB is to AC, so is AC to CE.

Therefore to two given straight lines AB, AC a third proportional to them, CE, has been found. Q. E. F. [5]

PROPOSITION 12.

To three given straight lines to find a fourth proportional.

Let A, B, C be the three given straight lines; thus it is required to find a fourth proportional to A, B, C.

Let two straight lines DE, DF be set out containing any angle EDF; let DG be made equal to A, GE equal to B, and further DH equal to C; let GH be joined, and let EF be drawn through E parallel to it. {I. 31}

Since, then, GH has been drawn parallel to EF, one of the sides of the triangle DEF, therefore, as DG is to GE, so is DH to HF. {VI. 2}

But DG is equal to A, GE to B, and DH to C; therefore, as A is to B, so is C to HF.

Therefore to the three given straight lines A, B, C a fourth proportional HF has been found. Q. E. F.

PROPOSITION 13.

To two given straight lines to find a mean proportional.

Let AB, BC be the two given straight lines; thus it is required to find a mean proportional to AB, BC.

Let them be placed in a straight line, and let the semicircle ADC be described on AC; let BD be drawn from the point B at right angles to the straight line AC, and let AD, DC be joined.

Since the angle ADC is an angle in a semicircle, it is right. {III. 31}

And, since, in the right-angled triangle ADC, DB has been drawn from the right angle perpendicular to the base, therefore DB is a mean proportional between the segments of the base, AB, BC. {VI. 8, Por.}

Therefore to the two given straight lines AB, BC a mean proportional DB has been found. Q. E. F.

PROPOSITION 14.

In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional; and equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal.

Let AB, BC be equal and equiangular parallelograms having the angles at B equal, and let DB, BE be placed in a straight line;

therefore FB, BG are also in a straight line. {I. 14}

I say that, in AB, BC, the sides about the equal angles are reciprocally proportional, that is to say, that, as DB is to BE, so is GB to BF.

For let the parallelogram FE be completed.

Since, then, the parallelogram AB is equal to the parallelogram BC,

and FE is another area,

therefore, as AB is to FE, so is BC to FE. {V. 7}

But, as AB is to FE, so is DB to BE, {VI. 1} and, as BC is to FE, so is GB to BF. {id.} therefore also, as DB is to BE, so is GB to BF. {V. 11}

Therefore in the parallelograms AB, BC the sides about the equal angles are reciprocally proportional.

Next, let GB be to BF as DB to BE; I say that the parallelogram AB is equal to the parallelogram BC.

For since, as DB is to BE, so is GB to BF, while, as DB is to BE, so is the parallelogram AB to the parallelogram FE, {VI. 1} and, as GB is to BF, so is the parallelogram BC to the parallelogram FE, {VI. 1} therefore also, as AB is to FE, so is BC to FE; {V. 11} therefore the parallelogram AB is equal to the parallelogram BC. {V. 9}

Therefore etc. Q. E. D.

PROPOSITION 15.

In equal triangles which have one angle equal to one angle the sides about the equal angles are reciprocally proportional; and those triangles which have one angle equal to one angle, and in which the sides about the equal angles are reciprocally proportional, are equal.

Let ABC, ADE be equal triangles having one angle equal to one angle, namely the angle BAC to the angle DAE; I say that in the triangles ABC, ADE the sides about the equal angles are reciprocally proportional, that is to say, that,

as CA is to AD, so is EA to AB.

For let them be placed so that CA is in a straight line with AD; therefore EA is also in a straight line with AB. {I. 14}

Let BD be joined.

Since then the triangle ABC is equal to the triangle ADE, and BAD is another area, therefore, as the triangle CAB is to the triangle BAD, so is the triangle EAD to the triangle BAD. {V. 7}

But, as CAB is to BAD, so is CA to AD, {VI. 1} and, as EAD is to BAD, so is EA to AB. {id.}

Therefore also, as CA is to AD, so is EA to AB. {V. 11}

Therefore in the triangles ABC, ADE the sides about the equal angles are reciprocally proportional.

Next, let the sides of the triangles ABC, ADE be reciprocally proportional, that is to say, let EA be to AB as CA to AD; I say that the triangle ABC is equal to the triangle ADE.

For, if BD be again joined, since, as CA is to AD, so is EA to AB, while, as CA is to AD, so is the triangle ABC to the triangle BAD, and, as EA is to AB, so is the triangle EAD to the triangle BAD, {VI. 1} therefore, as the

triangle ABC is to the triangle BAD, so is the triangle EAD to the triangle BAD. {V. 11}

Therefore each of the triangles ABC, EAD has the same ratio to BAD.

Therefore the triangle ABC is equal to the triangle EAD. {V. 9}

Therefore etc. Q. E. D.

PROPOSITION 16.

If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means; and, if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines will be proportional.

Let the four straight lines AB, CD, E, F be proportional, so that, as AB is to CD, so is E to F; I say that the rectangle contained by AB, F is equal to the rectangle contained by CD, E.

Let AG, CH be drawn from the points A, C at right angles to the straight lines AB, CD, and let AG be made equal to F, and CH equal to E.

Let the parallelograms BG, DH be completed.

Then since, as AB is to CD, so is E to F, while E is equal to CH, and F to AG, therefore, as AB is to CD, so is CH to AG.

Therefore in the parallelograms BG, DH the sides about the equal angles are reciprocally proportional.

But those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal; {VI. 14} therefore the parallelogram BG is equal to the parallelogram DH.

And BG is the rectangle AB, F, for AG is equal to F; and DH is the rectangle CD, E, for E is equal to CH; therefore the rectangle contained by AB, F is equal to the rectangle contained by CD, E.

Next, let the rectangle contained by AB, F be equal to the rectangle contained by CD, E; I say that the four straight lines will be proportional, so that, as AB is to CD, so is E to F.

For, with the same construction, since the rectangle AB, F is equal to the rectangle CD, E, and the rectangle AB, F is BG, for AG is equal to F, and the rectangle CD, E is DH, for CH is equal to E,

therefore BG is equal to DH.

And they are equiangular

But in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional. {VI. 14}

Therefore, as AB is to CD, so is CH to AG.

But CH is equal to E, and AG to F; therefore, as AB is to CD, so is E to F.

Therefore etc. Q. E. D.

PROPOSITION 17

If three straight lines be proportional, the rectangle contained by the extremes is equal to the square on the mean; and, if the rectangle contained by the extremes be equal to the square on the mean, the three straight lines will be proportional.

Let the three straight lines A, B, C be proportional, so that, as A is to B, so is B to C; I say that the rectangle contained by A, C is equal to the square on B.

Let D be made equal to B.

Then, since, as A is to B, so is B to C, and B is equal to D, therefore, as A is to B, so is D to C.

But, if four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means. {VI. 16}

Therefore the rectangle A, C is equal to the rectangle B, D.

But the rectangle B, D is the square on B, for B is equal to D; therefore the rectangle contained by A, C is equal to the square on B.

Next, let the rectangle A, C be equal to the square on B; I say that, as A is to B, so is B to C.

For, with the same construction, since the rectangle A, C is equal to the square on B, while the square on B is the rectangle B, D, for B is equal to D, therefore the rectangle A, C is equal to the rectangle B, D.

But, if the rectangle contained by the extremes be equal to that contained by the means, the four straight lines are proportional. {VI. 16}

Therefore, as A is to B, so is D to C.

But B is equal to D;

therefore, as A is to B, so is B to C.

Therefore etc. Q. E. D.

PROPOSITION 18.

On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure.

Let AB be the given straight line and CE the given rectilineal figure; thus it is required to describe on the straight line AB a rectilineal figure similar and similarly situated to the rectilineal figure CE.

Let DF be joined, and on the straight line AB, and at the points A, B on it, let the angle GAB be constructed equal to the angle at C, and the angle ABG equal to the angle CDF. {I. 23}

Therefore the remaining angle CFD is equal to the angle AGB; {I. 32}

therefore the triangle FCD is equiangular with the triangle GAB.

Therefore, proportionally, as FD is to GB, so is FC to GA, and CD to AB.

Again, on the straight line BG, and at the points B, G on it, let the angle BGH be constructed equal to the angle DFE, and the angle GBH equal to the angle FDE. {I. 23}

Therefore the remaining angle at E is equal to the remaining angle at H; {I. 32}

therefore the triangle FDE is equiangular with the triangle GBH; therefore, proportionally, as FD is to GB, so is FE to GH, and ED to HB. {VI. 4}

But it was also proved that, as FD is to GB, so is FC to GA, and CD to AB;

therefore also, as FC is to AG, so is CD to AB, and FE to GH, and further ED to HB.

And, since the angle CFD is equal to the angle AGB, and the angle DFE to the angle BGH, therefore the whole angle CFE is equal to the whole angle AGH.

For the same reason

the angle CDE is also equal to the angle ABH.

And the angle at C is also equal to the angle at A,

and the angle at E to the angle at H.

Therefore AH is equiangular with CE; and they have the sides about their equal angles proportional;

therefore the rectilineal figure AH is similar to the rectilineal figure CE. {VI. Def. 1}

Therefore on the given straight line AB the rectilineal figure AH has been described similar and similarly situated to the given rectilineal figure CE. Q. E. F.

PROPOSITION 19.

Similar triangles are to one another in the duplicate ratio of the corresponding sides.

Let ABC, DEF be similar triangles having the angle at B equal to the angle at E, and such that, as AB is to BC, so
is DE to EF, so that BC corresponds to EF; {V. Def. 11} I say that the triangle ABC has to the triangle DEF a ratio duplicate of that which BC has to EF.

For let a third proportional BG be taken to BC, EF, so that, as BC is to EF, so is EF to BG; {VI. 11}
and let AG be joined.

Since then, as AB is to BC, so is DE to EF, therefore, alternately, as AB is to DE, so is BC to EF. {V. 16}

But, as BC is to EF, so is EF to BG; therefore also, as AB is to DE, so is EF to BG. {V. 11}

Therefore in the triangles ABG, DEF the sides about the equal angles are reciprocally proportional.

But those triangles which have one angle equal to one angle, and in which the sides about the equal angles are reciprocally proportional, are equal; {VI. 15} therefore the triangle ABG is equal to the triangle DEF.

Now since, as BC is to EF, so is EF to BG, and, if three straight lines be proportional, the first has to the third a ratio duplicate of that which it has to the second, {V. Def. 9} therefore BC has to BG a ratio duplicate of that which CB
has to EF.

But, as CB is to BG, so is the triangle ABC to the triangle ABG; {VI. 1} therefore the triangle ABC also has to the triangle ABG a ratio duplicate of that which BC has to EF.

But the triangle ABG is equal to the triangle DEF; therefore the triangle ABC also has to the triangle DEF a ratio duplicate of that which BC has to EF.

Therefore etc.

PORISM.

From this it is manifest that, if three straight
lines be proportional, then, as the first is to the third, so is the figure described on the first to that which is similar and similarly described on the second. Q. E. D. [6]

PROPOSITION 20.

Similar polygons are divided into similar triangles, and into triangles equal in multitude and in the same ratio as the wholes, and the polygon has to the polygon a ratio duplicate of that which the corresponding side has to the

corresponding
side.

Let ABCDE, FGHKL be similar polygons, and let AB correspond to FG; I say that the polygons ABCDE, FGHKL are divided into similar triangles, and into triangles equal in multitude and in
the same ratio as the wholes, and the polygon ABCDE has to the polygon FGHKL a ratio duplicate of that which AB has to FG.

Let BE, EC, GL, LH be joined.

Now, since the polygon ABCDE is similar to the polygon
FGHKL, the angle BAE is equal to the angle GFL;

and, as BA is to AE, so is GF to FL. {VI. Def. 1}

Since then ABE, FGL are two triangles having one angle equal to one angle and the sides about the equal angles
proportional, therefore the triangle ABE is equiangular with the triangle FGL; {VI. 6}

so that it is also similar; {VI. 4 and Def. 1}

therefore the angle ABE is equal to the angle FGL.

But the whole angle ABC is also equal to the whole angle FGH because of the similarity of the polygons; therefore the remaining angle EBC is equal to the angle LGH.

And, since, because of the similarity of the triangles ABE,
FGL,

as EB is to BA, so is LG to GF,

and moreover also, because of the similarity of the polygons,

as AB is to BC, so is FG to GH,

therefore, ex aequali, as EB is to BC, so is LG to GH; {V. 22}
that is, the sides about the equal angles EBC, LGH are proportional; therefore the triangle EBC is equiangular with the triangle LGH, {VI. 6}

so that the triangle EBC is also similar to the triangle LGH. {VI. 4 and Def. 1}

For the same reason the triangle ECD is also similar to the triangle LHK.

Therefore the similar polygons ABCDE, FGHKL have been divided into similar triangles, and into triangles equal in
multitude.

I say that they are also in the same ratio as the wholes, that is, in such manner that the triangles are proportional, and ABE, EBC, ECD are antecedents, while FGL, LGH, LHK are their consequents, and that the polygon ABCDE has to the polygon FGHKL a ratio duplicate of that which the corresponding side has to the corresponding side, that is AB to FG.

For let AC, FH be joined.

Then since, because of the similarity of the polygons,
the angle ABC is equal to the angle FGH, and, as AB is to BC, so is FG to GH,

the triangle ABC is equiangular with the triangle FGH; {VI. 6} therefore the angle BAC is equal to the angle GFH, and the angle BCA to the angle GHF.

And, since the angle BAM is equal to the angle GFN, and the angle ABM is also equal to the angle FGN, therefore the remaining angle AMB is also equal to the remaining angle FNG; {I. 32} therefore the triangle ABM is equiangular with the triangle
FGN.

Similarly we can prove that the triangle BMC is also equiangular with the triangle GNH.

Therefore, proportionally, as AM is to MB, so is FN to NG,
and, as BM is to MC, so is GN to NH; so that, in addition, ex aequali,

as AM is to MC, so is FN to NH.

But, as AM is to MC, so is the triangle ABM to MBC, and AME to EMC; for they are to one another as their
bases. {VI. 1}

Therefore also, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents; {V. 12} therefore, as the triangle AMB is to BMC, so is ABE to CBE.

But, as AMB is to BMC, so is AM to MC; therefore also, as AM is to MC, so is the triangle ABE to the triangle EBC.

For the same reason also, as FN is to NH, so is the triangle FGL to the triangle
GLH.

And, as AM is to MC, so is FN to NH; therefore also, as the triangle ABE is to the triangle BEC, so is the triangle FGL to the triangle GLH; and,

alternately, as the triangle ABE is to the triangle FGL,
so is the triangle BEC to the triangle GLH.

Similarly we can prove, if BD, GK be joined, that, as the triangle BEC is to the triangle LGH, so also is the triangle ECD to the triangle LHK.

And since, as the triangle ABE is to the triangle FGL,
so is EBC to LGH, and further ECD to LHK, therefore also, as one of the antecedents is to one of the consequents so are all the antecedents to all the consequents; {V. 12 therefore, as the triangle ABE is to the triangle FGL, so is the polygon ABCDE to the polygon FGHKL.

But the triangle ABE has to the triangle FGL a ratio duplicate of that which the corresponding side AB has to the corresponding side FG; for similar triangles are in the duplicate ratio of the corresponding sides. {VI. 19}

Therefore the polygon ABCDE also has to the polygon
FGHKL a ratio duplicate of that which the corresponding side AB has to the corresponding side FG.

Therefore etc.

PORISM.

Similarly also it can be proved in the case of quadrilaterals that they are in the duplicate ratio of the
corresponding sides. And it was also proved in the case of triangles; therefore also, generally, similar rectilineal figures are to one another in the duplicate ratio of the corresponding sides. Q. E. D. [7] [8]

PROPOSITION 21.

Figures which are similar to the same rectilineal figure are also similar to one another.

For let each of the rectilineal figures A, B be similar to C; I say that A is also similar to B.

For, since A is similar to C, it is equiangular with it and has the sides about the equal angles proportional. {VI. Def. 1}

Again, since B is similar to C, it is equiangular with it and has the sides about the equal angles proportional.

Therefore each of the figures A, B is equiangular with C and with C has the sides about the equal angles proportional;

therefore A is similar to B. Q. E. D.

PROPOSITION 22.

If four straight lines be proportional, the rectilineal figures similar and similarly described upon them will also be proportional; and, if the rectilineal figures similar and similarly described upon them be proportional, the straight lines will themselves also be proportional.

Let the four straight lines AB, CD, EF, GH be proportional, so that, as AB is to CD, so is EF to GH, and let there be described on AB, CD the similar and similarly situated rectilineal figures KAB, LCD, and on EF, GH the similar and similarly situated rectilineal figures MF, NH; I say that, as KAB is to LCD, so is MF to NH.

For let there be taken a third proportional O to AB, CD, and a third proportional P to EF, GH. {VI. 11}

Then since, as AB is to CD, so is EF to GH,

and, as CD is to O, so is GH to P,

therefore, ex aequali, as AB is to O, so is EF to P. {V. 22}

But, as AB is to O, so is KAB to LCD, {VI. 19, Por.}

and, as EF is to P, so is MF to NH;

therefore also, as KAB is to LCD, so is MF to NH. {V. 11}

Next, let MF be to NH as KAB is to LCD; I say also that, as AB is to CD, so is EF to GH.

For, if EF is not to GH as AB to CD,

let EF be to QR as AB to CD, {VI. 12}

and on QR let the rectilineal figure SR be described similar and similarly situated to either of the two MF, NH. {VI. 18}

Since then, as AB is to CD, so is EF to QR, and there have been described on AB, CD the similar and similarly situated figures KAB, LCD, and on EF, QR the similar and similarly situated figures MF, SR, therefore, as KAB is to LCD, so is MF to SR.

But also, by hypothesis,

as KAB is to LCD, so is MF to NH; therefore also, as MF is to SR, so is MF to NH. {V. 11}

Therefore MF has the same ratio to each of the figures NH, SR;

therefore NH is equal to SR. {V. 9}

But it is also similar and similarly situated to it;

therefore GH is equal to QR.

And, since, as AB is to CD, so is EF to QR, while QR is equal to GH, therefore, as AB is to CD, so is EF to GH.

Therefore etc. Q. E. D.

PROPOSITION 23.

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.

Let AC, CF be equiangular parallelograms having the angle BCD equal to the angle ECG;
I say that the parallelogram AC has to the parallelogram CF the ratio compounded of the ratios of the sides.

For let them be placed so that BC is in a straight line with CG;

therefore DC is also in a straight line with CE.

Let the parallelogram DG be completed; let a straight line K be set out, and let it be contrived that,

as BC is to CG, so is K to L,

and, as DC is to CE, so is L to M. {VI. 12}

Then the ratios of K to L and of L to M are the same
as the ratios of the sides, namely of BC to CG and of DC to CE.

But the ratio of K to M is compounded of the ratio of K to L and of that of L to M; so that K has also to M the ratio compounded of the ratios
of the sides.

Now since, as BC is to CG, so is the parallelogram AC to the parallelogram CH, {VI. 1} while, as BC is to CG, so is K to L, therefore also, as K is to L, so is AC to CH. {V. 11}

Again, since, as DC is to CE, so is the parallelogram CH to CF, {VI. 1} while, as DC is to CE, so is L to M, therefore also, as L is to M, so is the parallelogram CH to the parallelogram CF. {V. 11}

Since then it was proved that, as K is to L, so is the parallelogram AC to the parallelogram CH, and, as L is to M, so is the parallelogram CH to the parallelogram CF, therefore, ex aequali, as K is to M, so is AC to the parallelogram
CF.

But K has to M the ratio compounded of the ratios of the sides;

therefore AC also has to CF the ratio compounded of the ratios of the sides.

Therefore etc. Q. E. D. [9] [10]

PROPOSITION 24.

In any parallelogram the parallelograms about the diameter are similar both to the whole and to one another.

Let ABCD be a parallelogram, and AC its diameter, and let EG, HK be parallelograms about AC; I say that each of the parallelograms EG, HK is similar both to the whole ABCD and to the other.

For, since EF has been drawn parallel to BC, one of the sides of the triangle ABC,

proportionally, as BE is to EA, so is CF to FA. {VI. 2}

Again, since FG has been drawn parallel to CD, one of the sides of the triangle ACD,

proportionally, as CF is to FA, so is DG to GA. {VI. 2}

But it was proved that,

as CF is to FA, so also is BE to EA; therefore also, as BE is to EA, so is DG to GA,

and therefore, componendo,

as BA is to AE, so is DA to AG, {V. 18}

and, alternately,

as BA is to AD, so is EA to AG. {V. 16}

Therefore in the parallelograms ABCD, EG, the sides about the common angle BAD are proportional.

And, since GF is parallel to DC,

the angle AFG is equal to the angle DCA;

and the angle DAC is common to the two triangles ADC, AGF;

therefore the triangle ADC is equiangular with the triangle AGF.

For the same reason

the triangle ACB is also equiangular with the triangle AFE,

and the whole parallelogram ABCD is equiangular with the parallelogram EG.

Therefore, proportionally,

as AD is to DC, so is AG to GF, as DC is to CA, so is GF to FA, as AC is to CB, so is AF to FE,

and further, as CB is to BA, so is FE to EA.

And, since it was proved that,

as DC is to CA, so is GF to FA,

and, as AC is to CB, so is AF to FE, therefore, ex aequali, as DC is to CB, so is GF to FE. {V. 22}

Therefore in the parallelograms ABCD, EG the sides about the equal angles are proportional; therefore the parallelogram ABCD is similar to the parallelogram EG. {VI. Def. 1}

For the same reason the parallelogram ABCD is also similar to the parallelogram KH; therefore each of the parallelograms EG, HK is similar to ABCD.

But figures similar to the same rectilineal figure are also similar to one another; {VI. 21} therefore the parallelogram EG is also similar to the parallelogram HK.

Therefore etc. Q. E. D.

PROPOSITION 25.

To construct one and the same figure similar to a given rectilineal figure and equal to another given rectilineal figure.

Let ABC be the given rectilineal figure to which the figure to be constructed must be similar, and D that to which it must be equal; thus it is required to construct one and the same figure similar to ABC and equal to D.

Let there be applied to BC the parallelogram BE equal to the triangle ABC {I. 44}, and to CE the parallelogram CM equal to D in the angle FCE which is equal to the angle CBL. {I. 45}

Therefore BC is in a straight line with CF, and LE with EM.

Now let GH be taken a mean proportional to BC, CF {VI. 13}, and on GH let KGH be described similar and similarly situated to ABC. {VI. 18}

Then, since, as BC is to GH, so is GH to CF, and, if three straight lines be proportional, as the first is to the third, so is the figure on the first to the similar and similarly situated figure described on the second, {VI. 19, Por.} therefore, as BC is to CF, so is the triangle ABC to the triangle KGH.

But, as BC is to CF, so also is the parallelogram BE to the parallelogram EF. {VI. 1}

Therefore also, as the triangle ABC is to the triangle KGH, so is the parallelogram BE to the parallelogram EF; therefore, alternately, as the triangle ABC is to the parallelogram BE, so is the triangle KGH to the parallelogram EF. {V. 16}

But the triangle ABC is equal to the parallelogram BE; therefore the triangle KGH is also equal to the parallelogram EF.

But the parallelogram EF is equal to D; therefore KGH is also equal to D.

And KGH is also similar to ABC.

Therefore one and the same figure KGH has been constructed similar to the given rectilineal figure ABC and equal to the other given figure D. Q. E. D. [11]

PROPOSITION 26.

If from a parallelogram there be taken away a parallelogram similar and similarly situated to the whole and having a common angle with it, it is about the same diameter with the whole

For from the parallelogram ABCD let there be taken away the parallelogram AF similar and similarly situated to ABCD, and having the angle DAB common with it; I say that ABCD is about the same diameter with AF.

For suppose it is not, but, if possible, let AHC be the diameter < of ABCD >, let GF be produced and carried through to H, and let HK be drawn through H parallel to either of the straight lines AD, BC. {I. 31}

Since, then, ABCD is about the same diameter with KG, therefore, as DA is to AB, so is GA to AK. {VI. 24}

But also, because of the similarity of ABCD, EG,

as DA is to AB, so is GA to AE;

therefore also, as GA is to AK, so is GA to AE. {V. 11}

Therefore GA has the same ratio to each of the straight lines AK, AE.

Therefore AE is equal to AK {V. 9}, the less to the greater : which is impossible.

Therefore ABCD cannot but be about the same diameter with AF; therefore the parallelogram ABCD is about the same diameter with the parallelogram AF.

Therefore etc. Q. E. D.

PROPOSITION 27.

Of all the parallelograms applied to the same straight line and deficient by parallelogrammic figures similar and similarly situated to that described on the half of the straight line, that parallelogram is greatest which is applied to the half of the straight line and is similar to the defect.

Let AB be a straight line and let it be bisected at C; let there be applied to the straight line AB the parallelogram AD deficient by the parallelogrammic figure DB described on the half of AB, that is, CB; I say that, of all the parallelograms applied to AB and deficient by parallelogrammic figures similar and similarly situated to DB, AD is greatest.

For let there be applied to the straight line AB the parallelogram AF deficient by the parallelogrammic figure FB similar and similarly situated to DB; I say that AD is greater than AF.

For, since the parallelogram DB is similar to the parallelogram FB,

they are about the same diameter. {VI. 26}

Let their diameter DB be drawn, and let the figure be described.

Then, since CF is equal to FE, {I. 43} and FB is common, therefore the whole CH is equal to the whole KE.

But CH is equal to CG, since AC is also equal to CB. {I. 36}

Therefore GC is also equal to EK.

Let CF be added to each;

therefore the whole AF is equal to the gnomon LMN;

so that the parallelogram DB, that is, AD, is greater than the parallelogram AF.

Therefore etc.

PROPOSITION 28.

To a given straight line to apply a parallelogram equal to a given rectilineal figure and deficient by a parallelogrammic figure similar to a given one : thus the given rectilineal figure must not be greater than the parallelogram described on the half of the straight line and similar to the defect.

Let AB be the given straight line, C the given rectilineal figure to which the figure to be applied to AB is required to be equal, not being greater than the parallelogram described on the half of AB and similar to the defect, and D the parallelogram to which the defect is required to be similar; thus it is required to apply to the given straight line AB a parallelogram equal to the

given rectilineal figure C and deficient by a parallelogrammic figure which is similar to D.

Let AB be bisected at the point E, and on EB let EBFG be described similar and similarly situated to D; {VI. 18} let the parallelogram AG be completed.

If then AG is equal to C, that which was enjoined will have been done;

for there has been applied to the given straight line AB the parallelogram AG equal to the given rectilineal figure C and deficient by a parallelogrammic figure GB which is similar to D.

But, if not, let HE be greater than C.

Now HE is equal to GB;

therefore GB is also greater than C.

Let KLMN be constructed at once equal to the excess by which GB is greater than C and similar and similarly situated to D. {VI. 25}

But D is similar to GB;

therefore KM is also similar to GB. {VI. 21}

Let, then, KL correspond to GE, and LM to GF.

Now, since GB is equal to C, KM,

therefore GB is greater than KM; therefore also GE is greater than KL, and GF than LM.

Let GO be made equal to KL, and GP equal to LM; and let the parallelogram OGPQ be completed;

therefore it is equal and similar to KM.

Therefore GQ is also similar to GB; {VI. 21} therefore GQ is about the same diameter with GB. {VI. 26}

Let GQB be their diameter, and let the figure be described.

Then, since BG is equal to C, KM, and in them GQ is equal to KM, therefore the remainder, the gnomon UWV, is equal to the remainder C.

And, since PR is equal to OS,

let QB be added to each;

therefore the whole PB is equal to the whole OB.

But OB is equal to TE, since the side AE is also equal to the side EB; {I. 36}

therefore TE is also equal to PB.

Let OS be added to each;

therefore the whole TS is equal to the whole, the gnomon VWU.

But the gnomon VWU was proved equal to C;

therefore TS is also equal to C.

Therefore to the given straight line AB there has been applied the parallelogram ST equal to the given rectilineal figure C and deficient by a parallelogrammic figure QB which is similar to D. Q. E. F.

PROPOSITION 29.

To a given straight line to apply a parallelogram equal to a given rectilineal figure and exceeding by a parallelogrammic figure similar to a given one.

Let AB be the given straight line, C the given rectilineal figure to which the figure to be applied to AB is required to be equal, and D that to which the excess is required to be similar; thus it is required to apply to the straight line

AB a parallelogram equal to the rectilineal figure C and exceeding by a parallelogrammic figure similar to D.

Let AB be bisected at E; let there be described on EB the parallelogram BF similar and similarly situated to D; and let GH be constructed at once equal to the sum of BF, C and similar and similarly situated to D. {VI. 25}

Let KH correspond to FL and KG to FE.

Now, since GH is greater than FB, therefore KH is also greater than FL, and KG than FE.

Let FL, FE be produced, let FLM be equal to KH, and FEN to KG, and let MN be completed;

therefore MN is both equal and similar to GH.

But GH is similar to EL;

therefore MN is also similar to EL; {VI. 21}

therefore EL is about the same diameter with MN. {VI. 26}

Let their diameter FO be drawn, and let the figure be described.

Since GH is equal to EL, C, while GH is equal to MN, therefore MN is also equal to EL, C.

Let EL be subtracted from each;

therefore the remainder, the gnomon XWV, is equal to C.

Now, since AE is equal to EB,

AN is also equal to NB {I. 36}, that is, to LP {I. 43}.

Let EO be added to each;

therefore the whole AO is equal to the gnomon VWX.

But the gnomon VWX is equal to C;

therefore AO is also equal to C.

Therefore to the given straight line AB there has been applied the parallelogram AO equal to the given rectilineal figure C and exceeding by a parallelogrammic figure QP which is similar to D, since PQ is also similar to EL {VI. 24}. Q. E. F.

PROPOSITION 30.

To cut a given finite straight line in extreme and mean ratio.

Let AB be the given finite straight line;

thus it is required to cut AB in extreme and mean ratio.

On AB let the square BC be described; and let there be applied to AC the parallelogram CD equal to BC and exceeding by the figure AD similar to BC. {VI. 29}

Now BC is a square;

therefore AD is also a square.

And, since BC is equal to CD, let CE be subtracted from each;

therefore the remainder BF is equal to the remainder AD.

But it is also equiangular with it; therefore in BF, AD the sides about the equal angles are reciprocally proportional; {VI. 14}

therefore, as FE is to ED, so is AE to EB.

But FE is equal to AB, and ED to AE.

Therefore, as BA is to AE, so is AE to EB.

And AB is greater than AE;

therefore AE is also greater than EB.

Therefore the straight line AB has been cut in extreme and mean ratio at E, and the greater segment of it is AE. Q. E. F.

PROPOSITION 31.

In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.

Let ABC be a right-angled triangle having the angle BAC right; I say that the figure on BC is equal to the similar and similarly described figures on BA, AC.

Let AD be drawn perpendicular.

Then since, in the right-angled triangle ABC, AD has been drawn from the right angle at A perpendicular to the base BC, the triangles ABD, ADC adjoining the perpendicular are similar both to the whole ABC and to one another. {VI. 8}

And, since ABC is similar to ABD, therefore, as CB is to BA, so is AB to BD. {VI. Def. 1}

And, since three straight lines are proportional, as the first is to the third, so is the figure on the first to the similar and similarly described figure on the second. {VI. 19, Por.}

Therefore, as CB is to BD, so is the figure on CB to the similar and similarly described figure on BA.

For the same reason also, as BC is to CD, so is the figure on BC to that on CA; so that, in addition, as BC is to BD, DC, so is the figure on BC to the similar and similarly described figures on BA, AC.

But BC is equal to BD, DC; therefore the figure on BC is also equal to the similar and similarly described figures on BA, AC.

Therefore etc. Q. E. D.

PROPOSITION 32.

If two triangles having two sides proportional to two sides be placed together at one angle so that their corresponding sides are also parallel, the remaining sides of the triangles will be in a straight line.

Let ABC, DCE be two triangles having the two sides BA, AC proportional to the two sides DC, DE, so that, as AB is to AC, so is DC to DE, and AB parallel to DC, and AC to DE; I say that BC is in a straight line with CE.

For, since AB is parallel to DC, and the straight line AC has fallen upon them, the alternate angles BAC, ACD are equal to one another. {I. 29}

For the same reason

the angle CDE is also equal to the angle ACD;

so that the angle BAC is equal to the angle CDE.

And, since ABC, DCE are two triangles having one angle, the angle at A, equal to one angle, the angle at D,

and the sides about the equal angles proportional,

so that, as BA is to AC, so is CD to DE,

therefore the triangle ABC is equiangular with the triangle DCE; {VI. 6}
therefore the angle ABC is equal to the angle DCE.

But the angle ACD was also proved equal to the angle BAC;

therefore the whole angle ACE is equal to the two angles ABC, BAC.

Let the angle ACB be added to each; therefore the angles ACE, ACB are equal to the angles BAC, ACB, CBA.

But the angles BAC, ABC, ACB are equal to two right angles; {I. 32}

therefore the angles ACE, ACB are also equal to two right angles.

Therefore with a straight line AC, and at the point C on it, the two straight lines BC, CE not lying on the same side make the adjacent angles ACE, ACB equal to two right angles;

therefore BC is in a straight line with CE. {I. 14}

Therefore etc. Q. E. D.

PROPOSITION 33.

In equal circles angles have the same ratio as the circumferences on which they stand, whether they stand at the centres or at the circumferences.

Let ABC, DEF be equal circles, and let the angles BGC, EHF be angles at their centres G, H, and the angles BAC, EDF angles at the circumferences; I say that, as the circumference BC is to the circumference EF, so is the angle BGC to the angle EHF, and the angle BAC to the angle EDF.

For let any number of consecutive circumferences CK, KL be made equal to the circumference BC, and any number of consecutive circumferences FM, MN equal to the circumference EF; and let GK, GL, HM, HN be joined.

Then, since the circumferences BC, CK, KL are equal to one another, the angles BGC, CGK, KGL are also equal to one another; {III. 27} therefore, whatever multiple the circumference BL is of BC, that multiple also is the angle BGL of the angle BGC.

For the same reason also, whatever multiple the circumference NE is of EF, that multiple also is the angle NHE of the angle EHF.

If then the circumference BL is equal to the circumference EN, the angle BGL is also equal to the angle EHN; {III. 27} if the circumference BL is

greater than the circumference EN, the angle BGL is also greater than the angle EHN; and, if less, less.

There being then four magnitudes, two circumferences BC, EF, and two angles BGC, EHF, there have been taken, of the circumference BC and the angle BGC equimultiples, namely the circumference BL and the angle BGL, and of the circumference EF and the angle EHF equimultiples, namely the circumference EN and the angle EHN.

And it has been proved that, if the circumference BL is in excess of the circumference EN, the angle BGL is also in excess of the angle EHN; if equal, equal; and if less, less.

Therefore, as the circumference BC is to EF, so is the angle BGC to the angle EHF. {V. Def. 5}

But, as the angle BGC is to the angle EHF, so is the angle BAC to the angle EDF; for they are doubles respectively.

Therefore also, as the circumference BC is to the circumference EF, so is the angle BGC to the angle EHF, and the angle BAC to the angle EDF.

Therefore etc. Q. E. D.

ENDNOTES.

[1] Under the same height. The Greek text has "under the same height AC," with a figure in which the side AC common to the two triangles is perpendicular to the base and is therefore itself the "height." But, even if the two triangles are placed contiguously so as to have a common side AC, it is quite gratuitous to require it to be perpendicular to the base. Theon, on this occasion making an improvement, altered to "which are (ὄντα) under the same height, (namely) the perpendicular drawn from A to BD." I have ventured to alter so far as to omit "AC" and to draw the figure in the usual way.

[2] ABC, AGB, AHG. Euclid, indifferent to exact order, writes "AHG, AGB, ABC."

[3] Since then it was proved that, as the base BC is to CD, so is the triangle ABC to the triangle ACD. Here again words have to be supplied in translating the extremely terse Greek ἐπεὶ οὐν ἐδείχθη, ὡς μὲν

ἡ βάσις ΒΓ πρὸς τὴν ΓΔ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον, literally "since was proved, as the base BC to CD, so the triangle ABC to the triangle ACD." Cf. note on V. 16, p. 165.

[4] any angle. The expression here and in the two following propositions is τυχοῦσα γωνία, corresponding exactly to τυχὸν σημεῖον which I have translated as "a point (taken) at random" ; but "an angle (taken) at random" would not be so appropriate where it is a question, not of taking any angle at all, but of drawing a straight line casually so as to make any angle with another straight line.

[5] to find. The Greek word, here and in the next two propositions, is προσευρεῖν, literally "to find in addition."

[6] and such that, as AB is to BC, so is DE to EF, literally "(triangles) having the angle at B equal to the angle at E, and (having), as AB to BC, so DE to EF."

[7] in the same ratio as the wholes. The same word ὁμόλογος is used which I have generally translated by "corresponding." But here it is followed by a dative, ὁμόλογα τοῖς ὅλοις "homologous with the wholes," instead of being used absolutely. The meaning can therefore here be nothing else but "in the same ratio with" or "proportional to the wholes" ; and Euclid seems to recognise that he is making a special use of the word, because he explains it lower down (I. 46): "the triangles are homologous to the wholes, that is, in such manner that the triangles are proportional, and ABE, EBC, ECD are antecedents, while FGL, LGH, LHK are their consequents."

[8] ἑπόμενα αὐτῶν, "their consequents," is a little awkward, but may be supposed to indicate which triangles correspond to which as consequent to antecedent.

[9] the ratio compounded of the ratios of the sides, λόγον τὸν συγκείμενον ἐκ τῶν πλευρῶν which, meaning literally "the ratio compounded of the sides," is negligently written here and commonly for λόγον τὸν συγκείμενον ἐκ τῶν τῶν πλευρῶν (sc. λόγων).

[10] let it be contrived that, as BC is to CG, so is K to L. The Greek phrase is of the usual terse kind, untranslatable literally : καὶ γεγονέτω ὡς μὲν ἡ ΒΓ πρὸς τὴν ΓΗ, οὕτως ἡ Κ πρὸς τὸ Λ, the words meaning "and let (there) be made, as BC to CG, so K to L," where L is the straight line which has to be constructed.

[11] to which the figure to be constructed must be similar, literally "to which it is required to construct (one) similar," ᾧ δεῖ ὅμοιον συστήσασθαι.

BOOK VII. DEFINITIONS.

1

An unit is that by virtue of which each of the things that exist is called one.

2

A number is a multitude composed of units.

3

A number is a part of a number, the less of the greater, when it measures the greater;

4

but parts when it does not measure it.

5

The greater number is a multiple of the less when it is measured by the less.

6

An even number is that which is divisible into two equal parts.

7

An odd number is that which is not divisible into two equal parts, or that which differs by an unit from an even number.

8

An even-times even number is that which is measured by an even number according to an even number.

9

An even-times odd number is that which is measured by an even number according to an odd number.

10

An odd-times odd number is that which is measured by an odd number according to an odd number.

11

A prime number is that which is measured by an unit alone.

12

Numbers prime to one another are those which are measured by an unit alone as a common measure.

13

A composite number is that which is measured by some number.

14

Numbers composite to one another are those which are measured by some number as a common measure.

15

A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.

16

And, when two numbers having multiplied one another make some number, the number so produced is called plane, and its sides are the numbers which have multiplied one another.

17

And, when three numbers having multiplied one another make some number, the number so produced is solid, and its sides are the numbers which have multiplied one another.

18

A square number is equal multiplied by equal, or a number which is contained by two equal numbers.

19

And a cube is equal multiplied by equal and again by equal, or a number which is contained by three equal numbers.

20

Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

21

Similar plane and solid numbers are those which have their sides proportional.

22

A perfect number is that which is equal to its own parts.

PROPOSITIONS.

PROPOSITION I.

Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another.

For, the less of two unequal numbers *AB*, *CD* being continually subtracted from the greater, let the number which is left never measure the one before it until an unit is left; I say that *AB*, *CD* are prime to one another, that is, that an unit alone measures *AB*, *CD*.

For, if *AB*, *CD* are not prime to one another, some number will measure them.

Let a number measure them, and let it be *E*; let *CD*, measuring *BF*, leave *FA* less than itself, let *AF*, measuring *DG*, leave *GC* less than itself, and let *GC*, measuring *FH*, leave an unit *HA*.

Since, then, *E* measures *CD*, and *CD* measures *BF*, therefore *E* also measures *BF*.

But it also measures the whole *BA*; therefore it will also measure the remainder *AF*.

But *AF* measures *DG*; therefore *E* also measures *DG*.

But it also measures the whole *DC* therefore it will also measure the remainder *CG*.

But *CG* measures *FH*; therefore *E* also measures *FH*.

But it also measures the whole *FA*; therefore it will also measure the remainder, the unit *AH*, though it is a number: which is impossible.

Therefore no number will measure the numbers *AB*, *CD*; therefore *AB*, *CD* are prime to one another. {VII. Def. 12} Q. E. D.

PROPOSITION 2.

Given two numbers not prime to one another, to find their greatest common measure.

Let *AB*, *CD* be the two given numbers not prime to one another.

Thus it is required to find the greatest common measure of *AB*, *CD*.

If now *CD* measures *AB* — and it also measures itself — *CD* is a common measure of *CD*, *AB*.

And it is manifest that it is also the greatest; for no greater number than *CD* will measure *CD*.

But, if *CD* does not measure *AB*, then, the less of the numbers *AB*, *CD* being continually subtracted from the greater, some number will be left which will measure the one before it.

For an unit will not be left; otherwise *AB*, *CD* will be prime to one another {VII. 1}, which is contrary to the hypothesis.

Therefore some number will be left which will measure the one before it.

Now let *CD*, measuring *BE*, leave *EA* less than itself, let *EA*, measuring *DF*, leave *FC* less than itself, and let *CF* measure *AE*.

Since then, *CF* measures *AE*, and *AE* measures *DF*, therefore *CF* will also measure *DF*.

But it also measures itself; therefore it will also measure the whole *CD*.

But *CD* measures *BE*; therefore *CF* also measures *BE*.

But it also measures *EA*; therefore it will also measure the whole *BA*.

But it also measures *CD*; therefore *CF* measures *AB*, *CD*.

Therefore *CF* is a common measure of *AB*, *CD*.

I say next that it is also the greatest.

For, if *CF* is not the greatest common measure of *AB*, *CD*, some number which is greater than *CF* will measure the numbers *AB*, *CD*.

Let such a number measure them, and let it be *G*.

Now, since *G* measures *CD*, while *CD* measures *BE*, *G* also measures *BE*.

But it also measures the whole *BA*; therefore it will also measure the remainder *AE*.

But *AE* measures *DF*; therefore *G* will also measure *DF*.

But it also measures the whole *DC*; therefore it will also measure the remainder *CF*, that is, the greater will measure the less: which is impossible.

Therefore no number which is greater than *CF* will measure the numbers *AB*, *CD*;

therefore CF is the greatest common measure of AB, CD.

PORISM.

From this it is manifest that, if a number measure two numbers, it will also measure their greatest common measure.

Q. E. D.

PROPOSITION 3.

Given three numbers not prime to one another, to find their greatest common measure.

Let *A*, *B*, *C* be the three given numbers not prime to one another; thus it is required to find the greatest common measure of *A*, *B*, *C*.

For let the greatest common measure, *D*, of the two numbers *A*, *B* be taken; {VII. 2} then *D* either measures, or does not measure, *C*.

First, let it measure it.

But it measures *A*, *B* also; therefore *D* measures *A*, *B*, *C*; therefore *D* is a common measure of *A*, *B*, *C*.

I say that it is also the greatest.

For, if *D* is not the greatest common measure of *A*, *B*, *C*, some number which is greater than *D* will measure the numbers *A*, *B*, *C*.

Let such a number measure them, and let it be *E*.

Since then *E* measures *A*, *B*, *C*, it will also measure *A*, *B*; therefore it will also measure the greatest common measure of *A*, *B*. {VII. 2, Por.}

But the greatest common measure of *A*, *B* is *D*; therefore *E* measures *D*, the greater the less: which is impossible.

Therefore no number which is greater than *D* will measure the numbers *A*, *B*, *C*;

therefore D is the greatest common measure of A, B, C.

Next, let *D* not measure *C*; I say first that *C*, *D* are not prime to one another.

For, since *A*, *B*, *C* are not prime to one another, some number will measure them.

Now that which measures *A*, *B*, *C* will also measure *A*, *B*, and will measure *D*, the greatest common measure of *A*, *B*. {VII. 2, Por.}

But it measures *C* also; therefore some number will measure the numbers *D*, *C*; therefore *D*, *C* are not prime to one another.

Let then their greatest common measure *E* be taken. {VII. 2}

Then, since *E* measures *D*, and *D* measures *A*, *B*, therefore *E* also measures *A*, *B*.

But it measures *C* also; therefore *E* measures *A*, *B*, *C*; therefore *E* is a common measure of *A*, *B*, *C*.

I say next that it is also the greatest.

For, if *E* is not the greatest common measure of *A*, *B*, *C*, some number which is greater than *E* will measure the numbers *A*, *B*, *C*.

Let such a number measure them, and let it be *F*.

Now, since *F* measures *A*, *B*, *C*, it also measures *A*, *B*; therefore it will also measure the greatest common measure of *A*, *B*. {VII. 2, Por.}

But the greatest common measure of *A*, *B* is *D*; therefore *F* measures *D*.

And it measures *C* also; therefore *F* measures *D*, *C*; therefore it will also measure the greatest common measure of *D*, *C*. {VII. 2, Por.}

But the greatest common measure of *D*, *C* is *E*; therefore *F* measures *E*, the greater the less: which is impossible.

Therefore no number which is greater than *E* will measure the numbers *A*, *B*, *C*; therefore *E* is the greatest common measure of *A*, *B*, *C*. Q. E. D.

PROPOSITION 4.

Any number is either a part or parts of any number, the less of the greater.

Let *A*, *BC* be two numbers, and let *BC* be the less; I say that *BC* is either a part, or parts, of *A*.

For *A*, *BC* are either prime to one another or not.

First, let *A*, *BC* be prime to one another.

Then, if *BC* be divided into the units in it, each unit of those in *BC* will be some part of *A*; so that *BC* is parts of *A*.

Next let *A*, *BC* not be prime to one another; then *BC* either measures, or does not measure, *A*.

If now *BC* measures *A*, *BC* is a part of *A*.

But, if not, let the greatest common measure *D* of *A*, *BC* be taken; {VII. 2} and let *BC* be divided into the numbers equal to *D*, namely *BE*, *EF*, *FC*.

Now, since *D* measures *A*, *D* is a part of *A*.

But *D* is equal to each of the numbers *BE*, *EF*, *FC*; therefore each of the numbers *BE*, *EF*, *FC* is also a part of *A*; so that *BC* is parts of *A*.

Therefore etc. Q. E. D.

PROPOSITION 5.

If a number be a part of a number, and another be the same part of another, the sum will also be the same part of the sum that the one is of the one.

For let the number *A* be a part of *BC*, and another, *D*, the same part of another *EF* that *A* is of *BC*; I say that the sum of *A*, *D* is also the same part of the sum of *BC*, *EF* that *A* is of *BC*.

For since, whatever part *A* is of *BC*, *D* is also the same part of *EF*, therefore, as many numbers as there are in *BC* equal to *A*, so many numbers are there also in *EF* equal to *D*.

Let *BC* be divided into the numbers equal to *A*, namely *BG*, *GC*, and *EF* into the numbers equal to *D*, namely *EH*, *HF*; then the multitude of *BG*, *GC* will be equal to the multitude of *EH*, *HF*.

And, since *BG* is equal to *A*, and *EH* to *D*, therefore *BG*, *EH* are also equal to *A*, *D*.

For the same reason *GC*, *HF* are also equal to *A*, *D*.

Therefore, as many numbers as there are in *BC* equal to *A*, so many are there also in *BC*, *EF* equal to *A*, *D*.

Therefore, whatever multiple *BC* is of *A*, the same multiple also is the sum of *BC*, *EF* of the sum of *A*, *D*.

Therefore, whatever part *A* is of *BC*, the same part also is the sum of *A*, *D* of the sum of *BC*, *EF*. Q. E. D.

PROPOSITION 6.

If a number be parts of a number, and another be the same parts of another, the sum will also be the same parts of the sum that the one is of the one.

For let the number *AB* be parts of the number *C*, and another, *DE*, the same parts of another, *F*, that *AB* is of *C*; I say that the sum of *AB*, *DE* is also the same parts of the sum of *C*, *F* that *AB* is of *C*.

For since, whatever parts *AB* is of *C*, *DE* is also the same parts of *F*, therefore, as many parts of *C* as there are in *AB*, so many parts of *F* are there also in *DE*.

Let *AB* be divided into the parts of *C*, namely *AG*, *GB*, and *DE* into the parts of *F*, namely *DH*, *HE*; thus the multitude of *AG*, *GB* will be equal to the multitude of *DH*, *HE*.

And since, whatever part *AG* is of *C*, the same part is *DH* of *F* also, therefore, whatever part *AG* is of *C*, the same part also is the sum of *AG*, *DH* of the sum of *C*, *F*. {VII. 5}

For the same reason, whatever part *GB* is of *C*, the same part also is the sum of *GB*, *HE* of the sum of *C*, *F*.

Therefore, whatever parts *AB* is of *C*, the same parts also is the sum of *AB*, *DE* of the sum of *C*, *F*. Q. E. D.

PROPOSITION 7.

If a number be that part of a number, which a number subtracted is of a number subtracted, the remainder will also be the same part of the remainder that the whole is of the whole.

For let the number *AB* be that part of the number *CD* which *AE* subtracted is of *CF* subtracted; I say that the remainder *EB* is also the same part of the remainder *FD* that the whole *AB* is of the whole *CD*.

For, whatever part *AE* is of *CF*, the same part also let *EB* be of *CG*.

Now since, whatever part *AE* is of *CF*, the same part also is *EB* of *CG*, therefore, whatever part *AE* is of *CF*, the same part also is *AB* of *GF*. {VII. 5}

But, whatever part *AE* is of *CF*, the same part also, by hypothesis, is *AB* of *CD*; therefore, whatever part *AB* is of *GF*, the same part is it of *CD* also; therefore *GF* is equal to *CD*.

Let *CF* be subtracted from each; therefore the remainder *GC* is equal to the remainder *FD*.

Now since, whatever part *AE* is of *CF*, the same part also is *EB* of *GC*, while *GC* is equal to *FD*, therefore, whatever part *AE* is of *CF*, the same part also is *EB* of *FD*.

But, whatever part *AE* is of *CF*, the same part also is *AB* of *CD*; therefore also the remainder *EB* is the same part of the remainder *FD* that the whole *AB* is of the whole *CD*. Q. E. D.

PROPOSITION 8.

If a number be the same parts of a number that a number subtracted is of a number subtracted, the remainder will also be the same parts of the remainder that the whole is of the whole.

For let the number *AB* be the same parts of the number *CD* that *AE* subtracted is of *CF* subtracted; I say that the remainder *EB* is also the same parts of the remainder *FD* that the whole *AB* is of the whole *CD*.

For let *GH* be made equal to *AB*.

Therefore, whatever parts *GH* is of *CD*, the same parts also is *AE* of *CF*.

Let *GH* be divided into the parts of *CD*, namely *GK*, *KH*, and *AE* into the parts of *CF*, namely *AL*, *LE*; thus the multitude of *GK*, *KH* will be equal to the multitude of *AL*, *LE*.

Now since, whatever part *GK* is of *CD*, the same part also is *AL* of *CF*, while. *CD* is greater than *CF*, therefore *GK* is also greater than *AL*.

Let *GM* be made equal to *AL*.

Therefore, whatever part *GK* is of *CD*, the same part also is *GM* of *CF*; therefore also the remainder *MK* is the same part of the remainder *FD* that the whole *GK* is of the whole *CD*. {VII. 7}

Again, since, whatever part *KH* is of *CD*, the same part also is *EL* of *CF*, while *CD* is greater than *CF*, therefore *HK* is also greater than *EL*.

Let *KN* be made equal to *EL*.

Therefore, whatever part *KH* is of *CD*, the same part also is *KN* of *CF*; therefore also the remainder *NH* is the same part of the remainder *FD* that the whole *KH* is of the whole *CD*. {VII. 7}

But the remainder *MK* was also proved to be the same part of the remainder *FD* that the whole *GK* is of the whole *CD*; therefore also the sum of *MK*, *NH* is the same parts of *DF* that the whole *HG* is of the whole *CD*.

But the sum of *MK*, *NH* is equal to *EB*, and *HG* is equal to *BA*; therefore the remainder *EB* is the same parts of the remainder *FD* that the whole *AB* is of the whole *CD*. Q. E. D.

PROPOSITION 9.

If a number be a part of a number, and another be the same part of another, alternately also, whatever part or parts the first is of the third, the same part, or the same parts, will the second also be of the fourth.

For let the number *A* be a part of the number *BC*, and another, *D*, the same part of another, *EF*, that *A* is of *BC*; I say that, alternately also, whatever part or parts *A* is of *D*, the same part or parts is *BC* of *EF* also.

For since, whatever part *A* is of *BC*, the same part also is *D* of *EF*, therefore, as many numbers as there are in *BC* equal to *A*, so many also are there in *EF* equal to *D*.

Let *BC* be divided into the numbers equal to *A*, namely *BG*, *GC*, and *EF* into those equal to *D*, namely *EH*, *HF*; thus the multitude of *BG*, *GC* will be equal to the multitude of *EH*, *HF*.

Now, since the numbers *BG*, *GC* are equal to one another, and the numbers *EH*, *HF* are also equal to one another, while the multitude of *BG*, *GC* is equal to the multitude of *EH*, *HF*, therefore, whatever part or parts *BG* is of *EH*, the same part or the same parts is *GC* of *HF* also; so that, in addition, whatever part or parts *BG* is of *EH*, the same part also, or the same parts, is the sum *BC* of the sum *EF*. {VII. 5, 6}

But *BG* is equal to *A*, and *EH* to *D*; therefore, whatever part or parts *A* is of *D*, the same part or the same parts is *BC* of *EF* also. Q. E. D.

PROPOSITION 10.

If a number be parts of a number, and another be the same parts of another, alternately also, whatever parts or part the first is of the third, the same parts or the same part will the second also be of the fourth.

For let the number *AB* be parts of the number *C*, and another, *DE*, the same parts of another, *F*; I say that, alternately also, whatever parts or part *AB* is of *DE*, the same parts or the same part is *C* of *F* also.

For since, whatever parts *AB* is of *C*, the same parts also is *DE* of *F*, therefore, as many parts of *C* as there are in *AB*, so many parts also of *F* are there in *DE*.

Let *AB* be divided into the parts of *C*, namely *AG*, *GB*, and *DE* into the parts of *F*, namely *DH*, *HE*; thus the multitude of *AG*, *GB* will be equal to the

multitude of *DH*, *HE*.

Now since, whatever part *AG* is of *C*, the same part also is *DH* of *F*, alternately also, whatever part or parts *AG* is of *DH*, the same part or the same parts is *C* of *F* also. {VII. 9}

For the same reason also, whatever part or parts *GB* is of *HE*, the same part or the same parts is *C* of *F* also; so that, in addition, whatever parts or part *AB* is of *DE*, the same parts also, or the same part, is *C* of *F*. {VII. 5, 6} Q. E. D.

PROPOSITION 11.

If, as whole is to whole, so is a number subtracted to a number subtracted, the remainder will also be to the remainder as whole to whole.

As the whole *AB* is to the whole *CD*, so let *AE* subtracted be to *CF* subtracted; I say that the remainder *EB* is also to the remainder *FD* as the whole *AB* to the whole *CD*.

Since, as *AB* is to *CD*, so is *AE* to *CF*, whatever part or parts *AB* is of *CD*, the same part or the same parts is *AE* of *CF* also; {VII. Def. 20}

Therefore also the remainder *EB* is the same part or parts of *FD* that *AB* is of *CD*. {VII. 7, 8}

Therefore, as *EB* is to *FD*, so is *AB* to *CD*. {VII. Def. 20} Q. E. D.

PROPOSITION 12.

If there be as many numbers as we please in proportion, then, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents.

Let *A*, *B*, *C*, *D* be as many numbers as we please in proportion, so that,

as A is to B, so is C to D;

I say that, as *A* is to *B*, so are *A*, *C* to *B*, *D*.

For since, as *A* is to *B*, so is *C* to *D*, whatever part or parts *A* is of *B*, the same part or parts is *C* of *D* also. {VII. Def. 20}

Therefore also the sum of *A*, *C* is the same part or the same parts of the sum of *B*, *D* that *A* is of *B*. {VII. 5, 6}

Therefore, as *A* is to *B*, so are *A*, *C* to *B*, *D*. {VII. Def. 20}

PROPOSITION 13.

If four numbers be proportional, they will also be proportional alternately.

Let the four numbers *A*, *B*, *C*, *D* be proportional, so that,

as A is to B, so is C to D;

I say that they will also be proportional alternately, so that,

as A is to C, so will B be to D.

For since, as *A* is to *B*, so is *C* to *D*, therefore, whatever part or parts *A* is of *B*, the same part or the same parts is *C* of *D* also. {VII. Def. 20}

Therefore, alternately, whatever part or parts *A* is of *C*, the same part or the same parts is *B* of *D* also. {VII. 10}

Therefore, as *A* is to *C*, so is *B* to *D*. {VII. Def. 20} Q. E. D.

PROPOSITION 14.

If there be as many numbers as we please, and others equal to them in multitude, which taken two and two are in the same ratio, they will also be in the same ratio ex aequali.

Let there be as many numbers as we please *A*, *B*, *C*, and others equal to them in multitude *D*, *E*, *F*, which taken two and two are in the same ratio, so that,

as A is to B, so is D to E,

and, as *B* is to *C*, so is *E* to *F*; I say that, *ex aequali*,

as A is to C, so also is D to F.

For, since, as *A* is to *B*, so is *D* to *E*, therefore, alternately,

as A is to D, so is B to E. {VII. 13}

Again, since, as *B* is to *C*, so is *E* to *F*, therefore, alternately,

as B is to E, so is C to F. {VII. 13}

But, as *B* is to *E*, so is *A* to *D*; therefore also, as *A* is to *D*, so is *C* to *F*.

Therefore, alternately,

as A is to C, so is D to F. {id.}

PROPOSITION 15.

If an unit measure any number, and another number measure any other number the same number of times, alternately also, the unit will measure the third number the same number of times that the second measures the fourth.

For let the unit *A* measure any number *BC*, and let another number *D* measure any other number *EF* the same number of times; I say that, alternately also, the unit *A* measures the number *D* the same number of times that *BC* measures *EF*.

For, since the unit *A* measures the number *BC* the same number of times that *D* measures *EF*, therefore, as many units as there are in *BC*, so many numbers equal to *D* are there in *EF* also.

Let *BC* be divided into the units in it, *BG*, *GH*, *HC*, and *EF* into the numbers *EK*, *KL*, *LF* equal to *D*.

Thus the multitude of *BG*, *GH*, *HC* will be equal to the multitude of *EK*, *KL*, *LF*.

And, since the units *BG*, *GH*, *HC* are equal to one another, and the numbers *EK*, *KL*, *LF* are also equal to one another, while the multitude of the units *BG*, *GH*, *HC* is equal to the multitude of the numbers *EK*, *KL*, *LF*, therefore, as the unit *BG* is to the number *EK*, so will the unit *GH* be to the number *KL*, and the unit *HC* to the number *LF*.

Therefore also, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents; {VII. 12} therefore, as the unit *BG* is to the number *EK*, so is *BC* to *EF*.

But the unit *BG* is equal to the unit *A*, and the number *EK* to the number *D*.

Therefore, as the unit *A* is to the number *D*, so is *BC* to *EF*.

Therefore the unit *A* measures the number *D* the same number of times that *BC* measures *EF*. Q. E. D.

PROPOSITION 16.

If two numbers by multiplying one another make certain numbers, the numbers so produced will be equal to one another.

Let *A*, *B* be two numbers, and let *A* by multiplying *B* make *C*, and *B* by multiplying *A* make *D*; I say that *C* is equal to *D*.

For, since *A* by multiplying *B* has made *C*, therefore *B* measures *C* according to the units in *A*.

But the unit *E* also measures the number *A* according to the units in it;

therefore the unit E measures A the same number of times that B measures C.

Therefore, alternately, the unit *E* measures the number *B* the same number of times that *A* measures *C*. {VII. 15}

Again, since *B* by multiplying *A* has made *D*, therefore *A* measures *D* according to the units in *B*.

But the unit *E* also measures *B* according to the units in it;

therefore the unit E measures the number B the same number of times that A measures D.

But the unit *E* measured the number *B* the same number of times that *A* measures *C*;

therefore A measures each of the numbers C, D the same number of times.

Therefore *C* is equal to *D*. Q. E. D. [1]

PROPOSITION 17.

If a number by multiplying two numbers make certain numbers, the numbers so produced will have the same ratio as the numbers multiplied.

For let the number *A* by multiplying the two numbers *B*, *C* make *D*, *E*; I say that, as *B* is to *C*, so is *D* to *E*.

For, since *A* by multiplying *B* has made *D*, therefore *B* measures *D* according to the units in *A*.

But the unit *F* also measures the number *A* according to the units in it; therefore the unit *F* measures the number *A* the same number of times that *B* measures *D*.

Therefore, as the unit *F* is to the number *A*, so is *B* to *D*. {VII. Def. 20}

For the same reason, as the unit *F* is to the number *A*, so also is *C* to *E*; therefore also, as *B* is to *D*, so is *C* to *E*.

Therefore, alternately, as *B* is to *C*, so is *D* to *E*. {VII. 13} Q. E. D.

PROPOSITION 18.

If two numbers by multiplying any number make certain numbers, the numbers so produced will have the same ratio as the multipliers.

For let two numbers *A*, *B* by multiplying any number *C* make *D*, *E*; I say that, as *A* is to *B*, so is *D* to *E*.

For, since *A* by multiplying *C* has made *D*, therefore also *C* by multiplying *A* has made *D*. {VII. 16} For the same reason also *C* by multiplying *B* has made *E*.

Therefore the number *C* by multiplying the two numbers *A*, *B* has made *D*, *E*.

Therefore, as *A* is to *B*, so is *D* to *E*. {VII. 17}

PROPOSITION 19.

If four numbers be proportional, the number produced from the first and fourth will be equal to the number produced from the second and third; and, if the number produced from the first and fourth be equal to that produced from the second and third, the four numbers will be proportional.

Let *A*, *B*, *C*, *D* be four numbers in proportion, so that,

as A is to B, so is C to D;

and let *A* by multiplying *D* make *E*, and let *B* by multiplying *C* make *F*; I say that *E* is equal to *F*.

For let *A* by multiplying *C* make *G*.

Since, then, *A* by multiplying *C* has made *G*, and by multiplying *D* has made *E*, the number *A* by multiplying the two numbers *C*, *D* has made *G*, *E*.

Therefore, as *C* is to *D*, so is *G* to *E*. {VII. 17}

But, as *C* is to *D*, so is *A* to *B*; therefore also, as *A* is to *B*, so is *G* to *E*.

Again, since *A* by multiplying *C* has made *G*, but, further, *B* has also by multiplying *C* made *F*, the two numbers *A*, *B* by multiplying a certain number *C* have made *G*, *F*.

Therefore, as *A* is to *B*, so is *G* to *F*. {VII. 18}

But further, as *A* is to *B*, so is *G* to *E* also; therefore also, as *G* is to *E*, so is *G* to *F*.

Therefore *G* has to each of the numbers *E*, *F* the same ratio;

therefore E is equal to F. {cf. V. 9}

Again, let *E* be equal to *F*; I say that, as *A* is to *B*, so is *C* to *D*.

For, with the same construction, since *E* is equal to *F*, therefore, as *G* is to *E*, so is *G* to *F*. {cf. V. 7}

But, as *G* is to *E*, so is *C* to *D*, {VII. 17}

and, as G is to F, so is A to B. {VII. 18}

Therefore also, as *A* is to *B*, so is *C* to *D*. Q. E. D.

PROPOSITION 20.

The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less.

For let *CD*, *EF* be the least numbers of those which have the same ratio with *A*, *B*; I say that *CD* measures *A* the same number of times that *EF*

measures *B*.

Now *CD* is not parts of *A*.

For, if possible, let it be so; therefore *EF* is also the same parts of *B* that *CD* is of *A*. {VII. 13 and Def. 20}

Therefore, as many parts of *A* as there are in *CD*, so many parts of *B* are there also in *EF*.

Let *CD* be divided into the parts of *A*, namely *CG*, *GD*, and *EF* into the parts of *B*, namely *EH*, *HF*; thus the multitude of *CG*, *GD* will be equal to the multitude of *EH*, *HF*.

Now, since the numbers *CG*, *GD* are equal to one another, and the numbers *EH*, *HF* are also equal to one another, while the multitude of *CG*, *GD* is equal to the multitude of *EH*, *HF*, therefore, as *CG* is to *EH*, so is *GD* to *HF*.

Therefore also, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents. {VII. 12}

Therefore, as *CG* is to *EH*, so is *CD* to *EF*.

Therefore *CG*, *EH* are in the same ratio with *CD*, *EF*, being less than they: which is impossible, for by hypothesis *CD*, *EF* are the least numbers of those which have the same ratio with them.

Therefore *CD* is not parts of *A*; therefore it is a part of it. {VII. 4}

And *EF* is the same part of *B* that *CD* is of *A*; {VII. 13 and Def. 20} therefore *CD* measures *A* the same number of times that *EF* measures *B*. Q. E. D.

PROPOSITION 21.

Numbers prime to one another are the least of those which have the same ratio with them.

Let *A*, *B* be numbers prime to one another; I say that *A*, *B* are the least of those which have the same ratio with them.

For, if not, there will be some numbers less than *A*, *B* which are in the same ratio with *A*, *B*.

Let them be *C*, *D*.

Since, then, the least numbers of those which have the same ratio measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent, {VII. 20} therefore *C* measures *A* the same number of times that *D* measures *B*.

Now, as many times as *C* measures *A*, so many units let there be in *E*.

Therefore *D* also measures *B* according to the units in *E*.

And, since *C* measures *A* according to the units in *E*, therefore *E* also measures *A* according to the units in *C*. {VII. 16}

For the same reason *E* also measures *B* according to the units in *D*. {VII. 16}

Therefore *E* measures *A*, *B* which are prime to one another: which is impossible. {VII. Def. 12}

Therefore there will be no numbers less than *A*, *B* which are in the same ratio with *A*, *B*.

Therefore *A*, *B* are the least of those which have the same ratio with them. Q. E. D.

PROPOSITION 22.

The least numbers of those which have the same ratio with them are prime to one another.

Let *A*, *B* be the least numbers of those which have the same ratio with them; I say that *A*, *B* are prime to one another.

For, if they are not prime to one another, some number will measure them.

Let some number measure them, and let it be *C*.

And, as many times as *C* measures *A*, so many units let there be in *D*, and, as many times as *C* measures *B*, so many units let there be in *E*

Since *C* measures *A* according to the units in *D*, therefore *C* by multiplying *D* has made *A*. {VII. Def. 15}

For the same reason also *C* by multiplying *E* has made *B*.

Thus the number *C* by multiplying the two numbers *D*, *E* has made *A*, *B*; therefore, as *D* is to *E*, so is *A* to *B*; {VII. 17} therefore *D*, *E* are in the same ratio with *A*, *B*, being less than they: which is impossible.

Therefore no number will measure the numbers *A*, *B*.

Therefore *A*, *B* are prime to one another. Q. E. D.

PROPOSITION 23.

If two number be prime to one another, the number which measures the one of them will be prime to the remaining number.

Let *A*, *B* be two numbers prime to one another, and let any number *C* measure *A*; I say that *C*, *B* are also prime to one another.

For, if *C*, *B* are not prime to one another, some number will measure *C*, *B*.

Let a number measure them, and let it be *D*.

Since *D* measures *C*, and *C* measures *A*, therefore *D* also measures *A*.

But it also measures *B*; therefore *D* measures *A*, *B* which are prime to one another: which is impossible. {VII. Def. 12}

Therefore no number will measure the numbers *C*, *B*.

Therefore *C*, *B* are prime to one another. Q. E. D.

PROPOSITION 24.

If two numbers be prime to any number, their product also will be prime to the same.

For let the two numbers *A*, *B* be prime to any number *C*, and let *A* by multiplying *B* make *D*; I say that *C*, *D* are prime to one another.

For, if *C*, *D* are not prime to one another, some number will measure *C*, *D*.

Let a number measure them, and let it be *E*.

Now, since *C*, *A* are prime to one another, and a certain number *E* measures *C*, therefore *A*, *E* are prime to one another. {VII. 23}

As many times, then, as *E* measures *D*, so many units let there be in *F*; therefore *F* also measures *D* according to the units in *E*. {VII. 16}

Therefore *E* by multiplying *F* has made *D*. {VII. Def. 15}

But, further, *A* by multiplying *B* has also made *D*; therefore the product of *E*, *F* is equal to the product of *A*, *B*.

But, if the product of the extremes be equal to that of the means, the four numbers are proportional; {VII. 19} therefore, as *E* is to *A*, so is *B* to *F*.

But *A*, *E* are prime to one another, numbers which are prime to one another are also the least of those which have the same ratio, {VII. 21} and the least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent; {VII. 20} therefore *E* measures *B*.

But it also measures *C*; therefore *E* measures *B*, *C* which are prime to one another: which is impossible. {VII. Def. 12}

Therefore no number will measure the numbers *C*, *D*.

Therefore *C*, *D* are prime to one another. Q. E. D. [2]

PROPOSITION 25.

If two numbers be prime to one another, the product of one of them into itself will be prime to the remaining one.

Let *A*, *B* be two numbers prime to one another, and let *A* by multiplying itself make *C*: I say that *B*, *C* are prime to one another.

For let *D* be made equal to *A*.

Since *A*, *B* are prime to one another, and *A* is equal to *D*, therefore *D*, *B* are also prime to one another.

Therefore each of the two numbers *D*, *A* is prime to *B*; therefore the product of *D*, *A* will also be prime to *B*. {VII. 24}

But the number which is the product of *D*, *A* is *C*.

Therefore *C*, *B* are prime to one another. Q. E. D. [3]

PROPOSITION 26.

If two numbers be prime to two numbers, both to each, their products also will be prime to one another.

For let the two numbers *A*, *B* be prime to the two numbers *C*, *D*; both to each, and let *A* by multiplying *B* make *E*, and let *C* by multiplying *D* make *F*; I say that *E*, *F* are prime to one another.

For, since each of the numbers *A*, *B* is prime to *C*, therefore the product of *A*, *B* will also be prime to *C*. {VII. 24}

But the product of *A*, *B* is *E*; therefore *E*, *C* are prime to one another.

For the same reason *E*, *D* are also prime to one another.

Therefore each of the numbers *C*, *D* is prime to *E*.

Therefore the product of *C*, *D* will also be prime to *E*. {VII. 24}

But the product of C, D is F.

Therefore E, F are prime to one another. Q. E. D.

PROPOSITION 27.

If two numbers be prime to one another, and each by multiplying itself make a certain number, the products will be prime to one another; and, if the original numbers by multiplying the products make certain numbers, the latter will also be prime to one another {and this is always the case with the extremes}.

Let A, B be two numbers prime to one another, let A by multiplying itself make C, and by multiplying C make D, and let B by multiplying itself make E, and by multiplying E make F; I say that both C, E and D, F are prime to one another.

For, since A, B are prime to one another, and A by multiplying itself has made C, therefore C, B are prime to one another. {VII. 25}

Since then C, B are prime to one another, and B by multiplying itself has made E, therefore C, E are prime to one another. {*id.*}

Again, since A, B are prime to one another, and B by multiplying itself has made E, therefore A, E are prime to one another. {*id.*}

Since then the two numbers A, C are prime to the two numbers B, E, both to each, therefore also the product of A, C is prime to the product of B, E. {VII. 26}

And the product of A, C is D, and the product of B, E is F.

Therefore D, F are prime to one another. Q. E. D.

PROPOSITION 28.

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another.

For let two numbers *AB*, *BC* prime to one another be added; I say that the sum *AC* is also prime to each of the numbers *AB*, *BC*.

For, if *CA*, *AB* are not prime to one another, some number will measure *CA*, *AB*.

Let a number measure them, and let it be *D*.

Since then *D* measures *CA*, *AB*, therefore it will also measure the remainder *BC*.

But it also measures *BA*; therefore *D* measures *AB*, *BC* which are prime to one another: which is impossible. {VII. Def. 12}

Therefore no number will measure the numbers *CA*, *AB*; therefore *CA*, *AB* are prime to one another.

For the same reason *AC*, *CB* are also prime to one another.

Therefore *CA* is prime to each of the numbers *AB*, *BC*.

Again, let *CA*, *AB* be prime to one another; I say that *AB*, *BC* are also prime to one another.

For, if *AB*, *BC* are not prime to one another, some number will measure *AB*, *BC*.

Let a number measure them, and let it be *D*.

Now, since *D* measures each of the numbers *AB*, *BC*, it will also measure the whole *CA*.

But it also measures *AB*; therefore *D* measures *CA*, *AB* which are prime to one another: which is impossible. {VII. Def. 12}

Therefore no number will measure the numbers *AB*, *BC*.

Therefore *AB*, *BC* are prime to one another. Q. E. D.

PROPOSITION 29.

Any prime number is prime to any number which it does not measure.

Let *A* be a prime number, and let it not measure *B*; I say that *B*, *A* are prime to one another.

For, if *B*, *A* are not prime to one another, some number will measure them.

Let *C* measure them.

Since *C* measures *B*, and *A* does not measure *B*, therefore *C* is not the same with *A*.

Now, since *C* measures *B*, *A*, therefore it also measures *A* which is prime, though it is not the same with it: which is impossible.

Therefore no number will measure *B*, *A*.

Therefore *A*, *B* are prime to one another. Q. E. D.

PROPOSITION 30.

If two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the original numbers.

For let the two numbers *A*, *B* by multiplying one another make *C*, and let any prime number *D* measure *C*; I say that *D* measures one of the numbers *A*, *B*.

For let it not measure *A*.

Now *D* is prime; therefore *A*, *D* are prime to one another. {VII. 29}

And, as many times as *D* measures *C*, so many units let there be in *E*.

Since then *D* measures *C* according to the units in *E*, therefore *D* by multiplying *E* has made *C*. {VII. Def. 15}

Further, *A* by multiplying *B* has also made *C*; therefore the product of *D*, *E* is equal to the product of *A*, *B*.

Therefore, as *D* is to *A*, so is *B* to *E*. {VII. 19}

But *D*, *A* are prime to one another, primes are also least, {VII. 21} and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent; {VII. 20} therefore *D* measures *B*.

Similarly we can also show that, if *D* do not measure *B*, it will measure *A*.

Therefore *D* measures one of the numbers *A*, *B*. Q. E. D.

PROPOSITION 31.

Any composite number is measured by some prime number.

Let *A* be a composite number; I say that *A* is measured by some prime number.

For, since *A* is composite,
some number will measure it.

Let a number measure it, and let it be *B*.

Now, if *B* is prime, what was enjoined will have been done.

But if it is composite, some number will measure it.

Let a number measure it, and let it be *C*.

Then, since *C* measures *B*, and *B* measures *A*, therefore *C* also measures *A*.

And, if *C* is prime, what was enjoined will have been done.

But if it is composite, some number will measure it.

Thus, if the investigation be continued in this way, some prime number will be found which will measure the number
before it, which will also measure *A*.

For, if it is not found, an infinite series of numbers will measure the number *A*, each of which is less than the other: which is impossible in numbers.

Therefore some prime number will be found which will
measure the one before it, which will also measure *A*.

Therefore any composite number is measured by some prime number. [4] [5]

PROPOSITION 32.

Any number either is prime or is measured by some prime number.

Let *A* be a number; I say that *A* either is prime or is measured by some prime number.

If now *A* is prime, that which was enjoined will have been done.

But if it is composite, some prime number will measure it. {VII. 31}

Therefore any number either is prime or is measured by some prime number. Q. E. D.

PROPOSITION 33.

Given as many numbers as we please, to find the least of those which have the same ratio with them.

Let *A*, *B*, *C* be the given numbers, as many as we please; thus it is required to find the least of
those which have the same ratio with *A*, *B*, *C*.

A, *B*, *C* are either prime to one another or not.

Now, if *A*, *B*, *C* are prime to one
another, they are the least of those which have the same ratio with them. {VII. 21}

But, if not, let *D* the greatest common measure of *A*, *B*, *C* be taken, {VII. 3} and, as many times as *D* measures the numbers *A*, *B*, *C*
respectively, so many units let there be in the numbers *E*, *F*, *G* respectively.

Therefore the numbers *E*, *F*, *G* measure the numbers *A*, *B*, *C* respectively according to the units in *D*. {VII. 16}

Therefore *E*, *F*, *G* measure *A*, *B*, *C* the same number of
times; therefore *E*, *F*, *G* are in the same ratio with *A*, *B*, *C*. {VII. Def. 20}

I say next that they are the least that are in that ratio.

For, if *E*, *F*, *G* are not the least of those which have the same ratio with *A*, *B*, *C*,
there will be numbers less than *E*, *F*, *G* which are in the same ratio with *A*, *B*, *C*.

Let them be *H*, *K*, *L*; therefore *H* measures *A* the same number of times that the numbers *K*, *L* measure the numbers *B*, *C* respectively.

Now, as many times as *H* measures *A*, so many units let there be in *M*; therefore the numbers *K*, *L* also measure the numbers *B*, *C* respectively

according to the units in *M*.

And, since *H* measures *A* according to the units in *M*,
therefore *M* also measures *A* according to the units in *H*. {VII. 16}

For the same reason *M* also measures the numbers *B*, *C* according to the units in the numbers *K*, *L* respectively;

Therefore *M* measures *A*, *B*, *C*.

Now, since *H* measures *A* according to the units in *M*, therefore *H* by multiplying *M* has made *A*. {VII. Def. 15}

For the same reason also *E* by multiplying *D* has made *A*.

Therefore the product of *E*, *D* is equal to the product of
H, *M*.

Therefore, as *E* is to *H*, so is *M* to *D*. {VII. 19}

But *E* is greater than *H*; therefore *M* is also greater than *D*.

And it measures *A*, *B*, *C*:
which is impossible, for by hypothesis *D* is the greatest common measure of *A*, *B*, *C*.

Therefore there cannot be any numbers less than *E*, *F*, *G* which are in the same ratio with *A*, *B*, *C*.

Therefore *E*, *F*, *G* are the least of those which have the
same ratio with *A*, *B*, *C*. Q. E. D. [6]

PROPOSITION 34.

Given two numbers, to find the least number which they measure.

Let *A*, *B* be the two given numbers; thus it is required to find the least number which they measure.

Now *A*, *B* are either prime to one another or not.

First, let *A*, *B* be prime to one another, and let *A* by multiplying *B* make *C*; therefore also *B* by multiplying *A* has made *C*. {VII. 16}

Therefore *A*, *B* measure *C*

I say next that it is also the least number they measure.

For, if not, *A*, *B* will measure some number which is less than *C*.

Let them measure *D*.

Then, as many times as *A* measures *D*, so many units let there be in *E*, and, as many times as *B* measures *D*, so many units let there be in *F*; therefore *A* by multiplying *E* has made *D*, and *B* by multiplying *F* has made *D*; {VII. Def. 15} therefore the product of *A*, *E* is equal to the product of *B*, *F*.

Therefore, as *A* is to *B*, so is *F E*. {VII. 19}

But *A*, *B* are prime, primes are also least, {VII. 21} and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less; {VII. 20} therefore *B* measures *E*, as consequent consequent.

And, since *A* by multiplying *B*, *E* has made *C*, *D*, therefore, as *B* is to *E*, so is *C* to *D*. {VII. 17}

But *B* measures *E*; therefore *C* also measures *D*, the greater the less: which is impossible.

Therefore *A*, *B* do not measure any number less than *C*; therefore *C* is the least that is measured by *A*, *B*.

Next, let *A*, *B* not be prime to one another, and let *F*, *E*, the least numbers of those which have the same ratio with *A*, *B*, be taken; {VII. 33} therefore the product of *A*, *E* is equal to the product of *B*, *F*. {VII. 19}

And let *A* by multiplying *E* make *C*; therefore also *B* by multiplying *F* has made *C*; therefore *A*, *B* measure *C*.

I say next that it is also the least number that they measure.

For, if not, *A*, *B* will measure some number which is less than *C*.

Let them measure *D*.

And, as many times as *A* measures *D*, so many units let there be in *G*, and, as many times as *B* measures *D*, so many units let there be in *H*.

Therefore *A* by multiplying *G* has made *D*, and *B* by multiplying *H* has made *D*.

Therefore the product of *A*, *G* is equal to the product of *B*, *H*; therefore, as *A* is to *B*, so is *H* to *G*. {VII. 19}

But, as *A* is to *B*, so is *F* to *E*.

Therefore also, as *F* is to *E*, so is *H* to *G*.

But *F*, *E* are least, and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less; {VII. 20} therefore *E* measures *G*.

And, since *A* by multiplying *E*, *G* has made *C*, *D*, therefore, as *E* is to *G*, so is *C* to *D*. {VII. 17}

But *E* measures *G*; therefore *C* also measures *D*, the greater the less: which is impossible.

Therefore *A*, *B* will not measure any number which is less than *C*.

Therefore *C* is the least that is measured by *A*, *B*. Q. E. D.

PROPOSITION 35.

If two numbers measure any number, the least number measured by them will also measure the same.

For let the two numbers *A*, *B* measure any number *CD*, and let *E* be the least that they measure; I say that *E* also measures *CD*.

For, if *E* does not measure *CD*, let *E*, measuring *DF*, leave *CF* less than itself.

Now, since *A*, *B* measure *E*, and *E* measures *DF*, therefore *A*, *B* will also measure *DF*.

But they also measure the whole *CD*; therefore they will also measure the remainder *CF* which is less than *E*: which is impossible.

Therefore *E* cannot fail to measure *CD*; therefore it measures it. Q. E. D.

PROPOSITION 36.

Given three numbers, to find the least number which they measure.

Let *A*, *B*, *C* be the three given numbers; thus it is required to find the least number which they measure.

Let *D*, the least number measured by the two numbers *A*, *B*, be taken. {VII. 34}

Then *C* either measures, or does not measure, *D*.

First, let it measure it.

But *A*, *B* also measure *D*; therefore *A*, *B*, *C* measure *D*.

I say next that it is also the least that they measure.

For, if not, *A*, *B*, *C* will measure some number which is less than *D*.

Let them measure *E*.

Since *A*, *B*, *C* measure *E*, therefore also *A*, *B* measure *E*.

Therefore the least number measured by *A*, *B* will also measure *E*. {VII. 35}

But *D* is the least number measured by *A*, *B*; therefore *D* will measure *E*, the greater the less: which is impossible.

Therefore *A*, *B*, *C* will not measure any number which is less than *D*;

therefore D is the least that A, B, C measure.

Again, let *C* not measure *D*, and let *E*, the least number measured by *C*, *D*, be taken. {VII. 34}

Since *A*, *B* measure *D*, and *D* measures *E*, therefore also *A*, *B* measure *E*.

But *C* also measures *E*; therefore also *A*, *B*, *C* measure *E*.

I say next that it is also the least that they measure.

For, if not, *A*, *B*, *C* will measure some number which is less than *E*.

Let them measure *F*.

Since *A*, *B*, *C* measure *F*, therefore also *A*, *B* measure *F*; therefore the least number measured by *A*, *B* will also measure *F*. {VII. 35}

But *D* is the least number measured by *A*, *B*; therefore *D* measures *F*.

But *C* also measures *F*; therefore *D*, *C* measure *F*, so that the least number measured by *D*, *C* will also measure *F*.

But *E* is the least number measured by *C*, *D*; therefore *E* measures *F*, the greater the less: which is impossible.

Therefore *A*, *B*, *C* will not measure any number which is less than *E*.

Therefore *E* is the least that is measured by *A*, *B*, *C*. Q. E. D.

PROPOSITION 37.

If a number be measured by any number, the number which is measured will have a part called by the same name as the measuring number.

For let the number *A* be measured by any number *B*; I say that *A* has a part called by the same name as *B*.

For, as many times as *B* measures *A*, so many units let there be in *C*.

Since *B* measures *A* according to the units in *C*, and the unit *D* also measures the number *C* according to the units in it, therefore the unit *D* measures the number *C* the same number of times as *B* measures *A*.

Therefore, alternately, the unit *D* measures the number *B* the same number of times as *C* measures *A*; {VII. 15} therefore, whatever part the unit *D* is of the number *B*, the same part is *C* of *A* also.

But the unit *D* is a part of the number *B* called by the same name as it; therefore *C* is also a part of *A* called by the same name as *B*, so that *A* has a part *C* which is called by the same name as *B*. Q. E. D.

PROPOSITION 38.

If a number have any part whatever, it will be measured by a number called by the same name as the part.

For let the number *A* have any part whatever, *B*, and let *C* be a number called by the same name as the part *B*; I say that *C* measures *A*.

For, since *B* is a part of *A* called by the same name as *C*, and the unit *D* is also a part of *C* called by the same name as it, therefore, whatever part the unit *D* is of the number *C*, the same part is *B* of *A* also; therefore the unit *D* measures the number *C* the same number of times that *B* measures *A*.

Therefore, alternately, the unit *D* measures the number *B* the same number of times that *C* measures *A*. {VII. 15}

Therefore *C* measures *A*. Q. E. D.

PROPOSITION 39.

To find the number which is the least that will have given parts.

Let *A*, *B*, *C* be the given parts; thus it is required to find the number which is the least that will have the parts *A*, *B*, *C*.

Let *D*, *E*, *F* be numbers called by the same name as the parts *A*, *B*, *C*, and let *G*, the least number measured by *D*, *E*, *F*, be taken. {VII. 36}

Therefore *G* has parts called by the same name as *D*, *E*, *F*. {VII. 37}

But *A*, *B*, *C* are parts called by the same name as *D*, *E*, *F*; therefore *G* has the parts *A*, *B*, *C*.

I say next that it is also the least number that has.

For, if not, there will be some number less than *G* which will have the parts *A*, *B*, *C*.

Let it be *H*.

Since *H* has the parts *A*, *B*, *C*, therefore *H* will be measured by numbers called by the same name as the parts *A*, *B*, *C*. {VII. 38}

But *D*, *E*, *F* are numbers called by the same name as the parts *A*, *B*, *C*; therefore *H* is measured by *D*, *E*, *F*.

And it is less than *G* : which is impossible.

Therefore there will be no number less than *G* that will have the parts *A*, *B*, *C*. Q. E. D.

ENDNOTES.

[1] The Greek has οἱ γενόμενοι ἐξ αὐτῶν, "the (numbers) produced *from them*." By "from them" Euclid means "from the original numbers," though this is not very clear even in the Greek. I think ambiguity is best avoided by leaving out the words.

[2] ὁ ἐξ αὐτῶν γενόμενος, literally "the (number) produced from them," will henceforth be translated as "their product."

[3] The Greek, ὁ ἐκ τοῦ ἑνὸς αὐτῶν γενόμενος, literally "the number produced from the one of them," leaves "multiplied into itself" to be understood.

[4] i.e. the implied *problem* of finding a prime number which measures *A*.

[5] In the Greek the sentence stops here, but it is necessary to add the words “the number before it, which will also measure *A*,” which are found a few lines further down. It is possible that the words may have fallen out of P here by a simple mistake due to ὁμοιοτέλευτον (Heiberg).

[6] literally (as usual) “each of the numbers *E*, *F*, *G* measures each of the numbers *A*, *B*, *C*.”

BOOK VIII. PROPOSITIONS

PROPOSITION 1.

If there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, the numbers are the least of those which have the same ratio with them.

Let there be as many numbers as we please, *A*, *B*, *C*, *D*, in continued proportion, and let the extremes of them *A*, *D* be prime to one another; I say that *A*, *B*, *C*, *D* are the least of those which have the same ratio with them.

For, if not, let *E*, *F*, *G*, *H* be less than *A*, *B*, *C*, *D*, and in the same ratio with them.

Now, since *A*, *B*, *C*, *D* are in the same ratio with *E*, *F*, *G*, *H*, and the multitude of the numbers *A*, *B*, *C*, *D* is equal to the multitude of the numbers *E*, *F*, *G*, *H*, therefore, *ex aequali*,

as A is to D, so is E to H. {VII. 14}

But *A*, *D* are prime, primes are also least, {VII. 21} and the least numbers measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent. {VII. 20}

Therefore *A* measures *E*, the greater the less: which is impossible.

Therefore *E*, *F*, *G*, *H* which are less than *A*, *B*, *C*, *D* are not in the same ratio with them.

Therefore *A*, *B*, *C*, *D* are the least of those which have the same ratio with them. Q. E. D.

PROPOSITION 2.

To find numbers in continued proportion, as many as may be prescribed, and the least that are in a given ratio.

Let the ratio of *A* to *B* be the given ratio in least numbers; thus it is required to find numbers in continued proportion, as many as may be prescribed, and the least that are in the ratio of *A* to *B*.

Let four be prescribed; let *A* by multiplying itself make *C*, and by multiplying *B* let it make *D*; let *B* by multiplying itself make *E*; further, let *A* by multiplying *C*, *D*, *E* make *F*, *G*, *H*, and let *B* by multiplying *E* make *K*.

Now, since *A* by multiplying itself has made *C*, and by multiplying *B* has made *D*, therefore, as *A* is to *B*, so is *C* to *D*. {VII. 17}

Again, since *A* by multiplying *B* has made *D*, and *B* by multiplying itself has made *E*, therefore the numbers *A*, *B* by multiplying *B* have made the numbers *D*, *E* respectively.

Therefore, as *A* is to *B*, so is *D* to *E*. {VII. 18}

But, as *A* is to *B*, so is *C* to *D*; therefore also, as *C* is to *D*, so is *D* to *E*.

And, since *A* by multiplying *C*, *D* has made *F*, *G*, therefore, as *C* is to *D*, so is *F* to *G*. {VII. 17}

But, as *C* is to *D*, so was *A* to *B*; therefore also, as *A* is to *B*, so is *F* to *G*.

Again, since *A* by multiplying *D*, *E* has made *G*, *H*, therefore, as *D* is to *E*, so is *G* to *H*. {VII. 17}

But, as *D* is to *E*, so is *A* to *B*.

Therefore also, as *A* is to *B*, so is *G* to *H*.

And, since *A*, *B* by multiplying *E* have made *H*, *K*, therefore, as *A* is to *B*, so is *H* to *K*. {VII. 18}

But, as *A* is to *B*, so is *F* to *G*, and *G* to *H*.

Therefore also, as *F* is to *G*, so is *G* to *H*, and *H* to *K*; therefore *C*, *D*, *E*, and *F*, *G*, *H*, *K* are proportional in the ratio of *A* to *B*.

I say next that they are the least numbers that are so.

For, since A, B are the least of those which have the same ratio with them, and the least of those which have the same ratio are prime to one another, {VII. 22} therefore A, B are prime to one another.

And the numbers A, B by multiplying themselves respectively have made the numbers C, E, and by multiplying the numbers C, E respectively have made the numbers F, K; therefore C, E and F, K are prime to one another respectively. {VII. 27}

But, if there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, they are the least of those which have the same ratio with them. {VIII. 1}

Therefore C, D, E and F, G, H, K are the least of those which have the same ratio with A, B. Q. E. D.

PORISM.

From this it is manifest that, if three numbers in continued proportion be the least of those which have the same ratio with them, the extremes of them are squares, and, if four numbers, cubes.

PROPOSITION 3.

If as many numbers as we please in continued proportion be the least of those which have the same ratio with them, the extremes of them are prime to one another.

Let as many numbers as we please, A, B, C, D, in continued proportion be the least of those which have the same ratio with them; I say that the extremes of them A, D are prime to one another.

For let two numbers *E*, *F*, the least that are in the ratio of *A*, *B*, *C*, *D*, be taken, {VII. 33} then three others *G*, *H*, *K* with the same property; and others, more by one continually, {VIII. 2} until the multitude taken becomes equal to the multitude of the numbers *A*, *B*, *C*, *D*.

Let them be taken, and let them be *L*, *M*, *N*, *O*.

Now, since *E*, *F* are the least of those which have the same ratio with them, they are prime to one another. {VII. 22}

And, since the numbers *E*, *F* by multiplying themselves respectively have made the numbers *G*, *K*, and by multiplying the numbers *G*, *K* respectively have made the numbers *L*, *O*, {VIII. 2, Por.} therefore both *G*, *K* and *L*, *O* are prime to one another. {VII. 27}

And, since *A*, *B*, *C*, *D* are the least of those which have the same ratio with them, while *L*, *M*, *N*, *O* are the least that are in the same ratio with *A*, *B*, *C*, *D*, and the multitude of the numbers *A*, *B*, *C*, *D* is equal to the multitude of the numbers *L*, *M*, *N*, *O*, therefore the numbers *A*, *B*, *C*, *D* are equal to the numbers *L*, *M*, *N*, *O* respectively; therefore *A* is equal to *L*, and *D* to *O*.

And *L*, *O* are prime to one another.

Therefore *A*, *D* are also prime to one another. Q. E. D.

PROPOSITION 4.

Given as many ratios as we please in least numbers, to find numbers in continued proportion which are the least in the given ratios.

Let the given ratios in least numbers be that of *A* to *B*,
that of *C* to *D*, and that of *E* to *F*; thus it is required to find numbers in continued proportion which are the least that are in the ratio of *A* to *B*, in the ratio of *C* to *D*, and in the ratio of *E* to *F*.

Let *G*, the least number measured by *B*, *C*, be taken. {VII. 34}

And, as many times as *B* measures *G*, so many times also let *A* measure *H*, and, as many times as *C* measures *G*, so many times also let *D* measure *K*.

Now *E* either measures or does not measure *K*.

First, let it measure it.

And, as many times as *E* measures *K*, so many times let *F* measure *L* also.

Now, since *A* measures *H* the same number of times that *B* measures *G*, therefore, as *A* is to *B*, so is *H* to *G*. {VII. Def. 20, VII. 13}

For the same reason also,

as C is to D, so is G to K,

and further, as *E* is to *F*, so is *K* to *L*; therefore *H*, *G*, *K*, *L* are continuously proportional in the
ratio of *A* to *B*, in the ratio of *C* to *D*, and in the ratio of *E* to *F*.

I say next that they are also the least that have this property.

For, if *H*, *G*, *K*, *L* are not the least numbers continuously
proportional in the ratios of *A* to *B*, of *C* to *D*, and of *E* to *F*, let them be *N*, *O*, *M*, *P*.

Then since, as *A* is to *B*, so is *N* to *O*, while *A*, *B* are least, and the least numbers measure those which have the same
ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent; therefore *B* measures *O*. {VII. 20}

For the same reason
C also measures *O*; therefore *B*, *C* measure *O*; therefore the least number

measured by *B*, *C* will also measure *O*. {VII. 35}

But *G* is the least number measured by *B*, *C*;
therefore *G* measures *O*, the greater the less: which is impossible.

Therefore there will be no numbers less than *H*, *G*, *K*, *L* which are continuously in the ratio of *A* to *B*, of *C* to *D*, and of *E* to *F*.

Next, let *E* not measure *K*.

Let *M*, the least number measured by *E*, *K*, be taken.

And, as many times as *K* measures *M*, so many times let *H*, *G* measure *N*, *O* respectively, and, as many times as *E* measures *M*, so many times let *F* measure *P* also.

Since *H* measures *N* the same number of times that *G* measures *O*, therefore, as *H* is to *G*, so is *N* to *O*. {VII. 13 and Def. 20}

But, as *H* is to *G*, so is *A* to *B*;
therefore also, as *A* is to *B*, so is *N* to *O*.

For the same reason also,

as C is to D, so is O to M.

Again, since *E* measures *M* the same number of times that *F* measures *P*, therefore, as *E* is to *F*, so is *M* to *P*; {VII. 13 and Def. 20} therefore *N*, *O*, *M*, *P* are continuously proportional in the ratios of *A* to *B*, of *C* to *D*, and of *E* to *F*.

I say next that they are also the least that are in the ratios *A* : *B*, *C* : *D*, *E* : *F*.

For, if not, there will be some numbers less than N, O, M, P continuously proportional in the ratios $A : B$, $C : D$, $E : F$.

Let them be Q, R, S, T.

Now since, as Q is to R, so is A to B,
while A, B are least, and the least numbers measure those which have the same ratio with them the same number of times, the antecedent the antecedent and the consequent the consequent, {VII. 20} therefore B measures R.

For the same reason C also measures R; therefore B, C measure R.

Therefore the least number measured by B, C will also measure R. {VII. 35}

But G is the least number measured by B, C;
therefore G measures R.

And, as G is to R, so is K to S: {VII. 13} therefore K also measures S.

But E also measures S; therefore E, K measure S.

Therefore the least number measured by E, K will also measure S. {VII. 35}

But M is the least number measured by E, K; therefore M measures S, the greater the less: which is impossible.

Therefore there will not be any numbers less than N, O, M, P continuously proportional in the ratios of A to B, of C to D, and of E to F; therefore N, O, M, P are the least numbers continuously proportional in the ratios $A : B$, $C : D$, $E : F$. Q. E. D. [1]

PROPOSITION 5.

Plane numbers have to one another the ratio compounded of the ratios of their sides.

Let *A*, *B* be plane numbers, and let the numbers *C*, *D* be the sides of *A*, and *E*, *F* of *B*;
I say that *A* has to *B* the ratio compounded of the ratios of the sides.

For, the ratios being given which *C* has to *E* and *D* to *F*, let the least numbers *G*, *H*, *K* that are continuously
in the ratios *C* : *E*, *D* : *F* be taken, so that,

as C is to E, so is G to H,

and, as *D* is to *F*, so is *H* to *K*. {VIII. 4}

And let *D* by multiplying *E* make *L*.

Now, since *D* by multiplying *C* has made *A*, and by multiplying *E* has made *L*, therefore, as *C* is to *E*, so is *A* to *L*. {VII. 17}

But, as *C* is to *E*, so is *G* to *H*; therefore also, as *G* is to *H*, so is *A* to *L*.

Again, since *E* by multiplying *D* has made *L*, and further by multiplying *F* has made *B*, therefore, as *D* is to *F*, so is *L* to *B*. {VII. 17}

But, as *D* is to *F*, so is *H* to *K*; therefore also, as *H* is to *K*, so is *L* to *B*.

But it was also proved that,

as G is to H, so is A to L;

therefore, ex aequali,

as G is to K, so is A to B. {VII. 14}

But *G* has to *K* the ratio compounded of the ratios of the
sides; therefore *A* also has to *B* the ratio compounded of the ratios of the sides. Q. E. D. [2]

PROPOSITION 6.

If there be as many numbers as we please in continued proportion, and the first do not measure the second, neither will any other measure any other.

Let there be as many numbers as we please, *A*, *B*, *C*, *D*, *E*, in continued proportion, and let *A* not measure *B*; I say that neither will any other measure any other.

Now it is manifest that *A*, *B*, *C*, *D*, *E* do not measure one another in order; for *A* does not even measure *B*.

I say, then, that neither will any other measure any other.

For, if possible, let *A* measure *C*.

And, however many *A*, *B*, *C* are, let as many numbers *F*, *G*, *H*, the least of those which have the same ratio with *A*, *B*, *C*, be taken. {VII. 33}

Now, since *F*, *G*, *H* are in the same ratio with *A*, *B*, *C*, and the multitude of the numbers *A*, *B*, *C* is equal to the multitude of the numbers *F*, *G*, *H*, therefore, ex aequali, as *A* is to *C*, so is *F* to *H*. {VII. 14}

And since, as *A* is to *B*, so is *F* to *G*, while *A* does not measure *B*, therefore neither does *F* measure *G*; {VII. Def. 20} therefore *F* is not an unit, for the unit measures any number.

Now *F*, *H* are prime to one another. {VIII. 3}

And, as *F* is to *H*, so is *A* to *C*; therefore neither does *A* measure *C*.

Similarly we can prove that neither will any other measure any other. Q. E. D.

PROPOSITION 7.

If there be as many numbers as we please in continued proportion, and the first measure the last, it will measure the second also.

Let there be as many numbers as we please, *A*, *B*, *C*, *D*, in continued proportion; and let *A* measure *D*; I say that *A* also measures *B*.

For, if *A* does not measure *B*, neither will any other of the numbers measure any other. {VIII. 6}

But *A* measures *D*.

Therefore *A* also measures *B*. Q. E. D.

PROPOSITION 8.

If between two numbers there fall numbers in continued proportion with them, then, however many numbers fall between them in continued proportion, so many will also fall in continued proportion between the numbers which have the same ratio with the original numbers.

Let the numbers *C*, *D* fall between the two numbers *A*, *B* in continued proportion with them, and let *E* be made in the same ratio to *F* as *A* is to *B*; I say that, as many numbers as have fallen between *A*, *B* in continued proportion, so many will also fall between *E*, *F* in continued proportion.

For, as many as *A*, *B*, *C*, *D* are in multitude, let so many numbers *G*, *H*, *K*, *L*, the least of those which have the same ratio with *A*, *C*, *D*, *B*, be taken; {VII. 33} therefore the extremes of them *G*, *L* are prime to one another. {VIII. 3}

Now, since *A*, *C*, *D*, *B* are in the same ratio with *G*, *H*, *K*, *L*, and the multitude of the numbers *A*, *C*, *D*, *B* is equal to the multitude of the numbers *G*, *H*, *K*, *L*, therefore, ex aequali, as *A* is to *B*, so is *G* to *L*. {VII. 14}

But, as *A* is to *B*, so is *E* to *F*; therefore also, as *G* is to *L*, so is *E* to *F*.

But G, L are prime, primes are also least, {VII. 21} and the least numbers measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent. {VII. 20}

Therefore G measures E the same number of times as L measures F.

Next, as many times as G measures E, so many times let H, K also measure M, N respectively; therefore G, H, K, L measure E, M, N, F the same number of times.

Therefore G, H, K, L are in the same ratio with E, M, N, F. {VII. Def. 20}

But G, H, K, L are in the same ratio with A, C, D, B; therefore A, C, D, B are also in the same ratio with E, M, N, F.

But A, C, D, B are in continued proportion; therefore E, M, N, F are also in continued proportion.

Therefore, as many numbers as have fallen between A, B in continued proportion with them, so many numbers have also fallen between E, F in continued proportion. Q. E. D. [3]

PROPOSITION 9.

If two numbers be prime to one another, and numbers fall between them in continued proportion, then, however many numbers fall between them in continued proportion, so many will also fall between each of them and an unit in continued proportion.

Let A, B be two numbers prime to one another, and let C, D fall between them in continued proportion, and let the unit E be set out; I say that, as many numbers as fall between A, B in continued proportion, so many will also fall between either of the numbers A, B and the unit in continued proportion.

For let two numbers *F*, *G*, the least that are in the ratio of *A*, *C*, *D*, *B*, be taken, three numbers *H*, *K*, *L* with the same property, and others more by one continually, until their multitude is equal to the multitude of *A*, *C*, *D*, *B*. {VIII. 2}

Let them be taken, and let them be *M*, *N*, *O*, *P*.

It is now manifest that *F* by multiplying itself has made *H* and by multiplying *H* has made *M*, while *G* by multiplying itself has made *L* and by multiplying *L* has made *P*. {VIII. 2, Por.}

And, since *M*, *N*, *O*, *P* are the least of those which have the same ratio with *F*, *G*, and *A*, *C*, *D*, *B* are also the least of those which have the same ratio with *F*, *G*, {VIII. 1} while the multitude of the numbers *M*, *N*, *O*, *P* is equal to the multitude of the numbers *A*, *C*, *D*, *B*, therefore *M*, *N*, *O*, *P* are equal to *A*, *C*, *D*, *B* respectively; therefore *M* is equal to *A*, and *P* to *B*.

Now, since *F* by multiplying itself has made *H*, therefore *F* measures *H* according to the units in *F*.

But the unit *E* also measures *F* according to the units in it; therefore the unit *E* measures the number *F* the same number of times as *F* measures *H*.

Therefore, as the unit *E* is to the number *F*, so is *F* to *H*. {VII. Def. 20}

Again, since *F* by multiplying *H* has made *M*, therefore *H* measures *M* according to the units in *F*.

But the unit *E* also measures the number *F* according to the units in it; therefore the unit *E* measures the number *F* the same number of times as *H* measures *M*.

Therefore, as the unit *E* is to the number *F*, so is *H* to *M*.

But it was also proved that, as the unit *E* is to the number *F*, so is *F* to *H*; therefore also, as the unit *E* is to the number *F*, so is *F* to *H*, and *H* to *M*.

But *M* is equal to *A*; therefore, as the unit *E* is to the number *F*, so is *F* to *H*, and *H* to *A*.

For the same reason also, as the unit *E* is to the number *G*, so is *G* to *L* and *L* to *B*.

Therefore, as many numbers as have fallen between *A*, *B* in continued proportion, so many numbers also have fallen between each of the numbers *A*, *B* and the unit *E* in continued proportion. Q. E. D.

PROPOSITION 10.

If numbers fall between each of two numbers and an unit in continued proportion, however many numbers fall between each of them and an unit in continued proportion, so many also will fall between the numbers themselves in continued proportion.

For let the numbers *D*, *E* and *F*, *G* respectively fall between the two numbers *A*, *B* and the unit *C* in continued proportion; I say that, as many numbers as have fallen between each of the numbers *A*, *B* and the unit *C* in continued proportion, so many numbers will also fall between *A*, *B* in continued proportion.

For let *D* by multiplying *F* make *H*, and let the numbers *D*, *F* by multiplying *H* make *K*, *L* respectively.

Now, since, as the unit *C* is to the number *D*, so is *D* to *E*, therefore the unit *C* measures the number *D* the same number of times as *D* measures *E*. {VII. Def. 20}

But the unit *C* measures the number *D* according to the units in *D*; therefore the number *D* also measures *E* according to the units in *D*; therefore *D* by multiplying itself has made *E*.

Again, since, as *C* is to the number *D*, so is *E* to *A*, therefore the unit *C* measures the number *D* the same number of times as *E* measures *A*.

But the unit *C* measures the number *D* according to the units in *D*; therefore *E* also measures *A* according to the units in *D*; therefore *D* by multiplying *E* has made *A*.

For the same reason also *F* by multiplying itself has made *G*, and by multiplying *G* has made *B*.

And, since *D* by multiplying itself has made *E* and by multiplying *F* has made *H*, therefore, as *D* is to *F*, so is *E* to *H*. {VII. 17}

For the same reason also,

as D is to F, so is H to G. {VII. 18}

Therefore also, as *E* is to *H*, so is *H* to *G*.

Again, since *D* by multiplying the numbers *E*, *H* has made *A*, *K* respectively, therefore, as *E* is to *H*, so is *A* to *K*. {VII. 17}

But, as *E* is to *H*, so is *D* to *F*; therefore also, as *D* is to *F*, so is *A* to *K*.

Again, since the numbers *D*, *F* by multiplying *H* have made *K*, *L* respectively, therefore, as *D* is to *F*, so is *K* to *L*. {VII. 18}

But, as *D* is to *F*, so is *A* to *K*; therefore also, as *A* is to *K*, so is *K* to *L*.

Further, since *F* by multiplying the numbers *H*, *G* has made *L*, *B* respectively, therefore, as *H* is to *G*, so is *L* to *B*. {VII. 17}

But, as *H* is to *G*, so is *D* to *F*; therefore also, as *D* is to *F*, so is *L* to *B*.

But it was also proved that,

as D is to F, so is A to K and K to L;

therefore also, as *A* is to *K*, so is *K* to *L* and *L* to *B*.

Therefore *A*, *K*, *L*, *B* are in continued proportion.

Therefore, as many numbers as fall between each of the numbers *A*, *B* and the unit *C* in continued proportion, so many also will fall between *A*, *B* in continued proportion. Q. E. D.

PROPOSITION 11.

Between two square numbers there is one mean proportional number, and the square has to the square the ratio duplicate of that which the side has to the side.

Let *A*, *B* be square numbers, and let *C* be the side of *A*, and *D* of *B*; I say that between *A*, *B* there is one mean proportional number, and *A* has to *B* the ratio duplicate of that which *C* has to *D*.

For let *C* by multiplying *D* make *E*.

Now, since *A* is a square and *C* is its side, therefore *C* by multiplying itself has made *A*.

For the same reason also *D* by multiplying itself has made *B*.

Since then *C* by multiplying the numbers *C*, *D* has made *A*, *E* respectively, therefore, as *C* is to *D*, so is *A* to *E*. {VII. 17}

For the same reason also,

as C is to D, so is E to B. {VII. 18}

Therefore also, as *A* is to *E*, so is *E* to *B*.

Therefore between *A*, *B* there is one mean proportional number.

I say next that *A* also has to *B* the ratio duplicate of that which *C* has to *D*.

For, since *A*, *E*, *B* are three numbers in proportion, therefore *A* has to *B* the ratio duplicate of that which *A* has to *E*. {V. Def. 9}

But, as *A* is to *E*, so is *C* to *D*.

Therefore *A* has to *B* the ratio duplicate of that which the side *C* has to *D*. Q. E. D.

PROPOSITION 12.

Between two cube numbers there are two mean proportional numbers, and the cube has to the cube the ratio triplicate of that which the side has to the side.

Let *A*, *B* be cube numbers, and let *C* be the side of *A*, and *D* of *B*; I say that between *A*, *B* there are two mean proportional numbers, and *A* has to *B* the ratio triplicate of that which *C* has to *D*.

For let *C* by multiplying itself make *E*, and by multiplying *D* let it make *F*; let *D* by multiplying itself make *G*, and let the numbers *C*, *D* by multiplying *F* make *H*, *K* respectively.

Now, since *A* is a cube, and *C* its side, and *C* by multiplying itself has made *E*, therefore *C* by multiplying itself has made *E* and by multiplying *E* has made *A*.

For the same reason also *D* by multiplying itself has made *G* and by multiplying *G* has made *B*.

And, since *C* by multiplying the numbers *C*, *D* has made *E*, *F* respectively, therefore, as *C* is to *D*, so is *E* to *F*. {VII. 17}

For the same reason also,

as C is to D, so is F to G. {VII. 18}

Again, since *C* by multiplying the numbers *E*, *F* has made *A*, *H* respectively, therefore, as *E* is to *F*, so is *A* to *H*. {VII. 17}

But, as *E* is to *F*, so is *C* to *D*.

Therefore also, as *C* is to *D*, so is *A* to *H*.

Again, since the numbers *C*, *D* by multiplying *F* have made *H*, *K* respectively, therefore, as *C* is to *D*, so is *H* to *K*. {VII. 18}

Again, since *D* by multiplying each of the numbers *F*, *G* has made *K*, *B* respectively, therefore, as *F* is to *G*, so is *K* to *B*. {VII. 17}

But, as *F* is to *G*, so is *C* to *D*; therefore also, as *C* is to *D*, so is *A* to *H*, *H* to *K*, and *K* to *B*.

Therefore *H*, *K* are two mean proportionals between *A*, *B*.

I say next that *A* also has to *B* the ratio triplicate of that which *C* has to *D*.

For, since *A*, *H*, *K*, *B* are four numbers in proportion, therefore *A* has to *B* the ratio triplicate of that which *A* has to *H*. {V. Def. 10}

But, as *A* is to *H*, so is *C* to *D*; therefore *A* also has to *B* the ratio triplicate of that which *C* has to *D*. Q. E. D.

PROPOSITION 13.

If there be as many numbers as we please in continued proportion, and each by multiplying itself make some number, the products will be proportional; and, if the original numbers by multiplying the products make certain numbers, the latter will also be proportional.

Let there be as many numbers as we please, *A*, *B*, *C*, in continued proportion, so that, as *A* is to *B*, so is *B* to *C*; let *A*, *B*, *C* by multiplying themselves make *D*, *E*, *F*, and by multiplying *D*, *E*, *F* let them make *G*, *H*, *K*; I say that *D*, *E*, *F* and *G*, *H*, *K* are in continued proportion.

For let *A* by multiplying *B* make *L*, and let the numbers *A*, *B* by multiplying *L* make *M*. *N* respectively.

And again let *B* by multiplying *C* make *O*, and let the numbers *B*, *C* by multiplying *O* make *P*, *Q* respectively.

Then, in manner similar to the foregoing, we can prove that *D*, *L*, *E* and *G*, *M*, *N*, *H* are continuously proportional in the ratio of *A* to *B*, and further *E*, *O*, *F* and *H*, *P*, *Q*, *K* are continuously proportional in the ratio of *B* to *C*.

Now, as *A* is to *B*, so is *B* to *C*; therefore *D*, *L*, *E* are also in the same ratio with *E*, *O*, *F*, and further *G*, *M*, *N*, *H* in the same ratio with *H*, *P*, *Q*, *K*.

And the multitude of *D*, *L*, *E* is equal to the multitude of *E*, *O*, *F*, and that of *G*, *M*, *N*, *H* to that of *H*, *P*, *Q*, *K*; therefore, *ex acquali*,

as D is to E, so is E to F,

and, as *G* is to *H*, so is *H* to *K*. {VII. 14} Q. E. D.

PROPOSITION 14.

If a square measure a square, the side will also measure the side; and, if the side measure the side, the square will also measure the square.

Let *A*, *B* be square numbers, let *C*, *D* be their sides, and let *A* measure *B*; I say that *C* also measures *D*.

For let *C* by multiplying *D* make *E*; therefore *A*, *E*, *B* are continuously proportional in the ratio of *C* to *D*. {VIII. 11}

And, since *A*, *E*, *B* are continuously proportional, and *A* measures *B*, therefore *A* also measures *E*. {VIII. 7}

And, as *A* is to *E*, so is *C* to *D*; therefore also *C* measures *D*. {VII. Def. 20}

Again, let *C* measure *D*; I say that *A* also measures *B*.

For, with the same construction, we can in a similar manner prove that *A*, *E*, *B* are continuously proportional in the ratio of *C* to *D*.

And since, as *C* is to *D*, so is *A* to *E*, and *C* measures *D*, therefore *A* also measures *E*. {VII. Def. 20}

And *A*, *E*, *B* are continuously proportional; therefore *A* also measures *B*.

Therefore etc. Q. E. D.

PROPOSITION 15.

If a cube number measure a cube number, the side will also measure the side; and, if the side measure the side, the cube will also measure the cube.

For let the cube number *A* measure the cube *B*, and let *C* be the side of *A* and *D* of *B*; I say that *C* measures *D*.

For let *C* by multiplying itself make *E*, and let *D* by multiplying itself make *G*; further, let *C* by multiplying *D* make *F*, and let *C*, *D* by multiplying *F* make *H*, *K* respectively.

Now it is manifest that *E*, *F*, *G* and *A*, *H*, *K*, *B* are continuously proportional in the ratio of *C* to *D*. {VIII. 11, 12}

And, since *A*, *H*, *K*, *B* are continuously proportional, and *A* measures *B*, therefore it also measures *H*. {VIII. 7}

And, as *A* is to *H*, so is *C* to *D*; therefore *C* also measures *D*. {VII. Def. 20}

Next, let *C* measure *D*; I say that *A* will also measure *B*.

For, with the same construction, we can prove in a similar manner that *A*, *H*, *K*, *B* are continuously proportional in the ratio of *C* to *D*.

And, since *C* measures *D*, and, as *C* is to *D*, so is *A* to *H*, therefore *A* also measures *H*, {VII. Def. 20} so that *A* measures *B* also. Q. E. D.

PROPOSITION 16.

If a square number do not measure a square number, neither will the side measure the side; and, if the side do not measure the side, neither will the square measure the square.

Let *A*, *B* be square numbers, and let *C*, *D* be their sides; and let *A* not measure *B*; I say that neither does *C* measure *D*.

For, if *C* measures *D*, *A* will also measure *B*. {VIII. 14}

But *A* does not measure *B*; therefore neither will *C* measure *D*.

Again, let *C* not measure *D*; I say that neither will *A* measure *B*.

For, if *A* measures *B*, *C* will also measure *D*. {VIII. 14}

But *C* does not measure *D*; therefore neither will *A* measure *B*. Q. E. D.

PROPOSITION 17.

If a cube number do not measure a cube number, neither will the side measure the side; and, if the side do not measure the side, neither will the cube measure the cube.

For let the cube number *A* not measure the cube number *B*, and let *C* be the side of *A*, and *D* of *B*; I say that *C* will not measure *D*.

For if *C* measures *D*, *A* will also measure *B*. {VIII. 15}

But *A* does not measure *B*; therefore neither does *C* measure *D*.

Again, let *C* not measure *D*; I say that neither will *A* measure *B*.

For, if *A* measures *B*, *C* will also measure *D*. {VIII. 15}

But *C* does not measure *D*; therefore neither will *A* measure *B*. Q. E. D.

PROPOSITION 18.

Between two similar plane numbers there is one mean proportional number; and the plane number has to the plane number the ratio duplicate of that which the corresponding side has to the corresponding side.

Let *A*, *B* be two similar plane numbers, and let the numbers *C*, *D* be the sides of *A*, and *E*, *F* of *B*.

Now, since similar plane numbers are those which have their sides proportional, {VII. Def. 21} therefore, as *C* is to *D*, so is *E* to *F*.

I say then that between *A*, *B* there is one mean proportional number, and *A* has to *B* the ratio duplicate of that which *C* has to *E*, or *D* to *F*, that is, of that which the corresponding side has to the corresponding side.

Now since, as *C* is to *D*, so is *E* to *F*, therefore, alternately, as *C* is to *E*, so is *D* to *F*. {VII. 13}

And, since *A* is plane, and *C*, *D* are its sides, therefore *D* by multiplying *C* has made *A*.

For the same reason also *E* by multiplying *F* has made *B*.

Now let *D* by multiplying *E* make *G*.

Then, since *D* by multiplying *C* has made *A*, and by multiplying *E* has made *G*, therefore, as *C* is to *E*, so is *A* to *G*. {VII. 17}

But, as *C* is to *E*, so is *D* to *F*; therefore also, as *D* is to *F*, so is *A* to *G*.

Again, since *E* by multiplying *D* has made *G*, and by multiplying *F* has made *B*, therefore, as *D* is to *F*, so is *G* to *B*. {VII. 17}

But it was also proved that,

as D is to F, so is A to G;

therefore also, as *A* is to *G*, so is *G* to *B*.

Therefore *A*, *G*, *B* are in continued proportion.

Therefore between *A*, *B* there is one mean proportional number.

I say next that *A* also has to *B* the ratio duplicate of that which the corresponding side has to the corresponding side, that is, of that which *C* has to *E* or *D* to *F*.

For, since *A*, *G*, *B* are in continued proportion, *A* has to *B* the ratio duplicate of that which it has to *G*. {V. Def. 9}

And, as *A* is to *G*, so is *C* to *E*, and so is *D* to *F*.

Therefore *A* also has to *B* the ratio duplicate of that which *C* has to *E* or *D* to *F*. Q. E. D.

PROPOSITION 19.

Between two similar solid numbers there fall two mean proportional numbers; and the solid number has to the similar solid number the ratio triplicate of that which the corresponding side has to the corresponding side.

Let *A*, *B* be two similar solid numbers, and let *C*, *D*, *E* be the sides of *A*, and *F*, *G*, *H* of *B*.

Now, since similar solid numbers are those which have their sides proportional, {VII. Def. 21} therefore, as *C* is to *D*, so is *F* to *G*,

and, as D is to E, so is G to H.

I say that between *A*, *B* there fall two mean proportional numbers, and *A* has to *B* the ratio triplicate of that which *C* has to *F*, *D* to *G*, and also *E* to *H*.

For let *C* by multiplying *D* make *K*, and let *F* by multiplying *G* make *L*.

Now, since *C*, *D* are in the same ratio with *F*, *G*, and *K* is the product of *C*, *D*, and *L* the product of *F*, *G*, *K*, *L* are similar plane numbers; {VII. Def. 21} therefore between *K*, *L* there is one mean proportional number. {VIII. 18}

Let it be *M*

Therefore *M* is the product of *D*, *F*, as was proved in the theorem preceding this. {VIII. 18}

Now, since *D* by multiplying *C* has made *K*, and by multiplying *F* has made *M*, therefore, as *C* is to *F*, so is *K* to *M*. {VII. 17}

But, as *K* is to *M*, so is *M* to *L*.

Therefore *K*, *M*, *L* are continuously proportional in the ratio of *C* to *F*.

And since, as *C* is to *D*, so is *F* to *G*, alternately therefore, as *C* is to *F*, so is *D* to *G*. {VII. 13}

For the same reason also,

as D is to G, so is E to H.

Therefore *K*, *M*, *L* are continuously proportional in the ratio of *C* to *F*, in the ratio of *D* to *G*, and also in the ratio of *E* to *H*.

Next, let *E*, *H* by multiplying *M* make *N*, *O* respectively.

Now, since *A* is a solid number, and *C*, *D*, *E* are its sides, therefore *E* by multiplying the product of *C*, *D* has made *A*.

But the product of *C*, *D* is *K*; therefore *E* by multiplying *K* has made *A*.

For the same reason also

H by multiplying L has made B.

Now, since *E* by multiplying *K* has made *A*, and further also by multiplying *M* has made *N*, therefore, as *K* is to *M*, so is *A* to *N*. {VII. 17}

But, as *K* is to *M*, so is *C* to *F*, *D* to *G*, and also *E* to *H*; therefore also, as *C* is to *F*, *D* to *G*, and *E* to *H*, so is *A* to *N*.

Again, since *E*, *H* by multiplying *M* have made *N*, *O* respectively, therefore, as *E* is to *H*, so is *N* to *O*. {VII. 18}

But, as *E* is to *H*, so is *C* to *F* and *D* to *G*; therefore also, as *C* is to *F*, *D* to *G*, and *E* to *H*, so is *A* to *N* and *N* to *O*.

Again, since *H* by multiplying *M* has made *O*, and further also by multiplying *L* has made *B*, therefore, as *M* is to *L*, so is *O* to *B*. {VII. 17}

But, as *M* is to *L*, so is *C* to *F*, *D* to *G*, and *E* to *H*.

Therefore also, as *C* is to *F*, *D* to *G*, and *E* to *H*, so not only is *O* to *B*, but also *A* to *N* and *N* to *O*.

Therefore *A*, *N*, *O*, *B* are continuously proportional in the aforesaid ratios of the sides.

I say that *A* also has to *B* the ratio triplicate of that which the corresponding side has to the corresponding side, that is, of the ratio which the number *C* has to *F*, or *D* to *G*, and also *E* to *H*.

For, since *A*, *N*, *O*, *B* are four numbers in continued proportion, therefore *A* has to *B* the ratio triplicate of that which *A* has to *N*. {V. Def. 10}

But, as *A* is to *N*, so it was proved that *C* is to *F*, *D* to *G*, and also *E* to *H*.

Therefore *A* also has to *B* the ratio triplicate of that which the corresponding side has to the corresponding side, that is, of the ratio which the number *C* has to *F*, *D* to *G*, and also *E* to *H*. Q. E. D.

PROPOSITION 20.

If one mean proportional number fall between two numbers, the numbers will be similar plane numbers.

For let one mean proportional number *C* fall between the two numbers *A*, *B*;
I say that *A*, *B* are similar plane numbers.

Let *D*, *E*, the least numbers of those which have the same ratio with *A*, *C*, be taken; {VII. 33} therefore *D* measures *A* the same number of times that *E* measures *C*. {VII. 20}

Now, as many times as *D* measures *A*, so many units let there be in *F*; therefore *F* by multiplying *D* has made *A*, so that *A* is plane, and *D*, *F* are its sides.

Again, since *D*, *E* are the least of the numbers which have
the same ratio with *C*, *B*, therefore *D* measures *C* the same number of times that *E* measures *B*. {VII. 20}

As many times, then, as *E* measures *B*, so many units let there be in *G*; therefore *E* measures *B* according to the units in *G*; therefore *G* by multiplying *E* has made *B*.

Therefore *B* is plane, and *E*, *G* are its sides.
Therefore *A*, *B* are plane numbers.
I say next that they are also similar.

For, <*> since F by multiplying D has made A, and by multiplying E has made C, therefore, as D is to E, so is A to C, that is, C to B. {VII. 17}

Again, <*> since E by multiplying F, G has made C, B respectively, therefore, as F is to G, so is C to B. {VII. 17}

But, as C is to B, so is D to E; therefore also, as D is to E, so is F to G.

And alternately, as D is to F, so is E to G. {VII. 13}

Therefore A, B are similar plane numbers; for their sides are proportional. Q. E. D. [4]

PROPOSITION 21.

If two mean proportional numbers fall between two numbers, the numbers are similar solid numbers.

For let two mean proportional numbers C, D fall between the two numbers A, B; I say that A, B are similar solid numbers.

For let three numbers E, F, G, the least of those which have the same ratio with A, C, D, be taken; {VII. 33 or VIII. 2} therefore the extremes of them E, G are prime to one another. {VIII. 3}

Now, since one mean proportional number F has fallen between E, G, therefore E, G are similar plane numbers. {VIII. 20}

Let, then, H, K be the sides of E, and L, M of G.

Therefore it is manifest from the theorem before this that E, F, G are continuously proportional in the ratio of H to L and that of K to M.

Now, since E, F, G are the least of the numbers which have the same ratio with A, C, D, and the multitude of the numbers E, F, G is equal to the

multitude of the numbers *A*, *C*, *D*, therefore, ex aequali, as *E* is to *G*, so is *A* to *D*. {VII. 14}

But *E*, *G* are prime, primes are also least, {VII. 21} and the least measure those which have the same ratio with them the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent; {VII. 20} therefore *E* measures *A* the same number of times that *G* measures *D*.

Now, as many times as *E* measures *A*, so many units let there be in *N*.

Therefore *N* by multiplying *E* has made *A*.

But *E* is the product of *H*, *K*; therefore *N* by multiplying the product of *H*, *K* has made *A*.

Therefore *A* is solid, and *H*, *K*, *N* are its sides.

Again, since *E*, *F*, *G* are the least of the numbers which have the same ratio as *C*, *D*, *B*, therefore *E* measures *C* the same number of times that *G* measures *B*.

Now, as many times as *E* measures *C*, so many units let there be in *O*.

Therefore *G* measures *B* according to the units in *O*; therefore *O* by multiplying *G* has made *B*.

But *G* is the product of *L*, *M*; therefore *O* by multiplying the product of *L*, *M* has made *B*.

Therefore *B* is solid, and *L*, *M*, *O* are its sides; therefore *A*, *B* are solid.

I say that they are also similar.

For since *N*, *O* by multiplying *E* have made *A*, *C*, therefore, as *N* is to *O*, so is *A* to *C*, that is, *E* to *F*. {VII. 18}

But, as *E* is to *F*, so is *H* to *L* and *K* to *M*; therefore also, as *H* is to *L*, so is *K* to *M* and *N* to *O*.

And *H*, *K*, *N* are the sides of *A*, and *O*, *L*, *M* the sides of *B*.

Therefore *A*, *B* are similar solid numbers. Q. E. D.

PROPOSITION 22.

If three numbers be in continued proportion, and the first be square, the third will also be square.

Let *A*, *B*, *C* be three numbers in continued proportion, and let *A* the first be square; I say that *C* the third is also square.

For, since between *A*, *C* there is one mean proportional number, *B*, therefore *A*, *C* are similar plane numbers. {VIII. 20}

But *A* is square; therefore *C* is also square. Q. E. D.

PROPOSITION 23.

If four numbers be in continued proportion, and the first be cube, the fourth will also be cube.

Let *A*, *B*, *C*, *D* be four numbers in continued proportion, and let *A* be cube; I say that *D* is also cube.

For, since between *A*, *D* there are two mean proportional numbers *B*, *C*, therefore *A*, *D* are similar solid numbers. {VIII. 21}

But *A* is cube; therefore *D* is also cube. Q. E. D.

PROPOSITION 24.

If two numbers have to one another the ratio which a square number has to a square number, and the first be square, the second will also be square.

For let the two numbers *A*, *B* have to one another the ratio which the square number *C* has to the square number *D*, and let *A* be square; I say that

B is also square.

For, since *C*, *D* are square, *C*, *D* are similar plane numbers.

Therefore one mean proportional number falls between *C*, *D*. {VIII. 18}

And, as *C* is to *D*, so is *A* to *B*; therefore one mean proportional number falls between *A*, *B* also. {VIII. 8}

And *A* is square; therefore *B* is also square. {VIII. 22} Q. E. D.

PROPOSITION 25.

If two numbers have to one another the ratio which a cube number has to a cube number, and the first be cube, the second will also be cube.

For let the two numbers *A*, *B* have to one another the ratio which the cube number *C* has to the cube number *D*, and let *A* be cube; I say that *B* is also cube.

For, since *C*, *D* are cube, *C*, *D* are similar solid numbers.

Therefore two mean proportional numbers fall between *C*, *D*. {VIII. 19}

And, as many numbers as fall between *C*, *D* in continued proportion, so many will also fall between those which have the same ratio with them; {VIII. 8} so that two mean proportional numbers fall between *A*, *B* also.

Let *E*, *F* so fall.

Since, then, the four numbers *A*, *E*, *F*, *B* are in continued proportion, and *A* is cube, therefore *B* is also cube. {VIII. 23} Q. E. D.

PROPOSITION 26.

Similar plane numbers have to one another the ratio which a square number has to a square number.

Let *A*, *B* be similar plane numbers; I say that *A* has to *B* the ratio which a square number has to a square number.

For, since *A*, *B* are similar plane numbers, therefore one mean proportional number falls between *A*, *B*. {VIII. 18}

Let it so fall, and let it be *C*; and let *D*, *E*, *F*, the least numbers of those which have the same ratio with *A*, *C*, *B*, be taken; {VII. 33 or VIII. 2} therefore the extremes of them *D*, *F* are square. {VIII. 2, Por.}

And since, as *D* is to *F*, so is *A* to *B*, and *D*, *F* are square, therefore *A* has to *B* the ratio which a square number has to a square number. Q. E. D.

PROPOSITION 27.

Similar solid numbers have to one another the ratio which a cube number has to a cube number.

Let *A*, *B* be similar solid numbers; I say that *A* has to *B* the ratio which a cube number has to a cube number.

For, since *A*, *B* are similar solid numbers, therefore two mean proportional numbers fall between *A*, *B*. {VIII. 19}

Let *C*, *D* so fall, and let *E*, *F*, *G*, *H*, the least numbers of those which have the same ratio with *A*, *C*, *D*, *B*, and equal with them in multitude, be taken; {VII. 33 or VIII. 2} therefore the extremes of them *E*, *H* are cube. {VIII. 2, Por.}

And, as *E* is to *H*, so is *A* to *B*; therefore *A* also has to *B* the ratio which a cube number has to a cube number. Q. E. D.

ENDNOTES.

[1] 69, 71, 99. the ratios A : B, C : D, E : F. This abbreviated expression is in the Greek ὁ AB, ΓΔ, EZ λόγοι.

[2] 1, 5, 29, 31. compounded of the ratios of their sides. As in VI. 23, the Greek has the less exact phrase, "compounded of their sides."

[3] 1. fall. The Greek word is ἐμπίπτειν, "fall *in*" ="can be interpolated."

[4] 25. For, since F.....27. C to B. The text has clearly suffered corruption here. It is not necessary to *infer* from other facts that, as *D* is to *E*, so is *A* to *C*; for this is part of the hypotheses (ll. 6, 7). Again, there is no explanation of the statement (l. 25) that *F* by multiplying *E* has made *C*. It is the statement and explanation of this latter fact which are alone wanted; after which the proof proceeds as in l. 28. We might therefore substitute for ll. 25-28 the following.

"For, since *E* measures *C* the same number of times that *D* measures *A* {l. 8}, that is, according to the units in *F* {l. 10}, therefore *F* by multiplying *E* has made *C*.

And, since *E* by multiplying *F*, *G*," etc. etc.

BOOK IX. PROPOSITIONS

PROPOSITION 1.

If two similar plane numbers by multiplying one another make some number, the product will be square.

Let *A*, *B* be two similar plane numbers, and let *A* by multiplying *B* make *C*; I say that *C* is square.

For let *A* by multiplying itself make *D*.

Therefore *D* is square.

Since then *A* by multiplying itself has made *D*, and by multiplying *B* has made *C*, therefore, as *A* is to *B*, so is *D* to *C*. {VII. 17}

And, since *A*, *B* are similar plane numbers, therefore one mean proportional number falls between *A*, *B*. {VIII. 18}

But, if numbers fall between two numbers in continued proportion, as many as fall between them, so many also fall between those which have the same ratio; {VIII. 8} so that one mean proportional number falls between *D*, *C* also.

And *D* is square; therefore *C* is also square. {VIII. 22} Q. E. D.

PROPOSITION 2.

If two numbers by multiplying one another make a square number, they are similar plane numbers.

Let *A*, *B* be two numbers, and let *A* by multiplying *B* make the square number *C*; I say that *A*, *B* are similar plane numbers.

For let *A* by multiplying itself make *D*; therefore *D* is square.

Now, since *A* by multiplying itself has made *D*, and by multiplying *B* has made *C*, therefore, as *A* is to *B*, so is *D* to *C*. {VII. 17}

And, since *D* is square, and *C* is so also, therefore *D*, *C* are similar plane numbers.

Therefore one mean proportional number falls between *D*, *C*. {VIII. 18}

And, as *D* is to *C*, so is *A* to *B*; therefore one mean proportional number falls between *A*, *B* also. {VIII. 8}

But, if one mean proportional number fall between two numbers, they are similar plane numbers; {VIII. 20} therefore *A*, *B* are similar plane numbers. Q. E. D.

PROPOSITION 3.

If a cube number by multiplying itself make some number, the product will be cube.

For let the cube number *A* by multiplying itself make *B*; I say that *B* is cube.

For let *C*, the side of *A*, be taken, and let *C* by multiplying itself make *D*.

It is then manifest that *C* by multiplying *D* has made *A*.

Now, since *C* by multiplying itself has made *D*, therefore *C* measures *D* according to the units in itself.

But further the unit also measures *C* according to the units in it; therefore, as the unit is to *C*, so is *C* to *D*. {VII. Def. 20}

Again, since *C* by multiplying *D* has made *A*, therefore *D* measures *A* according to the units in *C*.

But the unit also measures *C* according to the units in it; therefore, as the unit is to *C*, so is *D* to *A*.

But, as the unit is to *C*, so is *C* to *D*; therefore also, as the unit is to *C*, so is *C* to *D*, and *D* to *A*.

Therefore between the unit and the number *A* two mean proportional numbers *C*, *D* have fallen in continued proportion.

Again, since *A* by multiplying itself has made *B*, therefore *A* measures *B* according to the units in itself.

But the unit also measures *A* according to the units in it; therefore, as the unit is to *A*, so is *A* to *B*. {VII. Def. 20}

But between the unit and *A* two mean proportional numbers have fallen; therefore two mean proportional numbers will also fall between *A*, *B*. {VIII. 8}

But, if two mean proportional numbers fall between two numbers, and the first be cube, the second will also be cube. {VIII. 23}

And *A* is cube; therefore *B* is also cube. Q. E. D.

PROPOSITION 4.

If a cube number by multiplying a cube number make some number, the product will be cube.

For let the cube number *A* by multiplying the cube number *B* make *C*; I say that *C* is cube.

For let *A* by multiplying itself make *D*; therefore *D* is cube. {IX. 3}

And, since *A* by multiplying itself has made *D*, and by multiplying *B* has made *C* therefore, as *A* is to *B*, so is *D* to *C*. {VII. 17}

And, since *A*, *B* are cube numbers, *A*, *B* are similar solid numbers.

Therefore two mean proportional numbers fall between *A*, *B*; {VIII. 19} so that two mean proportional numbers will fall between *D*, *C* also. {VIII. 8}

And *D* is cube; therefore *C* is also cube {VIII. 23} Q. E. D.

PROPOSITION 5.

If a cube number by multiplying any number make a cube number, the multiplied number will also be cube.

For let the cube number *A* by multiplying any number *B* make the cube number *C*; I say that *B* is cube.

For let *A* by multiplying itself make *D*; therefore *D* is cube. {IX. 3}

Now, since *A* by multiplying itself has made *D*, and by multiplying *B* has made *C*, therefore, as *A* is to *B*, so is *D* to *C*. {VII. 17}

And since *D*, *C* are cube, they are similar solid numbers.

Therefore two mean proportional numbers fall between *D*, *C*. {VIII. 19}

And, as *D* is to *C*, so is *A* to *B*; therefore two mean proportional numbers fall between *A*, *B* also. {VIII. 8}

And *A* is cube; therefore *B* is also cube. {VIII. 23}

PROPOSITION 6.

If a number by multiplying itself make a cube number, it will itself also be cube.

For let the number *A* by multiplying itself make the cube number *B*; I say that *A* is also cube.

For let *A* by multiplying *B* make *C*.

Since, then, *A* by multiplying itself has made *B*, and by multiplying *B* has made *C*, therefore *C* is cube.

And, since *A* by multiplying itself has made *B*, therefore *A* measures *B* according to the units in itself.

But the unit also measures *A* according to the units in it.

Therefore, as the unit is to *A*, so is *A* to *B*. {VII. Def. 20}

And, since A by multiplying B has made C, therefore B measures C according to the units in A.

But the unit also measures A according to the units in it.

Therefore, as the unit is to A, so is B to C. {VII. Def. 20}

But, as the unit is to A, so is A to B; therefore also, as A is to B, so is B to C.

And, since B, C are cube, they are similar solid numbers.

Therefore there are two mean proportional numbers between B, C. {VIII. 19}

And, as B is to C, so is A to B.

Therefore there are two mean proportional numbers between A, B also. {VIII. 8}

And B is cube; therefore A is also cube. {cf. VIII. 23} Q. E. D.

PROPOSITION 7.

If a composite number by multiplying any number make some number, the product will be solid.

For let the composite number A by multiplying any number B make C; I say that C is solid.

For, since A is composite, it will be measured by some number. {VII. Def. 13}

Let it be measured by D; and, as many times as D measures A, so many units let there be in E.

Since then D measures A according to the units in E, therefore E by multiplying D has made A. {VII. Def. 15}

And, since A by multiplying B has made C, and A is the product of D, E, therefore the product of D, E by multiplying B has made C.

Therefore *C* is solid, and *D*, *E*, *B* are its sides. Q. E. D.

PROPOSITION 8.

If as many numbers as we please beginning from an unit be in continued proportion, the third from the unit will be square, as will also those which successively leave out one; the fourth will be cube, as will also all those which leave out two; and the seventh will be at once cube and square, as will also those which leave out five.

Let there be as many numbers as we please, *A*, *B*, *C*, *D*, *E*, *F*, beginning from an unit and in continued proportion; I say that *B*, the third from the unit, is square, as are also all those which leave out one; *C*, the fourth, is cube, as are also all those which leave out two; and *F*, the seventh, is at once cube and square, as are also all those which leave out five.

For since, as the unit is to *A*, so is *A* to *B*, therefore the unit measures the number *A* the same number of times that *A* measures *B*. {VII. Def. 20}

But the unit measures the number *A* according to the units in it; therefore *A* also measures *B* according to the units in *A*.

Therefore *A* by multiplying itself has made *B*; therefore *B* is square.

And, since *B*, *C*, *D* are in continued proportion, and *B* is square, therefore *D* is also square. {VIII. 22}

For the same reason *F* is also square.

Similarly we can prove that all those which leave out one are square.

I say next that *C*, the fourth from the unit, is cube, as are also all those which leave out two.

For since, as the unit is to *A*, so is *B* to *C*, therefore the unit measures the number *A* the same number of times that *B* measures *C*.

But the unit measures the number *A* according to the units in *A*; therefore *B* also measures *C* according to the units in *A*.

Therefore *A* by multiplying *B* has made *C*.

Since then *A* by multiplying itself has made *B*, and by multiplying *B* has made *C*, therefore *C* is cube.

And, since *C*, *D*, *E*, *F* are in continued proportion, and *C* is cube, therefore *F* is also cube. {VIII. 23}

But it was also proved square; therefore the seventh from the unit is both cube and square.

Similarly we can prove that all the numbers which leave out five are also both cube and square. Q. E. D.

PROPOSITION 9.

If as many numbers as we please beginning from an unit be in continued proportion, and the number after the unit be square, all the rest will also be square. And, if the number after the unit be cube, all the rest will also be cube.

Let there be as many numbers as we please, *A*, *B*, *C*, *D*, *E*, *F*, beginning from an unit and in continued proportion, and let *A*, the number after the unit, be square; I say that all the rest will also be square.

Now it has been proved that *B*, the third from the unit, is square, as are also all those which leave out one; {IX. 8} I say that all the rest are also square.

For, since *A*, *B*, *C* are in continued proportion, and *A* is square, therefore *C* is also square. {VIII. 22}

Again, since *B*, *C*, *D* are in continued proportion, and *B* is square, *D* is also square. {VIII. 22}

Similarly we can prove that all the rest are also square.

Next, let *A* be cube; I say that all the rest are also cube.

Now it has been proved that *C*, the fourth from the unit, is cube, as also are all those which leave out two; {IX. 8} I say that all the rest are also cube.

For, since, as the unit is to *A*, so is *A* to *B*, therefore the unit measures *A* the same number of times as *A* measures *B*.

But the unit measures *A* according to the units in it; therefore *A* also measures *B* according to the units in itself; therefore *A* by multiplying itself has made *B*.

And *A* is cube.

But, if a cube number by multiplying itself make some number, the product is cube. {IX. 3}

Therefore *B* is also cube.

And, since the four numbers *A*, *B*, *C*, *D* are in continued proportion, and *A* is cube, *D* also is cube. {VIII. 23}

For the same reason *E* is also cube, and similarly all the rest are cube. Q. E. D.

PROPOSITION 10.

If as many numbers as we please beginning from an unit be in continued proportion, and the number after the unit be not square, neither will any other be square except the third from the unit and all those which leave out one. And, if the number after the unit be not cube, neither will any other be cube except the fourth from the unit and all those which leave out two.

Let there be as many numbers as we please, *A*, *B*, *C*, *D*, *E*, *F*, beginning from an unit and in continued proportion, and let *A*, the number after the unit,

not be square; I say that neither will any other be square except the third from the unit <and those which leave out one>.

For, if possible, let *C* be square.

But *B* is also square; {IX. 8} {therefore *B*, *C* have to one another the ratio which a square number has to a square number}.

And, as *B* is to *C*, so is *A* to *B*; therefore *A*, *B* have to one another the ratio which a square number has to a square number; {so that *A*, *B* are similar plane numbers}. {VIII. 26, converse}

And *B* is square; therefore *A* is also square: which is contrary to the hypothesis.

Therefore *C* is not square.

Similarly we can prove that neither is any other of the numbers square except the third from the unit and those which leave out one.

Next, let *A* not be cube.

I say that neither will any other be cube except the fourth from the unit and those which leave out two.

For, if possible, let *D* be cube.

Now *C* is also cube; for it is fourth from the unit. {IX. 8}

And, as *C* is to *D*, so is *B* to *C*; therefore *B* also has to *C* the ratio which a cube has to a cube.

And *C* is cube; therefore *B* is also cube. {VIII. 25}

And since, as the unit is to *A*, so is *A* to *B*, and the unit measures *A* according to the units in it, therefore *A* also measures *B* according to the units in itself; therefore *A* by multiplying itself has made the cube number *B*.

But, if a number by multiplying itself make a cube number, it is also itself cube. {IX. 6}

Therefore *A* is also cube: which is contrary to the hypothesis.

Therefore *D* is not cube.

Similarly we can prove that neither is any other of the numbers cube except the fourth from the unit and those which leave out two. Q. E. D.

PROPOSITION II.

If as many numbers as we please beginning from an unit be in continued proportion, the less measures the greater according to some one of the numbers which have place among the proportional numbers.

Let there be as many numbers as we please, *B*, *C*, *D*, *E*, beginning from the unit *A* and in continued proportion; I say that *B*, the least of the numbers *B*, *C*, *D*, *E*, measures *E* according to some one of the numbers *C*, *D*.

For since, as the unit *A* is to *B*, so is *D* to *E*, therefore the unit *A* measures the number *B* the same number of times as *D* measures *E*; therefore, alternately, the unit *A* measures *D* the same number of times as *B* measures *E*. {VII. 15}

But the unit *A* measures *D* according to the units in it; therefore *B* also measures *E* according to the units in *D*; so that *B* the less measures *E* the greater according to some number of those which have place among the proportional numbers. —

PORISM.

And it is manifest that, whatever place the measuring number has, reckoned from the unit, the same place also has the number according to which it measures, reckoned from the number measured, in the direction of the number before it. —

Q. E. D.

PROPOSITION 12.

If as many numbers as we please beginning from an unit be in continued proportion, by however many prime numbers the last is measured, the next to the unit will also be measured by the same.

Let there be as many numbers as we please, *A*, *B*, *C*, *D*, beginning from an unit, and in continued proportion; I say that, by however many prime numbers *D* is measured, *A* will also be measured by the same.

For let *D* be measured by any prime number *E*; I say that *E* measures *A*.

For suppose it does not; now *E* is prime, and any prime number is prime to any which it does not measure; {VII. 29} therefore *E*, *A* are prime to one another.

And, since *E* measures *D*, let it measure it according to *F*, therefore *E* by multiplying *F* has made *D*.

Again, since *A* measures *D* according to the units in *C*, {IX. 11 and Por.} therefore *A* by multiplying *C* has made *D*.

But, further, *E* has also by multiplying *F* made *D*; therefore the product of *A*, *C* is equal to the product of *E*, *F*.

Therefore, as *A* is to *E*, so is *F* to *C*. {VII. 19}

But *A*, *E* are prime, primes are also least, {VII. 21} and the least measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent; {VII. 20} therefore *E* measures *C*.

Let it measure it according to *G*; therefore *E* by multiplying *G* has made *C*.

But, further, by the theorem before this, *A* has also by multiplying *B* made *C*. {IX. 11 and Por.}

Therefore the product of *A*, *B* is equal to the product of *E*, *G*.

Therefore, as *A* is to *E*, so is *G* to *B*. {VII. 19}

But *A*, *E* are prime, primes are also least, {VII. 21} and the least numbers measure those which have the same ratio with them the same number of times, the antecedent the antecedent and the consequent the consequent: {VII. 20} therefore *E* measures *B*.

Let it measure it according to *H*; therefore *E* by multiplying *H* has made *B*.

But further *A* has also by multiplying itself made *B*; {IX. 8} therefore the product of *E*, *H* is equal to the square on *A*.

Therefore, as *E* is to *A*, so is *A* to *H*. {VII. 19}

But *A*, *E* are prime, primes are also least, {VII. 21} and the least measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent; {VII. 20} therefore *E* measures *A*, as antecedent antecedent.

But, again, it also does not measure it: which is impossible.

Therefore *E*, *A* are not prime to one another.

Therefore they are composite to one another.

But numbers composite to one another are measured by some number. {VII. Def. 14}

And, since *E* is by hypothesis prime, and the prime is not measured by any number other than itself, therefore *E* measures *A*, *E*, so that *E* measures *A*.

{But it also measures *D*; therefore *E* measures *A*, *D*.}

Similarly we can prove that, by however many prime numbers *D* is measured, *A* will also be measured by the same. Q. E. D.

PROPOSITION 13.

If as many numbers as we please beginning from an unit be in continued proportion, and the number after the unit be prime, the greatest will not be

measured by any except those which have a place among the proportional numbers.

Let there be as many numbers as we please, *A*, *B*, *C*, *D*, beginning from an unit and in continued proportion, and let *A*, the number after the unit, be prime; I say that *D*, the greatest of them, will not be measured by any other number except *A*, *B*, *C*.

For, if possible, let it be measured by *E*, and let *E* not be the same with any of the numbers *A*, *B*, *C*.

It is then manifest that *E* is not prime.

For, if *E* is prime and measures *D*, it will also measure *A* {IX. 12}, which is prime, though it is not the same with it: which is impossible.

Therefore *E* is not prime.

Therefore it is composite.

But any composite number is measured by some prime number; {VII. 31} therefore *E* is measured by some prime number.

I say next that it will not be measured by any other prime except *A*.

For, if *E* is measured by another, and *E* measures *D*, that other will also measure *D*; so that it will also measure *A* {IX. 12}, which is prime, though it is not the same with it: which is impossible.

Therefore *A* measures *E*.

And, since *E* measures *D*, let it measure it according to *F*.

I say that *F* is not the same with any of the numbers *A*, *B*, *C*.

For, if *F* is the same with one of the numbers *A*, *B*, *C*, and measures *D* according to *E*, therefore one of the numbers *A*, *B*, *C* also measures *D* according to *E*.

But one of the numbers *A*, *B*, *C* measures *D* according to some one of the numbers *A*, *B*, *C*; {IX. 11} therefore *E* is also the same with one of the numbers *A*, *B*, *C*: which is contrary to the hypothesis.

Therefore *F* is not the same as any one of the numbers *A*, *B*, *C*.

Similarly we can prove that *F* is measured by *A*, by proving again that *F* is not prime.

For, if it is, and measures *D*, it will also measure *A* {IX. 12}, which is prime, though it is not the same with it: which is impossible; therefore *F* is not prime.

Therefore it is composite.

But any composite number is measured by some prime number; {VII. 31} therefore *F* is measured by some prime number.

I say next that it will not be measured by any other prime except *A*.

For, if any other prime number measures *F*, and *F* measures *D*, that other will also measure *D*; so that it will also measure *A* {IX. 12}, which is prime, though it is not the same with it: which is impossible.

Therefore *A* measures *F*.

And, since *E* measures *D* according to *F*, therefore *E* by multiplying *F* has made *D*.

But, further, *A* has also by multiplying *C* made *D*; {IX. 11} therefore the product of *A*, *C* is equal to the product of *E*, *F*.

Therefore, proportionally, as *A* is to *E*, so is *F* to *C*. {VII. 19}

But *A* measures *E*; therefore *F* also measures *C*.

Let it measure it according to *G*.

Similarly, then, we can prove that *G* is not the same with any of the numbers *A*, *B*, and that it is measured by *A*.

And, since *F* measures *C* according to *G* therefore *F* by multiplying *G* has made *C*.

But, further, *A* has also by multiplying *B* made *C*; {IX. 11} therefore the product of *A*, *B* is equal to the product of *F*, *G*.

Therefore, proportionally, as *A* is to *F*, so is *G* to *B*. {VII. 19}

But *A* measures *F*; therefore *G* also measures *B*.

Let it measure it according to *H*.

Similarly then we can prove that *H* is not the same with *A*.

And, since *G* measures *B* according to *H*, therefore *G* by multiplying *H* has made *B*.

But further *A* has also by multiplying itself made *B*; {IX. 8} therefore the product of *H*, *G* is equal to the square on *A*.

Therefore, as *H* is to *A*, so is *A* to *G*. {VII. 19}

But *A* measures *G*;< therefore *H* also measures *A*, which is prime, though it is not the same with it: which is absurd.

Therefore *D* the greatest will not be measured by any other number except *A*, *B*, *C*. Q. E. D.

PROPOSITION 14.

If a number be the least that is measured by prime numbers, it will not be measured by any other prime number except those originally measuring it.

For let the number *A* be the least that is measured by the prime numbers *B*, *C*, *D*; I say that *A* will not be measured by any other prime number except *B*, *C*, *D*.

For, if possible, let it be measured by the prime number *E*, and let *E* not be the same with any one of the numbers *B*, *C*, *D*.

Now, since *E* measures *A*, let it measure it according to *F*;

therefore E by multiplying F has made A.

And *A* is measured by the prime numbers *B*, *C*, *D*.

But, if two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the

original numbers; {VII. 30} therefore *B*, *C*, *D* will measure one of the numbers *E*, *F*.

Now they will not measure *E*; for *E* is prime and not the same with any one of the numbers *B*, *C*, *D*.

Therefore they will measure *F*, which is less than *A*: which is impossible, for *A* is by hypothesis the least number measured by *B*, *C*, *D*.

Therefore no prime number will measure *A* except *B*, *C*, *D*. Q. E. D.

PROPOSITION 15.

If three numbers in continued proportion be the least of those which have the same ratio with them, any two whatever added together will be prime to the remaining number.

Let *A*, *B*, *C*, three numbers in continued proportion, be the least of those which have the same ratio with them; I say that any two of the numbers *A*, *B*, *C* whatever added together are prime to the remaining number, namely *A*, *B* to *C*; *B*, *C* to *A*; and further *A*, *C* to *B*.

For let two numbers *DE*, *EF*, the least of those which have the same ratio with *A*, *B*, *C*, be taken. {VIII. 2}

It is then manifest that *DE* by multiplying itself has made *A*, and by multiplying *EF* has made *B*, and, further, *EF* by multiplying itself has made *C*. {VIII. 2}

Now, since *DE*, *EF* are least, they are prime to one another. {VII. 22}

But, if two numbers be prime to one another, their sum is also prime to each; {VII. 28} therefore *DF* is also prime to each of the numbers *DE*, *EF*.

But further *DE* is also prime to *EF*; therefore *DF*, *DE* are prime to *EF*.

But, if two numbers be prime to any number, their product is also prime to the other; {VII. 24} so that the product of *FD*, *DE* is prime to *EF*; hence the

product of *FD*, *DE* is also prime to the square on *EF*. {VII. 25}

But the product of *FD*, *DE* is the square on *DE* together with the product of *DE*, *EF*; {II. 3} therefore the square on *DE* together with the product of *DE*, *EF* is prime to the square on *EF*.

And the square on *DE* is *A*, the product of *DE*, *EF* is *B*, and the square on *EF* is *C*; therefore *A*, *B* added together are prime to *C*.

Similarly we can prove that *B*, *C* added together are prime to *A*.

I say next that *A*, *C* added together are also prime to *B*.

For, since *DF* is prime to each of the numbers *DE*, *EF*, the square on *DF* is also prime to the product of *DE*, *EF*. {VII. 24, 25}

But the squares on *DE*, *EF* together with twice the product of *DE*, *EF* are equal to the square on *DF*; {II. 4} therefore the squares on *DE*, *EF* together with twice the product of *DE*, *EF* are prime to the product of *DE*, *EF*.

Separando, the squares on *DE*, *EF* together with once the product of *DE*, *EF* are prime to the product of *DE*, *EF*.

Therefore, separando again, the squares on *DE*, *EF* are prime to the product of *DE*, *EF*.

And the square on *DE* is *A*, the product of *DE*, *EF* is *B*, and the square on *EF* is *C*.

Therefore *A*, *C* added together are prime to *B*. Q. E. D.

PROPOSITION 16.

If two numbers be prime to one another, the second will not be to any other number as the first is to the second.

For let the two numbers *A*, *B* be prime to one another; I say that *B* is not to any other number as *A* is to *B*.

For, if possible, as *A* is to *B*, so let *B* be to *C*.

Now *A*, *B* are prime, primes are also least, {VII. 21} and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent; {VII. 20} therefore *A* measures *B* as antecedent antecedent.

But it also measures itself; therefore *A* measures *A*, *B* which are prime to one another: which is absurd.

Therefore *B* will not be to *C*, as *A* is to *B*. Q. E. D.

PROPOSITION 17.

If there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, the last will not be to any other number as the first to the second.

For let there be as many numbers as we please, *A*, *B*, *C*, *D*, in continued proportion, and let the extremes of them, *A*, *D*, be prime to one another; I say that *D* is not to any other number as *A* is to *B*.

For, if possible, as *A* is to *B*, so let *D* be to *E*; therefore, alternately, as *A* is to *D*, so is *B* to *E*. {VII. 13}

But *A*, *D* are prime, primes are also least, {VII. 21} and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent. {VII. 20}

Therefore *A* measures *B*.

And, as *A* is to *B*, so is *B* to *C*.

Therefore *B* also measures *C*; so that *A* also measures *C*.

And since, as *B* is to *C*, so is *C* to *D*, and *B* measures *C*, therefore *C* also measures *D*.

But *A* measured *C*; so that *A* also measures *D*.

But it also measures itself; therefore *A* measures *A*, *D* which are prime to one another : which is impossible.

Therefore *D* will not be to any other number as *A* is to *B*. Q. E. D.

PROPOSITION 18.

Given two numbers, to investigate whether it is possible to find a third proportional to them.

Let *A*, *B* be the given two numbers, and let it be required to investigate whether it is possible to find a third proportional to them.

Now *A*, *B* are either prime to one another or not.

And, if they are prime to one another, it has been proved that it is impossible to find a third proportional to them. {IX. 16}

Next, let *A*, *B* not be prime to one another, and let *B* by multiplying itself make *C*.

Then *A* either measures *C* or does not measure it.

First, let it measure it according to *D*; therefore *A* by multiplying *D* has made *C*.

But, further, *B* has also by multiplying itself made *C*; therefore the product of *A*, *D* is equal to the square on *B*.

Therefore, as *A* is to *B*, so is *B* to *D*; {VII. 19} therefore a third proportional number *D* has been found to *A*, *B*.

Next, let *A* not measure *C*; I say that it is impossible to find a third proportional number to *A*, *B*.

For, if possible, let *D*, such third proportional, have been found.

Therefore the product of *A*, *D* is equal to the square on *B*.

But the square on *B* is *C*; therefore the product of *A*, *D* is equal to *C*.

Hence *A* by multiplying *D* has made *C*; therefore *A* measures *C* according to *D*.

But, by hypothesis, it also does not measure it: which is absurd.

Therefore it is not possible to find a third proportional number to *A*, *B* when *A* does not measure *C*. Q. E. D.

PROPOSITION 19.

Given three numbers, to investigate when it is possible to find a fourth proportional to them.

Let *A*, *B*, *C* be the given three numbers, and let it be required to investigate when it is possible to find a fourth proportional to them.

Now either they are not in continued proportion, and the extremes of them are prime to one another; or they are in continued proportion, and the extremes of them are not prime to one another; or they are not in continued proportion, nor are the extremes of them prime to one another; or they are in continued proportion, and the extremes of them are prime to one another.

If then *A*, *B*, *C* are in continued proportion, and the extremes of them *A*, *C* are prime to one another, it has been proved that it is impossible to find a fourth proportional number to them. {IX. 17}

<*>Next, let *A*, *B*, *C* not be in continued proportion, the extremes being again prime to one another; I say that in this case also it is impossible to find a fourth proportional to them.

For, if possible, let *D* have been found, so that,

as A is to B, so is C to D,

and let it be contrived that, as *B* is to *C*, so is *D* to *E*.

Now, since, as *A* is to *B*, so is *C* to *D*, and, as *B* is to *C*, so is *D* to *E*, therefore, ex aequali, as *A* is to *C*, so is *C* to *E*. {VII. 14}

But *A*, *C* are prime, primes are also least, {VII. 21} and the least numbers measure those which have the same ratio, the antecedent the antecedent and the consequent the consequent. {VII. 20}

Therefore *A* measures *C* as antecedent antecedent.

But it also measures itself; therefore *A* measures *A*, *C* which are prime to one another: which is impossible.

Therefore it is not possible to find a fourth proportional to *A*, *B*, *C*.<*>

Next, let *A*, *B*, *C* be again in continued proportion, but let *A*, *C* not be prime to one another.

I say that it is possible to find a fourth proportional to them.

For let *B* by multiplying *C* make *D*; therefore *A* either measures *D* or does not measure it.

First, let it measure it according to *E*; therefore *A* by multiplying *E* has made *D*.

But, further, *B* has also by multiplying *C* made *D*; therefore the product of *A*, *E* is equal to the product of *B*, *C*; therefore, proportionally, as *A* is to *B*, so is *C* to *E*; {VII. 19} therefore *E* has been found a fourth proportional to *A*, *B*, *C*.

Next, let *A* not measure *D*; I say that it is impossible to find a fourth proportional number to *A*, *B*, *C*.

For, if possible, let *E* have been found; therefore the product of *A*, *E* is equal to the product of *B*, *C*. {VII. 19}

But the product of *B*, *C* is *D*; therefore the product of *A*, *E* is also equal to *D*.

Therefore *A* by multiplying *E* has made *D*; therefore *A* measures *D* according to *E*, so that *A* measures *D*.

But it also does not measure it: which is absurd.

Therefore it is not possible to find a fourth proportional number to *A*, *B*, *C* when *A* does not measure *D*.

Next, let *A*, *B*, *C* not be in continued proportion, nor the extremes prime to one another.

And let *B* by multiplying *C* make *D*.

Similarly then it can be proved that, if *A* measures *D*, it is possible to find a fourth proportional to them, but, if it does not measure it, impossible. Q. E. D.

PROPOSITION 20.

Prime numbers are more than any assigned multitude of prime numbers.

Let *A*, *B*, *C* be the assigned prime numbers; I say that there are more prime numbers than *A*, *B*, *C*.

For let the least number measured by *A*, *B*, *C* be taken, and let it be *DE*; let the unit *DF* be added to *DE*.

Then *EF* is either prime or not.

First, let it be prime; then the prime numbers *A*, *B*, *C*, *EF* have been found which are more than *A*, *B*, *C*.

Next, let *EF* not be prime; therefore it is measured by some prime number. {VII. 31}

Let it be measured by the prime number *G*.

I say that *G* is not the same with any of the numbers *A*, *B*, *C*.

For, if possible, let it be so.

Now *A*, *B*, *C* measure *DE*; therefore *G* also will measure *DE*.

But it also measures *EF*.

Therefore *G*, being a number, will measure the remainder, the unit *DF*: which is absurd.

Therefore *G* is not the same with any one of the numbers *A*, *B*, *C*.

And by hypothesis it is prime.

Therefore the prime numbers *A*, *B*, *C*, *G* have been found which are more than the assigned multitude of *A*, *B*, *C*. Q. E. D.

PROPOSITION 21.

If as many even numbers as we please be added together, the whole is even.

For let as many even numbers as we please, *AB*, *BC*, *CD*, *DE*, be added together; I say that the whole *AE* is even.

For, since each of the numbers *AB*, *BC*, *CD*, *DE* is even, it has a half part; {VII. Def. 6} so that the whole *AE* also has a half part.

But an even number is that which is divisible into two equal parts; {*id*.} therefore *AE* is even. Q. E. D.

PROPOSITION 22.

If as many odd numbers as we please be added together, and their multitude be even, the whole will be even.

For let as many odd numbers as we please, *AB*, *BC*, *CD*, *DE*, even in multitude, be added together; I say that the whole *AE* is even.

For, since each of the numbers *AB*, *BC*, *CD*, *DE* is odd, if an unit be subtracted from each, each of the remainders will be even; {VII. Def. 7} so that the sum of them will be even. {IX. 21}

But the multitude of the units is also even.

Therefore the whole *AE* is also even. {IX. 21} Q. E. D.

PROPOSITION 23.

If as many odd numbers as we please be added together, and their multitude be odd, the whole will also be odd.

For let as many odd numbers as we please, *AB*, *BC*, *CD*, the multitude of which is odd, be added together; I say that the whole *AD* is also odd.

Let the unit *DE* be subtracted from *CD*; therefore the remainder *CE* is even. {VII. Def. 7}

But *CA* is also even; {IX. 22} therefore the whole *AE* is also even. {IX. 21}

And *DE* is an unit.

Therefore *AD* is odd. {VII. Def. 7} Q. E. D. [1]

PROPOSITION 24.

If from an even number an even number be subtracted, the remainder will be even.

For from the even number *AB* let the even number *BC* be subtracted: I say that the remainder *CA* is even.

For, since *AB* is even, it has a half part. {VII. Def. 6}

For the same reason *BC* also has a half part; so that the remainder {*CA* also has a half part, and} *AC* is therefore even. Q. E. D.

PROPOSITION 25.

If from an even number an odd number be subtracted, the remainder will be odd.

For from the even number *AB* let the odd number *BC* be subtracted; I say that the remainder *CA* is odd.

For let the unit *CD* be subtracted from *BC*; therefore *DB* is even. {VII. Def. 7}

But *AB* is also even; therefore the remainder *AD* is also even. {IX. 24}

And *CD* is an unit; therefore *CA* is odd. {VII. Def. 7} Q. E. D.

PROPOSITION 26.

If from an odd number an odd number be subtracted, the remainder will be even.

For from the odd number *AB* let the odd number *BC* be subtracted; I say that the remainder *CA* is even.

For, since *AB* is odd, let the unit *BD* be subtracted; therefore the remainder *AD* is even. {VII. Def. 7}

For the same reason *CD* is also even; {VII. Def. 7} so that the remainder *CA* is also even. {IX. 24} Q. E. D.

PROPOSITION 27.

If from an odd number an even number be subtracted, the remainder will be odd.

For from the odd number *AB* let the even number *BC* be subtracted; I say that the remainder *CA* is odd.

Let the unit *AD* be subtracted; therefore *DB* is even. {VII. Def. 7}

But *BC* is also even; therefore the remainder *CD* is even. { IX. 24 }

Therefore *CA* is odd. {VII. Def. 7} Q. E. D.

PROPOSITION 28.

If an odd number by multiplying an even number make some number, the product will be even.

For let the odd number *A* by multiplying the even number *B* make *C*; I say that *C* is even.

For, since *A* by multiplying *B* has made *C*, therefore *C* is made up of as many numbers equal to *B* as there are units in *A*. {VII. Def. 15}

And *B* is even; therefore *C* is made up of even numbers.

But, if as many even numbers as we please be added together, the whole is even. {IX. 21}

Therefore *C* is even. Q. E. D.

PROPOSITION 29.

If an odd number by multiplying an odd number make some number, the product will be odd.

For let the odd number *A* by multiplying the odd number *B* make *C*; I say that *C* is odd.

For, since *A* by multiplying *B* has made *C*, therefore *C* is made up of as many numbers equal to *B* as there are units in *A*. {VII. Def. 15}

And each of the numbers *A*, *B* is odd; therefore *C* is made up of odd numbers the multitude of which is odd.

Thus *C* is odd. {IX. 23} Q. E. D.

PROPOSITION 30.

If an odd number measure an even number, it will also measure the half of it.

For let the odd number *A* measure the even number *B*; I say that it will also measure the half of it.

For, since *A* measures *B*, let it measure it according to *C*; I say that *C* is not odd.

For, if possible, let it be so.

Then, since *A* measures *B* according to *C*, therefore *A* by multiplying *C* has made *B*.

Therefore *B* is made up of odd numbers the multitude of which is odd.

Therefore *B* is odd: {IX. 23} which is absurd, for by hypothesis it is even.

Therefore *C* is not odd; therefore *C* is even.

Thus *A* measures *B* an even number of times.

For this reason then it also measures the half of it. Q. E. D.

PROPOSITION 31.

If an odd number be prime to any number, it will also be prime to the double of it.

For let the odd number *A* be prime to any number *B*, and let *C* be double of *B*; I say that *A* is prime to *C*.

For, if they are not prime to one another, some number will measure them.

Let a number measure them, and let it be *D*.

Now *A* is odd; therefore *D* is also odd.

And since *D* which is odd measures *C*, and *C* is even, therefore {*D*} will measure the half of *C* also. {IX. 30}

But *B* is half of *C*; therefore *D* measures *B*.

But it also measures *A*; therefore *D* measures *A*, *B* which are prime to one another: which is impossible.

Therefore *A* cannot but be prime to *C*.

Therefore *A*, *C* are prime to one another. Q. E. D.

PROPOSITION 32.

Each of the numbers which are continually doubled beginning from a dyad is even-times even only.

For let as many numbers as we please, *B*, *C*, *D*, have been continually doubled beginning from the dyad *A*; I say that *B*, *C*, *D* are eventimes even only.

Now that each of the numbers *B*, *C*, *D* is even-times even is manifest; for it is doubled from a dyad.

I say that it is also even-times even only.

For let an unit be set out.

Since then as many numbers as we please beginning from an unit are in continued proportion, and the number *A* after the unit is prime, therefore *D*, the greatest of the numbers *A*, *B*, *C*, *D*, will not be measured by any other number except *A*, *B*, *C*. {IX. 13}

And each of the numbers *A*, *B*, *C* is even; therefore *D* is even-times even only. {VII. Def. 8}

Similarly we can prove that each of the numbers *B*, *C* is even-times even only. Q. E. D.

PROPOSITION 33.

If a number have its half odd, it is even-times odd only.

For let the number *A* have its half odd; I say that *A* is even-times odd only.

Now that it is even-times odd is manifest; for the half of it, being odd, measures it an even number of times. {VII. Def. 9}

I say next that it is also even-times odd only.

For, if *A* is even-times even also, it will be measured by an even number according to an even number; {VII. Def. 8} so that the half of it will also be measured by an even number though it is odd: which is absurd.

Therefore *A* is even-times odd only. Q. E. D.

PROPOSITION 34.

If a number neither be one of those which are continually doubled from a dyad, nor have its half odd, it is both eventimes even and even-times odd.

For let the number *A* neither be one of those doubled from a dyad, nor have its half odd; I say that *A* is both even-times even and even-times odd.

Now that *A* is even-times even is manifest; for it has not its half odd. {VII. Def. 8}

I say next that it is also even-times odd.

For, if we bisect *A*, then bisect its half, and do this continually, we shall come upon some odd number which will measure *A* according to an even number.

For, if not, we shall come upon a dyad, and *A* will be among those which are doubled from a dyad: which is contrary to the hypothesis.

Thus *A* is even-times odd.

But it was also proved even-times even.

Therefore *A* is both even-times even and even-times odd. Q. E. D.

PROPOSITION 35.

If as many numbers as we please be in continued proportion, and there be subtracted from the second and the last numbers equal to the first, then, as the excess of the second is to the first, so will the excess of the last be to all those before it.

Let there be as many numbers as we please in continued proportion, *A*, *BC*, *D*, *EF*, beginning from *A* as least, and let there be subtracted from *BC* and *EF* the numbers *BG*, *FH*, each equal to *A*; I say that, as *GC* is to *A*, so is *EH* to *A*, *BC*, *D*.

For let *FK* be made equal to *BC*, and *FL* equal to *D*.

Then, since *FK* is equal to *BC*, and of these the part *FH* is equal to the part *BG*, therefore the remainder *HK* is equal to the remainder *GC*.

And since, as *EF* is to *D*, so is *D* to *BC*, and *BC* to *A*, while *D* is equal to *FL*, *BC* to *FK*, and *A* to *FH*, therefore, as *EF* is to *FL*, so is *LF* to *FK*, and *FK* to *FH*.

Separando, as *EL* is to *LF*, so is *LK* to *FK*, and *KH* to *FH*. {VII. 11, 13}

Therefore also, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents; {VII. 12} therefore, as *KH* is to *FH*, so are *EL*, *LK*. *KH* to *LF*, *FK*, *HF*.

But *KH* is equal to *CG*, *FH* to *A*, and *LF*, *FK*, *HF* to *D*, *BC*, *A*; therefore, as *CG* is to *A*, so is *EH* to *D*, *BC*, *A*.

Therefore, as the excess of the second is to the first, so is the excess of the last to all those before it. Q. E. D.

PROPOSITION 36.

If as many numbers as we please beginning from an unit be set out continuously in double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product will be perfect.

For let as many numbers as we please, *A*, *B*, *C*, *D*, beginning from an unit be set out in double proportion, until the sum of all becomes prime, let *E* be equal to the sum, and let *E* by multiplying *D* make *FG*; I say that *FG* is perfect.

For, however many *A*, *B*, *C*, *D* are in multitude, let so many *E*, *HK*, *L*, *M* be taken in double proportion beginning from *E*; therefore, ex aequali, as *A* is to *D*, so is *E* to *M*. {VII. 14}

Therefore the product of *E*, *D* is equal to the product of *A*, *M*. {VII. 19}

And the product of *E*, *D* is *FG*; therefore the product of *A*, *M* is also *FG*.

Therefore *A* by multiplying *M* has made *FG*; therefore *M* measures *FG* according to the units in *A*.

And *A* is a dyad; therefore *FG* is double of *M*.

But *M*, *L*, *HK*, *E* are continuously double of each other; therefore *E*, *HK*, *L*, *M*, *FG* are continuously proportional in double proportion.

Now let there be subtracted from the second *HK* and the last *FG* the numbers *HN*, *FO*, each equal to the first *E*; therefore, as the excess of the second is to the first, so is the excess of the last to all those before it. {IX. 35}

Therefore, as *NK* is to *E*, so is *OG* to *M*, *L*, *KH*, *E*.

And *NK* is equal to *E*; therefore *OG* is also equal to *M*, *L*, *HK*, *E*.

But *FO* is also equal to *E*, and *E* is equal to *A*, *B*, *C*, *D* and the unit.

Therefore the whole *FG* is equal to *E*, *HK*, *L*, *M* and *A*, *B*, *C*, *D* and the unit; and it is measured by them.

I say also that *FG* will not be measured by any other number except *A*, *B*, *C*, *D*, *E*, *HK*, *L*, *M* and the unit.

For, if possible, let some number *P* measure *FG*, and let *P* not be the same with any of the numbers *A*, *B*, *C*, *D*, *E*, *HK*, *L*, *M*.

And, as many times as *P* measures *FG*, so many units let there be in *Q*; therefore *Q* by multiplying *P* has made *FG*.

But, further, *E* has also by multiplying *D* made *FG*; therefore, as *E* is to *Q*, so is *P* to *D*. {VII. 19}

And, since *A*, *B*, *C*, *D* are continuously proportional beginning from an unit, therefore *D* will not be measured by any other number except *A*, *B*, *C*. {IX. 13}

And, by hypothesis, *P* is not the same with any of the numbers *A*, *B*, *C*; therefore *P* will not measure *D*.

But, as *P* is to *D*, so is *E* to *Q*; therefore neither does *E* measure *Q*. {VII. Def. 20}

And *E* is prime; and any prime number is prime to any number which it does not measure. {VII. 29}

Therefore *E*, *Q* are prime to one another.

But primes are also least, {VII. 21} and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent; {VII. 20} and, as *E* is to *Q*, so is *P* to *D*; therefore *E* measures *P* the same number of times that *Q* measures *D*.

But *D* is not measured by any other number except *A*, *B*, *C*; therefore *Q* is the same with one of the numbers *A*, *B*, *C*.

Let it be the same with *B*.

And, however many *B*, *C*, *D* are in multitude, let so many *E*, *HK*, *L* be taken beginning from *E*.

Now *E*, *HK*, *L* are in the same ratio with *B*, *C*, *D*; therefore, ex aequali, as *B* is to *D*, so is *E* to *L*. {VII. 14}

Therefore the product of B, L is equal to the product of D, E. {VII. 19}

But the product of D, E is equal to the product of Q, P; therefore the product of Q, P is also equal to the product of B, L.

Therefore, as Q is to B, so is L to P. {VII. 19}

And Q is the same with B; therefore L is also the same with P; which is impossible, for by hypothesis P is not the same with any of the numbers set out.

Therefore no number will measure FG except A, B, C, D, E, HK, L, M and the unit.

And FG was proved equal to A, B, C, D, E, HK, L, M and the unit; and a perfect number is that which is equal to its own parts; {VII. Def. 22} therefore FG is perfect. Q. E. D.

ENDNOTE.

[1] 3. Literally “let there be as many numbers as we please, of which *let* the multitude *be* odd.” This form, natural in Greek, is awkward in English.

BOOK X. DEFINITIONS I.

1

Those magnitudes are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure.

2

Straight lines are commensurable in square when the squares on them are measured by the same area, and incommensurable in square when the squares on them cannot possibly have any area as a common measure.

3

With these hypotheses, it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively, some in length only, and others in square also, with an assigned straight line. Let then the assigned straight line be called rational, and those straight lines which are commensurable with it, whether in length and in square or in square only, rational, but those which are incommensurable with it irrational.

4

And let the square on the assigned straight line be called rational and those areas which are commensurable with it rational, but those which are incommensurable with it irrational, and the straight lines which produce them irrational, that is, in case the areas are squares, the sides themselves, but in case they are any other rectilineal figures, the straight lines on which are described squares equal to them.

PROPOSITIONS 1-47.

PROPOSITION 1.

Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Let *AB, C* be two unequal magnitudes of which *AB* is the greater: I say that, if from *AB* there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the magnitude *C*.

For *C* if multiplied will sometime be greater than *AB*. {cf. v. Def. 4}

Let it be multiplied, and let *DE* be a multiple of *C*, and greater than. *AB*; let *DE* be divided into the parts *DF*, *FG*, *GE* equal to *C*, from *AB* let there be subtracted *BH* greater than its half, and, from *AH*, *HK* greater than its half, and let this process be repeated continually until the divisions in *AB* are equal in multitude with the divisions in *DE*.

Let, then, *AK*, *KH*, *HB* be divisions which are equal in multitude with *DF*, *FG*, *GE*.

Now, since *DE* is greater than *AB*, and from *DE* there has been subtracted *EG* less than its half, and, from *AB*, *BH* greater than its half, therefore the remainder *GD* is greater than the remainder *HA*.

And, since *GD* is greater than *HA*, and there has been subtracted, from *GD*, the half *GF*, and, from *HA*, *HK* greater than its half, therefore the remainder *DF* is greater than the remainder *AK*.

But *DF* is equal to *C*; therefore *C* is also greater than *AK*.

Therefore *AK* is less than *C*.

Therefore there is left of the magnitude *AB* the magnitude *AK* which is less than the lesser magnitude set out, namely *C*. Q. E. D.

And the theorem can be similarly proved even if the parts subtracted be halves.

PROPOSITION 2.

If, when the less of two unequal magnitudes is continually subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable.

For, there being two unequal magnitudes *AB*, *CD*, and *AB* being the less, when the less is continually subtracted in turn from the greater, let that which is left over never measure the one before it; I say that the magnitudes *AB*, *CD* are incommensurable.

For, if they are commensurable, some magnitude will measure them.

Let a magnitude measure them, if possible, and let it be *E*; let *AB*, measuring *FD*, leave *CF* less than itself, let *CF* measuring *BG*, leave *AG* less than itself, and let this process be repeated continually, until there is left some magnitude which is less than *E*.

Suppose this done, and let there be left *AG* less than *E*.

Then, since *E* measures *AB*, while *AB* measures *DF*, therefore *E* will also measure *FD*.

But it measures the whole *CD* also; therefore it will also measure the remainder *CF*.

But *CF* measures *BG*; therefore *E* also measures *BG*.

But it measures the whole *AB* also; therefore it will also measure the remainder *AG*, the greater the less: which is impossible.

Therefore no magnitude will measure the magnitudes *AB*, *CD*; therefore the magnitudes *AB*, *CD* are incommensurable.

Therefore etc. {X. Def. 1}

PROPOSITION 3.

Given two commensurable magnitudes, to find their greatest common measure.

Let the two given commensurable magnitudes be *AB*, *CD* of which *AB* is the less; thus it is required to find the greatest common measure of *AB*, *CD*.

Now the magnitude *AB* either measures *CD* or it does not.

If then it measures it — and it measures itself also — *AB* is a common measure of *AB*, *CD*.

And it is manifest that it is also the greatest; for a greater magnitude than the magnitude *AB* will not measure *AB*.

Next, let *AB* not measure *CD*.

Then, if the less be continually subtracted in turn from the greater, that which is left over will sometime measure the one before it, because *AB*, *CD* are not incommensurable; {cf. X. 2} let *AB*, measuring *ED*, leave *EC* less than itself, let *EC*, measuring *FB*, leave *AF* less than itself, and let *AF* measure *CE*.

Since, then, *AF* measures *CE*, while *CE* measures *FB*, therefore *AF* will also measure *FB*.

But it measures itself also; therefore *AF* will also measure the whole *AB*.

But *AB* measures *DE*; therefore *AF* will also measure *ED*.

But it measures *CE* also; therefore it also measures the whole *CD*.

Therefore *AF* is a common measure of *AB*, *CD*.

I say next that it is also the greatest.

For, if not, there will be some magnitude greater than *AF* which will measure *AB*, *CD*.

Let it be *G*.

Since then *G* measures *AB*, while *AB* measures *ED*, therefore *G* will also measure *ED*.

But it measures the whole *CD* also; therefore *G* will also measure the remainder *CE*.

But *CE* measures *FB*; therefore *G* will also measure *FB*.

But it measures the whole *AB* also, and it will therefore measure the remainder *AF*, the greater the less: which is impossible.

Therefore no magnitude greater than *AF* will measure *AB*, *CD*; therefore *AF* is the greatest common measure of *AB*, *CD*.

Therefore the greatest common measure of the two given commensurable magnitudes *AB*, *CD* has been found. Q. E. D.

PORISM.

From this it is manifest that, if a magnitude measure two magnitudes, it will also measure their greatest common measure.

PROPOSITION 4.

Given three commensurable magnitudes, *to find their greatest common measure*.

Let *A*, *B*, *C* be the three given commensurable magnitudes; thus it is required to find the greatest common measure of *A*, *B*, *C*.

Let the greatest common measure of the two magnitudes *A*, *B* be taken, and let it be *D*; {X. 3} then *D* either measures *C*, or does not measure it.

First, let it measure it.

Since then D measures C, while it also measures A, B, therefore D is a common measure of A, B, C.

And it is manifest that it is also the greatest; for a greater magnitude than the magnitude D does not measure A, B.

Next, let D not measure C.

I say first that C, D are commensurable.

For, since A, B, C are commensurable, some magnitude will measure them, and this will of course measure A, B also; so that it will also measure the greatest common measure of A, B, namely D. {X. 3, Por.}

But it also measures C; so that the said magnitude will measure C, D; therefore C, D are commensurable.

Now let their greatest common measure be taken, and let it be E. {X. 3}

Since then E measures D, while D measures A, B, therefore E will also measure A, B.

But it measures C also; therefore E measures A, B, C; therefore E is a common measure of A, B, C.

I say next that it is also the greatest.

For, if possible, let there be some magnitude F greater than E, and let it measure A, B, C.

Now, since F measures A, B, C, it will also measure A, B, and will measure the greatest common measure of A, B. {X. 3, Por.}

But the greatest common measure of A, B is D; therefore F measures D.

But it measures C also; therefore F measures C, D; therefore F will also measure the greatest common measure of C, D. {X. 3, Por.}

But that is E; therefore F will measure E, the greater the less: which is impossible.

Therefore no magnitude greater than the magnitude *E* will measure *A*, *B*, *C*; therefore *E* is the greatest common measure of *A*, *B*, *C* if *D* do not measure *C*, and, if it measure it, *D* is itself the greatest common measure.

Therefore the greatest common measure of the three given commensurable magnitudes has been found.

PORISM.

From this it is manifest that, if a magnitude measure three magnitudes, it will also measure their greatest common measure.

Similarly too, with more magnitudes, the greatest common measure can be found, and the porism can be extended. Q. E. D.

PROPOSITION 5.

Commensurable magnitudes have to one another the ratio which a number has to a number.

Let *A*, *B* be commensurable magnitudes; I say that *A* has to *B* the ratio which a number has to a number.

For, since *A*, *B* are commensurable, some magnitude will measure them.

Let it measure them, and let it be *C*.

And, as many times as *C* measures *A*, so many units let there be in *D*; and, as many times as *C* measures *B*, so many units let there be in *E*.

Since then *C* measures *A* according to the units in *D*, while the unit also measures *D* according to the units in it, therefore the unit measures the number *D* the same number of times as the magnitude *C* measures *A*; therefore, as *C* is to *A*, so is the unit to *D*; {VII. Def. 20} therefore, inversely, as *A* is to *C*, so is *D* to the unit. {cf. V. 7, Por.}

Again, since *C* measures *B* according to the units in *E*, while the unit also measures *E* according to the units in it, therefore the unit measures *E* the same number of times as *C* measures *B*; therefore, as *C* is to *B*, so is the unit to *E*.

But it was also proved that,

as A is to C, so is D to the unit;

therefore, ex aequali,

as A is to B, so is the number D to E. {V. 22}

Therefore the commensurable magnitudes *A*, *B* have to one another the ratio which the number *D* has to the number *E*. Q. E. D.

PROPOSITION 6.

If two magnitudes have to one another the ratio which a number has to a number, *the magnitudes will be commensurable*.

For let the two magnitudes *A*, *B* have to one another the ratio which the number *D* has to the number *E*;
I say that the magnitudes *A*, *B* are commensurable.

For let *A* be divided into as many equal parts as there are units in *D*, and let *C* be equal to one of them; and let *F* be made up of as many magnitudes equal to *C* as
there are units in *E*.

Since then there are in *A* as many magnitudes equal to *C* as there are units in *D*, whatever part the unit is of *D*, the same part is *C* of *A* also; therefore, as *C* is to *A*, so is the unit to *D*. {VII. Def. 20}

But the unit measures the number *D*; therefore *C* also measures *A*.

And since, as *C* is to *A*, so is the unit to *D*, therefore, inversely, as *A* is to *C*, so is the number *D* to the unit. {cf. V. 7, Por.}

Again, since there are in *F* as many magnitudes equal to *C* as there are units in *E*, therefore, as *C* is to *F*, so is the unit to *E*. {VII. Def. 20}

But it was also proved that,

as A is to C, so is D to the unit;

therefore, ex aequali, as *A* is to *F*, so is *D* to *E*. {v. 22}

But, as *D* is to *E*, so is *A* to *B*; therefore also, as *A* is to *B*, so is it to *F* also. {V. 11}

Therefore *A* has the same ratio to each of the magnitudes *B*, *F*;
therefore *B* is equal to *F*. {V. 9}

But *C* measures *F*; therefore it measures *B* also.

Further it measures *A* also; therefore *C* measures *A*, *B*.

Therefore *A* is commensurable with *B*.

Therefore etc.

PORISM.

From this it is manifest that, if there be two numbers, as *D*, *E*, and a straight line, as *A*, it is possible to make a straight line {*F*} such that the given straight line is to
it as the number *D* is to the number *E*.

And, if a mean proportional be also taken between *A*, *F*, as *B*,

as *A* is to *F*, so will the square on *A* be to the square on *B*, that is, as the first is to the third, so is the figure on the first
to that which is similar and similarly described on the second. {VI. 19, Por.}

But, as *A* is to *F*, so is the number *D* to the number *E*; therefore it has been contrived that, as the number *D* is to the number *E*, so also is the figure on the straight line *A* to the figure on the straight line *B*. Q. E. D. [1]

PROPOSITION 7.

Incommensurable magnitudes have not to one another the ratio which a number has to a number.

Let *A*, *B* be incommensurable magnitudes; I say that *A* has not to *B* the ratio which a number has to a number.

For, if *A* has to *B* the ratio which a number has to a number, *A* will be commensurable with *B*. {X. 6}

But it is not; therefore *A* has not to *B* the ratio which a number has to a number.

Therefore etc.

PROPOSITION 8.

If two magnitudes have not to one another the ratio which a number has to a number, *the magnitudes will be incommensurable*.

For let the two magnitudes *A*, *B* not have to one another the ratio which a number has to a number; I say that the magnitudes *A*, *B* are incommensurable.

For, if they are commensurable, A will have to B the ratio which a number has to a number. {X. 5}

But it has not; therefore the magnitudes A, B are incommensurable.

Therefore etc.

PROPOSITION 9.

The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number; and squares which have to one another the ratio which a square number has to a square number will also have their sides commensurable in length. But the squares on straight lines incommensurable in length have not to one another the ratio which a square number has to a square number; and squares which have not to one another the ratio which a square number has to a square number will not have their sides commensurable in length either.

For let A, B be commensurable in length; I say that the square on A has to the square on B the ratio which a square number has to a square number.

For, since A is commensurable in length with B, therefore A has to B the ratio which a number has to a number. {X. 5}

Let it have to it the ratio which C has to D.

Since then, as A is to B, so is C to D, while the ratio of the square on A to the square on B is duplicate of the ratio of A to B, for similar figures are in the duplicate ratio of their corresponding sides; {VI. 20, Por.} and the ratio of the square on C to the square on D is duplicate of the ratio of C to D, for between two square numbers there is one mean proportional number, and the square number has to the square number the ratio duplicate of that which the side has to the side; {VIII. 11} therefore also, as the square on A is to the square on B, so is the square on C to the square on D.

Next, as the square on *A* is to the square on *B*, so let the square on *C* be to the square on *D*; I say that *A* is commensurable in length with *B*.

For since, as the square on *A* is to the square on *B*, so is the square on *C* to the square on *D*, while the ratio of the square on *A* to the square on *B* is duplicate of the ratio of *A* to *B*, and the ratio of the square on *C* to the square on *D* is duplicate of the ratio of *C* to *D*, therefore also, as *A* is to *B*, so is *C* to *D*.

Therefore *A* has to *B* the ratio which the number *C* has to the number *D*; therefore *A* is commensurable in length with *B*. {X. 6}

Next, let *A* be incommensurable in length with *B*; I say that the square on *A* has not to the square on *B* the ratio which a square number has to a square number.

For, if the square on *A* has to the square on *B* the ratio which a square number has to a square number, *A* will be commensurable with *B*.

But it is not; therefore the square on *A* has not to the square on *B* the ratio which a square number has to a square number.

Again, let the square on *A* not have to the square on *B* the ratio which a square number has to a square number; I say that *A* is incommensurable in length with *B*.

For, if *A* is commensurable with *B*, the square on *A* will have to the square on *B* the ratio which a square number has to a square number.

But it has not; therefore *A* is not commensurable in length with *B*.

Therefore etc.

PORISM.

And it is manifest from what has been proved that straight lines commensurable in length are always commensurable in square also, but those

commensurable in square are not always commensurable in length also.

LEMMA.

{It has been proved in the arithmetical books that similar plane numbers have to one another the ratio which a square number has to a square number, {VIII. 26} and that, if two numbers have to one another the ratio which a square number has to a square number, they are similar plane numbers. {Converse of VIII. 26}

And it is manifest from these propositions that numbers which are not similar plane numbers, that is, those which have not their sides proportional, have not to one another the ratio which a square number has to a square number.

For, if they have, they will be similar plane numbers: which is contrary to the hypothesis.

Therefore numbers which are not similar plane numbers have not to one another the ratio which a square number has to a square number.}

{PROPOSITION 10.

To find two straight lines incommensurable, the one in length only, and the other in square also, with an assigned straight line.

Let *A* be the assigned straight line; thus it is required to find two straight lines incommensurable, the one in length only, and the other in square also, with *A*.

Let two numbers *B*, *C* be set out which have not to one another the ratio which a square number has to a square number, that is, which are not similar plane numbers; and let it be contrived that,

as B is to C, so is the square on A to the square on D

— for we have learnt how to do this — {X. 6, Por.} therefore the square on *A* is commensurable with the square on *D*. {X. 6}

And, since *B* has not to *C* the ratio which a square number has to a square number, therefore neither has the square on *A* to the square on *D* the ratio which a square number has to a square number; therefore *A* is incommensurable in length with *D*. {X. 9}

Let *E* be taken a mean proportional between *A*, *D*; therefore, as *A* is to *D*, so is the square on *A* to the square on *E*. {V. Def. 9}

But *A* is incommensurable in length with *D*; therefore the square on *A* is also incommensurable with the square on *E*; {X. 11} therefore *A* is incommensurable in square with *E*.

Therefore two straight lines *D*, *E* have been found incommensurable, *D* in length only, and *E* in square and of course in length also, with the assigned straight line *A*.}

PROPOSITION 11.

If four magnitudes be proportional, and the first be commensurable with the second, the third will also be commensurable with the fourth; and, if the first be incommensurable with the second, the third will also be incommensurable with the fourth.

Let *A*, *B*, *C*, *D* be four magnitudes in proportion, so that, as *A* is to *B*, so is *C* to *D*, and let *A* be commensurable with *B*; I say that *C* will also be commensurable with *D*.

For, since *A* is commensurable with *B*, therefore *A* has to *B* the ratio which a number has to a number. {X. 5}

And, as *A* is to *B*, so is *C* to *D*; therefore *C* also has to *D* the ratio which a number has to a number; therefore *C* is commensurable with *D*. {X. 6}

Next, let *A* be incommensurable with *B*; I say that *C* will also be incommensurable with *D*.

For, since *A* is incommensurable with *B*, therefore *A* has not to *B* the ratio which a number has to a number. {X. 7}

And, as *A* is to *B*, so is *C* to *D*; therefore neither has *C* to *D* the ratio which a number has to a number; therefore *C* is incommensurable with *D*. {X. 8}

Therefore etc.

PROPOSITION 12.

Magnitudes commensurable with the same magnitude are commensurable with one another also.

For let each of the magnitudes *A*, *B* be commensurable with *C*; I say that *A* is also commensurable with *B*.

For, since *A* is commensurable with *C*, therefore *A* has to *C* the ratio which a number has to a number. {X. 5}

Let it have the ratio which *D* has to *E*.

Again, since *C* is commensurable with *B*, therefore *C* has to *B* the ratio which a number has to a number. {X. 5}

Let it have the ratio which *F* has to *G*.

And, given any number of ratios we please, namely the ratio which *D* has to *E* and that which *F* has to *G*, let the numbers *H*, *K*, *L* be taken continuously in the given ratios; {cf. VIII. 4} so that, as *D* is to *E*, so is *H* to *K*,

and, as F is to G, so is K to L.

Since, then, as *A* is to *C*, so is *D* to *E*, while, as *D* is to *E*, so is *H* to *K*, therefore also, as *A* is to *C*, so is *H* to *K*. {V. 11}

Again, since, as *C* is to *B*, so is *F* to *G*, while, as *F* is to *G*, so is *K* to *L*, therefore also, as *C* is to *B*, so is *K* to *L*. {V. 11}

But also, as *A* is to *C*, so is *H* to *K*; therefore, ex aequali, as *A* is to *B*, so is *H* to *L*. {V. 22}

Therefore *A* has to *B* the ratio which a number has to a number; therefore *A* is commensurable with *B*. {X. 6}

Therefore etc. Q. E. D.

PROPOSITION 13.

If two magnitudes be commensurable, and the one of them be incommensurable with any magnitude, the remaining one will also be incommensurable with the same.

Let *A*, *B* be two commensurable magnitudes, and let one of them, *A*, be incommensurable with any other magnitude *C*; I say that the remaining one, *B*, will also be incommensurable with *C*.

For, if *B* is commensurable with *C*, while *A* is also commensurable with *B*, *A* is also commensurable with *C*. {X. 12}

But it is also incommensurable with it: which is impossible.

Therefore *B* is not commensurable with *C*; therefore it is incommensurable with it.

Therefore etc.

LEMMA.

Given two unequal straight lines, to find by what square the square on the greater is greater than the square on the less.

Let *AB*, *C* be the given two unequal straight lines, and let *AB* be the greater of them; thus it is required to find by what square the square on *AB* is greater than the square on *C*.

Let the semicircle *ADB* be described on *AB*, and let *AD* be fitted into it equal to *C*; {IV. 1} let *DB* be joined.

It is then manifest that the angle *ADB* is right, {III. 31} and that the square on *AB* is greater than the square on *AD*, that is, *C*, by the square on *DB*. {I. 47}

Similarly also, if two straight lines be given, the straight line the square on which is equal to the sum of the squares on them is found in this manner.

Let *AD*, *DB* be the given two straight lines, and let it be required to find the straight line the square on which is equal to the sum of the squares on them.

Let them be placed so as to contain a right angle, that formed by *AD*, *DB*; and let *AB* be joined.

It is again manifest that the straight line the square on which is equal to the sum of the squares on *AD*, *DB* is *AB*. {I. 47} Q. E. D.

PROPOSITION 14.

If four straight lines be proportional, and the square on the first be greater than the square on the second by the square on a straight line commensurable with the first, the square on the third will also be greater than the square on the fourth by the square on a straight line commensurable with the third.

And, if the square on the first be greater than the square on the second by the square on a straight line incommensurable with the first, the square on the third will also be greater than the square on the fourth by the square on a straight line in-
commensurable with the third.

Let *A*, *B*, *C*, *D* be four straight lines in proportion, so that, as *A* is to *B*, so is *C* to *D*; and let the square on *A* be greater than the square on *B* by the square on *E*, and
let the square on *C* be greater than the square on *D* by the square on *F*; I say that, if *A* is commensurable with *E*, *C* is also commensurable with *F*, and, if *A* is incommensurable with *E*, *C* is
also incommensurable with *F*.

For since, as *A* is to *B*, so is *C* to *D*, therefore also, as the square on *A* is to the square on *B*, so is the square on *C* to the square on *D*. {VI. 22}

But the squares on *E*, *B* are equal to the square on *A*,
and the squares on *D*, *F* are equal to the square on *C*.

Therefore, as the squares on *E*, *B* are to the square on *B*, so are the squares on *D*, *F* to the square on *D*; therefore, separando, as the square on *E* is to the square on *B*, so is the square on *F* to the square on *D*; {V. 17}
therefore also, as *E* is to *B*, so is *F* to *D*; {VI. 22} therefore, inversely, as *B* is to *E*, so is *D* to *F*.

But, as *A* is to *B*, so also is *C* to *D*; therefore, ex aequali, as *A* is to *E*, so is *C* to *F*. {V. 22}

Therefore, if *A* is commensurable with *E*, *C* is also commensurable with *F*, and, if *A* is incommensurable with *E*, *C* is also incommensurable with *F*. {X. 11}

Therefore etc. [2]

PROPOSITION 15.

If two commensurable magnitudes be added together, the whole will also be commensurable with each of them; and, if the whole be commensurable with one of them, the original magnitudes will also be commensurable.

For let the two commensurable magnitudes *AB*, *BC* be added together; I say that the whole *AC* is also commensurable with each of the magnitudes *AB*, *BC*.

For, since *AB*, *BC* are commensurable, some magnitude will measure them.

Let it measure them, and let it be *D*.

Since then *D* measures *AB*, *BC*, it will also measure the whole *AC*.

But it measures *AB*, *BC* also; therefore *D* measures *AB*, *BC*, *AC*; therefore *AC* is commensurable with each of the magnitudes *AB*, *BC*. {X. Def. 1}

Next, let *AC* be commensurable with *AB*; I say that *AB*, *BC* are also commensurable.

For, since *AC*, *AB* are commensurable, some magnitude will measure them.

Let it measure them, and let it be *D*.

Since then *D* measures *CA*, *AB*, it will also measure the remainder *BC*.

But it measures *AB* also; therefore *D* will measure *AB*, *BC*; therefore *AB*, *BC* are commensurable. {X. Def. 1}

Therefore etc.

PROPOSITION 16.

If two incommensurable magnitudes be added together, the whole will also be incommensurable with each of them; and, if the whole be incommensurable with one of them, the original magnitudes will also be incommensurable.

For let the two incommensurable magnitudes *AB*, *BC* be added together; I say that the whole *AC* is also incommensurable with each of the magnitudes *AB*, *BC*.

For, if *CA*, *AB* are not incommensurable, some magnitude will measure them.

Let it measure them, if possible, and let it be *D*.

Since then *D* measures *CA*, *AB*, therefore it will also measure the remainder *BC*.

But it measures *AB* also; therefore *D* measures *AB*, *BC*.

Therefore *AB*, *BC* are commensurable; but they were also, by hypothesis, incommensurable: which is impossible.

Therefore no magnitude will measure *CA*, *AB*; therefore *CA*, *AB* are incommensurable. {X. Def. 1}

Similarly we can prove that *AC*, *CB* are also incommensurable.

Therefore *AC* is incommensurable with each of the magnitudes *AB*, *BC*.

Next, let *AC* be incommensurable with one of the magnitudes *AB*, *BC*.

First, let it be incommensurable with *AB*; I say that *AB*, *BC* are also incommensurable.

For, if they are commensurable, some magnitude will measure them.

Let it measure them, and let it be *D*.

Since then *D* measures *AB*, *BC*. therefore it will also measure the whole *AC*.

But it measures *AB* also; therefore *D* measures *CA*, *AB*.

Therefore *CA*, *AB* are commensurable; but they were also, by hypothesis, incommensurable: which is impossible.

Therefore no magnitude will measure *AB*, *BC*; therefore *AB*, *BC* are incommensurable. {X. Def. 1}

Therefore etc.

LEMMA.

If to any straight line there be applied a parallelogram deficient by a square figure, the applied parallelogram is equal to the rectangle contained by the segments of the straight line resulting from the application.

For let there be applied to the straight line *AB* the parallelogram *AD* deficient by the square figure *DB*; I say that *AD* is equal to the rectangle contained by *AC*, *CB*.

This is indeed at once manifest; for, since *DB* is a square, *DC* is equal to *CB*; and *AD* is the rectangle *AC*, *CD*, that is, the rectangle *AC*, *CB*.

Therefore etc.

PROPOSITION 17.

If there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into parts which are commensurable in length, then

the square on the greater will be greater than the square on the less by the square on a straight line commensurable with the greater.

And, if the square on the greater be greater than the square on the less by the square on a straight line commensurable with
the greater, and if there be applied to the greater a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, it will divide it into parts which are commensurable in length.

Let *A*, *BC* be two unequal straight lines, of which *BC* is
the greater, and let there be applied to *BC* a parallelogram equal to the fourth part of the square on the less, *A*, that is, equal to the square on the half of *A*, and deficient
by a square figure. Let this be the rectangle *BD*, *DC*, {cf. Lemma} and let *BD* be commensurable in length with *DC*; I say that the square on *BC* is greater than the square on *A* by the square on a straight line commensurable with *BC*.

For let *BC* be bisected at the point *E*, and let *EF* be made equal to *DE*. Therefore the remainder *DC* is equal to *BF*.

And, since the straight line *BC* has been cut into equal parts at *E*, and into unequal parts at *D*,
therefore the rectangle contained by *BD*, *DC*, together with the square on *ED*, is equal to the square on *EC*; {II. 5}

And the same is true of their quadruples; therefore four times the rectangle *BD*, *DC*, together with four times the square on *DE*, is equal to four times the square
on *EC*.

But the square on *A* is equal to four times the rectangle *BD*, *DC*; and the square on *DF* is equal to four times the square on *DE*, for *DF* is double of *DE*.

And the square on *BC* is equal to four times the square on *EC*, for again *BC* is double of *CE*.

Therefore the squares on *A*, *DF* are equal to the square on *BC*, so that the square on *BC* is greater than the square on *A* by
the square on *DF*.

It is to be proved that *BC* is also commensurable with *DF*.

Since *BD* is commensurable in length with *DC*, therefore *BC* is also commensurable in length with *CD*. {X. 15}

But *CD* is commensurable in length with *CD*, *BF*, for
CD is equal to *BF*. {X. 6}

Therefore *BC* is also commensurable in length with *BF*, *CD*, {X. 12} so that *BC* is also commensurable in length with the remainder *FD*; {X. 15}
therefore the square on *BC* is greater than the square on *A* by the square on a straight line commensurable with *BC*.

Next, let the square on *BC* be greater than the square on *A* by the square on a straight line commensurable with *BC*, let a parallelogram be applied to *BC* equal to the fourth part
of the square on *A* and deficient by a square figure, and let it be the rectangle *BD*, *DC*.

It is to be proved that *BD* is commensurable in length with *DC*.

With the same construction, we can prove similarly that
the square on *BC* is greater than the square on *A* by the square on *FD*.

But the square on *BC* is greater than the square on *A* by the square on a straight line commensurable with *BC*.

Therefore *BC* is commensurable in length with *FD*,
so that *BC* is also commensurable in length with the remainder, the sum of *BF*, *DC*. {X. 15}

But the sum of *BF*, *DC* is commensurable with *DC*, {X. 6} so that *BC* is also commensurable in length with *CD*; {X. 12} and therefore, separando, *BD* is commensurable in length
with *DC*. {X. 15}

Therefore etc. [3]

PROPOSITION 18.

If there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into parts which are incommensurable, the square on the greater will be greater than the square on the less by the square on a straight line incommensurable with the greater.

And, if the square on the greater be greater than the square on the less by the square on a straight line incommensurable with the greater, and if there be applied to the greater a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, it divides it into parts which are incommensurable.

Let *A*, *BC* be two unequal straight lines, of which *BC* is the greater, and to *BC* let there be applied a parallelogram equal to the fourth part of the square on the less, *A*, and deficient by a square figure. Let this be the rectangle *BD*, *DC*, {cf. Lemma before X. 17} and let *BD* be incommensurable in length with *DC*; I say that the square on *BC* is greater than the square on *A* by the square on a straight line incommensurable with *BC*.

For, with the same construction as before, we can prove similarly that the square on *BC* is greater than the square on *A* by the square on *FD*.

It is to be proved that *BC* is incommensurable in length with *DF*.

Since *BD* is incommensurable in length with *DC*, therefore *BC* is also incommensurable in length with *CD*. {X. 16}

But *DC* is commensurable with the sum of *BF*, *DC*; {X. 6} therefore *BC* is also incommensurable with the sum of *BF*, *DC*; {X. 13} so that *BC* is also incommensurable in length with the remainder *FD*. {X. 16}

And the square on *BC* is greater than the square on *A* by the square on *FD*; therefore the square on *BC* is greater than the square on *A* by the square on a straight line incommensurable with *BC*.

Again, let the square on *BC* be greater than the square on *A* by the square on a straight line incommensurable with *BC*, and let there be applied to *BC* a parallelogram equal to the fourth part of the square on *A* and deficient by a square figure. Let this be the rectangle *BD*, *DC*.

It is to be proved that *BD* is incommensurable in length with *DC*.

For, with the same construction, we can prove similarly that the square on *BC* is greater than the square on *A* by the square on *FD*.

But the square on *BC* is greater than the square on *A* by the square on a straight line incommensurable with *BC*; therefore *BC* is incommensurable in length with *FD*. so that *BC* is also incommensurable with the remainder, the sum of *BF*, *DC*. {X. 16}

But the sum of *BF*, *DC* is commensurable in length with *DC*; {X. 6} therefore *BC* is also incommensurable in length with *DC*, {X. 13} so that, separando, *BD* is also incommensurable in length with *DC*. {X. 16}

Therefore etc.

LEMMA.

{Since it has been proved that straight lines commensurable in length are always commensurable in square also, while those commensurable in square are not always commensurable in length also, but can of course be either commensurable or incommensurable in length, it is manifest that, if any straight line be commensurable in length with a given rational straight line, it is called rational and commensurable with the other not only in length but in square also, since straight lines commensurable in length are always commensurable in square also.

But, if any straight line be commensurable in square with a given rational straight line, then, if it is also commensurable in length with it, it is called in this case also rational and commensurable with it both in length and in square; but, if again any straight line, being commensurable in square with a given rational straight line, be incommensurable in length with it, it is called in this case also rational but commensurable in square only.}

PROPOSITION 19.

The rectangle contained by rational straight lines commensurable in length is rational.

For let the rectangle *AC* be contained by the rational straight lines *AB*, *BC* commensurable in length; I say that *AC* is rational.

For on *AB* let the square *AD* be described; therefore *AD* is rational. {X. Def. 4}

And, since *AB* is commensurable in length with *BC*, while *AB* is equal to *BD*, therefore *BD* is commensurable in length with *BC*.

And, as *BD* is to *BC*, so is *DA* to *AC*. {VI. 1}

Therefore *DA* is commensurable with *AC*. {X. 11}

But *DA* is rational; therefore *AC* is also rational. {X. Def. 4}

Therefore etc.

PROPOSITION 20.

If a rational area be applied to a rational straight line, it produces as breadth a straight line rational and commensurable in length with the straight line to which it is applied.

For let the rational area *AC* be applied to *AB*, a straight line once more rational in any of the aforesaid ways, producing *BC* as breadth; I say that *BC* is rational and commensurable in length with *BA*. For on *AB* let the square *AD* be described; therefore *AD* is rational. {X. Def. 4}

But *AC* is also rational; therefore *DA* is commensurable with *AC*.

And, as *DA* is to *AC*, so is *DB* to *BC*. {VI. 1}

Therefore *DB* is also commensurable with *BC*; {X. 11} and *DB* is equal to *BA*; therefore *AB* is also commensurable with *BC*.

But *AB* is rational; therefore *BC* is also rational and commensurable in length with *AB*.

Therefore etc.

PROPOSITION 21.

The rectangle contained by rational straight lines commensurable in square only is irrational, and the side of the square equal to it is irrational. Let the latter be called medial.

For let the rectangle *AC* be contained by the rational straight lines *AB*, *BC* commensurable in square only; I say that *AC* is irrational, and the side of the square equal to it is irrational; and let the latter be called medial.

For on *AB* let the square *AD* be described; therefore *AD* is rational. {X. Def. 4}

And, since *AB* is incommensurable in length with *BC*, for by hypothesis they are commensurable in square only, while *AB* is equal to *BD*, therefore *DB* is also incommensurable in length with *BC*.

And, as *DB* is to *BC*, so is *AD* to *AC*; {VI. 1} therefore *DA* is incommensurable with *AC*. {X. 11}

But *DA* is rational; therefore *AC* is irrational, so that the side of the square equal to *AC* is also irrational. {X. Def. 4}

And let the latter be called medial. Q. E. D.

LEMMA.

If there be two straight lines, then, as the first is to the second, so is the square on the first to the rectangle contained by the two straight lines.

Let *FE*, *EG* be two straight lines.

I say that, as *FE* is to *EG*, so is the square on *FE* to the rectangle *FE*, *EG*.

For on *FE* let the square *DF* be described, and let *GD* be completed.

Since then, as *FE* is to *EG*, so is *FD* to *DG*, {VI. 1} and *FD* is the square on *FE*, and *DG* the rectangle *DE*, *EG*, that is, the rectangle *FE*, *EG*, therefore, as *FE* is to *EG*, so is the square on *FE* to the rectangle *FE*, *EG*.

Similarly also, as the rectangle *GE*, *EF* is to the square on *EF*, that is, as *GD* is to *FD*, so is *GE* to *EF*. Q. E. D.

PROPOSITION 22.

The square on a medial straight line, if applied to a rational straight line, produces as breadth a straight line rational and incommensurable in length with that to which it is applied.

Let *A* be medial and *CB* rational, and let a rectangular area *BD* equal to the square on *A* be applied to *BC*, producing *CD* as breadth; I say that *CD* is rational and incommensurable in length with *CB*.

For, since *A* is medial, the square on it is equal to a rectangular area contained by rational straight lines commensurable in square only. {X. 21}

Let the square on it be equal to *GF*.

But the square on it is also equal to *BD*; therefore *BD* is equal to *GF*.

But it is also equiangular with it; and in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional; {VI. 14} therefore, proportionally, as *BC* is to *EG*, so is *EF* to *CD*.

Therefore also, as the square on *BC* is to the square on *EG*, so is the square on *EF* to the square on *CD*. {VI. 22}

But the square on *CB* is commensurable with the square on *EG*, for each of these straight lines is rational; therefore the square on *EF* is also commensurable with the square on *CD*. {X. 11}

But the square on *EF* is rational; therefore the square on *CD* is also rational; {X. Def. 4} therefore *CD* is rational.

And, since *EF* is incommensurable in length with *EG*, for they are commensurable in square only, and, as *EF* is to *EG*, so is the square on *EF* to

the rectangle *FE*, *EG*, {Lemma} therefore the square on *EF* is incommensurable with the rectangle *FE*, *EG*. {X. 11}

But the square on *CD* is commensurable with the square on *EF*, for the straight lines are rational in square; and the rectangle *DC*, *CB* is commensurable with the rectangle *FE*, *EG*, for they are equal to the square on *A*; therefore the square on *CD* is also incommensurable with the rectangle *DC*, *CB*. {X. 13}

But, as the square on *CD* is to the rectangle *DC*, *CB*, so is *DC* to *CB*; {Lemma} therefore *DC* is incommensurable in length with *CB*. {X. 11}

Therefore *CD* is rational and incommensurable in length with *CB*. Q. E. D.

PROPOSITION 23.

A straight line commensurable with a medial straight line is medial.

Let *A* be medial, and let *B* be commensurable with *A*; I say that *B* is also medial.

For let a rational straight line *CD* be set out, and to *CD* let the rectangular area *CE* equal to the square on *A* be applied, producing *ED* as breadth; therefore *ED* is rational and incommensurable in length with *CD*. {X. 22}

And let the rectangular area *CF* equal to the square on *B* be applied to *CD*, producing *DF* as breadth.

Since then *A* is commensurable with *B*, the square on *A* is also commensurable with the square on *B*.

But *EC* is equal to the square on *A*, and *CF* is equal to the square on *B*; therefore *EC* is commensurable with *CF*.

And, as *EC* is to *CF*, so is *ED* to *DF*; {VI. 1} therefore *ED* is commensurable in length with *DF*. {X. 11}

But *ED* is rational and incommensurable in length with *DC*; therefore *DF* is also rational {X. Def. 3} and incommensurable in length with *DC*. {X. 13}

Therefore *CD*, *DF* are rational and commensurable in square only.

But the straight line the square on which is equal to the rectangle contained by rational straight lines commensurable in square only is medial; {X. 21} therefore the side of the square equal to the rectangle *CD*, *DF* is medial.

And *B* is the side of the square equal to the rectangle *CD*, *DF*; therefore *B* is medial.

PORISM.

From this it is manifest that an area commensurable with a medial area is medial.

{And in the same way as was explained in the case of rationals {Lemma following X. 18} it follows, as regards medials, that a straight line commensurable in length with a medial straight line is called medial and commensurable with it not only in length but in square also, since, in general, straight lines commensurable in length are always commensurable in square also.

But, if any straight line be commensurable in square with a medial straight line, then, if it is also commensurable in length with it, the straight lines are called, in this case too, medial and commensurable in length and in square, but, if in square only, they are called medial straight lines commensurable in square only.}

PROPOSITION 24.

The rectangle contained by medial straight lines commensurable in length is medial.

For let the rectangle *AC* be contained by the medial straight lines *AB*, *BC* which are commensurable in length; I say that *AC* is medial.

For on *AB* let the square *AD* be described; therefore *AD* is medial.

And, since *AB* is commensurable in length with *BC*, while *AB* is equal to *BD*, therefore *DB* is also commensurable in length with *BC*; so that *DA* is also commensurable with *AC*. {VI. 1, X. 11}

But *DA* is medial; therefore *AC* is also medial. {X. 23, Por.} Q. E. D.

PROPOSITION 25.

The rectangle contained by medial straight lines commensurable in square only is either rational or medial.

For let the rectangle *AC* be contained by the medial straight lines *AB*, *BC* which are commensurable in square only; I say that *AC* is either rational or medial.

For on *AB*, *BC* let the squares *AD*, *BE* be described; therefore each of the squares *AD*, *BE* is medial.

Let a rational straight line *FG* be set out, to *FG* let there be applied the rectangular parallelogram *GH* equal to *AD*, producing *FH* as breadth, to *HM* let there be applied the rectangular parallelogram *MK* equal to *AC*, producing *HK* as breadth, and further to *KN* let there be similarly applied *NL* equal to *BE*, producing *KL* as breadth; therefore *FH*, *HK*, *KL* are in a straight line.

Since then each of the squares *AD*, *BE* is medial, and *AD* is equal to *GH*, and *BE* to *NL*, therefore each of the rectangles *GH*, *NL* is also medial.

And they are applied to the rational straight line *FG*; therefore each of the straight lines *FH*, *KL* is rational and incommensurable in length with *FG*. {X.

22}

And, since *AD* is commensurable with *BE*, therefore *GH* is also commensurable with *NL*.

And, as *GH* is to *NL*, so is *FH* to *KL*; {VI. 1} therefore *FH* is commensurable in length with *KL*. {X. 11}

Therefore *FH*, *KL* are rational straight lines commensurable in length; therefore the rectangle *FH*, *KL* is rational. {X. 19}

And, since *DB* is equal to *BA*, and *OB* to *BC*, therefore, as *DB* is to *BC*, so is *AB* to *BO*.

But, as *DB* is to *BC*, so is *DA* to *AC*, {VI. 1} and, as *AB* is to *BO*, so is *AC* to *CO*; {*id*.} therefore, as *DA* is to *AC*, so is *AC* to *CO*.

But *AD* is equal to *GH*, *AC* to *MK* and *CO* to *NL*; therefore, as *GH* is to *MK*, so is *MK* to *NL*; therefore also, as *FH* is to *HK*, so is *HK* to *KL*; {VI. 1, V. 11} therefore the rectangle *FH*, *KL* is equal to the square on *HK*. {VI. 17}

But the rectangle *FH*, *KL* is rational; therefore the square on *HK* is also rational.

Therefore *HK* is rational.

And, if it is commensurable in length with *FG*, *HN* is rational; {X. 19} but, if it is incommensurable in length with *FG*, *KH*, *HM* are rational straight lines commensurable in square only, and therefore *HN* is medial. {X. 21}

Therefore *HN* is either rational or medial.

But *HN* is equal to *AC*; therefore *AC* is either rational or medial.

Therefore etc.

PROPOSITION 26.

4 medial area does not exceed a medial area by a rational area.

For, if possible, let the medial area *AB* exceed the medial area *AC* by the rational area *DB*, and let a rational straight line *EF* be set out; to *EF* let there be applied the rectangular parallelogram *FH* equal to *AB*, producing *EH* as breadth, and let the rectangle *FG* equal to *AC* be subtracted; therefore the remainder *BD* is equal to the remainder *KH*.

But *DB* is rational; therefore *KH* is also rational.

Since, then, each of the rectangles *AB*, *AC* is medial, and *AB* is equal to *FH*, and *AC* to *FG*, therefore each of the rectangles *FH*, *FG* is also medial.

And they are applied to the rational straight line *EF*; therefore each of the straight lines *HE*, *EG* is rational and incommensurable in length with *EF*. {X. 22}

And, since {*DB* is rational and is equal to *KH*, therefore} *KH* is {also} rational; and it is applied to the rational straight line *EF*; therefore *GH* is rational and commensurable in length with *EF*. {X. 20}

But *EG* is also rational, and is incommensurable in length with *EF*; therefore *EG* is incommensurable in length with *GH*. {X. 13}

And, as *EG* is to *GH*, so is the square on *EG* to the rectangle *EG*, *GH*; therefore the square on *EG* is incommensurable with the rectangle *EG*, *GH*. {X. 11}

But the squares on *EG*, *GH* are commensurable with the square on *EG*, for both are rational; and twice the rectangle *EG*, *GH* is commensurable with the rectangle *EG*, *GH*, for it is double of it; {X. 6} therefore the squares on *EG*, *GH* are incommensurable with twice the rectangle *EG*, *GH*; {X. 13} therefore also the sum of the squares on *EG*, *GH* and twice the rectangle *EG*, *GH*, that is, the square on *EH* {II. 4}, is incommensurable with the squares on *EG*, *GH*. {X. 16}

But the squares on *EG*, *GH* are rational; therefore the square on *EH* is irrational. {X. Def. 4}

Therefore *EH* is irrational.

But it is also rational: which is impossible.

Therefore etc. Q. E. D.

PROPOSITION 27.

To find medial straight lines commensurable in square only which contain a rational rectangle.

Let two rational straight lines *A*, *B* commensurable in square only be set out; let *C* be taken a mean proportional between *A*, *B*, {VI. 13} and let it be contrived that,

as A is to B, so is C to D. {VI. 12}

Then, since *A*, *B* are rational and commensurable in square only, the rectangle *A*, *B*, that is, the square on *C* {VI.17}, is medial. {X. 21}

Therefore *C* is medial. {X. 21}

And since, as *A* is to *B*, so is *C* to *D*, and *A*, *B* are commensurable in square only, therefore *C*, *D* are also commensurable in square only. {X. 11}

And *C* is medial; therefore *D* is also medial. {X. 23, addition}

Therefore *C*, *D* are medial and commensurable in square only.

I say that they also contain a rational rectangle.

For since, as *A* is to *B*, so is *C* to *D*, therefore, alternately, as *A* is to *C*, so is *B* to *D*. {V. 16}

But, as *A* is to *C*, so is *C* to *B*; therefore also, as *C* is to *B*, so is *B* to *D*; therefore the rectangle *C*, *D* is equal to the square on *B*.

But the square on *B* is rational; therefore the rectangle *C*, *D* is also rational.

Therefore medial straight lines commensurable in square only have been found which contain a rational rectangle. Q. E. D.

PROPOSITION 28.

To find medial straight lines commensurable in square only which contain a medial rectangle.

Let the rational straight lines *A*, *B*, *C* commensurable in square only be set out; let *D* be taken a mean proportional between *A*, *B*, {VI. 13} and let it be contrived that,

as B is to C, so is D to E. {VI. 12}

Since *A*, *B* are rational straight lines commensurable in square only, therefore the rectangle *A*, *B*, that is, the square on *D* {VI. 17}, is medial. {X. 21}

Therefore *D* is medial. {X. 21}

And since *B*, *C* are commensurable in square only, and, as *B* is to *C*, so is *D* to *E*, therefore *D*, *E* are also commensurable in square only. {X. 11}

But *D* is medial; therefore *E* is also medial. {X. 23, addition}

Therefore *D*, *E* are medial straight lines commensurable in square only.

I say next that they also contain a medial rectangle.

For since, as *B* is to *C*, so is *D* to *E*, therefore, alternately, as *B* is to *D*, so is *C* to *E*. {V. 16}

But, as *B* is to *D*, so is *D* to *A*; therefore also, as *D* is to *A*, so is *C* to *E*; therefore the rectangle *A*, *C* is equal to the rectangle *D*, *E*. {VI. 16}

But the rectangle *A*, *C* is medial; {X. 21} therefore the rectangle *D*, *E* is also medial.

Therefore medial straight lines commensurable in square only have been found which contain a medial rectangle. Q. E. D.

LEMMA I.

To find two square numbers such that their sum is also square.

Let two numbers *AB*, *BC* be set out, and let them be either both even or both odd.

Then since, whether an even number is subtracted from an even number, or an odd number from an odd number, the remainder is even, {IX. 24, 26} therefore the remainder *AC* is even.

Let *AC* be bisected at *D*.

Let *AB*, *BC* also be either similar plane numbers, or square numbers, which are themselves also similar plane numbers.

Now the product of *AB*, *BC* together with the square on *CD* is equal to the square on *BD*. {II. 6}

And the product of *AB*, *BC* is square, inasmuch as it was proved that, if two similar plane numbers by multiplying one another make some number the product is square. {IX. 1}

Therefore two square numbers, the product of *AB*, *BC*, and the square on *CD*, have been found which, when added together, make the square on *BD*.

And it is manifest that two square numbers, the square on *BD* and the square on *CD*, have again been found such that their difference, the product of *AB*, *BC*, is a square, whenever *AB*, *BC* are similar plane numbers.

But when they are not similar plane numbers, two square numbers, the square on *BD* and the square on *DC*, have been found such that their difference, the product of *AB*, *BC*, is not square. Q. E. D.

LEMMA 2.

To find two square numbers such that their sum is not square.

For let the product of *AB*, *BC*, as we said, be square, and *CA* even, and let *CA* be bisected by *D*.

It is then manifest that the square product of *AB*, *BC* together with the square on *CD* is equal to the square on *BD*. {See Lemma 1}

Let the unit *DE* be subtracted; therefore the product of *AB*, *BC* together with the square on *CE* is less than the square on *BD*.

I say then that the square product of *AB*, *BC* together with the square on *CE* will not be square.

For, if it is square, it is either equal to the square on *BE*, or less than the square on *BE*, but cannot any more be greater, lest the unit be divided.

First, if possible, let the product of *AB*, *BC* together with the square on *CE* be equal to the square on *BE*, and let *GA* be double of the unit *DE*.

Since then the whole *AC* is double of the whole *CD*, and in them *AG* is double of *DE*, therefore the remainder *GC* is also double of the remainder *EC*; therefore *GC* is bisected by *E*.

Therefore the product of *GB*, *BC* together with the square on *CE* is equal to the square on *BE*. {II. 6}

But the product of *AB*, *BC* together with the square on *CE* is also, by hypothesis, equal to the square on *BE*; therefore the product of *GB*, *BC* together with the square on *CE* is equal to the product of *AB*, *BC* together with the square on *CE*.

And, if the common square on *CE* be subtracted, it follows that *AB* is equal to *GB*: which is absurd.

Therefore the product of *AB*, *BC* together with the square on *CE* is not equal to the square on *BE*.

I say next that neither is it less than the square on *BE*.

For, if possible, let it be equal to the square on *BF*, and let *HA* be double of *DF*.

Now it will again follow that *HC* is double of *CF*; so that *CH* has also been bisected at *F*, and for this reason the product of *HB*, *BC* together with the square on *FC* is equal to the square on *BF*. {II. 6}

But, by hypothesis, the product of *AB*, *BC* together with the square on *CE* is also equal to the square on *BF*.

Thus the product of *HB*, *BC* together with the square on *CF* will also be equal to the product of *AB*, *BC* together with the square on *CE*: which is absurd.

Therefore the product of *AB*, *BC* together with the square on *CE* is not less than the square on *BE*.

And it was proved that neither is it equal to the square on *BE*.

Therefore the product of *AB*, *BC* together with the square on *CE* is not square. Q. E. D.

PROPOSITION 29.

To find two rational straight lines commensurable in square only and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater.

For let there be set out any rational straight line *AB*, and two square numbers *CD*, *DE* such that their difference *CE* is not square; {Lemma 1} let there be described on *AB* the semicircle *AFB*, and let it be contrived that,

as DC is to CE, so is the square on BA to the square on AF. {X. 6, Por.}

Let *FB* be joined.

Since, as the square on *BA* is to the square on *AF*, so is *DC* to *CE*, therefore the square on *BA* has to the square on *AF* the ratio which the number *DC* has to the number *CE*; therefore the square on *BA* is commensurable with the square on *AF*. {X. 6}

But the square on *AB* is rational; {X. Def. 4} therefore the square on *AF* is also rational; {*id.*} therefore *AF* is also rational.

And, since *DC* has not to *CE* the ratio which a square number has to a square number, neither has the square on *BA* to the square on *AF* the ratio which a square number has to a square number; therefore *AB* is incommensurable in length with *AF*. {X. 9}

Therefore *BA*, *AF* are rational straight lines commensurable in square only.

And since, as *DC* is to *CE*, so is the square on *BA* to the square on *AF*, therefore, convertendo, as *CD* is to *DE*, so is the square on *AB* to the square on *BF*. {V. 19, Por., III. 31, I. 47}

But *CD* has to *DE* the ratio which a square number has to a square number: therefore also the square on *AB* has to the square on *BF* the ratio which a square number has to a square number; therefore *AB* is commensurable in length with *BF*. {X. 9}

And the square on *AB* is equal to the squares on *AF*, *FB*; therefore the square on *AB* is greater than the square on *AF* by the square on *BF* commensurable with *AB*.

Therefore there have been found two rational straight lines *BA*, *AF* commensurable in square only and such that the square on the greater *AB* is greater than the square on the less *AF* by the square on *BF* commensurable in length with *AB*.

PROPOSITION 30.

To find two rational straight lines commensurable in square only and such that the square on the greater is greater is greater than the square on the less by the square on a straight line incommensurable in length with the greater.

Let there be set out a rational straight line *AB*, and two square numbers *CE*, *ED* such that their sum *CD* is not square; {Lemma 2} let there be described on *AB* the semicircle *AFB*, let it be contrived that, as *DC* is to *CE*, so is the square on *BA* to the square on *AF*, {X. 6, Por.} and let *FB* be joined.

Then, in a similar manner to the preceding, we can prove that *BA*, *AF* are rational straight lines commensurable in square only.

And since, as *DC* is to *CE*, so is the square on *BA* to the square on *AF*, therefore, convertendo, as *CD* is to *DE*, so is the square on *AB* to the square on *BF*. {V. 19, Por., III. 31, I. 47}

But *CD* has not to *DE* the ratio which a square number has to a square number; therefore neither has the square on *AB* to the square on *BF* the ratio which a square number has to a square number; therefore *AB* is incommensurable in length with *BF*. {X. 9}

And the square on *AB* is greater than the square on *AF* by the square on *FB* incommensurable with *AB*.

Therefore *AB*, *AF* are rational straight lines commensurable in square only, and the square on *AB* is greater than the square on *AF* by the square on *FB* incommensurable in length with *AB*. Q. E. D.

PROPOSITION 31.

To find two medial straight lines commensurable in square only, containing a rational rectangle, and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater.

Let there be set out two rational straight lines *A*, *B* commensurable in square only and such that the square on *A*, being the greater, is greater than the square on *B* the less by the square on a straight line commensurable in length with *A*. {X. 29}

And let the square on *C* be equal to the rectangle *A*, *B*.

Now the rectangle *A*, *B* is medial; {X. 21} therefore the square on *C* is also medial; therefore *C* is also medial. {X. 21}

Let the rectangle *C*, *D* be equal to the square on *B*.

Now the square on *B* is rational; therefore the rectangle *C*, *D* is also rational.

And since, as *A* is to *B*, so is the rectangle *A*, *B* to the square on *B*, while the square on *C* is equal to the rectangle *A*, *B*, and the rectangle *C*, *D* is equal to the square on *B*, therefore, as *A* is to *B*, so is the square on *C* to the rectangle *C*, *D*.

But, as the square on *C* is to the rectangle *C*, *D*, so is *C* to *D*; therefore also, as *A* is to *B*, so is *C* to *D*.

But *A* is commensurable with *B* in square only; therefore *C* is also commensurable with *D* in square only. {X. 11}

And *C* is medial; therefore *D* is also medial. {X. 23, addition}

And since, as *A* is to *B*, so is *C* to *D*, and the square on *A* is greater than the square on *B* by the square on a straight line commensurable with *A*, therefore also the square on *C* is greater than the square on *D* by the square on a straight line commensurable with *C*. {X. 14}

Therefore two medial straight lines *C*, *D*, commensurable in square only and containing a rational rectangle, have been found, and the square on *C* is greater than the square on *D* by the square on a straight line commensurable in length with *C*.

Similarly also it can be proved that the square on *C* exceeds the square on *D* by the square on a straight line incommensurable with *C*, when the square on *A* is greater than the square on *B* by the square on a straight line incommensurable with *A*. {X. 30}

PROPOSITION 32.

To find two medial straight lines commensurable in square only, containing a medial rectangle, and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater.

Let there be set out three rational straight lines *A*, *B*, *C* commensurable in square only, and such that the square on *A* is greater than the square on *C* by the square on a straight line commensurable with *A*, {X. 29} and let the square on *D* be equal to the rectangle *A*, *B*.

Therefore the square on *D* is medial; therefore *D* is also medial. {X. 21}

Let the rectangle *D*, *E* be equal to the rectangle *B*, *C*.

Then since, as the rectangle *A*, *B* is to the rectangle *B*, *C*, so is *A* to *C*; while the square on *D* is equal to the rectangle *A*, *B*, and the rectangle *D*, *E* is equal to the rectangle *B*, *C*, therefore, as *A* is to *C*, so is the square on *D* to the rectangle *D*, *E*.

But, as the square on *D* is to the rectangle *D*, *E*, so is *D* to *E*; therefore also, as *A* is to *C*, so is *D* to *E*.

But *A* is commensurable with *C* in square only; therefore *D* is also commensurable with *E* in square only. {X. 11}

But *D* is medial; therefore *E* is also medial. {X. 23, addition}

And, since, as *A* is to *C*, so is *D* to *E*, while the square on *A* is greater than the square on *C* by the square on a straight line commensurable with *A*,

therefore also the square on *D* will be greater than the square on *E* by the square on a straight line commensurable with *D*.{X. 14}

I say next that the rectangle *D*, *E* is also medial.

For, since the rectangle *B*, *C* is equal to the rectangle *D*, *E*, while the rectangle *B*, *C* is medial, {X. 21} therefore the rectangle *D*, *E* is also medial.

Therefore two medial straight lines *D*, *E*, commensurable in square only, and containing a medial rectangle, have been found such that the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater.

Similarly again it can be proved that the square on *D* is greater than the square on *E* by the square on a straight line incommensurable with *D*, when the square on *A* is greater than the square on *C* by the square on a straight line incommensurable with *A*. {X. 30}

LEMMA.

Let *ABC* be a right-angled triangle having the angle *A* right, and let the perpendicular *AD* be drawn; I say that the rectangle *CB*, *BD* is equal to the square on *BA*, the rectangle *BC*, *CD* equal to the square on *CA*, the rectangle *BD*, *DC* equal to the square on *AD*, and, further, the rectangle *BC*, *AD* equal to the rectangle *BA*, *AC*.

And first that the rectangle *CB*, *BD* is equal to the square on *BA*.

For, since in a right-angled triangle *AD* has been drawn from the right angle perpendicular to the base, therefore the triangles *ABD*, *ADC* are similar both to the whole *ABC* and to one another. {VI. 8}

And since the triangle *ABC* is similar to the triangle *ABD*, therefore, as *CB* is to *BA*, so is *BA* to *BD*; {VI. 4} therefore the rectangle *CB*, *BD* is equal to

the square on *AB*. {VI. 17}

For the same reason the rectangle *BC*, *CD* is also equal to the square on *AC*.

And since, if in a right-angled triangle a perpendicular be drawn from the right angle to the base, the perpendicular so drawn is a mean proportional between the segments of the base, {VI. 8, Por.} therefore, as *BD* is to *DA*, so is *AD* to *DC*; therefore the rectangle *BD*, *DC* is equal to the square on *AD*. {VI. 17}

I say that the rectangle *BC*, *AD* is also equal to the rectangle *BA*, *AC*.

For since, as we said, *ABC* is similar to *ABD*, therefore, as *BC* is to *CA*, so is *BA* to *AD*. {VI. 4}

Therefore the rectangle *BC*, *AD* is equal to the rectangle *BA*, *AC*. {VI. 16} Q. E. D.

PROPOSITION 33.

To find two straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial.

Let there be set out two rational straight lines *AB*, *BC* commensurable in square only and such that the square on the greater *AB* is greater than the square on the less *BC* by the square on a straight line incommensurable with *AB*, {X. 30} let *BC* be bisected at *D*, let there be applied to *AB* a parallelogram equal to the square on either of the straight lines *BD*, *DC* and deficient by a square figure, and let it be the rectangle *AE*, *EB*; {VI. 28} let the semicircle *AFB* be described on *AB*, let *EF* be drawn at right angles to *AB*, and let *AF*, *FB* be joined.

Then, since *AB*, *BC* are unequal straight lines, and the square on *AB* is greater than the square on *BC* by the square on a straight line

incommensurable with *AB*, while there has been applied to *AB* a parallelogram equal to the fourth part of the square on *BC*, that is, to the square on half of it, and deficient by a square figure, making the rectangle *AE*, *EB*, therefore *AE* is incommensurable with *EB*. {X. 18}

And, as *AE* is to *EB*, so is the rectangle *BA*, *AE* to the rectangle *AB*, *BE*, while the rectangle *BA*, *AE* is equal to the square on *AF*, and the rectangle *AB*, *BE* to the square on *BF*; therefore the square on *AF* is incommensurable with the square on *FB*; therefore *AF*, *FB* are incommensurable in square.

And, since *AB* is rational, therefore the square on *AB* is also rational; so that the sum of the squares on *AF*, *FB* is also rational. {I. 47}

And since, again, the rectangle *AE*, *EB* is equal to the square on *EF*, and, by hypothesis, the rectangle *AE*, *EB* is also equal to the square on *BD*, therefore *FE* is equal to *BD*; therefore *BC* is double of *FE*, so that the rectangle *AB*, *BC* is also commensurable with the rectangle *AB*, *EF*.

But the rectangle *AB*, *BC* is medial; {X. 21} therefore the rectangle *AB*, *EF* is also medial. {X. 23, Por.}

But the rectangle *AB*, *EF* is equal to the rectangle *AF*, *FB*; {Lemma} therefore the rectangle *AF*, *FB* is also medial.

But it was also proved that the sum of the squares on these straight lines is rational.

Therefore two straight lines *AF*, *FB* incommensurable in square have been found which make the sum of the squares on them rational, but the rectangle contained by them medial. Q. E. D.

PROPOSITION 34.

To find two straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational.

Let there be set out two medial straight lines *AB*, *BC*, commensurable in square only, such that the rectangle which they contain is rational, and the square on *AB* is greater than the square on *BC* by the square on a straight line incommensurable with *AB*; {X. 31, *ad fin*.} let the semicircle *ADB* be described on *AB*, let *BC* be bisected at *E*, let there be applied to *AB* a parallelogram equal to the square on *BE* and deficient by a square figure, namely the rectangle *AF*, *FB*; {VI. 28} therefore *AF* is incommensurable in length with *FB*. {X. 18}

Let *FD* be drawn from *F* at right angles to *AB*, and let *AD*, *DB* be joined.

Since *AF* is incommensurable in length with *FB*, therefore the rectangle *BA*, *AF* is also incommensurable with the rectangle *AB*, *BF*. {X. 11}

But the rectangle *BA*, *AF* is equal to the square on *AD*, and the rectangle *AB*, *BF* to the square on *DB*; therefore the square on *AD* is also incommensurable with the square on *DB*.

And, since the square on *AB* is medial, therefore the sum of the squares on *AD*, *DB* is also medial. {III. 31, I. 47}

And, since *BC* is double of *DF*, therefore the rectangle *AB*, *BC* is also double of the rectangle *AB*, *FD*.

But the rectangle *AB*, *BC* is rational; therefore the rectangle *AB*, *FD* is also rational. {X. 6}

But the rectangle *AB*, *FD* is equal to the rectangle *AD*, *DB*; {Lemma} so that the rectangle *AD*, *DB* is also rational.

Therefore two straight lines *AD*, *DB* incommensurable in square have been found which make the sum of the squares on them medial, but the rectangle contained by them rational. Q. E. D.

PROPOSITION 35.

To find two straight lines incommensurable in square which make the sum of the squares on them medial and the rectangle contained by them medial and moreover incommensurable with the sum of the squares on them.

Let there be set out two medial straight lines *AB*, *BC* commensurable in square only, containing a medial rectangle, and such that the square on *AB* is greater than the square on *BC* by the square on a straight line incommensurable with *AB*; {X. 32 , *ad fin*.} let the semicircle *ADB* be described on *AB*, and let the rest of the construction be as above.

Then, since *AF* is incommensurable in length with *FB*, {X. 18 } *AD* is also incommensurable in square with *DB*. {X. 11 }

And, since the square on *AB* is medial, therefore the sum of the squares on *AD*, *DB* is also medial. {III. 31 , I. 47 }

And, since the rectangle *AF*, *FB* is equal to the square on each of the straight lines *BE*, *DF*, therefore *BE* is equal to *DF*; therefore *BC* is double of *FD*, so that the rectangle *AB*, *BC* is also double of the rectangle *AB*, *FD*.

But the rectangle *AB*, *BC* is medial; therefore the rectangle *AB*, *FD* is also medial. {X. 32, Por.}

And it is equal to the rectangle *AD*, *DB*; {Lemma after X. 32 } therefore the rectangle *AD*, *DB* is also medial.

And, since *AB* is incommensurable in length with *BC*, while *CB* is commensurable with *BE*, therefore *AB* is also incommensurable in length with *BE*, {X. 13 } so that the square on *AB* is also incommensurable with the rectangle *AB*, *BE*. {X. 11 }

But the squares on *AD*, *DB* are equal to the square on *AB*, {I. 47 } and the rectangle *AB*, *FD*, that is, the rectangle *AD*, *DB*, is equal to the rectangle *AB*, *BE*; therefore the sum of the squares on *AD*, *DB* is incommensurable with the rectangle *AD*, *DB*.

Therefore two straight lines *AD*, *DB* incommensurable in square have been found which make the sum of the squares on them medial and the rectangle contained by them medial and moreover incommensurable with the sum of the squares on them. Q. E. D.

PROPOSITION 36.

If two rational straight lines commensurable in square only be added together, the whole is irrational; and let it be called binomial.

For let two rational straight lines *AB*, *BC* commensurable
in square only be added together; I say that the whole *AC* is irrational.

For, since *AB* is incommensurable in length with *BC* —
for they are commensurable in square only — and, as *AB* is to *BC*, so is the rectangle *AB*, *BC* to the square on *BC*, therefore the rectangle *AB*, *BC* is incommensurable with the square on *BC*. {X. 11 }

But twice the rectangle *AB*, *BC* is commensurable with the rectangle *AB*, *BC* {X. 6 }, and the squares on *AB*, *BC* are commensurable with the square on *BC* — for *AB*, *BC* are rational straight lines commensurable in square only — {X. 15 } therefore twice the rectangle *AB*, *BC* is incommensurable
with the squares on *AB*, *BC*. {X. 13 }

And, componendo, twice the rectangle *AB*, *BC* together with the squares on *AB*, *BC*, that is, the square on *AC* {II. 4 }, is incommensurable with the sum of the squares on *AB*, *BC*. {X. 16 }

But the sum of the squares on *AB*, *BC* is rational;
therefore the square on *AC* is irrational, so that *AC* is also irrational. {X. Def.

4 }

And let it be called binomial.

PROPOSITION 37.

If two medial straight lines commensurable in square only and containing a rational rectangle be added together, the whole is irrational; and let it be called a first bimedial *straight line*.

For let two medial straight lines *AB*, *BC* commensurable in square only and containing a rational rectangle be added together; I say that the whole *AC* is irrational.

For, since *AB* is incommensurable in length with *BC*, therefore the squares on *AB*, *BC* are also incommensurable with twice the rectangle *AB*, *BC*; {cf. X. 36, ll. 9-20} and, componendo, the squares on *AB*, *BC* together with twice the rectangle *AB*, *BC*, that is, the square on *AC* {II. 4}, is incommensurable with the rectangle *AB*, *BC*. {X. 16 }

But the rectangle *AB*, *BC* is rational, for, by hypothesis, *AB*, *BC* are straight lines containing a rational rectangle; therefore the square on *AC* is irrational; therefore *AC* is irrational. {X. Def. 4 }

And let it be called a first bimedial straight line. Q. E. D.

PROPOSITION 38.

If two medial straight lines commensurable in square only and containing a medial rectangle be added together, the whole is irrational; and let it be called a second bimedial *straight line*.

For let two medial straight lines *AB*, *BC* commensurable in square only and containing a medial rectangle be added together; I say that *AC* is irrational.

For let a rational straight line *DE* be set out, and let the parallelogram *DF* equal to the square on *AC* be applied to *DE*, producing *DG* as breadth. {I. 44 }

Then, since the square on *AC* is equal to the squares on *AB*, *BC* and twice the rectangle *AB*, *BC*, {II. 4 } let *EH*, equal to the squares on *AB*, *BC*, be applied to *DE*; therefore the remainder *HF* is equal to twice the rectangle *AB*, *BC*.

And, since each of the straight lines *AB*, *BC* is medial, therefore the squares on *AB*, *BC* are also medial.

But, by hypothesis, twice the rectangle *AB*, *BC* is also medial.

And *EH* is equal to the squares on *AB*, *BC*,
while *FH* is equal to twice the rectangle *AB*, *BC*; therefore each of the rectangle *EH*, *HF* is medial.

And they are applied to the rational straight line *DE*; therefore each of the straight lines *DH*, *HG* is rational and incommensurable in length with *DE*. {X. 22 }

Since then *AB* is incommensurable in length with *BC*, and, as *AB* is to *BC*, so is the square on *AB* to the rectangle *AB*, *BC*, therefore the square on *AB* is incommensurable with the rectangle *AB*, *BC*. {X. 11 }

But the sum of the squares on *AB*, *BC* is commensurable with the square on *AB*, {X. 15 } and twice the rectangle *AB*, *BC* is commensurable with the rectangle *AB*, *BC*. {X. 6 }

Therefore the sum of the squares on *AB*, *BC* is incommensurable with twice the rectangle *AB*, *BC*. {X. 13 }

But *EH* is equal to the squares on *AB*, *BC*, and *HF* is equal to twice the rectangle *AB*, *BC*.

Therefore *EH* is incommensurable with *HF*, so that *DH* is also incommensurable in length with *HG*. {VI. 1 , X. 11 }

Therefore *DH*, *HG* are rational straight lines commensurable in square only; so that *DG* is irrational. {X. 36 }

But *DE* is rational; and the rectangle contained by an irrational and a rational straight line is irrational; {cf. X. 20 } therefore the area *DF* is irrational, and the side of the square equal to it is irrational. {X. Def. 4 }

But *AC* is the side of the square equal to *DF*; therefore *AC* is irrational.

And let it be called a second bimedial straight line. Q. E. D.

PROPOSITION 39.

If two straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial, be added together, the whole straight line is irrational : and let it be called major.

For let two straight lines *AB*, *BC* incommensurable in square, and fulfilling the given conditions {X. 33 }, be added together; I say that *AC* is irrational.

For, since the rectangle *AB*, *BC* is medial, twice the rectangle *AB*, *BC* is also medial. {X. 6 and 23, Por.}

But the sum of the squares on *AB*, *BC* is rational; therefore twice the rectangle *AB*, *BC* is incommensurable with the sum of the squares on *AB*, *BC*,

so that the squares on *AB*, *BC* together with twice the rectangle *AB*, *BC* that is, the square on *AC*, is also incommensurable with the sum of the squares on *AB*, *BC*; {X. 16 } therefore the square on *AC* is irrational, so that *AC* is also irrational. {X. Def. 4 }

And let it be called major. Q. E. D.

PROPOSITION 40.

If two straight lines incommensurable in square which make the sum of the squares on them medial, *but the rectangle contained by them rational*, *be added together, the whole straight line is irrational; and let it be called the* side of a rational plus a medial area.

For let two straight lines *AB*, *BC* incommensurable in square, and fulfilling the given conditions {X. 34 }, be added together; I say that *AC* is irrational.

For, since the sum of the squares on *AB*, *BC* is medial, while twice the rectangle *AB*, *BC* is rational, therefore the sum of the squares on *AB*, *BC* is incommensurable with twice the rectangle *AB*, *BC*; so that the square on *AC* is also incommensurable with twice the rectangle *AB*, *BC*. {X. 16 }

But twice the rectangle *AB*, *BC* is rational; therefore the square on *AC* is irrational.

Therefore *AC* is irrational. {X. Def. 4 }

And let it be called the side of a rational plus a medial area. Q. E. D.

PROPOSITION 41.

If two straight lines incommensurable in square which make the sum of the squares on them medial, *and the rectangle contained by them medial and also incommensurable with the sum of the squares on them*, *be added*

together, *the whole straight line is irrational*; *and let it be called the* side of the sum of two medial areas.

For let two straight lines *AB*, *BC* incommensurable in square and satisfying the given conditions {X. 35 } be added together; I say that *AC* is irrational.

Let a rational straight line *DE* be set out, and let there be applied to *DE* the rectangle *DF* equal to the squares on *AB*, *BC*, and the rectangle *GH* equal to twice the rectangle *AB*, *BC*; therefore the whole *DH* is equal to the square on *AC*. {II. 4 }

Now, since the sum of the squares on *AB*, *BC* is medial, and is equal to *DF*, therefore *DF* is also medial.

And it is applied to the rational straight line *DE*; therefore *DG* is rational and incommensurable in length with *DE*. {X. 22 }

For the same reason *GK* is also rational and incommensurable in length with *GF*, that is, *DE*.

And, since the squares on *AB*, *BC* are incommensurable with twice the rectangle *AB*, *BC*, *DF* is incommensurable with *GH*; so that *DG* is also incommensurable with *GK*. {VI. 1 , X. 11 }

And they are rational; therefore *DG*, *GK* are rational straight lines commensurable in square only; therefore *DK* is irrational and what is called binomial. {X. 36 }

But *DE* is rational; therefore *DH* is irrational, and the side of the square which is equal to it is irrational. {X. Def. 4 }

But *AC* is the side of the square equal to *HD*; therefore *AC* is irrational.

And let it be called the side of the sum of two medial areas. Q. E. D.

LEMMA.

And that the aforesaid irrational straight lines are divided only in one way into the straight lines of which they are the sum and which produce the types in question, we will now prove after premising the following lemma.

Let the straight line *AB* be set out, let the whole be cut into unequal parts at each of the points *C*, *D*, and let *AC* be supposed greater than *DB*; I say that the squares on *AC*, *CB* are greater than the squares on *AD*, *DB*.

For let *AB* be bisected at *E*.

Then, since *AC* is greater than *DB*, let *DC* be subtracted from each; therefore the remainder *AD* is greater than the remainder *CB*.

But *AE* is equal to *EB*; therefore *DE* is less than *EC*; therefore the points *C*, *D* are not equidistant from the point of bisection.

And, since the rectangle *AC*, *CB* together with the square on *EC* is equal to the square on *EB*, {II. 5 } and, further, the rectangle *AD*, *DB* together with the square on *DE* is equal to the square on *EB*, {*id.*} therefore the rectangle *AC*, *CB* together with the square on *EC* is equal to the rectangle *AD*, *DB* together with the square on *DE*.

And of these the square on *DE* is less than the square on *EC*; therefore the remainder, the rectangle *AC*, *CB*, is also less than the rectangle *AD*, *DB*, so that twice the rectangle *AC*, *CB* is also less than twice the rectangle *AD*, *DB*.

Therefore also the remainder, the sum of the squares on *AC*, *CB*, is greater than the sum of the squares on *AD*, *DB*. Q. E. D. [4]

PROPOSITION 42.

A binomial straight line is divided into its terms at one point only

.

Let *AB* be a binomial straight line divided into its terms at *C*; therefore *AC*, *CB* are rational straight lines commensurable in square only. {X. 36 }

I say that *AB* is not divided at another point into two rational straight lines commensurable in square only.

For, if possible, let it be divided at *D* also, so that *AD*, *DB* are also rational straight lines commensurable in square only.

It is then manifest that *AC* is not the same with *DB*.

For, if possible, let it be so.

Then *AD* will also be the same as *CB*, and, as *AC* is to *CB*, so will *BD* be to *DA*; thus *AB* will be divided at *D* also in the same way as by the division at *C*: which is contrary to the hypothesis.

Therefore *AC* is not the same with *DB*.

For this reason also the points *C*, *D* are not equidistant from the point of bisection.

Therefore that by which the squares on *AC*, *CB* differ from the squares on *AD*, *DB* is also that by which twice the rectangle *AD*, *DB* differs from twice the rectangle *AC*, *CB*, because both the squares on *AC*, *CB* together with twice the rectangle *AC*, *CB*, and the squares on *AD*, *DB* together with twice the rectangle *AD*, *DB*, are equal to the square on *AB*. {II. 4 }

But the squares on *AC*, *CB* differ from the squares on *AD*, *DB* by a rational area, for both are rational; therefore twice the rectangle *AD*, *DB* also differs from twice the rectangle *AC*, *CB* by a rational area, though they are medial {X. 21 }: which is absurd, for a medial area does not exceed a medial by a rational area. {x. 26 }

Therefore a binomial straight line is not divided at different points; therefore it is divided at one point only. Q. E. D.

PROPOSITION 43.

A first bimedial straight line is divided at one point only.

Let *AB* be a first bimedial straight line divided at *C*, so that *AC*, *CB* are medial straight lines commensurable in square only and containing a rational rectangle; I say that *AB* is not so divided at another point.

For, if possible, let it be divided at *D* also, so that *AD*, *DB* are also medial straight lines commensurable in square only and containing a rational rectangle.

Since, then, that by which twice the rectangle *AD*, *DB* differs from twice the rectangle *AC*, *CB* is that by which the squares on *AC*, *CB* differ from the squares on *AD*, *DB*, while twice the rectangle *AD*, *DB* differs from twice the rectangle *AC*, *CB* by a rational area — for both are rational — therefore the squares on *AC*, *CB* also differ from the squares on *AD*, *DB* by a rational area, though they are medial: which is absurd. {x. 26 }

Therefore a first bimedial straight line is not divided into its terms at different points; therefore it is so divided at one point only.

PROPOSITION 44.

A second bimedial straight line is divided at one point only.

Let *AB* be a second bimedial straight line divided at *C*, so that *AC*, *CB* are medial straight lines commensurable in square only and containing a medial rectangle; {X. 38 } it is then manifest that *C* is not at the point of bisection, because the segments are not commensurable in length.

I say that *AB* is not so divided at another point.

For, if possible, let it be divided at *D* also, so that *AC* is not the same with *DB*, but *AC* is supposed greater; it is then clear that the squares on *AD*, *DB* are also, as we proved above {Lemma}, less than the squares on *AC*, *CB*; and

suppose that *AD*, *DB* are medial straight lines commensurable in square only and containing a medial rectangle.

Now let a rational straight line *EF* be set out, let there be applied to *EF* the rectangular parallelogram *EK* equal to the square on *AB*, and let *EG* equal to the squares on *AC*, *CB* be subtracted; therefore the remainder *HK* is equal to twice the rectangle *AC*, *CB*. {II. 4 }

Again, let there be subtracted *EL*, equal to the squares on *AD*, *DB*, which were proved less than the squares on *AC*, *CB* {Lemma }; therefore the remainder *MK* is also equal to twice the rectangle *AD*, *DB*.

Now, since the squares on *AC*, *CB* are medial, therefore *EG* is medial.

And it is applied to the rational straight line *EF*; therefore *EH* is rational and incommensurable in length with *EF*. {X. 22 }

For the same reason *HN* is also rational and incommensurable in length with *EF*.

And, since *AC*, *CB* are medial straight lines commensurable in square only, therefore *AC* is incommensurable in length with *CB*.

But, as *AC* is to *CB*, so is the square on *AC* to the rectangle *AC*, *CB*; therefore the square on *AC* is incommensurable with the rectangle *AC*, *CB*. {X. 11 }

But the squares on *AC*, *CB* are commensurable with the square on *AC*; for *AC*, *CB* are commensurable in square. {x. 15 }

And twice the rectangle *AC*, *CB* is commensurable with the rectangle *AC*, *CB*. {X. 6 }

Therefore the squares on *AC*, *CB* are also incommensurable with twice the rectangle *AC*, *CB*. {X. 13 }

But *EG* is equal to the squares on *AC*, *CB*, and *HK* is equal to twice the rectangle *AC*, *CB*; therefore *EG* is incommensurable with *HK*, so that *EH* is also incommensurable in length with *HN*. {VI. 1 , X. 11 }

And they are rational; therefore *EH*, *HN* are rational straight lines commensurable in square only.

But, if two rational straight lines commensurable in square only be added together, the whole is the irrational which is called binomial. {X. 36 }

Therefore *EN* is a binomial straight line divided at *H*.

In the same way *EM*, *MN* will also be proved to be rational straight lines commensurable in square only; and *EN* will be a binomial straight line divided at different points, *H* and *M*.

And *EH* is not the same with *MN*.

For the squares on *AC*, *CB* are greater than the squares on *AD*, *DB*.

But the squares on *AD*, *DB* are greater than twice the rectangle *AD*, *DB*; therefore also the squares on *AC*, *CB*, that is, *EG*, are much greater than twice the rectangle *AD*, *DB*, that is, *MK*, so that *EH* is also greater than *MN*.

Therefore *EH* is not the same with *MN*. Q. E. D.

PROPOSITION 45.

A major straight line is divided at one and the same point only.

Let *AB* be a major straight line divided at *C*, so that *AC*, *CB* are incommensurable in square and make the sum of the squares on *AC*, *CB* rational, but the rectangle *AC*, *CB* medial; {X. 39 } I say that *AB* is not so divided at another point.

For, if possible, let it be divided at *D* also, so that *AD*, *DB* are also incommensurable in square and make the sum of the squares on *AD*, *DB* rational, but the rectangle contained by them medial.

Then, since that by which the squares on *AC*, *CB* differ from the squares on *AD*, *DB* is also that by which twice the rectangle *AD*, *DB* differs from twice the rectangle *AC*, *CB*, while the squares on *AC*, *CB* exceed the squares

on *AD*, *DB* by a rational area — for both are rational — therefore twice the rectangle *AD*, *DB* also exceeds twice the rectangle *AC*, *CB* by a rational area, though they are medial: which is impossible. {X. 26 }

Therefore a major straight line is not divided at different points; therefore it is only divided at one and the same point. Q. E. D.

PROPOSITION 46.

The side of a rational plus a medial area is divided at one point only.

Let *AB* be the side of a rational plus a medial area divided at *C*, so that *AC*, *CB* are incommensurable in square and make the sum of the squares on *AC*, *CB* medial, but twice the rectangle *AC*, *CB* rational; {X. 40 } I say that *AB* is not so divided at another point.

For, if possible, let it be divided at *D* also, so that *AD*, *DB* are also incommensurable in square and make the sum of the squares on *AD*, *DB* medial, but twice the rectangle *AD*, *DB* rational.

Since then that by which twice the rectangle *AC*, *CB* differs from twice the rectangle *AD*, *DB* is also that by which the squares on *AD*, *DB* differ from the squares on *AC*, *CB*, while twice the rectangle *AC*, *CB* exceeds twice the rectangle *AD*, *DB* by a rational area, therefore the squares on *AD*, *DB* also exceed the squares on *AC*, *CB* by a rational area, though they are medial: which is impossible. {X. 26 }

Therefore the side of a rational plus a medial area is not divided at different points; therefore it is divided at one point only. Q. E. D.

PROPOSITION 47.

The side of the sum of two medial areas is divided at one point only.

Let *AB* be divided at *C*, so that *AC*, *CB* are incommensurable in square and make the sum of the squares on *AC*, *CB* medial, and the rectangle *AC*, *CB* medial and also incommensurable with the sum of the squares on them; I say that *AB* is not divided at another point so as to fulfil the given conditions.

For, if possible, let it be divided at *D*, so that again *AC* is of course not the same as *BD*, but *AC* is supposed greater; let a rational straight line *EF* be set out, and let there be applied to *EF* the rectangle *EG* equal to the squares on *AC*, *CB*, and the rectangle *HK* equal to twice the rectangle *AC*, *CB*; therefore the whole *EK* is equal to the square on *AB*. {II. 4 }

Again, let *EL*, equal to the squares on *AD*, *DB*, be applied to *EF*; therefore the remainder, twice the rectangle *AD*, *DB*, is equal to the remainder *MK*.

And since, by hypothesis, the sum of the squares on *AC*, *CB* is medial, therefore *EG* is also medial.

And it is applied to the rational straight line *EF*; therefore *HE* is rational and incommensurable in length with *EF*. {X. 22 }

For the same reason *HN* is also rational and incommensurable in length with *EF*.

And, since the sum of the squares on *AC*, *CB* is incommensurable with twice the rectangle *AC*, *CB*, therefore *EG* is also incommensurable with *GN*, so that *EH* is also incommensurable with *HN*. {VI. 1 , X. 11 }

And they are rational; therefore *EH*, *HN* are rational straight lines commensurable in square only; therefore *EN* is a binomial straight line divided at *H*. {X. 36 }

Similarly we can prove that it is also divided at *M*.

And *EH* is not the same with *MN*; therefore a binomial has been divided at different points: which is absurd. {X. 42 }

Therefore a side of the sum of two medial areas is not divided at different points; therefore it is divided at one point only.

DEFINITIONS II.

1

Given a rational straight line and a binomial, divided into its terms, such that the square on the greater term is greater than the square on the lesser by the square on a straight line commensurable in length with the greater, then, if the greater term be commensurable in length with the rational straight line set out, let the whole be called a first binomial straight line;

2

but if the lesser term be commensurable in length with the rational straight line set out, let the whole be called a second binomial;

3

and if neither of the terms be commensurable in length with the rational straight line set out, let the whole be called a third binomial.

4

Again, if the square on the greater term be greater than the square on the lesser by the square on a straight line incommensurable in length with the greater, then, if the greater term be commensurable in length with the rational straight line set out, let the whole be called a fourth binomial;

5

if the lesser, a fifth binomial;

6

and if neither, a sixth binomial.

PROPOSITIONS 48-84.

PROPOSITION 48.

To find the first binomial straight line.

Let two numbers *AC*, *CB* be set out such that the sum of them *AB* has to *BC* the ratio which a square number has to a square number, but has not to *CA* the ratio which a square number has to a square number; {Lemma I after X. 28} let any rational straight line *D* be set out, and let *EF* be commensurable in length with *D*.

Therefore *EF* is also rational.

Let it be contrived that, as the number *BA* is to *AC*, so is the square on *EF* to the square on *FG*. {X. 6, Por.}

But *AB* has to *AC* the ratio which a number has to a number; therefore the square on *EF* also has to the square on *FG* the ratio which a number has to a number, so that the square on *EF* is commensurable with the square on *FG*. {X. 6}

And *EF* is rational; therefore *FG* is also rational.

And, since *BA* has not to *AC* the ratio which a square number has to a square number. neither, therefore, has the square on *EF* to the square on *FG* the ratio which a square number has to a square number; therefore *EF* is incommensurable in length with *FG*. {X. 9}

Therefore *EF*, *FG* are rational straight lines commensurable in square only; therefore *EG* is binomial. {X. 36}

I say that it is also a first binomial straight line.

For since, as the number *BA* is to *AC*, so is the square on *EF* to the square on *FG*, while *BA* is greater than *AC*, therefore the square on *EF* is also greater than the square on *FG*.

Let then the squares on *FG*, *H* be equal to the square on *EF*.

Now since, as *BA* is to *AC*, so is the square on *EF* to the square on *FG*, therefore, *convertendo*, as *AB* is to *BC*, so is the square on *EF* to the square on *H*. {V. 19, Por.}

But *AB* has to *BC* the ratio which a square number has to a square number; therefore the square on *EF* also has to the square on *H* the ratio which a square number has to a square number.

Therefore *EF* is commensurable in length with *H*; {X. 9} therefore the square on *EF* is greater than the square on *FG* by the square on a straight line commensurable with *EF*.

And *EF*, *FG* are rational, and *EF* is commensurable in length with *D*.

Therefore *EF* is a first binomial straight line. Q. E. D.

PROPOSITION 49.

To find the second binomial straight line.

Let two numbers *AC*, *CB* be set out such that the sum of them *AB* has to *BC* the ratio which a square number has to a square number, but has not to *AC* the ratio which a square number has to a square number; let a rational straight line *D* be set out, and let *EF* be commensurable in length with *D*; therefore *EF* is rational.

Let it be contrived then that, as the number *CA* is to *AB*, so also is the square on *EF* to the square on *FG*; {X. 6, Por.} therefore the square on *EF* is commensurable with the square on *FG*. {X. 6}

Therefore *FG* is also rational.

Now, since the number *CA* has not to *AB* the ratio which a square number has to a square number, neither has the square on *EF* to the square on *FG* the ratio which a square number has to a square number.

Therefore *EF* is incommensurable in length with *FG*; {X. 9} therefore *EF*, *FG* are rational straight lines commensurable in square only; therefore *EG* is binomial. {X. 36}

It is next to be proved that it is also a second binomial straight line.

For since, inversely, as the number *BA* is to *AC*, so is the square on *GF* to the square on *FE*, while *BA* is greater than *AC*, therefore the square on *GF* is greater than the square on *FE*.

Let the squares on *EF*, *H* be equal to the square on *GF*; therefore, convertendo, as *AB* is to *BC*, so is the square on *FG* to the square on *H*. {V. 19, Por.}

But *AB* has to *BC* the ratio which a square number has to a square number; therefore the square on *FG* also has to the square on *H* the ratio which a square number has to a square number.

Therefore *FG* is commensurable in length with *H*; {X. 9} so that the square on *FG* is greater than the square on *FE* by the square on a straight line commensurable with *FG*.

And *FG*, *FE* are rational straight lines commensurable in square only, and *EF*, the lesser term, is commensurable in length with the rational straight line *D* set out.

Therefore *EG* is a second binomial straight line. Q. E. D.

PROPOSITION 50.

To find the third binomial straight line.

Let two numbers *AC*, *CB* be set out such that the sum of them *AB* has to *BC* the ratio which a square number has to a square number, but has not to *AC* the ratio which a square number has to a square number.

Let any other number *D*, not square, be set out also, and let it not have to either of the numbers *BA*. *AC* the ratio which a square number has to a square number.

Let any rational straight line *E* be set out, and let it be contrived that, as *D* is to *AB*, so is the square on *E* to the square on *FG*; {X. 6, Por.} therefore the square on *E* is commensurable with the square on *FG*. {X. 6}

And *E* is rational; therefore *FG* is also rational.

And, since *D* has not to *AB* the ratio which a square number has to a square number, neither has the square on *E* to the square on *FG* the ratio which a square number has to a square number; therefore *E* is incommensurable in length with *FG*. {X. 9}

Next let it be contrived that, as the number *BA* is to *AC*, so is the square on *FG* to the square on *GH*; {X. 6, Por.} therefore the square on *FG* is commensurable with the square on *GH*. {X. 6}

But *FG* is rational; therefore *GH* is also rational.

And, since *BA* has not to *AC* the ratio which a square number has to a square number, neither has the square on *FG* to the square on *HG* the ratio which a square number has to a square number; therefore *FG* is incommensurable in length with *GH*. {X. 9}

Therefore *FG*, *GH* are rational straight lines commensurable in square only; therefore *FH* is binomial. {X. 36}

I say next that it is also a third binomial straight line.

For since, as *D* is to *AB*, so is the square on *E* to the square on *FG*, and, as *BA* is to *AC*, so is the square on *FG* to the square on *GH*, therefore, ex aequali, as *D* is to *AC*, so is the square on *E* to the square on *GH*. {V. 22}

But *D* has not to *AC* the ratio which a square number has to a square number; therefore neither has the square on *E* to the square on *GH* the ratio

which a square number has to a square number; therefore *E* is incommensurable in length with *GH*. {X. 9}

And since, as *BA* is to *AC*, so is the square on *FG* to the square on *GH*, therefore the square on *FG* is greater than the square on *GH*.

Let then the squares on *GH*, *K* be equal to the square on *FG*; therefore, convertendo, as *AB* is to *BC*, so is the square on *FG* to the square on *K*. {V. 19, Por.}

But *AB* has to *BC* the ratio which a square number has to a square number; therefore the square on *FG* also has to the square on *K* the ratio which a square number has to a square number; therefore *FG* is commensurable in length with *K*. {X. 9}

Therefore the square on *FG* is greater than the square on *GH* by the square on a straight line commensurable with *FG*.

And *FG*, *GH* are rational straight lines commensurable in square only, and neither of them is commensurable in length with *E*.

Therefore *FH* is a third binomial straight line. Q. E. D.

PROPOSITION 51.

To find the fourth binomial straight line.

Let two numbers *AC*, *CB* be set out such that *AB* neither has to *BC*, nor yet to *AC*, the ratio which a square number has to a square number.

Let a rational straight line *D* be set out, and let *EF* be commensurable in length with *D*; therefore *EF* is also rational.

Let it be contrived that, as the number *BA* is to *AC*, so is the square on *EF* to the square on *FG*; {X. 6, Por.} therefore the square on *EF* is commensurable with the square on *FG*; {X. 6} therefore *FG* is also rational.

Now, since *BA* has not to *AC* the ratio which a square number has to a square number, neither has the square on *EF* to the square on *FG* the ratio which a square number has to a square number; therefore *EF* is incommensurable in length with *FG*. {X. 9}

Therefore *EF*, *FG* are rational straight lines commensurable in square only; so that *EG* is binomial.

I say next that it is also a fourth binomial straight line.

For since, as *BA* is to *AC*, so is the square on *EF* to the square on *FG*, therefore the square on *EF* is greater than the square on *FG*.

Let then the squares on *FG*, *H* be equal to the square on *EF*; therefore, convertendo, as the number *AB* is to *BC*, so is the square on *EF* to the square on *H*. {V. 19, Por.}

But *AB* has not to *BC* the ratio which a square number has to a square number; therefore neither has the square on *EF* to the square on *H* the ratio which a square number has to a square number.

Therefore *EF* is incommensurable in length with *H*; {X. 9} therefore the square on *EF* is greater than the square on *GF* by the square on a straight line incommensurable with *EF*.

And *EF*, *FG* are rational straight lines commensurable in square only, and *EF* is commensurable in length with *D*.

Therefore *EG* is a fourth binomial straight line. Q. E. D.

PROPOSITION 52.

To find the fifth binomial straight line.

Let two numbers *AC*, *CB* be set out such that *AB* has not to either of them the ratio which a square number has to a square number; let any rational

straight line *D* be set out, and let *EF* be commensurable with *D*; therefore *EF* is rational.

Let it be contrived that, as *CA* is to *AB*, so is the square on *EF* to the square on *FG*. {X. 6, Por.}

But *CA* has not to *AB* the ratio which a square number has to a square number; therefore neither has the square on *EF* to the square on *FG* the ratio which a square number has to a square number.

Therefore *EF*, *FG* are rational straight lines commensurable in square only; {X. 9} therefore *EG* is binomial. {X. 36}

I say next that it is also a fifth binomial straight line.

For since, as *CA* is to *AB*, so is the square on *EF* to the square on *FG*, inversely, as *BA* is to *AC*, so is the square on *FG* to the square on *FE*; therefore the square on *GF* is greater than the square on *FE*.

Let then the squares on *EF*, *H* be equal to the square on *GF*; therefore, convertendo, as the number *AB* is to *BC*, so is the square on *GF* to the square on *H*. {V. 19, Por.}

But *AB* has not to *BC* the ratio which a square number has to a square number; therefore neither has the square on *FG* to the square on *H* the ratio which a square number has to a square number.

Therefore *FG* is incommensurable in length with *H*; {X. 9} so that the square on *FG* is greater than the square on *FE* by the square on a straight line incommensurable with *FG*.

And *GF*, *FE* are rational straight lines commensurable in square only, and the lesser term *EF* is commensurable in length with the rational straight line *D* set out.

Therefore *EG* is a fifth binomial straight line. Q. E. D.

PROPOSITION 53.

To find the sixth binomial straight line.

Let two numbers *AC*, *CB* be set out such that *AB* has not to either of them the ratio which a square number has to a square number; and let there also be another number *D* which is not square and which has not to either of the numbers *BA*, *AC* the ratio which a square number has to a square number.

Let any rational straight line *E* be set out, and let it be contrived that, as *D* is to *AB*, so is the square on *E* to the square on *FG*; {X. 6, Por.} therefore the square on *E* is commensurable with the square on *FG*. {X. 6}

And *E* is rational; therefore *FG* is also rational.

Now, since *D* has not to *AB* the ratio which a square number has to a square number, neither has the square on *E* to the square on *FG* the ratio which a square number has to a square number; therefore *E* is incommensurable in length with *FG*. {X. 9}

Again, let it be contrived that, as *BA* is to *AC*, so is the square on *FG* to the square on *GH*. {X. 6, Por.}

Therefore the square on *FG* is commensurable with the square on *HG*. {X. 6}

Therefore the square on *HG* is rational; therefore *HG* is rational.

And, since *BA* has not to *AC* the ratio which a square number has to a square number, neither has the square on *FG* to the square on *GH* the ratio which a square number has to a square number; therefore *FG* is incommensurable in length with *GH*. {X. 9}

Therefore *FG*, *GH* are rational straight lines commensurable in square only; therefore *FH* is binomial. {X. 36}

It is next to be proved that it is also a sixth binomial straight line.

For since, as *D* is to *AB*, so is the square on *E* to the square on *FG*, and also, as *BA* is to *AC*, so is the square on *FG* to the square on *GH*, therefore, ex aequali, as *D* is to *AC*, so is the square on *E* to the square on *GH*. {V. 22}

But D has not to AC the ratio which a square number has to a square number; therefore neither has the square on E to the square on GH the ratio which a square number has to a square number; therefore E is incommensurable in length with GH. {X. 9}

But it was also proved incommensurable with FG; therefore each of the straight lines FG, GH is incommensurable in length with E.

And, since, as BA is to AC, so is the square on FG to the square on GH, therefore the square on FG is greater than the square on GH.

Let then the squares on GH, K be equal to the square on FG; therefore, convertendo, as AB is to BC, so is the square on FG to the square on K. {V. 19, Por.}

But AB has not to BC the ratio which a square number has to a square number; so that neither has the square on FG to the square on K the ratio which a square number has to a square number.

Therefore FG is incommensurable in length with K; {X. 9} therefore the square on FG is greater than the square on GH by the square on a straight line incommensurable with FG.

And FG, GH are rational straight lines commensurable in square only, and neither of them is commensurable in length with the rational straight line E set out.

Therefore FH is a sixth binomial straight line. Q. E. D.

LEMMA.

Let there be two squares AB, BC, and let them be placed so that DB is in a straight line with BE; therefore FB is also in a straight line with BG.

Let the parallelogram AC be completed; I say that AC is a square, that DG is a mean proportional between AB, BC, and further that DC is a mean

proportional between *AC*, *CB*.

For, since *DB* is equal to *BF*, and *BE* to *BG*, therefore the whole *DE* is equal to the whole *FG*.

But *DE* is equal to each of the straight lines *AH*, *KC*, and *FG* is equal to each of the straight lines *AK*, *HC*; {I. 34} therefore each of the straight lines *AH*, *KC* is also equal to each of the straight lines *AK*, *HC*.

Therefore the parallelogram *AC* is equilateral.

And it is also rectangular; therefore *AC* is a square.

And since, as *FB* is to *BG*, so is *DB* to *BE*, while, as *FB* is to *BG*, so is *AB* to *DG*, and, as *DB* is to *BE*, so is *DG* to *BC*, {VI. 1} therefore also, as *AB* is to *DG*, so is *DG* to *BC*. {V. 11}

Therefore *DG* is a mean proportional between *AB*, *BC*.

I say next that *DC* is also a mean proportional between *AC*, *CB*.

For since, as *AD* is to *DK*, so is *KG* to *GC* — for they are equal respectively — and, componendo, as *AK* is to *KD*, so is *KC* to *CG*, {V. 18} while, as *AK* is to *KD*, so is *AC* to *CD*, and, as *KC* is to *CG*, so is *DC* to *CB*, {VI. 1} therefore also, as *AC* is to *DC*, so is *DC* to *BC*. {V. 11}

Therefore *DC* is a mean proportional between *AC*, *CB*.

Being what it was proposed to prove.

PROPOSITION 54.

If an area be contained by a rational straight line and the first binomial, the "side" of the area is the irrational straight line which is called binomial.

For let the area *AC* be contained by the rational straight line *AB* and the first binomial *AD*; I say that the "side" of the area *AC* is the irrational straight line which is called binomial.

For, since *AD* is a first binomial straight line, let it be divided into its terms at *E*, and let *AE* be the greater term.

It is then manifest that *AE*, *ED* are rational straight lines commensurable in square only, the square on *AE* is greater than the square on *ED* by the square on a straight line commensurable with *AE*, and *AE* is commensurable in length with the rational straight line *AB* set out. {X. Deff. II. 1}

Let *ED* be bisected at the point *F*.

Then, since the square on *AE* is greater than the square on *ED* by the square on a straight line commensurable with *AE*, therefore, if there be applied to the greater *AE* a parallelogram equal to the fourth part of the square on the less, that is, to the square on *EF*, and deficient by a square figure, it divides it into commensurable parts. {X. 17}

Let then the rectangle *AG*, *GE* equal to the square on *EF* be applied to *AE*; therefore *AG* is commensurable in length with *EG*.

Let *GH*, *EK*, *FL* be drawn from *G*, *E*, *F* parallel to either of the straight lines *AB*, *CD*; let the square *SN* be constructed equal to the parallelogram *AH*, and the square *NQ* equal to *GK*, {II. 14} and let them be placed so that *MN* is in a straight line with *NO*; therefore *RN* is also in a straight line with *NP*.

And let the parallelogram *SQ* be completed; therefore *SQ* is a square. {Lemma}

Now, since the rectangle *AG*, *GE* is equal to the square on *EF*, therefore, as *AG* is to *EF*, so is *FE* to *EG*; {VI. 17} therefore also, as *AH* is to *EL*, so is *EL* to *KG*; {VI. 1} therefore *EL* is a mean proportional between *AH*, *GK*.

But *AH* is equal to *SN*, and *GK* to *NQ*; therefore *EL* is a mean proportional between *SN*, *NQ*.

But *MR* is also a mean proportional between the same *SN*, *NQ*; {Lemma} therefore *EL* is equal to *MR*, so that it is also equal to *PO*.

But *AH*, *GK* are also equal to *SN*, *NQ*; therefore the whole *AC* is equal to the whole *SQ*, that is, to the square on *MO*; therefore *MO* is the “side” of *AC*.

I say next that *MO* is binomial.

For, since *AG* is commensurable with *GE*, therefore *AE* is also commensurable with each of the straight lines *AG*, *GE*. {X. 15}

But *AE* is also, by hypothesis, commensurable with *AB*; therefore *AG*, *GE* are also commensurable with *AB*. {X. 12}

And *AB* is rational; therefore each of the straight lines *AG*, *GE* is also rational; therefore each of the rectangles *AH*, *GK* is rational, {X. 19} and *AH* is commensurable with *GK*.

But *AH* is equal to *SN*, and *GK* to *NQ*; therefore *SN*, *NQ*, that is, the squares on *MN*, *NO*, are rational and commensurable.

And, since *AE* is incommensurable in length with *ED*, while *AE* is commensurable with *AG*, and *DE* is commensurable with *EF*, therefore *AG* is also incommensurable with *EF*, {X. 13} so that *AH* is also incommensurable with *EL*. {VI. 1, X. 11}

But *AH* is equal to *SN*, and *EL* to *MR*; therefore *SN* is also incommensurable with *MR*.

But, as *SN* is to *MR*, so is *PN* to *NR*; {VI. 1} therefore *PN* is incommensurable with *NR*. {X. 11}

But *PN* is equal to *MN*, and *NR* to *NO*; therefore *MN* is incommensurable with *NO*.

And the square on *MN* is commensurable with the square on *NO*, and each is rational; therefore *MN*, *NO* are rational straight lines commensurable in square only.

Therefore *MO* is binomial {X. 36} and the "side" of *AC*. Q. E. D. [5]

PROPOSITION 55.

If an area be contained by a rational straight line and the second binomial, the "side" of the area is the irrational straight line which is called a first

bimedial.

For let the area *ABCD* be contained by the rational
straight line *AB* and the second binomial *AD*; I say that the "side" of the area *AC* is a first bimedial straight line.

For, since *AD* is a second binomial straight line, let it be divided into its terms at *E*, so that *AE* is the greater term;
therefore *AE*, *ED* are rational straight lines commensurable in square only, the square on *AE* is greater than the square on *ED* by the square on a straight line commensurable with *AE*, and the lesser term *ED* is commensurable in length with *AB*. {X. Deff. II. 2}

Let *ED* be bisected at *F*, and let there be applied to *AE* the rectangle *AG*, *GE* equal to the square on *EF* and deficient by a square figure; therefore *AG* is commensurable in length with *GE*. {X. 17}

Through *G*, *E*, *F* let *GH*, *EK*, *FL* be drawn parallel to
AB, *CD*, let the square *SN* be constructed equal to the parallelogram *AH*, and the square *NQ* equal to *GK*, and let them be placed so that *MN* is in a straight line with *NO*;
therefore *RN* is also in a straight line with *NP*.

Let the square *SQ* be completed.

It is then manifest from what was proved before that *MR* is a mean proportional between *SN*, *NQ* and is equal to *EL*, and that *MO* is the "side" of the area *AC*.

It is now to be proved that *MO* is a first bimedial straight line.

Since *AE* is incommensurable in length with *ED*, while *ED* is commensurable with *AB*, therefore *AE* is incommensurable with *AB*. {X. 13}

And, since *AG* is commensurable with *EG*,
AE is also commensurable with each of the straight lines *AG*, *GE*. {X. 15}

But *AE* is incommensurable in length with *AB*; therefore *AG*, *GE* are also incommensurable with *AB*. {X. 13}

Therefore *BA*, *AG* and *BA*, *GE* are pairs of rational
straight lines commensurable in square only; so that each of the rectangles *AH*, *GK* is medial. {X. 21}

Hence each of the squares *SN*, *NQ* is medial.
Therefore *MN*, *NO* are also medial.

And, since *AG* is commensurable in length with *GE*,
AH is also commensurable with *GK*, {VI. 1. X. 11} that is, *SN* is commensurable with *NQ*, that is, the square on *MN* with the square on *NO*.

And, since *AE* is incommensurable in length with *ED*, while *AE* is commensurable with *AG*,
and *ED* is commensurable with *EF*, therefore *AG* is incommensurable with *EF*; {X. 13} so that *AH* is also incommensurable with *EL*, that is, *SN* is incommensurable with *MR*, that is, *PN* with *NR*, {VI. 1, X. 11}
that is, *MN* is incommensurable in length with *NO*.

But *MN*, *NO* were proved to be both medial and commensurable in square; therefore *MN*, *NO* are medial straight lines commensurable in square only.

I say next that they also contain a rational rectangle.

For, since *DE* is, by hypothesis, commensurable with each of the straight lines *AB*, *EF*, therefore *EF* is also commensurable with *EK*. {X. 12}

And each of them is rational;
therefore *EL*, that is, *MR* is rational, {X. 19} and *MR* is the rectangle *MN*, *NO*.

But, if two medial straight lines commensurable in square only and containing a rational rectangle be added together, the whole is irrational and is called a first bimedial straight line. {X. 37}

Therefore *MO* is a first bimedial straight line. Q. E. D. [6]

PROPOSITION 56.

If an area be contained by a rational straight line and the third binomial, the "side" of the area is the irrational straight line called a second bimedial.

For let the area *ABCD* be contained by the rational straight line *AB* and the third binomial *AD* divided into its terms at *E*, of which terms *AE* is the greater; I say that the "side" of the area *AC* is the irrational straight line called a second bimedial.

For let the same construction be made as before.

Now, since *AD* is a third binomial straight line, therefore *AE*, *ED* are rational straight lines commensurable in square only, the square on *AE* is greater than the square on *ED* by the square on a straight line commensurable with *AE*, and neither of the terms *AE*, *ED* is commensurable in length with *AB*. {X. Deff. II. 3}

Then, in manner similar to the foregoing, we shall prove that *MO* is the "side" of the area *AC*, and *MN*, *NO* are medial straight lines commensurable in square only; so that *MO* is bimedial.

It is next to be proved that it is also a second bimedial straight line.

Since *DE* is incommensurable in length with *AB*, that is, with *EK*, and *DE* is commensurable with *EF*, therefore *EF* is incommensurable in length with *EK*. {X. 13}

And they are rational; therefore *FE*, *EK* are rational straight lines commensurable in square only.

Therefore *EL*, that is, *MR*, is medial. {X. 21}

And it is contained by *MN*, *NO*; therefore the rectangle *MN*, *NO* is medial.

Therefore *MO* is a second bimedial straight line. {X. 38} Q. E. D.

PROPOSITION 57.

If an area be contained by a rational straight line and the fourth binomial, the "side" of the area is the irrational straight line called major.

For let the area *AC* be contained by the rational straight line *AB* and the fourth binomial *AD* divided into its terms at *E*, of which terms let *AE* be the greater; I say that the "side" of the area *AC* is the irrational straight line called major.

For, since *AD* is a fourth binomial straight line, therefore *AE*, *ED* are rational straight lines commensurable in square only, the square on *AE* is greater than the square on *ED* by the square on a straight line incommensurable with *AE*, and *AE* is commensurable in length with *AB*. {X. Deff. II. 4}

Let *DE* be bisected at *F*, and let there be applied to *AE* a parallelogram, the rectangle *AG*, *GE*, equal to the square on *EF*; therefore *AG* is incommensurable in length with *GE*. {X. 18}

Let *GH*, *EK*, *FL* be drawn parallel to *AB*, and let the rest of the construction be as before; it is then manifest that *MO* is the "side" of the area *AC*.

It is next to be proved that *MO* is the irrational straight line called major.

Since *AG* is incommensurable with *EG*, *AH* is also incommensurable with *GK*, that is, *SN* with *NQ*; {VI. 1, X. 11} therefore *MN*, *NO* are incommensurable in square.

And, since *AE* is commensurable with *AB*, *AK* is rational; {X. 19} and it is equal to the squares on *MN*, *NO*; therefore the sum of the squares on *MN*, *NO* is also rational.

And, since *DE* is incommensurable in length with *AB*, that is, with *EK*, while *DE* is commensurable with *EF*, therefore *EF* is incommensurable in length with *EK*. {X. 13}

Therefore *EK*, *EF* are rational straight lines commensurable in square only; therefore *LE*, that is, *MR*, is medial. {X. 21}

And it is contained by *MN*, *NO*; therefore the rectangle *MN*, *NO* is medial.

And the {sum} of the squares on *MN*, *NO* is rational, and *MN*, *NO* are incommensurable in square.

But, if two straight lines incommensurable in square and making the sum of the squares on them rational, but the rectangle contained by them medial, be added together, the whole is irrational and is called major. {X. 39}

Therefore *MO* is the irrational straight line called major and is the "side" of the area *AC*. Q. E. D.

PROPOSITION 58.

If an area be contained by a rational straight line and the fifth binomial, the "side" of the area is the irrational straight line called the side of a rational plus a medial area.

For let the area *AC* be contained by the rational straight line *AB* and the fifth binomial *AD* divided into its terms at *E*, so that *AE* is the greater term; I

say that the "side" of the area *AC* is the irrational straight line called the side of a rational plus a medial area.

For let the same construction be made as before shown; it is then manifest that *MO* is the "side" of the area *AC.*

It is then to be proved that *MO* is the side of a rational plus a medial area.

For, since *AG* is incommensurable with *GE*, {X. 18} therefore *AH* is also commensurable with *HE*, {VI. 1, X. 11} that is, the square on *MN* with the square on *NO*; therefore *MN*, *NO* are incommensurable in square.

And, since *AD* is a fifth binomial straight line, and *ED* the lesser segment, therefore *ED* is commensurable in length with *AB*. {X. Deff. II. 5}

But *AE* is incommensurable with *ED*; therefore *AB* is also incommensurable in length with *AE.* {X. 13}

Therefore *AK*, that is, the sum of the squares on *MN*, *NO*, is medial. {X. 21}

And, since *DE* is commensurable in length with *AB*, that is, with *EK*, while *DE* is commensurable with *EF*, therefore *EF* is also commensurable with *EK.* {X. 12}

And *EK* is rational; therefore *EL*, that is, *MR*, that is, the rectangle *MN*, *NO*, is also rational. {X. 19}

Therefore *MN*, *NO* are straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational.

Therefore *MO* is the side of a rational plus a medial area {X. 40} and is the "side" of the area *AC.* Q. E. D.

PROPOSITION 59.

If an area be contained by a rational straight line and the sixth binomial, the "side" of the area is the irrational straight line called the side of the sum of two medial areas.

For let the area *ABCD* be contained by the rational straight line *AB* and the sixth binomial *AD*, divided into its terms at *E*, so that *AE* is the greater term; I say that the "side" of *AC* is the side of the sum of two medial areas.

Let the same construction be made as before shown.

It is then manifest that *MO* is the "side" of *AC*, and that *MN* is incommensurable in square with *NO*.

Now, since *EA* is incommensurable in length with *AB*, therefore *EA*, *AB* are rational straight lines commensurable in square only; therefore *AK*, that is, the sum of the squares on *MN*, *NO*, is medial. {X. 21}

Again, since *ED* is incommensurable in length with *AB*, therefore *FE* is also incommensurable with *EK*; {X. 13} therefore *FE*, *EK* are rational straight lines commensurable in square only; therefore *EL*, that is, *MR*, that is, the rectangle *MN*, *NO*, is medial. {X. 21}

And, since *AE* is incommensurable with *EF*, *AK* is also incommensurable with *EL*. {VI. 1, X. 11}

But *AK* is the sum of the squares on *MN*, *NO*, and *EL* is the rectangle *MN*, *NO*; therefore the sum of the squares on *MN*, *NO* is incommensurable with the rectangle *MN*, *NO*.

And each of them is medial, and *MN*, *NO* are incommensurable in square.

Therefore *MO* is the side of the sum of two medial areas {X. 41}, and is the "side" of *AC*. Q. E. D.

{LEMMA.

If a straight line be cut into unequal parts, the squares on the unequal parts are greater than twice the rectangle contained by the unequal parts.

Let *AB* be a straight line, and let it be cut into unequal parts at *C*, and let *AC* be the greater; I say that the squares on *AC*, *CB* are greater than twice the rectangle *AC*, *CB*.

For let *AB* be bisected at *D*.

Since then a straight line has been cut into equal parts at *D*, and into unequal parts at *C*, therefore the rectangle *AC*, *CB* together with the square on *CD* is equal to the square on *AD*, {II. 5} so that the rectangle *AC*, *CB* is less than double of the square on *AD*.

But the squares on *AC*, *CB* are double of the squares on *AD*, *DC*; {II. 9} therefore the squares on *AC*, *CB* are greater than twice the rectangle *AC*, *CB*. Q. E. D.}

PROPOSITION 60.

The square on the binomial straight line applied to a rational straight line produces as breadth the first binomial.

Let *AB* be a binomial straight line divided into its terms at *C*, so that *AC* is the greater term; let a rational straight line *DE* be set out, and let *DEFG* equal to the square on *AB* be applied to *DE* producing *DG* as its breadth; I say that *DG* is a first binomial straight line.

For let there be applied to *DE* the rectangle *DH* equal to the square on *AC*, and *KL* equal to the square on *BC*; therefore the remainder, twice the rectangle *AC*, *CB*, is equal to *MF*.

Let *MG* be bisected at *N*, and let *NO* be drawn parallel {to *ML* or *GF*}.

Therefore each of the rectangles *MO*, *NF* is equal to once the rectangle *AC*, *CB*.

Now, since *AB* is a binomial divided into its terms at *C*, therefore *AC*, *CB* are rational straight lines commensurable in square only; {X. 36} therefore the squares on *AC*, *CB* are rational and commensurable with one another, so that the sum of the squares on *AC*, *CB* is also rational. {X. 15}

And it is equal to *DL*; therefore *DL* is rational.

And it is applied to the rational straight line *DE*; therefore *DM* is rational and commensurable in length with *DE*. {X. 20}

Again, since *AC*, *CB* are rational straight lines commensurable in square only, therefore twice the rectangle *AC*, *CB*, that is *MF*, is medial. {X. 21}

And it is applied to the rational straight line *ML*; therefore *MG* is also rational and incommensurable in length with *ML*, that is, *DE*. {X. 22}

But *MD* is also rational and is commensurable in length with *DE*; therefore *DM* is incommensurable in length with *MG*. {X. 13}

And they are rational; therefore *DM*, *MG* are rational straight lines commensurable in square only; therefore *DG* is binomial. {X. 36}

It is next to be proved that it is also a first binomial straight line.

Since the rectangle *AC*, *CB* is a mean proportional between the squares on *AC*, *CB*, {cf. Lemma after X. 53} therefore *MO* is also a mean proportional between *DH*, *KL*.

Therefore, as *DH* is to *MO*, so is *MO* to *KL*, that is, as *DK* is to *MN*, so is *MN* to *MK*; {VI. 1} therefore the rectangle *DK*, *KM* is equal to the square on *MN*. {VI. 17}

And, since the square on *AC* is commensurable with the square on *CB*, *DH* is also commensurable with *KL*, so that *DK* is also commensurable with *KM*. {VI. 1, X. 11}

And, since the squares on *AC*, *CB* are greater than twice the rectangle *AC*, *CB*, {Lemma} therefore *DL* is also greater than *MF*, so that *DM* is also greater than *MG*. {VI. 1}

And the rectangle *DK*, *KM* is equal to the square on *MN*, that is, to the fourth part of the square on *MG*, and *DK* is commensurable with *KM*.

But, if there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into commensurable parts, the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater; {X. 17} therefore the square on *DM* is greater than the square on *MG* by the square on a straight line commensurable with *DM*.

And *DM*, *MG* are rational, and *DM*, which is the greater term, is commensurable in length with the rational straight line *DE* set out.

Therefore *DG* is a first binomial straight line. {X. Deff. II. 1} Q. E. D.

PROPOSITION 61.

The square on the first bimedial straight line applied to a rational straight line produces as breadth the second binomial.

Let *AB* be a first bimedial straight line divided into its medials at *C*, of which medials *AC* is the greater; let a rational straight line *DE* be set out, and let there be applied to *DE* the parallelogram *DF* equal to the square on *AB*, producing *DG* as its breadth; I say that *DG* is a second binominal straight line.

For let the same construction as before be made.

Then, since *AB* is a first bimedial divided at *C*, therefore *AC*, *CB* are medial straight lines commensurable in square only, and containing a rational rectangle, {X. 37} so that the squares on *AC*, *CB* are also medial. {X. 21}

Therefore *DL* is medial. {X. 15 and 23, Por.}

And it has been applied to the rational straight line *DE*; therefore *MD* is rational and incommensurable in length with *DE*. {X. 22}

Again, since twice the rectangle *AC*, *CB* is rational, *MF* is also rational.

And it is applied to the rational straight line *ML*; therefore *MG* is also rational and commensurable in length with *ML*, that is, *DE*; {X. 20} therefore *DM* is incommensurable in length with *MG*. {X. 13}

And they are rational; therefore *DM*, *MG* are rational straight lines commensurable in square only; therefore *DG* is binomial. {X. 36}

It is next to be proved that it is also a second binomial straight line.

For, since the squares on *AC*, *CB* are greater than twice the rectangle *AC*, *CB*, therefore *DL* is also greater than *MF*, so that *DM* is also greater than *MG*. {VI. 1}

And, since the square on *AC* is commensurable with the square on *CB*, *DH* is also commensurable with *KL*, so that *DK* is also commensurable with *KM*. {VI. 1, X. 11}

And the rectangle *DK*, *KM* is equal to the square on *MN*; therefore the square on *DM* is greater than the square on *MG* by the square on a straight line commensurable with *DM*. {X. 17}

And *MG* is commensurable is length with *DE*.

Therefore *DG* is a second binomial straight line. {X. Deff. II. 2}

PROPOSITION 62.

The square on the second bimedial straight line applied to a rational straight line produces as breadth the third binomial.

Let *AB* be a second bimedial straight line divided into its medials at *C*, so that *AC* is the greater segment; let *DE* be any rational straight line, and to *DE*

let there be applied the parallelogram *DF* equal to the square on *AB* and producing *DG* as its breadth; I say that *DG* is a third binomial straight line.

Let the same construction be made as before shown.

Then, since *AB* is a second bimedial divided at *C*, therefore *AC*, *CB* are medial straight lines commensurable in square only and containing a medial rectangle, {X. 38} so that the sum of the squares on *AC*, *CB* is also medial. {X. 15 and 23 Por.}

And it is equal to *DL*; therefore *DL* is also medial.

And it is applied to the rational straight line *DE*; therefore *MD* is also rational and incommensurable in length with *DE*. {X. 22}

For the same reason, *MG* is also rational and incommensurable in length with *ML*, that is, with *DE*; therefore each of the straight lines *DM*, *MG* is rational and incommensurable in length with *DE*.

And, since *AC* is incommensurable in length with *CB*, and, as *AC* is to *CB*, so is the square on *AC* to the rectangle *AC*, *CB*, therefore the square on *AC* is also incommensurable with the rectangle *AC*, *CB*. {X. 11}

Hence the sum of the squares on *AC*, *CB* is incommensurable with twice the rectangle *AC*, *CB*, {X. 12, 13} that is, *DL* is incommensurable with *MF*, so that *DM* is also incommensurable with *MG*. {VI. 1, X. 11}

And they are rational; therefore *DG* is binomial. {X. 36}

It is to be proved that it is also a third binomial straight line.

In manner similar to the foregoing we may conclude that *DM* is greater than *MG*, and that *DK* is commensurable with *KM*.

And the rectangle *DK*, *KM* is equal to the square on *MN*; therefore the square on *DM* is greater than the square on *MG* by the square on a straight line commensurable with *DM*.

And neither of the straight lines *DM*, *MG* is commensurable in length with *DE*.

Therefore *DG* is a third binomial straight line. {X. Deff. II. 3} Q. E. D.

PROPOSITION 63.

The square on the major straight line applied to a rational straight line produces as breadth the fourth binomial.

Let *AB* be a major straight line divided at *C*, so that *AC* is greater than *CB*; let *DE* be a rational straight line, and to *DE* let there be applied the parallelogram *DF* equal to the square on *AB* and producing *DG* as its breadth; I say that *DG* is a fourth binomial straight line.

Let the same construction be made as before shown.

Then, since *AB* is a major straight line divided at *C*, *AC*, *CB* are straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial. {X. 39}

Since then the sum of the squares on *AC*, *CB* is rational, therefore *DL* is rational; therefore *DM* is also rational and commensurable in length with *DE*. {X. 20}

Again, since twice the rectangle *AC*, *CB*, that is, *MF*, is medial, and it is applied to the rational straight line *ML*, therefore *MG* is also rational and incommensurable in length with *DE*; {X. 22} therefore *DM* is also incommensurable in length with *MG*. {X. 13}

Therefore *DM*, *MG* are rational straight lines commensurable in square only; therefore *DG* is binomial. {X. 36}

It is to be proved that it is also a fourth binomial straight line.

In manner similar to the foregoing we can prove that *DM* is greater than *MG*, and that the rectangle *DK*, *KM* is equal to the square on *MN*.

Since then the square on *AC* is incommensurable with the square on *CB*, therefore *DH* is also incommensurable with *KL*, so that *DK* is also

incommensurable with *KM*. {VI. 1, X. 11}

But, if there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into incommensurable parts, then the square on the greater will be greater than the square on the less by the square on a straight line incommensurable in length with the greater; {X. 18} therefore the square on *DM* is greater than the square on *MG* by the square on a straight line incommensurable with *DM*.

And *DM*, *MG* are rational straight lines commensurable in square only, and *DM* is commensurable with the rational straight line *DE* set out.

Therefore *DG* is a fourth binomial straight line. {X. Deff. II. 4} Q. E. D.

PROPOSITION 64.

The square on the side of a rational plus a medial area applied to a rational straight line produces as breadth the fifth binomial.

Let *AB* be the side of a rational plus a medial area, divided into its straight lines at *C*, so that *AC* is the greater; let a rational straight line *DE* be set out, and let there be applied to *DE* the parallelogram *DF* equal to the square on *AB*, producing *DG* as its breadth; I say that *DG* is a fifth binomial straight line.

Let the same construction as before be made.

Since then *AB* is the side of a rational plus a medial area, divided at *C*, therefore *AC*, *CB* are straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational. {X. 40}

Since then the sum of the squares on *AC*, *CB* is medial, therefore *DL* is medial, so that *DM* is rational and incommensurable in length with *DE*. {X.

22}

Again, since twice the rectangle *AC*, *CB*, that is *MF*, is rational, therefore *MG* is rational and commensurable with *DE*. {X. 20}

Therefore *DM* is incommensurable with *MG*; {X. 13} therefore *DM*, *MG* are rational straight lines commensurable in square only; therefore *DG* is binomial. {X. 36}

I say next that it is also a fifth binomial straight line.

For it can be proved similarly that the rectangle *DK*, *KM* is equal to the square on *MN*, and that *DK* is incommensurable in length with *KM*; therefore the square on *DM* is greater than the square on *MG* by the square on a straight line incommensurable with *DM*. {X. 18}

And *DM*, *MG* are commensurable in square only, and the less, *MG*, is commensurable in length with *DE*.

Therefore *DG* is a fifth binomial. Q. E. D.

PROPOSITION 65.

The square on the side of the sum of two medial areas applied to a rational straight line produces as breadth the sixth binomial.

Let *AB* be the side of the sum of two medial areas, divided at *C*, let *DE* be a rational straight line, and let there be applied to *DE* the parallelogram *DF* equal to the square on *AB*, producing *DG* as its breadth; I say that *DG* is a sixth binomial straight line.

For let the same construction be made as before.

Then, since *AB* is the side of the sum of two medial areas, divided at *C*, therefore *AC*, *CB* are straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and moreover the sum of the squares on them incommensurable with

the rectangle contained by them, {X. 41} so that, in accordance with what was before proved, each of the rectangles *DL*, *MF* is medial.

And they are applied to the rational straight line *DE*; therefore each of the straight lines *DM*, *MG* is rational and incommensurable in length with *DE*. {X. 22}

And, since the sum of the squares on *AC*, *CB* is incommensurable with twice the rectangle *AC*, *CB*, therefore *DL* is incommensurable with *MF*.

Therefore *DM* is also incommensurable with *MG*; {VI. 1, X. 11} therefore *DM*, *MG* are rational straight lines commensurable in square only; therefore *DG* is binomial. {X. 36}

I say next that it is also a sixth binomial straight line.

Similarly again we can prove that the rectangle *DK*, *KM* is equal to the square on *MN*, and that *DK* is incommensurable in length with *KM*; and, for the same reason, the square on *DM* is greater than the square on *MG* by the square on a straight line incommensurable in length with *DM*.

And neither of the straight lines *DM*, *MG* is commensurable in length with the rational straight line *DE* set out.

Therefore *DG* is a sixth binomial straight line. Q. E. D.

PROPOSITION 66.

A straight line commensurable in length with a binomial straight line is itself also binomial and the same in order.

Let *AB* be binomial, and let *CD* be commensurable in length with *AB*; I say that *CD* is binomial and the same in order with *AB*.

For, since *AB* is binomial, let it be divided into its terms at *E*, and let *AE* be the greater term; therefore *AE*, *EB* are rational straight lines commensurable in square only. {X. 36}

Let it be contrived that, as *AB* is to *CD*, so is *AE* to *CF*; {VI. 12} therefore also the remainder *EB* is to the remainder *FD* as *AB* is to *CD*. {V. 19}

But *AB* is commensurable in length with *CD*; therefore *AE* is also commensurable with *CF*, and *EB* with *FD*. {X. 11}

And *AE*, *EB* are rational; therefore *CF*, *FD* are also rational.

And, as *AE* is to *CF*, so is *EB* to *FD*. {V. 11}

Therefore, alternately, as *AE* is to *EB*, so is *CF* to *FD*. {V. 16}

But *AE*, *EB* are commensurable in square only; therefore *CF*, *FD* are also commensurable in square only. {X. 11}

And they are rational; therefore *CD* is binomial. {X. 36}

I say next that it is the same in order with *AB*.

For the square on *AE* is greater than the square on *EB* either by the square on a straight line commensurable with *AE* or by the square on a straight line incommensurable with it.

If then the square on *AE* is greater than the square on *EB* by the square on a straight line commensurable with *AE*, the square on *CF* will also be greater than the square on *FD* by the square on a straight line commensurable with *CF*. {X. 14}

And, if *AE* is commensurable with the rational straight line set out, *CF* will also be commensurable with it, {X. 12} and for this reason each of the straight lines *AB*, *CD* is a first binomial, that is, the same in order. {X. Deff. II. 1}

But, if *EB* is commensurable with the rational straight line set out, *FD* is also commensurable with it, {X. 12} and for this reason again *CD* will be the same in order with *AB*, for each of them will be a second binomial. {X. Deff. II. 2}

But, if neither of the straight lines *AE*, *EB* is commensurable with the rational straight line set out, neither of the straight lines *CF*, *FD* will be

commensurable with it, {X. 13} and each of the straight lines *AB*, *CD* is a third binomial. {X. Deff. II. 3}

But, if the square on *AE* is greater than the square on *EB* by the square on a straight line incommensurable with *AE*, the square on *CF* is also greater than the square on *FD* by the square on a straight line incommensurable with *CF*. {X. 14}

And, if *AE* is commensurable with the rational straight line set out, *CF* is also commensurable with it, and each of the straight lines *AB*, *CD* is a fourth binomial. {X. Deff. II. 4}

But, if *EB* is so commensurable, so is *FD* also, and each of the straight lines *AB*, *CD* will be a fifth binomial. {X. Deff. II. 5}

But, if neither of the straight lines *AE*, *EB* is so commensurable, neither of the straight lines *CF*, *FD* is commensurable with the rational straight line set out, and each of the straight lines *AB*, *CD* will be a sixth binomial. {X. Deff. II. 6}

Hence a straight line commensurable in length with a binomial straight line is binomial and the same in order. Q. E. D.

PROPOSITION 67.

A straight line commensurable in length with a bimedial straight line is itself also bimedial and the same in order.

Let *AB* be bimedial, and let *CD* be commensurable in length with *AB*; I say that *CD* is bimedial and the same in order with *AB*.

For, since *AB* is bimedial, let it be divided into its medials at *E*; therefore *AE*, *EB* are medial straight lines commensurable in square only. {X. 37, 38}

And let it be contrived that, as *AB* is to *CD*, so is *AE* to *CF*; therefore also the remainder *EB* is to the remainder *FD* as *AB* is to *CD*. {V. 19}

But *AB* is commensurable in length with *CD*; therefore *AE*, *EB* are also commensurable with *CF*, *FD* respectively. {X. 11}

But *AE*, *EB* are medial; therefore *CF*, *FD* are also medial. {X. 23}

And since, as *AE* is to *EB*, so is *CF* to *FD*, {V. 11} and *AE*, *EB* are commensurable in square only, *CF*, *FD* are also commensurable in square only. {X. 11}

But they were also proved medial; therefore *CD* is bimedial.

I say next that it is also the same in order with *AB*.

For since, as *AE* is to *EB*, so is *CF* to *FD*, therefore also, as the square on *AE* is to the rectangle *AE*, *EB*, so is the square on *CF* to the rectangle *CF*, *FD*; therefore, alternately, as the square on *AE* is to the square on *CF*, so is the rectangle *AE*, *EB* to the rectangle *CF*, *FD*. {V. 16}

But the square on *AE* is commensurable with the square on *CF*; therefore the rectangle *AE*, *EB* is also commensurable with the rectangle *CF*, *FD*.

If therefore the rectangle *AE*, *EB* is rational, the rectangle *CF*, *FD* is also rational, {and for this reason *CD* is a first bimedial}; {X. 37} but if medial, medial, {X. 23, Por.} and each of the straight lines *AB*, *CD* is a second bimedial. {X. 38}

And for this reason *CD* will be the same in order with *AB*. Q. E. D.

PROPOSITION 68.

A straight line commensurable with a major straight line is itself also major.

Let *AB* be major, and let *CD* be commensurable with *AB*; I say that *CD* is major.

Let *AB* be divided at *E*; therefore *AE*, *EB* are straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial. {X. 39}

Let the same construction be made as before.

Then since, as *AB* is to *CD*, so is *AE* to *CF*, and *EB* to *FD*, therefore also, as *AE* is to *CF*, so is *EB* to *FD*. {V. 11}

But *AB* is commensurable with *CD*; therefore *AE*, *EB* are also commensurable with *CF*, *FD* respectively. {X. 11}

And since, as *AE* is to *CF*, so is *EB* to *FD*, alternately also, as *AE* is to *EB*, so is *CF* to *FD*; {V. 16} therefore also, componendo, as *AB* is to *BE*, so is *CD* to *DF*; {V. 18} therefore also, as the square on *AB* is to the square on *BE*, so is the square on *CD* to the square on *DF*. {VI. 20}

Similarly we can prove that, as the square on *AB* is to the square on *AE*, so also is the square on *CD* to the square on *CF*.

Therefore also, as the square on *AB* is to the squares on *AE*, *EB*, so is the square on *CD* to the squares on *CF*, *FD*; therefore also, alternately, as the square on *AB* is to the square on *CD*, so are the squares on *AE*, *EB* to the squares on *CF*, *FD*. {V. 16}

But the square on *AB* is commensurable with the square on *CD*; therefore the squares on *AE*, *EB* are also commensurable with the squares on *CF*, *FD*.

And the squares on *AE*, *EB* together are rational; therefore the squares on *CF*, *FD* together are rational.

Similarly also twice the rectangle *AE*, *EB* is commensurable with twice the rectangle *CF*, *FD*.

And twice the rectangle *AE*, *EB* is medial; therefore twice the rectangle *CF*, *FD* is also medial. {X. 23, Por.}

Therefore *CF*, *FD* are straight lines incommensurable in square which make, at the same time, the sum of the squares on them rational, but the rectangle contained by them medial; therefore the whole *CD* is the irrational straight line called major. {X. 39}

Therefore a straight line commensurable with the major straight line is major. Q. E. D.

PROPOSITION 69.

A straight line commensurable with the side of a rational plus a medial area is itself also the side of a rational plus a medial area.

Let *AB* be the side of a rational plus a medial area, and let *CD* be commensurable with *AB*; it is to be proved that *CD* is also the side of a rational plus a medial area.

Let *AB* be divided into its straight lines at *E*; therefore *AE*, *EB* are straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational. {X. 40}

Let the same construction be made as before.

We can then prove similarly that *CF*, *FD* are incommensurable in square, and the sum of the squares on *AE*, *EB* is commensurable with the sum of the squares on *CF*, *FD*, and the rectangle *AE*, *EB* with the rectangle *CF*, *FD*; so that the sum of the squares on *CF*, *FD* is also medial, and the rectangle *CF*, *FD* rational.

Therefore *CD* is the side of a rational plus a medial area. Q. E. D.

PROPOSITION 70.

A straight line commensurable with the side of the sum of two medial areas is the side of the sum of two medial areas.

Let *AB* be the side of the sum of two medial areas, and *CD* commensurable with *AB*; it is to be proved that *CD* is also the side of the sum of two medial areas.

For, since *AB* is the side of the sum of two medial areas, let it be divided into its straight lines at *E*; therefore *AE*, *EB* are straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and furthermore the sum of the squares on *AE*, *EB* incommensurable with the rectangle *AE*, *EB*. {X. 41}

Let the same construction be made as before.

We can then prove similarly that *CF*, *FD* are also incommensurable in square, the sum of the squares on *AE*, *EB* is commensurable with the sum of the squares on *CF*, *FD*, and the rectangle *AE*, *EB* with the rectangle *CF*, *FD*; so that the sum of the squares on *CF*, *FD* is also medial, the rectangle *CF*, *FD* is medial, and moreover the sum of the squares on *CF*, *FD* is incommensurable with the rectangle *CF*, *FD*.

Therefore *CD* is the side of the sum of two medial areas. Q. E. D.

PROPOSITION 71.

If a rational and a medial area be added together, four irrational straight lines arise, namely a binomial or a first bimedial or a major or a side of a rational plus a medial area.

Let. *AB* be rational, and *CD* medial; I say that the "side" of the area *AD* is a binomial or a first bimedial or a major or a side of a rational plus a medial area.

For *AB* is either greater or less than *CD*.

First, let it be greater; let a rational straight line *EF* be set out, let there be applied to *EF* the rectangle *EG* equal to *AB*, producing *EH* as breadth, and let *HI*, equal to *DC*, be applied to *EF*, producing *HK* as breadth.

Then, since *AB* is rational and is equal to *EG*, therefore *EG* is also rational.

And it has been applied to *EF*, producing *EH* as breadth; therefore *EH* is rational and commensurable in length with *EF*. {X. 20}

Again, since *CD* is medial and is equal to *HI*, therefore *HI* is also medial.

And it is applied to the rational straight line *EF*, producing *HK* as breadth; therefore *HK* is rational and incommensurable in length with *EF* {X. 22}

And, since *CD* is medial, while *AB* is rational, therefore *AB* is incommensurable with *CD*, so that *EG* is also incommensurable with *HI*.

But, as *EG* is to *HI*, so is *EH* to *HK*; {VI. 1} therefore *EH* is also incommensurable in length with *HK*. {X. 11}

And both are rational; therefore *EH*, *HK* are rational straight lines commensurable in square only; therefore *EK* is a binomial straight line, divided at *H*. {X. 36}

And, since *AB* is greater than *CD*, while *AB* is equal to *EG* and *CD* to *HI*, therefore *EG* is also greater than *HI*; therefore *EH* is also greater than *HK*.

The square, then, on *EH* is greater than the square on *HK* either by the square on a straight line commensurable in length with *EH* or by the square on a straight line incommensurable with it.

First, let the square on it be greater by the square on a straight line commensurable with itself.

Now the greater straight line *HE* is commensurable in length with the rational straight line *EF* set out; therefore *EK* is a first binomial. {X. Deff. II. 1}

But *EF* is rational; and, if an area be contained by a rational straight line and the first binomial, the side of the square equal to the area is binomial. {X. 54}

Therefore the “side” of *EI* is binomial; so that the “side” of *AD* is also binomial.

Next, let the square on *EH* be greater than the square on *HK* by the square on a straight line incommensurable with *EH*.

Now the greater straight line *EH* is commensurable in length with the rational straight line *EF* set out; therefore *EK* is a fourth binomial. {X. Deff. II. 4}

But *EF* is rational; and, if an area be contained by a rational straight line and the fourth binomial, the "side" of the area is the irrational straight line called major. {X. 57}

Therefore the "side" of the area *EI* is major; so that the "side" of the area *AD* is also major.

Next, let *AB* be less than *CD*; therefore *EG* is also less than *HI*, so that *EH* is also less than *HK*.

Now the square on *HK* is greater than the square on *EH* either by the square on a straight line commensurable with *HK* or by the square on a straight line incommensurable with it.

First, let the square on it be greater by the square on a straight line commensurable in length with itself.

Now the lesser straight line *EH* is commensurable in length with the rational straight line *EF* set out; therefore *EK* is a second binomial. {X. Deff. II. 2}

But *EF* is rational, and, if an area be contained by a rational straight line and the second binomial, the side of the square equal to it is a first bimedial; {X. 55} therefore the "side" of the area *EI* is a first bimedial, so that the "side" of *AD* is also a first bimedial.

Next, let the square on *HK* be greater than the square on *HE* by the square on a straight line incommensurable with *HK*.

Now the lesser straight line *EH* is commensurable with the rational straight line *EF* set out; therefore *EK* is a fifth binomial. {X. Deff. II. 5}

But *EF* is rational; and, if an area be contained by a rational straight line and the fifth binomial, the side of the square equal to the area is a side of a rational plus a medial area. {X. 58}

Therefore the "side" of the area *EI* is a side of a rational plus a medial area, so that the "side" of the area *AD* is also a side of a rational plus a medial area.

Therefore etc. Q. E. D.

PROPOSITION 72.

If two medial areas incommensurable with one another be added together, the remaining two irrational straight lines arise, namely either a second bimedial or a side of the sum of two medial areas.

For let two medial areas *AB*, *CD* incommensurable with one another be added together; I say that the "side" of the area *AD* is either a second bimedial or a side of the sum of two medial areas.

For *AB* is either greater or less than *CD*.

First, if it so chance, let *AB* be greater than *CD*.

Let the rational straight line *EF* be set out, and to *EF* let there be applied the rectangle *EG* equal to *AB* and producing *EH* as breadth, and the rectangle *HI* equal to *CD* and producing *HK* as breadth.

Now, since each of the areas *AB*, *CD* is medial, therefore each of the areas *EG*, *HI* is also medial.

And they are applied to the rational straight line *FE*, producing *EH*, *HK* as breadth; therefore each of the straight lines *EH*, *HK* is rational and incommensurable in length with *EF*. {X. 22}

And, since *AB* is incommensurable with *CD*, and *AB* is equal to *EG*, and *CD* to *HI*, therefore *EG* is also incommensurable with *HI*.

But, as *EG* is to *HI*, so is *EH* to *HK*; {VI. 1} therefore *EH* is incommensurable in length with *HK*. {X. 11}

Therefore *EH*, *HK* are rational straight lines commensurable in square only; therefore *EK* is binomial. {X. 36}

But the square on *EH* is greater than the square on *HK* either by the square on a straight line commensurable with *EH* or by the square on a straight line incommensurable with it.

First, let the square on it be greater by the square on a straight line commensurable in length with itself.

Now neither of the straight lines *EH*, *HK* is commensurable in length with the rational straight line *EF* set out; therefore *EK* is a third binomial. {X. Deff. II. 3}

But *EF* is rational; and, if an area be contained by a rational straight line and the third binomial, the "side" of the area is a second bimedial; {X. 56} therefore the "side" of *EI*, that is, of *AD*, is a second bimedial.

Next, let the square on *EH* be greater than the square on *HK* by the square on a straight line incommensurable in length with *EH*.

Now each of the straight lines *EH*, *HK* is incommensurable in length with *EF*; therefore *EK* is a sixth binomial. {X. Deff. II. 6}

But, if an area be contained by a rational straight line and the sixth binomial, the "side" of the area is the side of the sum of two medial areas; {X. 59} so that the "side" of the area *AD* is also the side of the sum of two medial areas.

Therefore etc. Q. E. D.

PROPOSITION 73.

If from a rational straight line there be subtracted a rational straight line commensurable with the whole in square only, the remainder is irrational; and let it be called an apotome.

For from the rational straight line *AB* let the rational straight line *BC*, commensurable with the whole in square only, be subtracted; I say that the remainder *AC* is the irrational straight line called apotome.

For, since *AB* is incommensurable in length with *BC*, and, as *AB* is to *BC*, so is the square on *AB* to the rectangle *AB*, *BC*, therefore the square on *AB* is incommensurable with the rectangle *AB*, *BC*. {X. 11}

But the squares on *AB*, *BC* are commensurable with the square on *AB*, {X. 15} and twice the rectangle *AB*, *BC* is commensurable with the rectangle *AB*, *BC*. {X. 6}

And, inasmuch as the squares on *AB*, *BC* are equal to twice the rectangle *AB*, *BC* together with the square on *CA*, {II. 7} therefore the squares on *AB*, *BC* are also incommensurable with the remainder, the square on *AC*. {X. 13, 16}

But the squares on *AB*, *BC* are rational; therefore *AC* is irrational. {X. Def. 4}

And let it be called an apotome. Q. E. D.

PROPOSITION 74.

If from a medial straight line there be subtracted a medial straight line which is commensurable with the whole in square only, and which contains with the whole a rational rectangle, the remainder is irrational. And let it be called a first apotome of a medial *straight line*.

For from the medial straight line *AB* let there be subtracted the medial straight line *BC* which is commensurable with *AB* in square only and with *AB*

makes the rectangle *AB*, *BC* rational; I say that the remainder *AC* is irrational; and let it be called a first apotome of a medial straight line.

For, since *AB*, *BC* are medial, the squares on *AB*, *BC* are also medial.

But twice the rectangle *AB*, *BC* is rational; therefore the squares on *AB*, *BC* are incommensurable with twice the rectangle *AB*, *BC*; therefore twice the rectangle *AB*, *BC* is also incommensurable with the remainder, the square on *AC*, {Cf. II. 7} since, if the whole is incommensurable with one of the magnitudes, the original magnitudes will also be incommensurable. {X. 16}

But twice the rectangle *AB*, *BC* is rational; therefore the square on *AC* is irrational; therefore *AC* is irrational. {X. Def. 4}

And let it be called a first apotome of a medial straight line.

PROPOSITION 75

If from a medial straight line there be subtracted a medial straight line which is commensurable with the whole in square only, and which contains with the whole a medial rectangle, the remainder is irrational; and let it be called a second apotome of a medial *straight line*.

For from the medial straight line *AB* let there be subtracted the medial straight line *CB* which is commensurable with the whole *AB* in square only and such that the rectangle *AB*, *BC*, which it contains with the whole *AB*, is medial; {X. 28} I say that the remainder *AC* is irrational; and let it be called a second apotome of a medial straight line.

For let a rational straight line *DI* be set out, let *DE* equal to the squares on *AB*, *BC* be applied to *DI*, producing *DG* as breadth, and let *DH* equal to twice the rectangle *AB*, *BC* be applied to *DI*, producing *DF* as breadth; therefore the remainder *FE* is equal to the square on *AC*. {II. 7}

Now, since the squares on *AB*, *BC* are medial and commensurable, therefore *DE* is also medial. {X. 15 and 23, Por.}

And it is applied to the rational straight line *DI*, producing *DG* as breadth; therefore *DG* is rational and incommensurable in length with *DI*. {X. 22}

Again, since the rectangle *AB*, *BC* is medial, therefore twice the rectangle *AB*, *BC* is also medial. {X. 23, Por.}

And it is equal to *DH*; therefore *DH* is also medial.

And it has been applied to the rational straight line *DI*, producing *DF* as breadth; therefore *DF* is rational and incommensurable in length with *DI*. {X. 22}

And, since *AB*, *BC* are commensurable in square only, therefore *AB* is incommensurable in length with *BC*; therefore the square on *AB* is also incommensurable with the rectangle *AB*, *BC*. {X. 11}

But the squares on *AB*, *BC* are commensurable with the square on *AB*, {X. 15} and twice the rectangle *AB*, *BC* is commensurable with the rectangle *AB*, *BC*; {X. 6} therefore twice the rectangle *AB*, *BC* is incommensurable with the squares on *AB*, *BC*. {X. 13}

But *DE* is equal to the squares on *AB*, *BC*, and *DH* to twice the rectangle *AB*, *BC*; therefore *DE* is incommensurable with *DH*.

But, as *DE* is to *DH*, so is *GD* to *DF*; {VI. 1} therefore *GD* is incommensurable with *DF*. {X. 11}

And both are rational; therefore *GD*, *DF* are rational straight lines commensurable in square only; therefore *FG* is an apotome. {X. 73}

But *DI* is rational, and the rectangle contained by a rational and an irrational straight line is irrational, {deduction from X. 20} and its ‘side’ is irrational.

And *AC* is the ‘side’ of *FE*; therefore *AC* is irrational.

And let it be called a second apotome of a medial straight line. Q. E. D.

PROPOSITION 76

If from a straight line there be subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the squares on them added together rational, but the rectangle contained by them medial, the remainder is irrational; and let it be called minor.

For from the straight line *AB* let there be subtracted the straight line *BC* which is incommensurable in square with the whole and fulfils the given conditions. {X. 33}

I say that the remainder *AC* is the irrational straight line called minor.

For, since the sum of the squares on *AB*, *BC* is rational, while twice the rectangle *AB*, *BC* is medial, therefore the squares on *AB*, *BC* are incommensurable with twice the rectangle *AB*, *BC*; and, convertendo, the squares on *AB*, *BC* are incommensurable with the remainder, the square on *AC*. {II. 7, X. 16}

But the squares on *AB*, *BC* are rational; therefore the square on *AC* is irrational; therefore *AC* is irrational.

And let it be called minor.

PROPOSITION 77

If from a straight line there be subtracted a straight line which is incommensurable in square with the whole, and which with the whole makes the sum of the squares on them medial, but twice the rectangle contained by them rational, the remainder is irrational: and let it be called that which produces with a rational area a medial whole.

For from the straight line *AB* let there be subtracted the straight line *BC* which is incommensurable in square with *AB* and fulfils the given conditions; {X. 34} I say that the remainder *AC* is the irrational straight line aforesaid.

For, since the sum of the squares on *AB*, *BC* is medial, while twice the rectangle *AB*, *BC* is rational, therefore the squares on *AB*, *BC* are incommensurable with twice the rectangle *AB*, *BC*; therefore the remainder also, the square on *AC*, is incommensurable with twice the rectangle *AB*, *BC*. {II. 7, X. 16}

And twice the rectangle *AB*, *BC* is rational; therefore the square on *AC* is irrational; therefore *AC* is irrational.

And let it be called that which produces with a rational area a medial whole. Q. E. D.

PROPOSITION 78

If from a straight line there be subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the sum of the squares on them medial, twice the rectangle contained by them medial, and further the squares on them incommensurable with twice the rectangle contained by them, the remainder is irrational; and let it be called that which produces with a medial area a medial whole.

For from the straight line *AB* let there be subtracted the straight line *BC* incommensurable in square with *AB* and fulfilling the given conditions; {X. 35} I say that the remainder *AC* is the irrational straight line called that which produces with a medial area a medial whole.

For let a rational straight line *DI* be set out, to *DI* let there be applied *DE* equal to the squares on *AB*, *BC*, producing *DG* as breadth, and let *DH* equal to twice the rectangle *AB*, *BC* be subtracted.

Therefore the remainder *FE* is equal to the square on *AC*, {II. 7} so that *AC* is the "side" of *FE*.

Now, since the sum of the squares on *AB*, *BC* is medial and is equal to *DE*, therefore *DE* is medial.

And it is applied to the rational straight line *DI*, producing *DG* as breadth; therefore *DG* is rational and incommensurable in length with *DI*. {X. 22}

Again, since twice the rectangle *AB*, *BC* is medial and is equal to *DH*, therefore *DH* is medial.

And it is applied to the rational straight line *DI*, producing *DF* as breadth; therefore *DF* is also rational and incommensurable in length with *DI*. {X. 22}

And, since the squares on *AB*, *BC* are incommensurable with twice the rectangle *AB*, *BC*, therefore *DE* is also incommensurable with *DH*.

But, as *DE* is to *DH*, so also is *DG* to *DF*; {VI. 1} therefore *DG* is incommensurable with *DF*. {X. 11}

And both are rational; therefore *GD*, *DF* are rational straight lines commensurable in square only.

Therefore *FG* is an apotome. {X. 73}

And *FH* is rational; but the rectangle contained by a rational straight line and an apotome is irrational, {deduction from X. 20} and its "side" is irrational.

And *AC* is the "side" of *FE*; therefore *AC* is irrational.

And let it be called that which produces with a medial area a medial whole. Q. E. D.

PROPOSITION 79

To an apotome only one rational straight line can be annexed which is commensurable with the whole in square only.

Let *AB* be an apotome, and *BC* an annex to it; therefore *AC*, *CB* are rational straight lines commensurable in square only. {X. 73}

I say that no other rational straight line can be annexed to *AB* which is commensurable with the whole in square only.

For, if possible, let *BD* be so annexed; therefore *AD*, *DB* are also rational straight lines commensurable in square only. {X. 73}

Now, since the excess of the squares on *AD*, *DB* over twice the rectangle *AD*, *DB* is also the excess of the squares on *AC*, *CB* over twice the rectangle *AC*, *CB*, for both exceed by the same, the square on *AB*, {II. 7} therefore, alternately, the excess of the squares on *AD*, *DB* over the squares on *AC*, *CB* is the excess of twice the rectangle *AD*, *DB* over twice the rectangle *AC*, *CB*.

But the squares on *AD*, *DB* exceed the squares on *AC*, *CB* by a rational area, for both are rational; therefore twice the rectangle *AD*, *DB* also exceeds twice the rectangle *AC*, *CB* by a rational area: which is impossible, for both are medial {X. 21}, and a medial area does not exceed a medial by a rational area. {X. 26}

Therefore no other rational straight line can be annexed to *AB* which is commensurable with the whole in square only.

Therefore only one rational straight line can be annexed to an apotome which is commensurable with the whole in square only. Q. E. D.

PROPOSITION 80.

To a first apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a rational rectangle.

For let *AB* be a first apotome of a medial straight line, and let *BC* be an annex to *AB*; therefore *AC*, *CB* are medial straight lines commensurable in

square only and such that the rectangle *AC*, *CB* which they contain is rational; {X. 74} I say that no other medial straight line can be annexed to *AB* which is commensurable with the whole in square only and which contains with the whole a rational area.

For, if possible, let *DB* also be so annexed; therefore *AD*, *DB* are medial straight lines commensurable in square only and such that the rectangle *AD*, *DB* which they contain is rational. {X. 74}

Now, since the excess of the squares on *AD*, *DB* over twice the rectangle *AD*, *DB* is also the excess of the squares on *AC*, *CB* over twice the rectangle *AC*, *CB*, for they exceed by the same, the square on *AB*, {II. 7} therefore, alternately, the excess of the squares on *AD*, *DB* over the squares on *AC*, *CB* is also the excess of twice the rectangle *AD*, *DB* over twice the rectangle *AC*, *CB*.

But twice the rectangle *AD*, *DB* exceeds twice the rectangle *AC*, *CB* by a rational area, for both are rational.

Therefore the squares on *AD*, *DB* also exceed the squares on *AC*, *CB* by a rational area. which is impossible, for both are medial {X. 15 and 23, Por.}, and a medial area does not exceed a medial by a rational area. {X. 26}

Therefore etc. Q. E. D.

PROPOSITION 81.

To a second apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a medial rectangle.

Let *AB* be a second apotome of a medial straight line and *BC* an annex to *AB*; therefore *AC*, *CB* are medial straight lines commensurable in square only and such that the rectangle *AC*, *CB* which they contain is medial. {X. 75}

I say that no other medial straight line can be annexed to *AB* which is commensurable with the whole in square only and which contains with the whole a medial rectangle.

For, if possible, let *BD* also be so annexed; therefore *AD*, *DB* are also medial straight lines commensurable in square only and such that the rectangle *AD*, *DB* which they contain is medial. {X. 75}

Let a rational straight line *EF* be set out, let *EG* equal to the squares on *AC*, *CB* be applied to *EF*, producing *EM* as breadth, and let *HG* equal to twice the rectangle *AC*, *CB* be subtracted, producing *HM* as breadth; therefore the remainder *EL* is equal to the square on *AB*, {II. 7} so that *AB* is the "side" of *EL*.

Again, let *EI* equal to the squares on *AD*, *DB* be applied to *EF*, producing *EN* as breadth.

But *EL* is also equal to the square on *AB*; therefore the remainder *HI* is equal to twice the rectangle *AD*, *DB*. {II. 7}

Now, since *AC*, *CB* are medial straight lines, therefore the squares on *AC*, *CB* are also medial.

And they are equal to *EG*; therefore *EG* is also medial. {X. 15 and 23, Por.}

And it is applied to the rational straight line *EF*, producing *EM* as breadth; therefore *EM* is rational and incommensurable in length with *EF*. {X. 22}

Again, since the rectangle *AC*, *CB* is medial, twice the rectangle *AC*, *CB* is also medial. {X. 23, Por.}

And it is equal to *HG*; therefore *HG* is also medial.

And it is applied to the rational straight line *EF*, producing *HM* as breadth; therefore *HM* is also rational and incommensurable in length with *EF*. {X. 22}

And, since *AC*, *CB* are commensurable in square only, therefore *AC* is incommensurable in length with *CB*.

But, as *AC* is to *CB*, so is the square on *AC* to the rectangle *AC*, *CB*; therefore the square on *AC* is incommensurable with the rectangle *AC*, *CB*. {X. 11}

But the squares on *AC*, *CB* are commensurable with the square on *AC*, while twice the rectangle *AC*, *CB* is commensurable with the rectangle *AC*, *CB*; {X. 6} therefore the squares on *AC*, *CB* are incommensurable with twice the rectangle *AC*, *CB*. {X. 13}

And *EG* is equal to the squares on *AC*, *CB*, while *GH* is equal to twice the rectangle *AC*, *CB*; therefore *EG* is incommensurable with *HG*.

But, as *EG* is to *HG*, so is *EM* to *HM*; {VI. 1} therefore *EM* is incommensurable in length with *MH*. {X. 11}

And both are rational; therefore *EM*, *MH* are rational straight lines commensurable in square only; therefore *EH* is an apotome, and *HM* an annex to it. {X. 73}

Similarly we can prove that *HN* is also an annex to it; therefore to an apotome different straight lines are annexed which are commensurable with the wholes in square only: which is impossible. {X. 79}

Therefore etc. Q. E. D.

PROPOSITION 82.

To a minor straight line only one straight line can be annexed which is incommensurable in square with the whole and which makes, with the whole, the sum of the squares on them rational but twice the rectangle contained by them medial.

Let *AB* be the minor straight line, and let *BC* be an annex to *AB*; therefore *AC*, *CB* are straight lines incommensurable in square which make the sum of the squares on them rational, but twice the rectangle contained by them medial. {X. 76}

I say that no other straight line can be annexed to *AB* fulfilling the same conditions.

For, if possible, let *BD* be so annexed; therefore *AD*, *DB* are also straight lines incommensurable in square which fulfil the aforesaid conditions. {X. 76}

Now, since the excess of the squares on *AD*, *DB* over the squares on *AC*, *CB* is also the excess of twice the rectangle *AD*, *DB* over twice the rectangle *AC*, *CB*, while the squares on *AD*, *DB* exceed the squares on *AC*, *CB* by a rational area, for both are rational, therefore twice the rectangle *AD*, *DB* also exceeds twice the rectangle *AC*, *CB* by a rational area: which is impossible, for both are medial. {X. 26}

Therefore to a minor straight line only one straight line can be annexed which is incommensurable in square with the whole and which makes the squares on them added together rational, but twice the rectangle contained by them medial. Q. E. D.

PROPOSITION 83.

To a straight line which produces with a rational area a medial whole only one straight line can be annexed which is incommensurable in square with the whole straight line and which with the whole straight line makes the sum of the squares on them medial, but twice the rectangle contained by them rational.

Let *AB* be the straight line which produces with a rational area a medial whole, and let *BC* be an annex to *AB*; therefore *AC*, *CB* are straight lines incommensurable in square which fulfil the given conditions. {X. 77}

I say that no other straight line can be annexed to *AB* which fulfils the same conditions.

For, if possible, let *BD* be so annexed; therefore *AD*, *DB* are also straight lines incommensurable in square which fulfil the given conditions. {X. 77}

Since then, as in the preceding cases, the excess of the squares on *AD*, *DB* over the squares on *AC*, *CB* is also the excess of twice the rectangle *AD*, *DB* over twice the rectangle *AC*, *CB*, while twice the rectangle *AD*, *DB* exceeds twice the rectangle *AC*, *CB* by a rational area, for both are rational, therefore the squares on *AD*, *DB* also exceed the squares on *AC*, *CB* by a rational area: which is impossible, for both are medial. {X. 26}

Therefore no other straight line can be annexed to *AB* which is incommensurable in square with the whole and which with the whole fulfils the aforesaid conditions; therefore only one straight line can be so annexed. Q. E. D.

PROPOSITION 84.

To a straight line which produces with a medial area a medial whole only one straight line can be annexed which is incommensurable in square with the whole straight line and which with the whole straight line makes the sum of the squares on them medial and twice the rectangle contained by them both medial and also incommensurable with the sum of the squares on them.

Let *AB* be the straight line which produces with a medial area a medial whole, and *BC* an annex to it; therefore *AC*, *CB* are straight lines incommensurable in square which fulfil the aforesaid conditions. {X. 78}

I say that no other straight line can be annexed to *AB* which fulfils the aforesaid conditions.

For, if possible, let *BD* be so annexed, so that *AD*, *DB* are also straight lines incommensurable in square which make the squares on *AD*, *DB* added together medial, twice the rectangle *AD*, *DB* medial, and also the squares on *AD*, *DB* incommensurable with twice the rectangle *AD*, *DB*. {X. 78}

Let a rational straight line *EF* be set out, let *EG* equal to the squares on *AC*, *CB* be applied to *EF*, producing *EM* as breadth, and let *HG* equal to twice the rectangle *AC*, *CB* be applied to *EF*, producing *HM* as breadth; therefore the remainder, the square on *AB* {II. 7}, is equal to *EL*; therefore *AB* is the "side" of *EL*.

Again, let *EI* equal to the squares on *AD*, *DB* be applied to *EF*, producing *EN* as breadth.

But the square on *AB* is also equal to *EL*; therefore the remainder, twice the rectangle *AD*, *DB* {II. 7}, is equal to *HI*.

Now, since the sum of the squares on *AC*, *CB* is medial and is equal to *EG*, therefore *EG* is also medial.

And it is applied to the rational straight line *EF*, producing *EM* as breadth; therefore *EM* is rational and incommensurable in length with *EF*. {X. 22}

Again, since twice the rectangle *AC*, *CB* is medial and is equal to *HG*, therefore *HG* is also medial.

And it is applied to the rational straight line *EF*, producing *HM* as breadth; therefore *HM* is rational and incommensurable in length with *EF*. {X. 22}

And, since the squares on *AC*, *CB* are incommensurable with twice the rectangle *AC*, *CB*, *EG* is also incommensurable with *HG*; therefore *EM* is also incommensurable in length with *MH*. {VI. 1, X. 11}

And both are rational; therefore *EM*, *MH* are rational straight lines commensurable in square only; therefore *EH* is an apotome, and *HM* an

annex to it. {X. 73}

Similarly we can prove that *EH* is again an apotome and *HN* an annex to it.

Therefore to an apotome different rational straight lines are annexed which are commensurable with the wholes in square only: which was proved impossible. {X. 79}

Therefore no other straight line can be so annexed to *AB*.

Therefore to *AB* only one straight line can be annexed which is incommensurable in square with the whole and which with the whole makes the squares on them added together medial, twice the rectangle contained by them medial, and also the squares on them incommensurable with twice the rectangle contained by them. Q. E. D.

DEFINITIONS III.

1

Given a rational straight line and an apotome, if the square on the whole be greater than the square on the annex by the square on a straight line commensurable in length with the whole, and the whole be commensurable in length with the rational straight line set out, let the apotome be called a first apotome.

2

But if the annex be commensurable in length with the rational straight line set out, and the square on the whole be greater than that on the annex by the square on a straight line commensurable with the whole, let the apotome be called a second apotome.

3

But if neither be commensurable in length with the rational straight line set out, and the square on the whole be greater than the square on the annex by the square on a straight line commensurable with the whole, let the apotome be called a third apotome.

4

Again, if the square on the whole be greater than the square on the annex by the square on a straight line incommensurable with the whole, then, if the whole be commensurable in length with the rational straight line set out, let the apotome be called a fourth apotome;

5

if the annex be so commensurable, a fifth;

6

and, if neither, a sixth.

PROPOSITIONS 85-115.

PROPOSITION 85.

To find the first apotome.

Let a rational straight line *A* be set out, and let *BG* be commensurable in length with *A*; therefore *BG* is also rational.

Let two square numbers *DE*, *EF* be set out, and let their difference *FD* not be square; therefore neither has *ED* to *DF* the ratio which a square number has to a square number.

Let it be contrived that, as *ED* is to *DF*, so is the square on *BG* to the square on *GC*; {X. 6, Por.} therefore the square on *BG* is commensurable with the square on *GC*. {X. 6}

But the square on *BG* is rational; therefore the square on *GC* is also rational; therefore *GC* is also rational.

And, since *ED* has not to *DF* the ratio which a square number has to a square number, therefore neither has the square on *BG* to the square on *GC* the ratio which a square number has to a square number; therefore *BG* is incommensurable in length with *GC*. {X. 9}

And both are rational; therefore *BG*, *GC* are rational straight lines commensurable in square only; therefore *BC* is an apotome. {X. 73}

I say next that it is also a first apotome.

For let the square on *H* be that by which the square on *BG* is greater than the square on *GC*.

Now since. as *ED* is to *FD*, so is the square on *BG* to the square on *GC*, therefore also, convertendo, {v. 19, Por.} as *DE* is to *EF*, so is the square on *GB* to the square on *H*.

But *DE* has to *EF* the ratio which a square number has to a square number, for each is square; therefore the square on *GB* also has to the square on *H* the

ratio which a square number has to a square number; therefore *BG* is commensurable in length with *H*. {X. 9}

And the square on *BG* is greater than the square on *GC* by the square on a straight line commensurable in length with *BG*.

And the whole *BG* is commensurable in length with the rational straight line *A* set out.

Therefore *BC* is a first apotome. {X. Deff. III. 1}

Therefore the first apotome *BC* has been found.

(Being) that which it was required to find.

PROPOSITION 86.

To find the second apotome.

Let a rational straight line *A* be set out, and *GC* commensurable in length with *A*; therefore *GC* is rational.

Let two square numbers *DE*, *EF* be set out, and let their difference *DF* not be square.

Now let it be contrived that, as *FD* is to *DE*, so is the square on *CG* to the square on *GB*. {X. 6, Por.}

Therefore the square on *CG* is commensurable with the square on *GB*. {X. 6}

But the square on *CG* is rational; therefore the square on *GB* is also rational; therefore *BG* is rational.

And, since the square on *GC* has not to the square on *GB* the ratio which a square number has to a square number, *CG* is incommensurable in length with *GB*. {X. 9}

And both are rational; therefore *CG*, *GB* are rational straight lines commensurable in square only; therefore *BC* is an apotome. {X. 73}

I say next that it is also a second apotome.

For let the square on *H* be that by which the square on *BG* is greater than the square on *GC*.

Since then, as the square on *BG* is to the square on *GC*, so is the number *ED* to the number *DF*, therefore, convertendo, as the square on *BG* is to the square on *H*, so is *DE* to *EF*. {V. 19, Por.}

And each of the numbers *DE*, *EF* is square; therefore the square on *BG* has to the square on *H* the ratio which a square number has to a square number; therefore *BG* is commensurable in length with *H*. {X. 9}

And the square on *BG* is greater than the square on *GC* by the square on *H*; therefore the square on *BG* is greater than the square on *GC* by the square on a straight line commensurable in length with *BG*.

And *CG*, the annex, is commensurable with the rational straight line *A* set out.

Therefore *BC* is a second apotome. {X. Deff. III. 2}

Therefore the second apotome *BC* has been found. Q. E. D.

PROPOSITION 87.

To find the third apotome.

Let a rational straight line *A* be set out, let three numbers *E*, *BC*, *CD* be set out which have not to one another the ratio which a square number has to a square number, but let *CB* have to *BD* the ratio which a square number has to a square number.

Let it be contrived that, as *E* is to *BC*, so is the square on *A* to the square on *FG*, and, as *BC* is to *CD*, so is the square on *FG* to the square on *GH*. {X. 6, Por.}

Since then, as *E* is to *BC*, so is the square on *A* to the square on *FG*, therefore the square on *A* is commensurable with the square on *FG*. {X. 6}

But the square on *A* is rational; therefore the square on *FG* is also rational; therefore *FG* is rational.

And, since *E* has not to *BC* the ratio which a square number has to a square number, therefore neither has the square on *A* to the square on *FG* the ratio which a square number has to a square number; therefore *A* is incommensurable in length with *FG*. {X. 9}

Again, since, as *BC* is to *CD*, so is the square on *FG* to the square on *GH*, therefore the square on *FG* is commensurable with the square on *GH*. {X. 6}

But the square on *FG* is rational; therefore the square on *GH* is also rational; therefore *GH* is rational.

And, since *BC* has not to *CD* the ratio which a square number has to a square number, therefore neither has the square on *FG* to the square on *GH* the ratio which a square number has to a square number; therefore *FG* is incommensurable in length with *GH*. {X. 9}

And both are rational; therefore *FG*, *GH* are rational straight lines commensurable in square only; therefore *FH* is an apotome. {X. 73}

I say next that it is also a third apotome.

For since, as *E* is to *BC*, so is the square on *A* to the square on *FG*, and, as *BC* is to *CD*, so is the square on *FG* to the square on *HG*, therefore, *ex aequali*, as *E* is to *CD*, so is the square on *A* to the square on *HG*. {V. 22}

But *E* has not to *CD* the ratio which a square number has to a square number; therefore neither has the square on *A* to the square on *GH* the ratio which a square number has to a square number; therefore *A* is incommensurable in length with *GH*. {X. 9}

Therefore neither of the straight lines *FG*, *GH* is commensurable in length with the rational straight line *A* set out.

Now let the square on *K* be that by which the square on *FG* is greater than the square on *GH*.

Since then, as *BC* is to *CD*, so is the square on *FG* to the square on *GH*, therefore, convertendo, as *BC* is to *BD*, so is the square on *FG* to the square on *K*. {V. 19, Por.}

But *BC* has to *BD* the ratio which a square number has to a square number; therefore the square on *FG* also has to the square on *K* the ratio which a square number has to a square number.

Therefore *FG* is commensurable in length with *K*, {X. 9} and the square on *FG* is greater than the square on *GH* by the square on a straight line commensurable with *FG*.

And neither of the straight lines *FG*, *GH* is commensurable in length with the rational straight line *A* set out; therefore *FH* is a third apotome. {X. Deff. III. 3}

Therefore the third apotome *FH* has been found. Q. E. D.

PROPOSITION 88.

To find the fourth apotome.

Let a rational straight line *A* be set out, and *BG* commensurable in length with it; therefore *BG* is also rational.

Let two numbers *DF*, *FE* be set out such that the whole *DE* has not to either of the numbers *DF*, *EF* the ratio which a square number has to a square number.

Let it be contrived that, as *DE* is to *EF*, so is the square on *BG* to the square on *GC*; {X. 6, Por.} therefore the square on *BG* is commensurable with the square on *GC*. {X. 6}

But the square on *BG* is rational; therefore the square on *GC* is also rational; therefore *GC* is rational.

Now, since *DE* has not to *EF* the ratio which a square number has to a square number, therefore neither has the square on *BG* to the square on *GC* the ratio which a square number has to a square number; therefore *BG* is incommensurable in length with *GC*. {X. 9}

And both are rational; therefore *BG*, *GC* are rational straight lines commensurable in square only; therefore *BC* is an apotome. {X. 73}

Now let the square on *H* be that by which the square on *BG* is greater than the square on *GC*.

Since then, as *DE* is to *EF*, so is the square on *BG* to the square on *GC*, therefore also, convertendo, as *ED* is to *DF*, so is the square on *GB* to the square on *H*. {v. 19, Por.}

But *ED* has not to *DF* the ratio which a square number has to a square number; therefore neither has the square on *GB* to the square on *H* the ratio which a square number has to a square number; therefore *BG* is incommensurable in length with *H*. {X. 9}

And the square on *BG* is greater than the square on *GC* by the square on *H*; therefore the square on *BG* is greater than the square on *GC* by the square on a straight line incommensurable with *BG*.

And the whole *BG* is commensurable in length with the rational straight line *A* set out.

Therefore *BC* is a fourth apotome. {X. Deff. III. 4}

Therefore the fourth apotome has been found. Q. E. D.

PROPOSITION 89.

To find the fifth apotome.

Let a rational straight line *A* be set out, and let *CG* be commensurable in length with *A*; therefore *CG* is rational.

Let two numbers *DF*, *FE* be set out such that *DE* again has not to either of the numbers *DF*, *FE* the ratio which a square number has to a square number; and let it be contrived that, as *FE* is to *ED*, so is the square on *CG* to the square on *GB*.

Therefore the square on *GB* is also rational; {X. 6} therefore *BG* is also rational.

Now since, as *DE* is to *EF*, so is the square on *BG* to the square on *GC*, while *DE* has not to *EF* the ratio which a square number has to a square number, therefore neither has the square on *BG* to the square on *GC* the ratio which a square number has to a square number; therefore *BG* is incommensurable in length with *GC*. {X. 9}

And both are rational; therefore *BG*, *GC* are rational straight lines commensurable in square only; therefore *BC* is an apotome. {X. 73}

I say next that it is also a fifth apotome.

For let the square on *H* be that by which the square on *BG* is greater than the square on *GC*.

Since then, as the square on *BG* is to the square on *GC*, so is *DE* to *EF*, therefore, convertendo, as *ED* is to *DF*, so is the square on *BG* to the square on *H*. {V. 19, Por.}

But *ED* has not to *DF* the ratio which a square number has to a square number; therefore neither has the square on *BG* to the square on *H* the ratio which a square number has to a square number; therefore *BG* is incommensurable in length with *H*. {X. 9}

And the square on *BG* is greater than the square on *GC* by the square on *H*; therefore the square on *GB* is greater than the square on *GC* by the square on a straight line incommensurable in length with *GB*.

And the annex *CG* is commensurable in length with the rational straight line *A* set out; therefore *BC* is a fifth apotome. {X. Deff. III. 5}

Therefore the fifth apotome *BC* has been found. Q. E. D.

PROPOSITION 90.

To find the sixth apotome.

Let a rational straight line *A* be set out, and three numbers *E*, *BC*, *CD* not having to one another the ratio which a square number has to a square number; and further let *CB* also not have to *BD* the ratio which a square number has to a square number.

Let it be contrived that, as *E* is to *BC*, so is the square on *A* to the square on *FG*, and, as *BC* is to *CD*, so is the square on *FG* to the square on *GH*. {X. 6, Por.}

Now since, as *E* is to *BC*, so is the square on *A* to the square on *FG*, therefore the square on *A* is commensurable with the square on *FG*. {X. 6}

But the square on *A* is rational; therefore the square on *FG* is also rational; therefore *FG* is also rational.

And, since *E* has not to *BC* the ratio which a square number has to a square number, therefore neither has the square on *A* to the square on *FG* the ratio which a square number has to a square number; therefore *A* is incommensurable in length with *FG*. {X. 9}

Again, since, as *BC* is to *CD*, so is the square on *FG* to the square on *GH*, therefore the square on *FG* is commensurable with the square on *GH*. {X. 6}

But the square on *FG* is rational; therefore the square on *GH* is also rational; therefore *GH* is also rational.

And, since *BC* has not to *CD* the ratio which a square number has to a square number, therefore neither has the square on *FG* to the square on *GH* the ratio which a square number has to a square number; therefore *FG* is incommensurable in length with *GH*. {X. 9}

And both are rational; therefore *FG*, *GH* are rational straight lines commensurable in square only; therefore *FH* is an apotome. {X. 73}

I say next that it is also a sixth apotome.

For since, as *E* is to *BC*, so is the square on *A* to the square on *FG*, and, as *BC* is to *CD*, so is the square on *FG* to the square on *GH*, therefore, ex aequali, as *E* is to *CD*, so is the square on *A* to the square on *GH*. {v. 22}

But *E* has not to *CD* the ratio which a square number has to a square number; therefore neither has the square on *A* to the square on *GH* the ratio which a square number has to a square number; therefore *A* is incommensurable in length with *GH*; {X. 9} therefore neither of the straight lines *FG*, *GH* is commensurable in length with the rational straight line *A*.

Now let the square on *K* be that by which the square on *FG* is greater than the square on *GH*.

Since then, as *BC* is to *CD*, so is the square on *FG* to the square on *GH*, therefore, convertendo, as *CB* is to *BD*, so is the square on *FG* to the square on *K*. {v. 19, Por.}

But *CB* has not to *BD* the ratio which a square number has to a square number; therefore neither has the square on *FG* to the square on *K* the ratio which a square number has to a square number; therefore *FG* is incommensurable in length with *K*. {X. 9}

And the square on *FG* is greater than the square on *GH* by the square on *K*; therefore the square on *FG* is greater than the square on *GH* by the square on a straight line incommensurable in length with *FG*.

And neither of the straight lines *FG*, *GH* is commensurable with the rational straight line *A* set out.

Therefore *FH* is a sixth apotome. {X. Deff. III. 6}

Therefore the sixth apotome *FH* has been found. Q. E. D.

PROPOSITION 91.

If an area be contained by a rational straight line and a first apotome, the "side" of the area is an apotome.

For let the area *AB* be contained by the rational straight line *AC* and the first apotome *AD*;

I say that the "side" of the area *AB* is an apotome.

For, since *AD* is a first apotome, let *DG* be its annex; therefore *AG*, *GD* are rational straight lines commensurable in square only. {X. 73}

And the whole *AG* is commensurable with the rational straight line *AC* set out, and the square on *AG* is greater than the square on *GD* by the square on a straight line commensurable in length with *AG*; {X. Deff. III. 1} if therefore there be applied to *AG* a parallelogram equal to the fourth part of the square on *DG* and deficient by a square figure, it divides it into commensurable parts. {X. 17}

Let *DG* be bisected at *E*, let there be applied to *AG* a parallelogram equal to the square on *EG* and deficient by a square figure, and let it be the rectangle *AF*, *FG*; therefore *AF* is commensurable with *FG*.

And through the points *E*, *F*, *G* let *EH*, *FI*, *GK* be drawn parallel to *AC*.

Now, since *AF* is commensurable in length with *FG*, therefore *AG* is also commensurable in length with each of the straight lines *AF*, *FG*. {X. 15}

But *AG* is commensurable with *AC*; therefore each of the straight lines *AF*, *FG* is commensurable in length with *AC*. {X. 12}

And *AC* is rational; therefore each of the straight lines *AF*, *FG* is also rational, so that each of the rectangles *AI*, *FK* is also rational. {X. 19}

Now, since *DE* is commensurable in length with *EG*, therefore *DG* is also commensurable in length with each of the straight lines *DE*, *EG*. {X. 15}

But *DG* is rational and incommensurable in length with *AC*; therefore each of the straight lines *DE*, *EG* is also rational and incommensurable in length

with *AC*; {X. 13} therefore each of the rectangles *DH*, *EK* is medial. {X. 21}

Now let the square *LM* be made equal to *AI*, and let there be subtracted the square *NO* having a common angle with it, the angle *LPM*, and equal to *FK*; therefore the squares *LM*, *NO* are about the same diameter. {VI. 26}

Let *PR* be their diameter, and let the figure be drawn.

Since then the rectangle contained by *AF*, *FG* is equal to the square on *EG*, therefore, as *AF* is to *EG*, so is *EG* to *FG*. {VI. 17}

But, as *AF* is to *EG*, so is *AI* to *EK*, and, as *EG* is to *FG*, so is *EK* to *KF*; {VI. 1} therefore *EK* is a mean proportional between *AI*, *KF*. {V. 11}

But *MN* is also a mean proportional between *LM*, *NO*, as was before proved, {Lemma after X. 53} and *AI* is equal to the square *LM*, and *KF* to *NO*; therefore *MN* is also equal to *EK*.

But *EK* is equal to *DH*, and *MN* to *LO*; therefore *DK* is equal to the gnomon *UVW* and *NO*.

But *AK* is also equal to the squares *LM*, *NO*; therefore the remainder *AB* is equal to *ST*.

But *ST* is the square on *LN*; therefore the square on *LN* is equal to *AB*; therefore *LN* is the "side" of *AB*.

I say next that *LN* is an apotome.

For, since each of the rectangles *AI*, *FK* is rational, and they are equal to *LM*, *NO*, therefore each of the squares *LM*, *NO*, that is, the squares on *LP*, *PN* respectively, is also rational; therefore each of the straight lines *LP*, *PN* is also rational.

Again, since *DH* is medial and is equal to *LO*, therefore *LO* is also medial.

Since then *LO* is medial, while *NO* is rational, therefore *LO* is incommensurable with *NO*.

But, as *LO* is to *NO*, so is *LP* to *PN*; {VI. 1} therefore *LP* is incommensurable in length with *PN*. {X. 11}

And both are rational; therefore *LP*, *PN* are rational straight lines commensurable in square only; therefore *LN* is an apotome. {X. 73}

And it is the "side" of the area *AB*; therefore the "side" of the area *AB* is an apotome.

Therefore etc.

PROPOSITION 92.

If an area be contained by a rational straight line and a second apotome, the "side" of the area is a first apotome of a medial straight line.

For let the area *AB* be contained by the rational straight line *AC* and the second apotome *AD*; I say that the "side" of the area *AB* is a first apotome of a medial straight line.

For let *DG* be the annex to *AD*; therefore *AG*, *GD* are rational straight lines commensurable in square only, {X. 73} and the annex *DG* is commensurable with the rational straight line *AC* set out, while the square on the whole *AG* is greater than the square on the annex *GD* by the square on a straight line commensurable in length with *AG*. {X. Deff. III. 2}

Since then the square on *AG* is greater than the square on *GD* by the square on a straight line commensurable with *AG*, therefore, if there be applied to *AG* a parallelogram equal to the fourth part of the square on *GD* and deficient by a square figure, it divides it into commensurable parts. {X. 17}

Let then *DG* be bisected at *E*, let there be applied to *AG* a parallelogram equal to the square on *EG* and deficient by a square figure, and let it be the rectangle *AF*, *FG*; therefore *AF* is commensurable in length with *FG*.

Therefore *AG* is also commensurable in length with each of the straight lines *AF*, *FG*. {X. 15}

But *AG* is rational and incommensurable in length with *AC*; therefore each of the straight lines *AF*, *FG* is also rational and incommensurable in length with *AC*; {X. 13} therefore each of the rectangles *AI*, *FK* is medial. {X. 21}

Again, since *DE* is commensurable with *EG*, therefore *DG* is also commensurable with each of the straight lines *DE*, *EG*. {X. 15}

But *DG* is commensurable in length with *AC*.

Therefore each of the rectangles *DH*, *EK* is rational. {X. 19}

Let then the square *LM* be constructed equal to *AI*, and let there be subtracted *NO* equal to *FK* and being about the same angle with *LM*, namely the angle *LPM*; therefore the squares *LM*, *NO* are about the same diameter. {VI. 26}

Let *PR* be their diameter, and let the figure be drawn.

Since then *AI*, *FK* are medial and are equal to the squares on *LP*, *PN*, the squares on *LP*, *PN* are also medial; therefore *LP*, *PN* are also medial straight lines commensurable in square only.

And, since the rectangle *AF*, *FG* is equal to the square on *EG*, therefore, as *AF* is to *EG*, so is *EG* to *FG*, {VI. 17} while, as *AF* is to *EG*, so is *AI* to *EK*, and, as *EG* is to *FG*, so is *EK* to *FK*; {VI. 1} therefore *EK* is a mean proportional between *AI*, *FK*. {V. 11}

But *MN* is also a mean proportional between the squares *LM*, *NO*, and *AI* is equal to *LM*, and *FK* to *NO*; therefore *MN* is also equal to *EK*.

But *DH* is equal to *EK*, and *LO* equal to *MN*; therefore the whole *DK* is equal to the gnomon *UVW* and *NO*.

Since then the whole *AK* is equal to *LM*, *NO*, and, in these, *DK* is equal to the gnomon *UVW* and *NO*, therefore the remainder *AB* is equal to *TS*.

But *TS* is the square on *LN*; therefore the square on *LN* is equal to the area *AB*; therefore *LN* is the “side” of the area *AB*.

I say that *LN* is a first apotome of a medial straight line.

For, since *EK* is rational and is equal to *LO*, therefore *LO*, that is, the rectangle *LP*, *PN*, is rational.

But *NO* was proved medial; therefore *LO* is incommensurable with *NO*.

But, as *LO* is to *NO*, so is *LP* to *PN*; {VI. 1} therefore *LP*, *PN* are incommensurable in length. {X. 11}

Therefore *LP*, *PN* are medial straight lines commensurable in square only which contain a rational rectangle; therefore *LN* is a first apotome of a medial straight line. {X. 74}

And it is the "side" of the area *AB*.

Therefore the "side" of the area *AB* is a first apotome of a medial straight line. Q. E. D.

PROPOSITION 93.

If an area be contained by a rational straight line and a third apotome, the "side" of the area is a second apotome of a medial straight line.

For let the area *AB* be contained by the rational straight line *AC* and the third apotome *AD*; I say that the "side" of the area *AB* is a second apotome of a medial straight line.

For let *DG* be the annex to *AD*; therefore *AG*, *GD* are rational straight lines commensurable in square only, and neither of the straight lines *AG*, *GD* is commensurable in length with the rational straight line *AC* set out, while the square on the whole *AG* is greater than the square on the annex *DG* by the square on a straight line commensurable with *AG*. {X. Deff. III. 3}

Since then the square on *AG* is greater than the square on *GD* by the square on a straight line commensurable with *AG*, therefore, if there be applied to *AG* a parallelogram equal to the fourth part of the square on *DG*

and deficient by a square figure, it will divide it into commensurable parts. {X. 17}

Let then *DG* be bisected at *E*, let there be applied to *AG* a parallelogram equal to the square on *EG* and deficient by a square figure, and let it be the rectangle *AF*, *FG*.

Let *EH*, *FI*, *GK* be drawn through the points *E*, *F*, *G* parallel to *AC*.

Therefore *AF*, *FG* are commensurable; therefore *AI* is also commensurable with *FK*. {VI. 1, X. 11}

And, since *AF*, *FG* are commensurable in length, therefore *AG* is also commensurable in length with each of the straight lines *AF*, *FG*. {X. 15}

But *AG* is rational and incommensurable in length with *AC*; so that *AF*, *FG* are so also. {X. 13}

Therefore each of the rectangles *AI*, *FK* is medial. {X. 21}

Again, since *DE* is commensurable in length with *EG*, therefore *DG* is also commensurable in length with each of the straight lines *DE*, *EG*. {X. 15}

But *GD* is rational and incommensurable in length with *AC*; therefore each of the straight lines *DE*, *EG* is also rational and incommensurable in length with *AC*; {X. 13} therefore each of the rectangles *DH*, *EK* is medial. {X. 21}

And, since *AG*, *GD* are commensurable in square only, therefore *AG* is incommensurable in length with *GD*.

But *AG* is commensurable in length with *AF*, and *DG* with *EG*; therefore *AF* is incommensurable in length with *EG*. {X. 13}

But, as *AF* is to *EG*, so is *AI* to *EK*; {VI. 1} therefore *AI* is incommensurable with *EK*. {X. 11}

Now let the square *LM* be constructed equal to *AI*, and let there be subtracted *NO* equal to *FK* and being about the same angle with *LM*; therefore *LM*, *NO* are about the same diameter. {VI. 26}

Let *PR* be their diameter, and let the figure be drawn.

Now, since the rectangle *AF*, *FG* is equal to the square on *EG*, therefore, as *AF* is to *EG*, so is *EG* to *FG*. {VI. 17}

But, as *AF* is to *EG*, so is *AI* to *EK*, and, as *EG* is to *FG*, so is *EK* to *FK*; {VI. 1} therefore also, as *AI* is to *EK*, so is *EK* to *FK*; {V. 11} therefore *EK* is a mean proportional between *AI*, *FK*.

But *MN* is also a mean proportional between the squares *LM*, *NO*, and *AI* is equal to *LM*, and *FK* to *NO*; therefore *EK* is also equal to *MN*.

But *MN* is equal to *LO*, and *EK* equal to *DH*; therefore the whole *DK* is also equal to the gnomon *UVW* and *NO*.

But *AK* is also equal to *LM*, *NO*; therefore the remainder *AB* is equal to *ST*, that is, to the square on *LN*; therefore *LN* is the "side" of the area *AB*.

I say that *LN* is a second apotome of a medial straight line.

For, since *AI*, *FK* were proved medial, and are equal to the squares on *LP*, *PN*, therefore each of the squares on *LP*, *PN* is also medial; therefore each of the straight lines *LP*, *PN* is medial.

And, since *AI* is commensurable with *FK*, {VI. 1, X. 11} therefore the square on *LP* is also commensurable with the square on *PN*.

Again, since *AI* was proved incommensurable with *EK*, therefore *LM* is also incommensurable with *MN*, that is, the square on *LP* with the rectangle *LP*, *PN*; so that *LP* is also incommensurable in length with *PN*; {VI. 1, X. 11} therefore *LP*, *PN* are medial straight lines commensurable in square only.

I say next that they also contain a medial rectangle.

For, since *EK* was proved medial, and is equal to the rectangle *LP*, *PN*, therefore the rectangle *LP*, *PN* is also medial, so that *LP*, *PN* are medial straight lines commensurable in square only which contain a medial rectangle.

Therefore *LN* is a second apotome of a medial straight line; {X. 75} and it is the "side" of the area *AB*.

Therefore the "side" of the area *AB* is a second apotome of a medial straight line. Q. E. D.

PROPOSITION 94.

If an area be contained by a rational straight line and a fourth apotome, the "side" of the area is minor.

For let the area *AB* be contained by the rational straight line *AC* and the fourth apotome *AD*; I say that the "side" of the area *AB* is minor.

For let *DG* be the annex to *AD*; therefore *AG*, *GD* are rational straight lines commensurable in square only, *AG* is commensurable in length with the rational straight line *AC* set out, and the square on the whole *AG* is greater than the square on the annex *DG* by the square on a straight line incommensurable in length with *AG*, {X. Deff. III. 4}

Since then the square on *AG* is greater than the square on *GD* by the square on a straight line incommensurable in length with *AG*, therefore, if there be applied to *AG* a parallelogram equal to the fourth part of the square on *DG* and deficient by a square figure, it will divide it into incommensurable parts. {X. 18}

Let then *DG* be bisected at *E*, let there be applied to *AG* a parallelogram equal to the square on *EG* and deficient by a square figure, and let it be the rectangle *AF*, *FG*; therefore *AF* is incommensurable in length with *FG*.

Let *EH*, *FI*, *GK* be drawn through *E*, *F*, *G* parallel to *AC*, *BD*.

Since then *AG* is rational and commensurable in length with *AC*, therefore the whole *AK* is rational. {X. 19}

Again, since *DG* is incommensurable in length with *AC*, and both are rational, therefore *DK* is medial. {X. 21}

Again, since *AF* is incommensurable in length with *FG*, therefore *AI* is also incommensurable with *FK*. {VI. 1, X. 11}

Now let the square *LM* be constructed equal to *AI*, and let there be subtracted *NO* equal to *FK* and about the same angle, the angle *LPM*.

Therefore the squares *LM*, *NO* are about the same diameter. {VI. 26}

Let *PR* be their diameter, and let the figure be drawn.

Since then the rectangle *AF*, *FG* is equal to the square on *EG*, therefore, proportionally, as *AF* is to *EG*, so is *EG* to *FG*. {VI. 17}

But, as *AF* is to *EG*, so is *AI* to *EK*, and, as *EG* is to *FG*, so is *EK* to *FK*; {VI. 1} therefore *EK* is a mean proportional between *AI*, *FK*. {V. 11}

But *MN* is also a mean proportional between the squares *LM*, *NO*, and *AI* is equal to *LM*, and *FK* to *NO*; therefore *EK* is also equal to *MN*.

But *DH* is equal to *EK*, and *LO* is equal to *MN*; therefore the whole *DK* is equal to the gnomon *UVW* and *NO*.

Since, then, the whole *AK* is equal to the squares *LM*, *NO*, and, in these, *DK* is equal to the gnomon *UVW* and the square *NO*, therefore the remainder *AB* is equal to *ST*, that is, to the square on *LN*; therefore *LN* is the "side" of the area *AB*.

I say that *LN* is the irrational straight line called minor.

For, since *AK* is rational and is equal to the squares on *LP*, *PN*, therefore the sum of the squares on *LP*, *PN* is rational.

Again, since *DK* is medial, and *DK* is equal to twice the rectangle *LP*, *PN*, therefore twice the rectangle *LP*, *PN* is medial.

And, since *AI* was proved incommensurable with *FK*, therefore the square on *LP* is also incommensurable with the square on *PN*.

Therefore *LP*, *PN* are straight lines incommensurable in square which make the sum of the squares on them rational, but twice the rectangle contained by them medial.

Therefore *LN* is the irrational straight line called minor; {X. 76} and it is the "side" of the area *AB*.

Therefore the "side" of the area *AB* is minor. Q. E. D.

PROPOSITION 95.

If an area be contained by a rational straight line and a fifth apotome, the "side" of the area is a straight line which produces with a rational area a medial whole.

For let the area *AB* be contained by the rational straight line *AC* and the fifth apotome *AD*; I say that the "side" of the area *AB* is a straight line which produces with a rational area a medial whole.

For let *DG* be the annex to *AD*; therefore *AG*, *GD* are rational straight lines commensurable in square only, the annex *GD* is commensurable in length with the rational straight line *AC* set out, and the square on the whole *AG* is greater than the square on the annex *DG* by the square on a straight line incommensurable with *AG*. {X. Deff. III. 5}

Therefore, if there be applied to *AG* a parallelogram equal to the fourth part of the square on *DG* and deficient by a square figure, it will divide it into incommensurable parts. {X. 18}

Let then *DG* be bisected at the point *E*, let there be applied to *AG* a parallelogram equal to the square on *EG* and deficient by a square figure, and let it be the rectangle *AF*, *FG*; therefore *AF* is incommensurable in length with *FG*.

Now, since *AG* is incommensurable in length with *CA*, and both are rational, therefore *AK* is medial. {X. 21}

Again, since *DG* is rational and commensurable in length with *AC*, *DK* is rational. {X. 19}

Now let the square *LM* be constructed equal to *AI*, and let the square *NO* equal to *FK* and about the same angle, the angle *LPM*, be subtracted; therefore the squares *LM*, *NO* are about the same diameter. {VI. 26}

Let *PR* be their diameter, and let the figure be drawn.

Similarly then we can prove that *LN* is the "side" of the area *AB*.

I say that *LN* is the straight line which produces with a rational area a medial whole.

For, since *AK* was proved medial and is equal to the squares on *LP*, *PN*, therefore the sum of the squares on *LP*, *PN* is medial.

Again, since *DK* is rational and is equal to twice the rectangle *LP*, *PN*, the latter is itself also rational.

And, since *AI* is incommensurable with *FK*, therefore the square on *LP* is also incommensurable with the square on *PN*; therefore *LP*, *PN* are straight lines incommensurable in square which make the sum of the squares on them medial but twice the rectangle contained by them rational.

Therefore the remainder *LN* is the irrational straight line called that which produces with a rational area a medial whole; {X. 77} and it is the "side" of the area *AB*.

Therefore the "side" of the area *AB* is a straight line which produces with a rational area a medial whole. Q. E. D.

PROPOSITION 96.

If an area be contained by a rational straight line and a sixth apotome, the "side" of the area is a straight line which produces with a medial area a medial whole.

For let the area *AB* be contained by the rational straight line *AC* and the sixth apotome *AD*; I say that the "side" of the area *AB* is a straight line which

produces with a medial area a medial whole.

For let *DG* be the annex to *AD*; therefore *AG*, *GD* are rational straight lines commensurable in square only, neither of them is commensurable in length with the rational straight line *AC* set out, and the square on the whole *AG* is greater than the square on the annex *DG* by the square on a straight line incommensurable in length with *AG*. {X. Deff. III. 6}

Since then the square on *AG* is greater than the square on *GD* by the square on a straight line incommensurable in length with *AG*, therefore, if there be applied to *AG* a parallelogram equal to the fourth part of the square on *DG* and deficient by a square figure, it will divide it into incommensurable parts. {X. 18}

Let then *DG* be bisected at *E*, let there be applied to *AG* a parallelogram equal to the square on *EG* and deficient by a square figure, and let it be the rectangle *AF*, *FG*; therefore *AF* is incommensurable in length with *FG*.

But, as *AF* is to *FG*, so is *AI* to *FK*. {VI. 1} therefore *AI* is incommensurable with *FK*. {X. 11}

And, since *AG*, *AC* are rational straight lines commensurable in square only, *AK* is medial. {X. 21}

Again, since *AC*, *DG* are rational straight lines and incommensurable in length, *DK* is also medial. {X. 21}

Now, since *AG*, *GD* are commensurable in square only, therefore *AG* is incommensurable in length with *GD*.

But, as *AG* is to *GD*, so is *AK* to *KD*; {VI. 1} therefore *AK* is incommensurable with *KD*. {X. 11}

Now let the square *LM* be constructed equal to *AI*, and let *NO* equal to *FK*, and about the same angle, be subtracted; therefore the squares *LM*, *NO* are about the same diameter. {VI. 26}

Let *PR* be their diameter, and let the figure be drawn.

Then in manner similar to the above we can prove that *LN* is the "side" of the area *AB*.

I say that *LN* is a straight line which produces with a medial area a medial whole.

For, since *AK* was proved medial and is equal to the squares on *LP*, *PN*, therefore the sum of the squares on *LP*, *PN* is medial.

Again, since *DK* was proved medial and is equal to twice the rectangle *LP*, *PN*, twice the rectangle *LP*, *PN* is also medial.

And, since *AK* was proved incommensurable with *DK*, the squares on *LP*, *PN* are also incommensurable with twice the rectangle *LP*, *PN*.

And, since *AI* is incommensurable with *FK*, therefore the square on *LP* is also incommensurable with the square on *PN*; therefore *LP*, *PN* are straight lines incommensurable in square which make the sum of the squares on them medial, twice the rectangle contained by them medial, and further the squares on them incommensurable with twice the rectangle contained by them.

Therefore *LN* is the irrational straight line called that which produces with a medial area a medial whole; {X. 78} and it is the "side" of the area *AB*.

Therefore the "side" of the area is a straight line which produces with a medial area a medial whole. Q. E. D.

PROPOSITION 97.

The square on an apotome applied to a rational straight line produces as breadth a first apotome.

Let *AB* be an apotome, and *CD* rational, and to *CD* let there be applied *CE* equal to the square on *AB* and producing *CF* as breadth; I say that *CF* is a first apotome.

For let *BG* be the annex to *AB*; therefore *AG*, *GB* are rational straight lines commensurable in square only. {X. 73}

To *CD* let there be applied *CH* equal to the square on *AG*, and *KL* equal to the square on *BG*.

Therefore the whole *CL* is equal to the squares on *AG*, *GB*, and, in these, *CE* is equal to the square on *AB*; therefore the remainder *FL* is equal to twice the rectangle *AG*, *GB*. {II. 7}

Let *FM* be bisected at the point *N*, and let *NO* be drawn through *N* parallel to *CD*; therefore each of the rectangles *FO*, *LN* is equal to the rectangle *AG*, *GB*.

Now, since the squares on *AG*, *GB* are rational, and *DM* is equal to the squares on *AG*, *GB*,. therefore *DM* is rational.

And it has been applied to the rational straight line *CD*, producing *CM* as breadth; therefore *CM* is rational and commensurable in length with *CD*. {X. 20}

Again, since twice the rectangle *AG*, *GB* is medial, and *FL* is equal to twice the rectangle *AG*, *GB*, therefore *FL* is medial.

And it is applied to the rational straight line *CD*, producing *FM* as breadth; therefore *FM* is rational and incommensurable in length with *CD*. {X. 22}

And, since the squares on *AG*, *GB* are rational, while twice the rectangle *AG*, *GB* is medial, therefore the squares on *AG*, *GB* are incommensurable with twice the rectangle *AG*, *GB*.

And *CL* is equal to the squares on *AG*, *GB*, and *FL* to twice the rectangle *AG*, *GB*; therefore *DM* is incommensurable with *FL*.

But, as *DM* is to *FL*, so is *CM* to *FM*; {VI. 1} therefore *CM* is incommensurable in length with *FM*. {X. 11}

And both are rational; therefore *CM*, *MF* are rational straight lines commensurable in square only; therefore *CF* is an apotome. {X. 73}

I say next that it is also a first apotome.

For, since the rectangle *AG*, *GB* is a mean proportional between the squares on *AG*, *GB*, and *CH* is equal to the square on *AG*, *KL* equal to the square on *BG*, and *NL* equal to the rectangle *AG*, *GB*, therefore *NL* is also a mean proportional between *CH*, *KL*; therefore, as *CH* is to *NL*, so is *NL* to *KL*.

But, as *CH* is to *NL*, so is *CK* to *NM*, and, as *NL* is to *KL*, so is *NM* to *KM*; {VI. 1} therefore the rectangle *CK*, *KM* is equal to the square on *NM* {VI. 17}, that is, to the fourth part of the square on *FM*.

And, since the square on *AG* is commensurable with the square on *GB*, *CH* is also commensurable with *KL*.

But, as *CH* is to *KL*, so is *CK* to *KM*; {VI. 1} therefore *CK* is commensurable with *KM*. {X. 11}

Since then *CM*, *MF* are two unequal straight lines, and to *CM* there has been applied the rectangle *CK*, *KM* equal to the fourth part of the square on *FM* and deficient by a square figure, while *CK* is commensurable with *KM*, therefore the square on *CM* is greater than the square on *MF* by the square on a straight line commensurable in length with *CM*. {X. 17}

And *CM* is commensurable in length with the rational straight line *CD* set out; therefore *CF* is a first apotome. {X. Deff. III. 1}

Therefore etc. Q. E. D.

PROPOSITION 98.

The square on a first apotome of a medial straight line applied to a rational straight line produces as breadth a second apotome.

Let *AB* be a first apotome of a medial straight line and *CD* a rational straight line, and to *CD* let there be applied *CE* equal to the square on *AB*,

producing *CF* as breadth; I say that *CF* is a second apotome.

For let *BG* be the annex to *AB*;. therefore *AG*, *GB* are medial straight lines commensurable in square only which contain a rational rectangle. {X. 74}

To *CD* let there be applied *CH* equal to the square on *AG*, producing *CK* as breadth, and *KL* equal to the square on *GB*, producing *KM* as breadth; therefore the whole *CL* is equal to the squares on *AG*, *GB*; therefore *CL* is also medial. {X. 15 and 23, Por.}

And it is applied to the rational straight line *CD*, producing *CM* as breadth; therefore *CM* is rational and incommensurable in length with *CD*. {X. 22}

Now, since *CL* is equal to the squares on *AG*, *GB*, and, in these, the square on *AB* is equal to *CE*, therefore the remainder, twice the rectangle *AG*, *GB*, is equal to *FL*. {II. 7}

But twice the rectangle *AG*, *GB* is rational; therefore *FL* is rational.

And it is applied to the rational straight line *FE*, producing *FM* as breadth; therefore *FM* is also rational and commensurable in length with *CD*. {X. 20}

Now, since the sum of the squares on *AG*, *GB*, that is, *CL*, is medial, while twice the rectangle *AG*, *GB*, that is, *FL*, is rational, therefore *CL* is incommensurable with *FL*.

But, as *CL* is to *FL*, so is *CM* to *FM*; {VI. 1} therefore *CM* is incommensurable in length with *FM*. {X. 11}

And both are rational; therefore *CM*, *MF* are rational straight lines commensurable in square only; therefore *CF* is an apotome. {X. 73}

I say next that it is also a second apotome.

For let *FM* be bisected at *N*, and let *NO* be drawn through *N* parallel to *CD*; therefore each of the rectangles *FO*, *NL* is equal to the rectangle *AG*, *GB*.

Now, since the rectangle *AG*, *GB* is a mean proportional between the squares on *AG*, *GB*, and the square on *AG* is equal to *CH*, the rectangle *AG*,

GB to *NL*, and the square on *BG* to *KL*, therefore *NL* is also a mean proportional between *CH*, *KL*; therefore, as *CH* is to *NL*, so is *NL* to *KL*.

But, as *CH* is to *NL*, so is *CK* to *NM*, and, as *NL* is to *KL*, so is *NM* to *MK*; {VI. 1} therefore, as *CK* is to *NM*, so is *NM*, so is *KM*; {V. 11} therefore the rectangle *CK*, *KM* is equal to the square on *NM* {VI. 17}, that is, to the fourth part of the square on *FM*.

Since the *CM*, *MF* are two unequal straight lines, and the rectangle *CK*, *KM* equal to the fourth part of the square on *MF* and deficient by a square figure has been applied to the greater, *CM*, and divides it into commensurable parts, therefore the square on *CM* is greater than the square on *MF* by the square on a straight line commensurable in length with *CM*. {X. 17}

And the annex *FM* is commensurable in length with the rational straight line *CD* set out; therefore *CF* is a second apotome. {X. Deff. III. 2}

Therefore etc. Q. E. D.

PROPOSITION 99.

The square on a second apotome of a medial straight line applied to a rational straight line produces as breadth a third apotome.

Let *AB* be a second apotome of a medial straight line, and *CD* rational, and to *CD* let there be applied *CE* equal to the square on *AB*, producing *CF* as breadth; I say that *CF* is a third apotome.

For let *BG* be the annex to *AB*; therefore *AG*, *GB* are medial straight lines commensurable in square only which contain a medial rectangle. {X. 75}

Let *CH* equal to the square on *AG* be applied to *CD*, producing *CK* as breadth, and let *KL* equal to the square on *BG* be applied to *KH*, producing *KM* as breadth; therefore the whole *CL* is equal to the squares on *AG*, *GB*; therefore *CL* is also medial. {X. 15 and 23, Por.}

And it is applied to the rational straight line *CD*, producing *CM* as breadth; therefore *CM* is rational and incommensurable in length with *CD*. {X. 22}

Now, since the whole *CL* is equal to the squares on *AG*, *GB*, and, in these, *CE* is equal to the square on *AB*, therefore the remainder *LF* is equal to twice the rectangle *AG*, *GB*. {II. 7}

Let then *FM* be bisected at the point *N*, and let *NO* be drawn parallel to *CD*; therefore each of the rectangles *FO*, *NL* is equal to the rectangle *AG*, *GB*.

But the rectangle *AG*, *GB* is medial; therefore *FL* is also medial.

And it is applied to the rational straight line *EF*, producing *FM* as breadth; therefore *FM* is also rational and incommensurable in length with *CD*. {X. 22}

And, since *AG*, *GB* are commensurable in square only, therefore *AG* is incommensurable in length with *GB*; therefore the square on *AG* is also incommensurable with the rectangle *AG*, *GB*. {VI. 1, X. 11}

But the squares on *AG*, *GB* are commensurable with the square on *AG*, and twice the rectangle *AG*, *GB* with the rectangle *AG*, *GB*; therefore the squares on *AG*, *GB* are incommensurable with twice the rectangle *AG*, *GB*. {X. 13}

But *CL* is equal to the squares on *AG*, *GB*, and *FL* is equal to twice the rectangle *AG*, *GB*; therefore *CL* is also incommensurable with *FL*.

But, as *CL* is to *FL*, so is *CM* to *FM*; {VI. 1} therefore *CM* is incommensurable in length with *FM*. {X. 11}

And both are rational; therefore *CM*, *MF* are rational straight lines commensurable in square only; therefore *CF* is an apotome. {X. 73}

I say next that it is also a third apotome.

For, since the square on *AG* is commensurable with the square on *GB*, therefore *CH* is also commensurable with *KL*, so that *CK* is also commensurable with *KM*. {VI. 1, X. 11}

And, since the rectangle *AG*, *GB* is a mean proportional between the squares on *AG*, *GB*, and *CH* is equal to the square on *AG*, *KL* equal to the square on *GB*, and *NL* equal to the rectangle *AG*, *GB*, therefore *NL* is also a mean proportional between *CH*, *KL*; therefore, as *CH* is to *NL*, so is *NL* to *KL*.

But, as *CH* is to *NL*, so is *CK* to *NM*, and, as *NL* is to *KL*, so is *NM* to *KM*; {VI. 1} therefore, as *CK* is to *MN*, so is *MN* to *KM*; {V. 11} therefore the rectangle *CK*, *KM* is equal to {the square on *MN*, that is, to} the fourth part of the square on *FM*.

Since then *CM*, *MF* are two unequal straight lines, and a parallelogram equal to the fourth part of the square on *FM* and deficient by a square figure has been applied to *CM*, and divides it into commensurable parts, therefore the square on *CM* is greater than the square on *MF* by the square on a straight line commensurable with *CM*. {X. 17}

And neither of the straight lines *CM*, *MF* is commensurable in length with the rational straight line *CD* set out; therefore *CF* is a third apotome. {X. Deff. III. 3}

Therefore etc. Q. E. D.

PROPOSITION 100.

The square on a minor straight line applied to a rational straight line produces as breadth a fourth apotome.

Let *AB* be a minor and *CD* a rational straight line, and to the rational straight line *CD* let *CE* be applied equal to the square on *AB* and producing *CF* as breadth; I say that *CF* is a fourth apotome.

For let *BG* be the annex to *AB*; therefore *AG*, *GB* are straight lines incommensurable in square which make the sum of the squares on *AG*, *GB*

rational, but twice the rectangle *AG*, *GB* medial. {X. 76}

To *CD* let there be applied *CH* equal to the square on *AG* and producing *CK* as breadth, and *KL* equal to the square on *BG*, producing *KM* as breadth; therefore the whole *CL* is equal to the squares on *AG*, *GB*.

And the sum of the squares on *AG*, *GB* is rational; therefore *CL* is also rational.

And it is applied to the rational straight line *CD*, producing *CM* as breadth; therefore *CM* is also rational and commensurable in length with *CD*. {X. 20}

And, since the whole *CL* is equal to the squares on *AG*, *GB*, and, in these, *CE* is equal to the square on *AB*, therefore the remainder *FL* is equal to twice the rectangle *AG*, *GB*. {II. 7}

Let then *FM* be bisected at the point *N*, and let *NO* be drawn through *N* parallel to either of the straight lines *CD*, *ML*; therefore each of the rectangles *FO*, *NL* is equal to the rectangle *AG*, *GB*.

And, since twice the rectangle *AG*, *GB* is medial and is equal to *FL*, therefore *FL* is also medial.

And it is applied to the rational straight line *FE*, producing *FM* as breadth; therefore *FM* is rational and incommensurable in length with *CD*. {X. 22}

And, since the sum of the squares on *AG*, *GB* is rational, while twice the rectangle *AG*, *GB* is medial, the squares on *AG*, *GB* are incommensurable with twice the rectangle *AG*, *GB*.

But *CL* is equal to the squares on *AG*, *GB*, and *FL* equal to twice the rectangle *AG*, *GB*; therefore *CL* is incommensurable with *FL*.

But, as *CL* is to *FL*, so is *CM* to *MF*; {VI. 1} therefore *CM* is incommensurable in length with *MF*. {X. 11}

And both are rational; therefore *CM*, *MF* are rational straight lines commensurable in square only; therefore *CF* is an apotome. {X. 73}

I say that it is also a fourth apotome.

For, since *AG*, *GB* are incommensurable in square, therefore the square on *AG* is also incommensurable with the square on *GB*.

And *CH* is equal to the square on *AG*, and *KL* equal to the square on *GB*; therefore *CH* is incommensurable with *KL*.

But, as *CH* is to *KL*, so is *CK* to *KM*; {VI. 1} therefore *CK* is incommensurable in length with *KM*. {X. 11}

And, since the rectangle *AG*, *GB* is a mean proportional between the squares on *AG*, *GB*, and the square on *AG* is equal to *CH*, the square on *GB* to *KL*, and the rectangle *AG*, *GB* to *NL*, therefore *NL* is a mean proportional between *CH*, *KL*; therefore, as *CH* is to *NL*, so is *NL* to *KL*.

But, as *CH* is to *NL*, so is *CK* to *NM*, and, as *NL* is to *KL*, so is *NM* to *KM*; {VI. 1} therefore, as *CK* is to *MN*, so is *MN* to *KM*; {V. 11} therefore the rectangle *CK*, *KM* is equal to the square on *MN* {VI. 17}, that is, to the fourth part of the square on *FM*.

Since then *CM*, *MF* are two unequal straight lines, and the rectangle *CK*, *KM* equal to the fourth part of the square on *MF* and deficient by a square figure has been applied to *CM* and divides it into incommensurable parts, therefore the square on *CM* is greater than the square on *MF* by the square on a straight line incommensurable with *CM*. {X. 18}

And the whole *CM* is commensurable in length with the rational straight line *CD* set out; therefore *CF* is a fourth apotome. {X. Deff. III. 4}

Therefore etc. Q. E. D.

PROPOSITION 101.

The square on the straight line which produces with a rational area a medial whole, if applied to a rational straight line, produces as breadth a fifth apotome.

Let *AB* be the straight line which produces with a rational area a medial whole, and *CD* a rational straight line, and to *CD* let *CE* be applied equal to the square on *AB* and producing *CF* as breadth; I say that *CF* is a fifth apotome.

For let *BG* be the annex to *AB*; therefore *AG*, *GB* are straight lines incommensurable in square which make the sum of the squares on them medial but twice the rectangle contained by them rational. {X. 77}

To *CD* let there be applied *CH* equal to the square on *AG*, and *KL* equal to the square on *GB*; therefore the whole *CL* is equal to the squares on *AG*, *GB*.

But the sum of the squares on *AG*, *GB* together is medial; therefore *CL* is medial.

And it is applied to the rational straight line *CD*, producing *CM* as breadth; therefore *CM* is rational and incommensurable with *CD*. {X. 22}

And, since the whole *CL* is equal to the squares on *AG*, *GB*, and, in these, *CE* is equal to the square on *AB*, therefore the remainder *FL* is equal to twice the rectangle *AG*, *GB*. {II. 7}

Let then *FM* be bisected at *N*, and through *N* let *NO* be drawn parallel to either of the straight lines *CD*, *ML*; therefore each of the rectangles *FO*, *NL* is equal to the rectangle *AG*, *GB*:

And, since twice the rectangle *AG*, *GB* is rational and equal to *FL*, therefore *FL* is rational.

And it is applied to the rational straight line *EF*, producing *FM* as breadth; therefore *FM* is rational and commensurable in length with *CD*. {X. 20}

Now, since *CL* is medial, and *FL* rational, therefore *CL* is incommensurable with *FL*.

But, as *CL* is to *FL*, so is *CM* to *MF*; {VI. 1} therefore *CM* is incommensurable in length with *MF*. {X. 11}

And both are rational; therefore *CM*, *MF* are rational straight lines commensurable in square only; therefore *CF* is an apotome. {X. 73}

I say next that it is also a fifth apotome.

For we can prove similarly that the rectangle *CK*, *KM* is equal to the square on *NM*, that is, to the fourth part of the square on *FM*.

And, since the square on *AG* is incommensurable with the square on *GB*, while the square on *AG* is equal to *CH*, and the square on *GB* to *KL*, therefore *CH* is incommensurable with *KL*.

But, as *CH* is to *KL*, so is *CK* to *KM*; {VI. 1} therefore *CK* is incommensurable in length with *KM*. {X. 11}

Since then *CM*, *MF* are two unequal straight lines, and a parallelogram equal to the fourth part of the square on *FM* and deficient by a square figure has been applied to *CM*, and divides it into incommensurable parts, therefore the square on *CM* is greater than the square on *MF* by the square on a straight line incommensurable with *CM*. {X. 18}

And the annex *FM* is commensurable with the rational straight line *CD* set out; therefore *CF* is a fifth apotome. {X. Deff. III. 5} Q. E. D.

PROPOSITION 102.

The square on the straight line which produces with a medial area a medial whole, if applied to a rational straight line, produces as breadth a sixth apotome.

Let *AB* be the straight line which produces with a medial area a medial whole, and *CD* a rational straight line, and to *CD* let *CE* be applied equal to the square on *AB* and producing *CF* as breadth; I say that *CF* is a sixth apotome.

For let *BG* be the annex to *AB*; therefore *AG*, *GB* are straight lines incommensurable in square which make the sum of the squares on them medial, twice the rectangle *AG*, *GB* medial, and the squares on *AG*, *GB* incommensurable with twice the rectangle *AG*, *GB*. {X. 78}

Now to *CD* let there be applied *CH* equal to the square on *AG* and producing *CK* as breadth, and *KL* equal to the square on *BG*; therefore the whole *CL* is equal to the squares on *AG*, *GB*; therefore *CL* is also medial.

And it is applied to the rational straight line *CD*, producing *CM* as breadth; therefore *CM* is rational and incommensurable in length with *CD*. {X. 22}

Since now *CL* is equal to the squares on *AG*, *GB*, and, in these, *CE* is equal to the square on *AB*, therefore the remainder *FL* is equal to twice the rectangle *AG*, *GB*. {II. 7}

And twice the rectangle *AG*, *GB* is medial; therefore *FL* is also medial.

And it is applied to the rational straight line *FE*, producing *FM* as breadth; therefore *FM* is rational and incommensurable in length with *CD*. {X. 22}

And, since the squares on *AG*, *GB* are incommensurable with twice the rectangle *AG*, *GB*, and *CL* is equal to the squares on *AG*, *GB*, and *FL* equal to twice the rectangle *AG*, *GB*, therefore *CL* is incommensurable with *FL*.

But, as *CL* is to *FL*, so is *CM* to *MF*; {VI. 1} therefore *CM* is incommensurable in length with *MF*. {X. 11}

And both are rational.

Therefore *CM*, *MF* are rational straight lines commensurable in square only; therefore *CF* is an apotome. {X. 73}

I say next that it is also a sixth apotome.

For, since *FL* is equal to twice the rectangle *AG*, *GB*, let *FM* be bisected at *N*, and let *NO* be drawn through *N* parallel to *CD*; therefore each of the rectangles *FO*, *NL* is equal to the rectangle *AG*, *GB*.

And, since *AG*, *GB* are incommensurable in square, therefore the square on *AG* is incommensurable with the square on *GB*.

But *CH* is equal to the square on *AG*, and *KL* is equal to the square on *GB*; therefore *CH* is incommensurable with *KL*.

But, as *CH* is to *KL*, so is *CK* to *KM*; {VI. 1} therefore *CK* is incommensurable with *KM*. {X. 11}

And, since the rectangle *AG*, *GB* is a mean proportional between the squares on *AG*, *GB*, and *CH* is equal to the square on *AG*, *KL* equal to the square on *GB*, and *NL* equal to the rectangle *AG*, *GB*, therefore *NL* is also a mean proportional between *CH*, *KL*; therefore, as *CH* is to *NL*, so is *NL* to *KL*.

And for the same reason as before the square on *CM* is greater than the square on *MF* by the square on a straight line incommensurable with *CM*. {X. 18}

And neither of them is commensurable with the rational straight line *CD* set out; therefore *CF* is a sixth apotome. {X. Deff. III. 6} Q. E. D.

PROPOSITION 103.

A straight line commensurable in length with an apotome is an apotome and the same in order.

Let *AB* be an apotome, and let *CD* be commensurable in length with *AB*; I say that *CD* is also an apotome and the same in order with *AB*.

For, since *AB* is an apotome, let *BE* be the annex to it; therefore *AE*, *EB* are rational straight lines commensurable in square only. {X. 73}

Let it be contrived that the ratio of *BE* to *DF* is the same as the ratio of *AB* to *CD*; {VI. 12} therefore also, as one is to one, so are all to all; {V. 12} therefore also, as the whole *AE* is to the whole *CF*, so is *AB* to *CD*.

But *AB* is commensurable in length with *CD*.

Therefore *AE* is also commensurable with *CF*, and *BE* with *DF*. {X. 11}

And *AE*, *EB* are rational straight lines commensurable in square only; therefore *CF*, *FD* are also rational straight lines commensurable in square only. {X. 13}

Now since, as *AE* is to *CF*, so is *BE* to *DF*, alternately therefore, as *AE* is to *EB*, so is *CF* to *FD*. {V. 16}

And the square on *AE* is greater than the square on *EB* either by the square on a straight line commensurable with *AE* or by the square on a straight line incommensurable with it.

If then the square on *AE* is greater than the square on *EB* by the square on a straight line commensurable with *AE*, the square on *CF* will also be greater than the square on *FD* by the square on a straight line commensurable with *CF*. {X. 14}

And, if *AE* is commensurable in length with the rational straight line set out, *CF* is so also, {X. 12} if *BE*, then *DF* also, {id.} and, if neither of the straight lines *AE*, *EB*, then neither of the straight lines *CF*, *FD*. {X. 13}

But, if the square on *AE* is greater than the square on *EB* by the square on a straight line incommensurable with *AE*, the square on *CF* will also be greater than the square on *FD* by the square on a straight line incommensurable with *CF*. {X. 14}

And, if *AE* is commensurable in length with the rational straight line set out, *CF* is so also, if *BE*, then *DF* also, {X. 12} and, if neither of the straight lines *AE*, *EB*, then neither of the straight lines *CF*, *FD*. {X. 13}

Therefore *CD* is an apotome and the same in order with *AB*. Q. E. D.

PROPOSITION 104.

A straight line commensurable with an apotome of a medial straight line is an apotome of a medial straight line and the same in order.

Let *AB* be an apotome of a medial straight line, and let *CD* be commensurable in length with *AB*; I say that *CD* is also an apotome of a medial straight line and the same in order with *AB*.

For, since *AB* is an apotome of a medial straight line, let *EB* be the annex to it.

Therefore *AE*, *EB* are medial straight lines commensurable in square only. {X. 74, 75}

Let it be contrived that, as *AB* is to *CD*, so is *BE* to *DF*; {VI. 12} therefore *AE* is also commensurable with *CF*, and *BE* with *DF*. {V. 12, X. 11}

But *AE*, *EB* are medial straight lines commensurable in square only; therefore *CF*, *FD* are also medial straight lines {X. 23} commensurable in square only; {X. 13} therefore *CD* is an apotome of a medial straight line. {X. 74, 75}

I say next that it is also the same in order with *AB*.

Since, as *AE* is to *EB*, so is *CF* to *FD*, therefore also, as the square on *AE* is to the rectangle *AE*, *EB*, so is the square on *CF* to the rectangle *CF*, *FD*.

But the square on *AE* is commensurable with the square on *CF*; therefore the rectangle *AE*, *EB* is also commensurable with the rectangle *CF*, *FD*. {V. 16, X. 11}

Therefore, if the rectangle *AE*, *EB* is rational, the rectangle *CF*, *FD* will also be rational, {X. Def. 4} and if the rectangle *AE*, *EB* is medial, the rectangle *CF*, *FD* is also medial. {X. 23, Por.}

Therefore *CD* is an apotome of a medial straight line and the same in order with *AB*. {X. 74, 75} Q. E. D.

PROPOSITION 105.

A straight line commensurable with a minor straight line is minor.

Let *AB* be a minor straight line, and *CD* commensurable with *AB*; I say that *CD* is also minor.

Let the same construction be made as before; then, since *AE*, *EB* are incommensurable in square, {X. 76} therefore *CF*, *FD* are also incommensurable in square. {X. 13}

Now since, as *AE* is to *EB*, so is *CF* to *FD*, {V. 12, V. 16} therefore also, as the square on *AE* is to the square on *EB*, so is the square on *CF* to the square on *FD*. {VI. 22}

Therefore, componendo, as the squares on *AE*, *EB* are to the square on *EB*, so are the squares on *CF*, *FD* to the square on *FD*. {V. 18}

But the square on *BE* is commensurable with the square on *DF*; therefore the sum of the squares on *AE*, *EB* is also commensurable with the sum of the squares on *CF*, *FD*. {V. 16, X. 11}

But the sum of the squares on *AE*, *EB* is rational; {X. 76} therefore the sum of the squares on *CF*, *FD* is also rational. {X. Def. 4}

Again, since, as the square on *AE* is to the rectangle *AE*, *EB*, so is the square on *CF* to the rectangle *CF*, *FD*, while the square on *AE* is commensurable with the square on *CF*, therefore the rectangle *AE*, *EB* is also commensurable with the rectangle *CF*, *FD*.

But the rectangle *AE*, *EB* is medial; {X. 76} therefore the rectangle *CF*, *FD* is also medial; {X. 23, Por.} therefore *CF*, *FD* are straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial.

Therefore *CD* is minor. {X. 76} Q. E. D.

PROPOSITION 106.

A straight line commensurable with that which produces with a rational area a medial whole is a straight line which produces with a rational area a medial whole.

Let *AB* be a straight line which produces with a rational area a medial whole, and *CD* commensurable with *AB*; I say that *CD* is also a straight line which produces with a rational area a medial whole.

For let *BE* be the annex to *AB*; therefore *AE*, *EB* are straight lines incommensurable in square which make the sum of the squares on *AE*, *EB* medial, but the rectangle contained by them rational. {X. 77}

Let the same construction be made.

Then we can prove, in manner similar to the foregoing, that *CF*, *FD* are in the same ratio as *AE*, *EB*, the sum of the squares on *AE*, *EB* is commensurable with the sum of the squares on *CF*, *FD*, and the rectangle *AE*, *EB* with the rectangle *CF*, *FD*; so that *CF*, *FD* are also straight lines incommensurable in square which make the sum of the squares on *CF*, *FD* medial, but the rectangle contained by them rational.

Therefore *CD* is a straight line which produces with a rational area a medial whole. {X. 77} Q. E. D.

PROPOSITION 107.

A straight line commensurable with that which produces with a medial area a medial whole is itself also a straight line which produces with a medial area a medial whole.

Let *AB* be a straight line which produces with a medial area a medial whole, and let *CD* be commensurable with *AB*; I say that *CD* is also a straight line which produces with a medial area a medial whole.

For let *BE* be the annex to *AB*, and let the same construction be made; therefore *AE*, *EB* are straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and further the sum of the squares on them incommensurable with the rectangle contained by them. {X. 78}

Now, as was proved, *AE*, *EB* are commensurable with *CF*, *FD*, the sum of the squares on *AE*, *EB* with the sum of the squares on *CF*, *FD*, and the rectangle *AE*, *EB* with the rectangle *CF*, *FD*; therefore *CF*, *FD* are also straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and further the sum of the squares on them incommensurable with the rectangle contained by them.

Therefore *CD* is a straight line which produces with a medial area a medial whole. {X. 78}

PROPOSITION 108.

If from a rational area a medial area be subtracted, the "side" of the remaining area becomes one of two irrational straight lines, either an apotome or a minor straight line.

For from the rational area *BC* let the medial area *BD* be subtracted; I say that the "side" of the remainder *EC* becomes one of two irrational straight lines, either an apotome or a minor straight line.

For let a rational straight line *FG* be set out, to *FG* let there be applied the rectangular parallelogram *GH* equal to *BC*, and let *GK* equal to *DB* be subtracted; therefore the remainder *EC* is equal to *LH*.

Since then *BC* is rational, and *BD* medial, while *BC* is equal to *GH*, and *BD* to *GK*, therefore *GH* is rational, and *GK* medial.

And they are applied to the rational straight line *FG*; therefore *FH* is rational and commensurable in length with *FG*, {X. 20} while *FK* is rational and incommensurable in length with *FG*; {X. 22} therefore *FH* is incommensurable in length with *FK*. {X. 13}

Therefore *FH*, *FK* are rational straight lines commensurable in square only; therefore *KH* is an apotome {X. 73}, and *KF* the annex to it.

Now the square on *HF* is greater than the square on *FK* by the square on a straight line either commensurable with *HF* or not commensurable.

First, let the square on it be greater by the square on a straight line commensurable with it.

Now the whole *HF* is commensurable in length with the rational straight line *FG* set out; therefore *KH* is a first apotome. {X. Deff. III. 1}

But the "side" of the rectangle contained by a rational straight line and a first apotome is an apotome. {X. 91}

Therefore the "side" of *LH*, that is, of *EC*, is an apotome.

But, if the square on *HF* is greater than the square on *FK* by the square on a straight line incommensurable with *HF*, while the whole *FH* is commensurable in length with the rational straight line *FG* set out, *KH* is a fourth apotome. {X. Deff. III. 4}

But the "side" of the rectangle contained by a rational straight line and a fourth apotome is minor. {X. 94} Q. E. D.

PROPOSITION 109.

If from a medial area a rational area be subtracted, there arise two other irrational straight lines, either a first apotome of a medial straight line or a straight line which produces with a rational area a medial whole.

For from the medial area *BC* let the rational area *BD* be subtracted.

I say that the "side" of the remainder *EC* becomes one of two irrational straight lines, either a first apotome of a medial straight line or a straight line which produces with a rational area a medial whole.

For let a rational straight line *FG* be set out, and let the areas be similarly applied.

It follows then that *FH* is rational and incommensurable in length with *FG*, while *KF* is rational and commensurable in length with *FG*; therefore *FH*, *FK* are rational straight lines commensurable in square only; {X. 13} therefore *KH* is an apotome, and *FK* the annex to it. {X. 73}

Now the square on *HF* is greater than the square on *FK* either by the square on a straight line commensurable with *HF* or by the square on a straight line incommensurable with it.

If then the square on *HF* is greater than the square on *FK* by the square on a straight line commensurable with *HF*, while the annex *FK* is commensurable in length with the rational straight line *FG* set out, *KH* is a second apotome. {X. Deff. III. 2}

But *FG* is rational; so that the "side" of *LH*, that is, of *EC*, is a first apotome of a medial straight line. {X. 92}

But, if the square on *HF* is greater than the square on *FK* by the square on a straight line incommensurable with *HF*, while the annex *FK* is commensurable in length with the rational straight line *FG* set out, *KH* is a fifth apotome; {X. Deff. III. 5} so that the "side" of *EC* is a straight line which produces with a rational area a medial whole. {X. 95}

PROPOSITION 110.

If from a medial area there be subtracted a medial area incommensurable with the whole, the two remaining irrational straight lines arise, either a

second apotome of a medial straight line or a straight line which produces with a medial area a medial whole.

For, as in the foregoing figures, let there be subtracted from the medial area *BC* the medial area *BD* incommensurable with the whole; I say that the "side" of *EC* is one of two irrational straight lines, either a second apotome of a medial straight line or a straight line which produces with a medial area a medial whole.

For, since each of the rectangles *BC*, *BD* is medial, and *BC* is incommensurable with *BD*, it follows that each of the straight lines *FH*, *FK* will be rational and incommensurable in length with *FG*. {X. 22}

And, since *BC* is incommensurable with *BD*, that is, *GH* with *GK*, *HF* is also incommensurable with *FK*; {VI. 1, X. 11} therefore *FH*, *FK* are rational straight lines commensurable in square only; therefore *KH* is an apotome. {X. 73}

If then the square on *FH* is greater than the square on *FK* by the square on a straight line commensurable with *FH*, while neither of the straight lines *FH*, *FK* is commensurable in length with the rational straight line *FG* set out, *KH* is a third apotome. {X. Deff. III. 3}

But *KL* is rational, and the rectangle contained by a rational straight line and a third apotome is irrational, and the "side" of it is irrational, and is called a second apotome of a medial straight line; {X. 93} so that the "side" of *LH*, that is, of *EC*, is a second apotome of a medial straight line.

But, if the square on *FH* is greater than the square on *FK* by the square on a straight line incommensurable with *FH*, while neither of the straight lines *HF*, *FK* is commensurable in length with *FG*, *KH* is a sixth apotome. {X. Deff. III. 6}

But the "side" of the rectangle contained by a rational straight line and a sixth apotome is a straight line which produces with a medial area a medial

whole. {X. 96}

Therefore the "side" of *LH*, that is, of *EC*, is a straight line which produces with a medial area a medial whole. Q. E. D.

PROPOSITION 111.

The apotome is not the same with the binomial straight line.

Let *AB* be an apotome; I say that *AB* is not the same with the binomial straight line.

For, if possible, let it be so; let a rational straight line *DC* be set out, and to *CD* let there be applied the rectangle *CE* equal to the square on *AB* and producing *DE* as breadth.

Then, since *AB* is an apotome, *DE* is a first apotome. {X. 97}

Let *EF* be the annex to it; therefore *DF*, *FE* are rational straight lines commensurable in square only, the square on *DF* is greater than the square on *FE* by the square on a straight line commensurable with *DF*, and *DF* is commensurable in length with the rational straight line *DC* set out. {X. Deff. III. 1}

Again, since *AB* is binomial, therefore *DE* is a first binomial straight line. {X. 60}

Let it be divided into its terms at *G*, and let *DG* be the greater term; therefore *DG*, *GE* are rational straight lines commensurable in square only, the square on *DG* is greater than the square on *GE* by the square on a straight line commensurable with *DG*, and the greater term *DG* is commensurable in length with the rational straight line *DC* set out. {X. Deff. II. 1}

Therefore *DF* is also commensurable in length with *DG*; {X. 12} therefore the remainder *GF* is also commensurable in length with *DF*. {X. 15}

But *DF* is incommensurable in length with *EF*; therefore *FG* is also incommensurable in length with *EF*. {X. 13}

Therefore *GF*, *FE* are rational straight lines commensurable in square only; therefore *EG* is an apotome. {X. 73}

But it is also rational: which is impossible.

Therefore the apotome is not the same with the binomial straight line. Q. E. D.

PROPOSITION 112.

The square on a rational straight line applied to the binomial straight line produces as breadth an apotome the terms of which are commensurable with the terms of the binomial and moreover in the same ratio; and further the apotome so arising will have the same order as the binomial straight line.

Let *A* be a rational straight line, let *BC* be a binomial, and let *DC* be its greater term; let the rectangle *BC*, *EF* be equal to the square on *A*; I say that *EF* is an apotome the terms of which are commensurable with *CD*, *DB*, and in the same ratio, and further *EF* will have the same order as *BC*.

For again let the rectangle *BD*, *G* be equal to the square on *A*.

Since then the rectangle *BC*, *EF* is equal to the rectangle *BD*, *G*, therefore, as *CB* is to *BD*, so is *G* to *EF*. {VI. 16}

But *CB* is greater than *BD*; therefore *G* is also greater than *EF*. {V. 16, V. 14}

Let *EH* be equal to *G*; therefore, as *CB* is to *BD*, so is *HE* to *EF*; therefore, separando, as *CD* is to *BD*, so is *HF* to *FE*. {V. 17}

Let it be contrived that, as *HF* is to *FE*, so is *FK* to *KE*; therefore also the whole *HK* is to the whole *KF* as *FK* is to *KE*; for, as one of the antecedents is

to one of the consequents, so are all the antecedents to all the consequents. {V. 12}

But, as *FK* is to *KE*, so is *CD* to *DB*; {V. 11} therefore also, as *HK* is to *KF*, so is *CD* to *DB*. {id.}

But the square on *CD* is commensurable with the square on *DB*; {X. 36} therefore the square on *HK* is also commensurable with the square on *KF*. {VI. 22, X. 11}

And, as the square on *HK* is to the square on *KF*, so is *HK* to *KE*, since the three straight lines *HK*, *KF*, *KE* are proportional. {V. Def. 9}

Therefore *HK* is commensurable in length with *KE*, so that *HE* is also commensurable in length with *EK*. {X. 15}

Now, since the square on *A* is equal to the rectangle *EH*, *BD*, while the square on *A* is rational, therefore the rectangle *EH*, *BD* is also rational.

And it is applied to the rational straight line *BD*; therefore *EH* is rational and commensurable in length with *BD*; {X. 20} so that *EK*, being commensurable with it, is also rational and commensurable in length with *BD*.

Since, then, as *CD* is to *DB*, so is *FK* to *KE*, while *CD*, *DB* are straight lines commensurable in square only, therefore *FK*, *KE* are also commensurable in square only. {X. 11}

But *KE* is rational; therefore *FK* is also rational.

Therefore *FK*, *KE* are rational straight lines commensurable in square only; therefore *EF* is an apotome. {X. 73}

Now the square on *CD* is greater than the square on *DB* either by the square on a straight line commensurable with *CD* or by the square on a straight line incommensurable with it.

If then the square on *CD* is greater than the square on *DB* by the square on a straight line commensurable with *CD*, the square on *FK* is also greater than

the square on *KE* by the square on a straight line commensurable with *FK*. {X. 14}

And, if *CD* is commensurable in length with the rational straight line set out, so also is *FK*; {X. 11, 12} if *BD* is so commensurable, so also is *KE*; {X. 12} but, if neither of the straight lines *CD*, *DB* is so commensurable, neither of the straight lines *FK*, *KE* is so.

But, if the square on *CD* is greater than the square on *DB* by the square on a straight line incommensurable with *CD*, the square on *FK* is also greater than the square on *KE* by the square on a straight line incommensurable with *FK*. {X. 14}

And, if *CD* is commensurable with the rational straight line set out, so also is *FK*; if *BD* is so commensurable, so also is *KE*; but, if neither of the straight lines *CD*, *DB* is so commensurable, neither of the straight lines *FK*, *KE* is so; so that *FE* is an apotome, the terms of which *FK*, *KE* are commensurable with the terms *CD*, *DB* of the binomial straight line and in the same ratio, and it has the same order as *BC*. Q. E. D.

PROPOSITION 113.

The square on a rational straight line, if applied to an apotome, produces as, breadth the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio; and further the binomial so arising has the same order as the apotome.

Let *A* be a rational straight line and *BD* an apotome, and let the rectangle *BD*, *KH* be equal to the square on *A*, so that the square on the rational straight line *A* when applied to the apotome *BD* produces *KH* as breadth; I say that *KH* is a binomial straight line the terms of which are commensurable with the terms of *BD* and in the same ratio; and further *KH* has the same order as *BD*.

For let *DC* be the annex to *BD*; therefore *BC*, *CD* are rational straight lines commensurable in square only. {X. 73}

Let the rectangle *BC*, *G* be also equal to the square on *A*.

But the square on *A* is rational; therefore the rectangle *BC*, *G* is also rational.

And it has been applied to the rational straight line *BC*; therefore *G* is rational and commensurable in length with *BC*. {X. 20}

Since now the rectangle *BC*, *G* is equal to the rectangle *BD*, *KH*, therefore, proportionally, as *CB* is to *BD*, so is *KH* to *G*. {VI. 16}

But *BC* is greater than *BD*; therefore *KH* is also greater than *G*. {V. 16, V. 14}

Let *KE* be made equal to *G*; therefore *KE* is commensurable in length with *BC*.

And since, as *CB* is to *BD*, so is *HK* to *KE*, therefore, convertendo, as *BC* is to *CD*, so is *KH* to *HE*. {V. 19, Por.}

Let it be contrived that, as *KH* is to *HE*, so is *HF* to *FE*; therefore also the remainder *KF* is to *FH* as *KH* is to *HE*, that is, as *BC* is to *CD*. {V. 19}

But *BC*, *CD* are commensurable in square only; therefore *KF*, *FH* are also commensurable in square only. {X. 11}

And since, as *KH* is to *HE*, so is *KF* to *FH*, while, as *KH* is to *HE*, so is *HF* to *FE*, therefore also, as *KF* is to *FH*, so is *HF* to *FE*, {V. 11} so that also, as the first is to the third, so is the square on the first to the square on the second; {V. Def. 9} therefore also, as *KF* is to *FE*, so is the square on *KF* to the square on *FH*.

But the square on *KF* is commensurable with the square on *FH*, for *KF*, *FH* are commensurable in square; therefore *KF* is also commensurable in length with *FE*, {X. 11} so that *KF* is also commensurable in length with *KE*. {X. 15}

But *KE* is rational and commensurable in length with *BC*; therefore *KF* is also rational and commensurable in length with *BC*. {X. 12}

And, since, as *BC* is to *CD*, so is *KF* to *FH*, alternately, as *BC* is to *KF*, so is *DC* to *FH*. {V. 16}

But *BC* is commensurable with *KF*; therefore *FH* is also commensurable in length with *CD*. {X. 11}

But *BC*, *CD* are rational straight lines commensurable in square only; therefore *KF*, *FH* are also rational straight lines {X. Def. 3} commensurable in square only; therefore *KH* is binomial. {X. 36}

If now the square on *BC* is greater than the square on *CD* by the square on a straight line commensurable with *BC*, the square on *KF* will also be greater than the square on *FH* by the square on a straight line commensurable with *KF*. {X 14}

And, if *BC* is commensurable in length with the rational straight line set out, so also is *KF*; if *CD* is commensurable in length with the rational straight line set out, so also is *FH*, but, if neither of the straight lines *BC*, *CD*, then neither of the straight lines *KF*, *FH*.

But, if the square on *BC* is greater than the square on *CD* by the square on a straight line incommensurable with *BC*, the square on *KF* is also greater than the square on *FH* by the square on a straight line incommensurable with *KF*. {X. 14}

And, if *BC* is commensurable with the rational straight line set out, so also is *KF*; if *CD* is so commensurable, in length with the rational straight line set out, so also is *FH*; but, if neither of the straight lines *BC*, *CD*, then neither of the straight lines *KF*, *FH*.

Therefore *KH* is a binomial straight line, the terms of which *KF*, *FH* are commensurable with the terms *BC*, *CD* of the apotome and in the same ratio, and further *KH* has the same order as *BD*. Q. E. D.

PROPOSITION 114.

If an area be contained by an apotome and the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio, the "side" of the area is rational.

For let an area, the rectangle *AB*, *CD*, be contained by the apotome *AB* and the binomial straight line *CD*, and let *CE* be the greater term of the latter; let the terms *CE*, *ED* of the binomial straight line be commensurable with the terms *AF*, *FB* of the apotome and in the same ratio; and let the "side" of the rectangle *AB*, *CD* be *G*; I say that *G* is rational.

For let a rational straight line *H* be set out, and to *CD* let there be applied a rectangle equal to the square on *H* and producing *KL* as breadth.

Therefore *KL* is an apotome.

Let its terms be *KM*, *ML* commensurable with the terms *CE*, *ED* of the binomial straight line and in the same ratio. {X. 112}

But *CE*, *ED* are also commensurable with *AF*, *FB* and in the same ratio; therefore, as *AF* is to *FB*, so is *KM* to *ML*.

Therefore, alternately, as *AF* is to *KM*, so is *BF* to *LM*; therefore also the remainder *AB* is to the remainder *KL* as *AF* is to *KM*. {V. 19}

But *AF* is commensurable with *KM*; {X. 12} therefore *AB* is also commensurable with *KL*. {X. 11}

And, as *AB* is to *KL*, so is the rectangle *CD*, *AB* to the rectangle *CD*, *KL*; {VI. 1} therefore the rectangle *CD*, *AB* is also commensurable with the rectangle *CD*, *KL*. {X. 11}

But the rectangle *CD*, *KL* is equal to the square on *H*; therefore the rectangle *CD*, *AB* is commensurable with the square on *H*.

But the square on *G* is equal to the rectangle *CD*, *AB*; therefore the square on *G* is commensurable with the square on *H*.

But the square on *H* is rational; therefore the square on *G* is also rational; therefore *G* is rational.

And it is the "side" of the rectangle *CD*, *AB*.

Therefore etc.

PORISM.

And it is made manifest to us by this also that it is possible for a rational area to be contained by irrational straight lines. Q. E. D.

PROPOSITION 115.

From a medial straight line there arise irrational straight lines infinite in number, and none of them is the same as any of the preceding.

Let *A* be a medial straight line; I say that from *A* there arise irrational straight lines infinite in number, and none of them is the same as any of the preceding.

Let a rational straight line *B* be set out, and let the square on *C* be equal to the rectangle *B*, *A*; therefore *C* is irrational; {X. Def. 4} for that which is contained by an irrational and a rational straight line is irrational. {deduction from X. 20}

And it is not the same with any of the preceding; for the square on none of the preceding, if applied to a rational straight line produces as breadth a medial straight line.

Again, let the square on *D* be equal to the rectangle *B*, *C*; therefore the square on *D* is irrational. {deduction from X. 20}

Therefore *D* is irrational; {X. Def. 4} and it is not the same with any of the preceding, for the square on none of the preceding, if applied to a rational straight line, produces *C* as breadth.

Similarly, if this arrangement proceeds ad infinitum, it is manifest that from the medial straight line there arise irrational straight lines infinite in number, and none is the same with any of the preceding. Q. E. D.

ENDNOTES.

[1] 15. But the unit measures the number D; therefore C also measures A. These words are redundant, though they are apparently found in all the MSS.

[2] 3, 5, 8, 10. Euclid speaks of the square on the first (third) being greater than the square on the second (fourth) by the square on a straight line commensurable (incommensurable) "with *itself* (ἑαυτῇ)," and similarly in all like phrases throughout the Book. For clearness' sake I substitute "the first," "the third," or whatever it may be, for "itself" in these cases.

[3] 45. After saying literally that "the square on *BC* is greater than the square on *A* by the square on *DF*," Euclid adds the equivalent expression with δύναται in its technical sense, ἡ ΒΓ ἄρα τῆς Α μεῖζον δύναται τῇ ΔΖ. As this is untranslatable in English except by a paraphrase in practically the same words as have preceded, I have not attempted to reproduce it.

[4] 3. and which produce the types in question. The Greek is ποιουσῶν τὰ προκείμενα εἴδη, and I have taken εἴδη to mean "types (of irrational straight lines)," though the expression might perhaps mean "satisfying the *conditions* in question."

[5] 2. "side." I use the word "side" in the sense explained in the note on X. Def. 4 (P. 13 above), i.e. as short for "side of a square equal to." The Greek is ἡ τὸ χωίον δυναμένη.

[6] 39. Therefore *BA*, *AG* and *BA*, *GE* are pairs of rational straight lines commensurable in square only. The text has "Therefore *BA*, *AG*, *GE* are rational straight lines commensurable in square only," which I have altered because it would naturally convey the impression that *any two* of the three straight lines are commensurable in square only, whereas *AG*, *GE* are commensurable in length (I. 18), and it is only the other two pairs which are commensurable in square only.

BOOK XI. DEFINITIONS.

1

A solid is that which has length, breadth, and depth.

2

An extremity of a solid is a surface.

3

A straight line is at right angles to a plane, when it makes right angles with all the straight lines which meet it and are in the plane.

4

A plane is at right angles to a plane when the straight lines drawn, in one of the planes, at right angles to the common section of the planes are at right angles to the remaining plane.

5

The inclination of a straight line to a plane is, assuming a perpendicular drawn from the extremity of the straight line which is elevated above the plane to the plane, and a straight line joined from the point thus arising to the extremity of the straight line which is in the plane, the angle contained by the straight line so drawn and the straight line standing up.

6

The inclination of a plane to a plane is the acute angle contained by the straight lines drawn at right angles to the common section at the same point,

one in each of the planes.

7

A plane is said to be similarly inclined to a plane as another is to another when the said angles of the inclinations are equal to one another.

8

Parallel planes are those which do not meet.

9

Similar solid figures are those contained by similar planes equal in multitude.

10

Equal and similar solid figures are those contained by similar planes equal in multitude and in magnitude.

11

A solid angle is the inclination constituted by more than two lines which meet one another and are not in the same surface, towards all the lines.

Otherwise: A solid angle is that which is contained by more than two plane angles which are not in the same plane and are constructed to one point.

12

A pyramid is a solid figure, contained by planes, which is constructed from one plane to one point.

13

A prism is a solid figure contained by planes two of which, namely those which are opposite, are equal, similar and parallel, while the rest are parallelograms.

14

When, the diameter of a semicircle remaining fixed, the semicircle is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a sphere.

15

The axis of the sphere is the straight line which remains fixed and about which the semicircle is turned.

16

The centre of the sphere is the same as that of the semicircle.

17

A diameter of the sphere is any straight line drawn through the centre and terminated in both directions by the surface of the sphere.

18

When, one side of those about the right angle in a right-angled triangle remaining fixed, the triangle is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a cone.

And, if the straight line which remains fixed be equal to the remaining side about the right angle which is carried round, the cone will be right-angled; if

less, obtuse-angled; and if greater, acute-angled.

19

The axis of the cone is the straight line which remains fixed and about which the triangle is turned.

20

And the base is the circle described by the straight line which is carried round.

21

When, one side of those about the right angle in a rectangular parallelogram remaining fixed, the parallelogram is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a cylinder.

22

The axis of the cylinder is the straight line which remains fixed and about which the parallelogram is turned.

23

And the bases are the circles described by the two sides opposite to one another which are carried round.

24

Similar cones and cylinders are those in which the axes and the diameters of the bases are proportional.

25

A cube is a solid figure contained by six equal squares.

26

An octahedron is a solid figure contained by eight equal and equilateral triangles.

27

An icosahedron is a solid figure contained by twenty equal and equilateral triangles.

28

A dodecahedron is a solid figure contained by twelve equal, equilateral, and equiangular pentagons.

PROPOSITIONS.

PROPOSITION 1.

A part of a straight line cannot be in the plane of reference and a part in a plane more elevated.

For, if possible, let a part *AB* of the straight line *ABC* be in the plane of reference, and a part *BC* in a plane more elevated.

There will then be in the plane of reference some straight line continuous with *AB* in a straight line.

Let it be *BD*; therefore *AB* is a common segment of the two straight lines *ABC*, *ABD*: which is impossible, inasmuch as, if we describe a circle with centre *B* and distance *AB*, the diameters will cut off unequal circumferences of the circle.

Therefore a part of a straight line cannot be in the plane of reference, and a part in a plane more elevated. Q. E. D. [1] [2]

PROPOSITION 2.

If two straight lines cut one another, they are in one plane, and every triangle is in one plane.

For let the two straight lines *AB*, *CD* cut one another at the point *E*; I say that *AB*, *CD* are in one plane, and every triangle is in one plane.

For let points *F*, *G* be taken at random on *EC*, *EB*, let *CB*, *FG* be joined, and let *FH*, *GK* be drawn across; I say first that the triangle *ECB* is in one plane.

For, if part of the triangle *ECB*, either *FHC* or *GBK*, is in the plane of reference, and the rest in another, a part also of one of the straight lines *EC*,

EB will be in the plane of reference, and a part in another.

But, if the part *FCBG* of the triangle *ECB* be in the plane of reference, and the rest in another, a part also of both the straight lines *EC*, *EB* will be in the plane of reference and a part in another: which was proved absurd. {XI. 1}

Therefore the triangle *ECB* is in one plane.

But, in whatever plane the triangle *ECB* is, in that plane also is each of the straight lines *EC*, *EB*, and, in whatever plane each of the straight lines *EC*, *EB* is, in that plane are *AB*, *CD* also. {XI. 1}

Therefore the straight lines *AB*, *CD* are in one plane, and every triangle is in one plane. Q. E. D.

PROPOSITION 3.

If two planes cut one another, *their common section is a straight line.*

For let the two planes *AB*, *BC* cut one another, and let the line *DB* be their common section; I say that the line *DB* is a straight line.

For, if not, from *D* to *B* let the straight line *DEB* be joined in the plane *AB*, and in the plane *BC* the straight line *DFB*.

Then the two straight lines *DEB*, *DFB* will have the same extremities, and will clearly enclose an area: which is absurd.

Therefore *DEB*, *DFB* are not straight lines.

Similarly we can prove that neither will there be any other straight line joined from *D* to *B* except *DB* the common section of the planes *AB*, *BC*.

Therefore etc. Q. E. D.

PROPOSITION 4.

If a straight line be set up at right angles to two straight lines which cut one another, at their common point of section, it will also be at right angles to the plane through them.

For let a straight line *EF* be set up at right angles to the two straight lines *AB*, *CD*, which cut one another at the point *E*, from *E*; I say that *EF* is also at right angles to the plane through *AB*, *CD*.

For let *AE*, *EB*, *CE*, *ED* be cut off equal to one another, and let any straight line *GEH* be drawn across through *E*, at random; let *AD*, *CB* be joined, and further let *FA*, *FG*, *FD*, *FC*, *FH*, *FB* be joined from the point *F* taken at random <on *EF*>.

Now, since the two straight lines *AE*, *ED* are equal to the two straight lines *CE*, *EB*, and contain equal angles, {I. 15} therefore the base *AD* is equal to the base *CB*, and the triangle *AED* will be equal to the triangle *CEB*; {I. 4} so that the angle *DAE* is also equal to the angle *EBC*.

But the angle *AEG* is also equal to the angle *BEH*; {I. 15} therefore *AGE*, *BEH* are two triangles which have two angles equal to two angles respectively, and one side equal to one side, namely that adjacent to the equal angles, that is to say, *AE* to *EB*; therefore they will also have the remaining sides equal to the remaining sides. {I. 26}

Therefore *GE* is equal to *EH*, and *AG* to *BH*.

And, since *AE* is equal to *EB*, while *FE* is common and at right angles, therefore the base *FA* is equal to the base *FB*. {I. 4}

For the same reason *FC* is also equal to *FD*.

And, since *AD* is equal to *CB*, and *FA* is also equal to *FB*, the two sides *FA*, *AD* are equal to the two sides *FB*, *BC* respectively; and the base *FD* was proved equal to the base *FC*; therefore the angle *FAD* is also equal to the angle *FBC*. {I. 8}

And since, again, *AG* was proved equal to *BH*, and further *FA* also equal to *FB*, the two sides *FA*, *AG* are equal to the two sides *FB*, *BH*.

And the angle *FAG* was proved equal to the angle *FBH*; therefore the base *FG* is equal to the base *FH*. {I. 4}

Now since, again, *GE* was proved equal to *EH*, and *EF* is common, the two sides *GE*, *EF* are equal to the two sides *HE*, *EF*; and the base *FG* is equal to the base *FH*; therefore the angle *GEF* is equal to the angle *HEF*. {I. 8}

Therefore each of the angles *GEF*, *HEF* is right.

Therefore *FE* is at right angles to *GH* drawn at random through *E*.

Similarly we can prove that *FE* will also make right angles with all the straight lines which meet it and are in the plane of reference.

But a straight line is at right angles to a plane when it makes right angles with all the straight lines which meet it and are in that same plane; {XI. Def. 3} therefore *FE* is at right angles to the plane of reference.

But the plane of reference is the plane through the straight lines *AB*, *CD*.

Therefore *FE* is at right angles to the plane through *AB*, *CD*.

Therefore etc. Q. E. D.

PROPOSITION 5.

If a straight line be set up at right angles to three straight lines which meet one another, at their common point of section, the three straight lines are in one plane.

For let a straight line *AB* be set up at right angles to the three straight lines *BC*, *BD*, *BE*, at their point of meeting at *B*; I say that *BC*, *BD*, *BE* are in one plane.

For suppose they are not, but, if possible, let *BD*, *BE* be in the plane of reference and *BC* in one more elevated; let the plane through *AB*, *BC* be

produced; it will thus make, as common section in the plane of reference, a straight line. {XI. 3}

Let it make *BF*.

Therefore the three straight lines *AB*, *BC*, *BF* are in one plane, namely that drawn through *AB*, *BC*.

Now, since *AB* is at right angles to each of the straight lines *BD*, *BE*, therefore *AB* is also at right angles to the plane through *BD*, *BE*. {XI. 4}

But the plane through *BD*, *BE* is the plane of reference; therefore *AB* is at right angles to the plane of reference.

Thus *AB* will also make right angles with all the straight lines which meet it and are in the plane of reference. {XI. Def. 3}

But *BF* which is in the plane of reference meets it; therefore the angle *ABF* is right.

But, by hypothesis, the angle *ABC* is also right; therefore the angle *ABF* is equal to the angle *ABC*.

And they are in one plane: which is impossible.

Therefore the straight line *BC* is not in a more elevated plane; therefore the three straight lines *BC*, *BD*, *BE* are in one plane.

Therefore, if a straight line be set up at right angles to three straight lines, at their point of meeting, the three straight lines are in one plane. Q. E. D.

PROPOSITION 6.

If two straight lines be at right angles to the same plane, the straight lines will be parallel.

For let the two straight lines *AB*, *CD* be at right angles to the plane of reference; I say that *AB* is parallel to *CD*.

For let them meet the plane of reference at the points *B*, *D*, let the straight line *BD* be joined, let *DE* be drawn, in the plane of reference, at right angles to *BD*, let *DE* be made equal to *AB*, and let *BE*, *AE*, *AD* be joined.

Now, since *AB* is at right angles to the plane of reference, it will also make right angles with all the straight lines which meet it and are in the plane of reference. {XI. Def. 3}

But each of the straight lines *BD*, *BE* is in the plane of reference and meets *AB*; therefore each of the angles *ABD*, *ABE* is right.

For the same reason each of the angles *CDB*, *CDE* is also right.

And, since *AB* is equal to *DE*, and *BD* is common, the two sides *AB*, *BD* are equal to the two sides *ED*, *DB*; and they include right angles; therefore the base *AD* is equal to the base *BE*. {I. 4}

And, since *AB* is equal to *DE*, while *AD* is also equal to *BE*, the two sides *AB*, *BE* are equal to the two sides *ED*, *DA*; and *AE* is their common base; therefore the angle *ABE* is equal to the angle *EDA*. {I. 8}

But the angle *ABE* is right; therefore the angle *EDA* is also right; therefore *ED* is at right angles to *DA*.

But it is also at right angles to each of the straight lines *BD*, *DC*; therefore *ED* is set up at right angles to the three straight lines *BD*, *DA*, *DC* at their point of meeting; therefore the three straight lines *BD*, *DA*, *DC* are in one plane. {XI. 5}

But, in whatever plane *DB*, *DA* are, in that plane is *AB* also, for every triangle is in one plane; {XI. 2} therefore the straight lines *AB*, *BD*, *DC* are in one plane.

And each of the angles *ABD*, *BDC* is right; therefore *AB* is parallel to *CD*. {I. 28}

Therefore etc. Q. E. D.

PROPOSITION 7.

If two straight lines be parallel and points be taken at random on each of them, the straight line joining the points is in the same plane with the parallel straight lines.

Let *AB*, *CD* be two parallel straight lines, and let points *E*, *F* be taken at random on them respectively; I say that the straight line joining the points *E*, *F* is in the same plane with the parallel straight lines.

For suppose it is not, but, if possible, let it be in a more elevated plane as *EGF*, and let a plane be drawn through *EGF*; it will then make, as section in the plane of reference, a straight line. {XI. 3}

Let it make it, as *EF*; therefore the two straight lines *EGF*, *EF* will enclose an area: which is impossible.

Therefore the straight line joined from *E* to *F* is not in a plane more elevated; therefore the straight line joined from *E* to *F* is in the plane through the parallel straight lines *AB*, *CD*.

Therefore etc. Q. E. D.

PROPOSITION 8.

If two straight lines be parallel, and one of them be at right angles to any plane, the remaining one will also be at right angles to the same plane.

Let *AB*, *CD* be two parallel straight lines, and let one of them, *AB*, be at right angles to the plane of reference; I say that the remaining one, *CD*, will also be at right angles to the same plane.

For let *AB*, *CD* meet the plane of reference at the points *B*, *D*, and let *BD* be joined; therefore *AB*, *CD*, *BD* are in one plane. {XI. 7}

Let *DE* be drawn, in the plane of reference, at right angles to *BD*, let *DE* be made equal to *AB*, and let *BE*, *AE*, *AD* be joined.

Now, since *AB* is at right angles to the plane of reference, therefore *AB* is also at right angles to all the straight lines which meet it and are in the plane of reference; {XI. Def. 3} therefore each of the angles *ABD*, *ABE* is right.

And, since the straight line *BD* has fallen on the parallels *AB*, *CD*, therefore the angles *ABD*, *CDB* are equal to two right angles. {I. 29}

But the angle *ABD* is right; therefore the angle *CDB* is also right; therefore *CD* is at right angles to *BD*.

And, since *AB* is equal to *DE*, and *BD* is common, the two sides *AB*, *BD* are equal to the two sides *ED*, *DB*; and the angle *ABD* is equal to the angle *EDB*, for each is right; therefore the base *AD* is equal to the base *BE*.

And, since *AB* is equal to *DE*, and *BE* to *AD*, the two sides *AB*, *BE* are equal to the two sides *ED*, *DA* respectively, and *AE* is their common base; therefore the angle *ABE* is equal to the angle *EDA*.

But the angle *ABE* is right; therefore the angle *EDA* is also right; therefore *ED* is at right angles to *AD*.

But it is also at right angles to *DB*; therefore *ED* is also at right angles to the plane through *BD*, *DA*. {XI. 4}

Therefore *ED* will also make right angles with all the straight lines which meet it and are in the plane through *BD*, *DA*.

But *DC* is in the plane through *BD*, *DA*, inasmuch as *AB*, *BD* are in the plane through *BD*, *DA*, {XI. 2} and *DC* is also in the plane in which *AB*, *BD* are.

Therefore *ED* is at right angles to *DC*, so that *CD* is also at right angles to *DE*.

But *CD* is also at right angles to *BD*.

Therefore *CD* is set up at right angles to the two straight lines *DE*, *DB* which cut one another, from the point of section at *D*; so that *CD* is also at right angles to the plane through *DE*, *DB*. {XI. 4}

But the plane through *DE*, *DB* is the plane of reference; therefore *CD* is at right angles to the plane of reference.

Therefore etc. Q. E. D.

PROPOSITION 9.

Straight lines which are parallel to the same straight line and are not in the same plane with it are also parallel to one another.

For let each of the straight lines *AB*, *CD* be parallel to *EF*, not being in the same plane with it; I say that *AB* is parallel to *CD*.

For let a point *G* be taken at random on *EF*, and from it let there be drawn *GH*, in the plane through *EF*, *AB*, at right angles to *EF*, and *GK* in the plane through *FE*, *CD* again at right angles to *EF*.

Now, since *EF* is at right angles to each of the straight lines *GH*, *GK*, therefore *EF* is also at right angles to the plane through *GH*, *GK*. {XI. 4}

And *EF* is parallel to *AB*; therefore *AB* is also at right angles to the plane through *HG*, *GK*. {XI. 8}

For the same reason *CD* is also at right angles to the plane through *HG*, *GK*; therefore each of the straight lines *AB*, *CD* is at right angles to the plane through *HG*, *GK*.

But if two straight lines be at right angles to the same plane, the straight lines are parallel; {XI. 6} therefore *AB* is parallel to *CD*.

PROPOSITION 10.

If two straight lines meeting one another be parallel to two straight lines meeting one another not in the same plane, they will contain equal angles.

For let the two straight lines *AB*, *BC* meeting one another be parallel to the two straight lines *DE*, *EF* meeting one another, not in the same plane; I say that the angle *ABC* is equal to the angle *DEF*.

For let *BA*, *BC*, *ED*, *EF* be cut off equal to one another, and let *AD*, *CF*, *BE*, *AC*, *DF* be joined.

Now, since *BA* is equal and parallel to *ED*, therefore *AD* is also equal and parallel to *BE*. {I. 33}

For the same reason *CF* is also equal and parallel to *BE*.

Therefore each of the straight lines *AD*, *CF* is equal and parallel to *BE*.

But straight lines which are parallel to the same straight line and are not in the same plane with it are parallel to one another; {XI. 9} therefore *AD* is parallel and equal to *CF*.

And *AC*, *DF* join them; therefore *AC* is also equal and parallel to *DF*. {I. 33}

Now, since the two sides *AB*, *BC* are equal to the two sides *DE*, *EF*, and the base *AC* is equal to the base *DF*, therefore the angle *ABC* is equal to the angle *DEF*. {I. 8}

Therefore etc.

PROPOSITION 11.

From a given elevated point to draw a straight line perpendicular to a given plane.

Let *A* be the given elevated point, and the plane of reference the given plane; thus it is required to draw from the point *A* a straight line perpendicular to the plane of reference.

Let any straight line *BC* be drawn, at random, in the plane of reference, and let *AD* be drawn from the point *A* perpendicular to *BC*. {I. 12}

If then *AD* is also perpendicular to the plane of reference, that which was enjoined will have been done.

But, if not, let *DE* be drawn from the point *D* at right angles to *BC* and in the plane of reference, {I. 11} let *AF* be drawn from *A* perpendicular to *DE*, {I. 12} and let *GH* be drawn through the point *F* parallel to *BC*. {I. 31}

Now, since *BC* is at right angles to each of the straight lines *DA*, *DE*, therefore *BC* is also at right angles to the plane through *ED*, *DA*. {XI. 4}

And *GH* is parallel to it; but, if two straight lines be parallel, and one of them be at right angles to any plane, the remaining one will also be at right angles to the same plane; {XI. 8} therefore *GH* is also at right angles to the plane through *ED*, *DA*.

Therefore *GH* is also at right angles to all the straight lines which meet it and are in the plane through *ED*, *DA*. {XI. Def. 3}

But *AF* meets it and is in the plane through *ED*, *DA*; therefore *GH* is at right angles to *FA*, so that *FA* is also at right angles to *GH*.

But *AF* is also at right angles to *DE*; therefore *AF* is at right angles to each of the straight lines *GH*, *DE*.

But, if a straight line be set up at right angles to two straight lines which cut one another, at the point of section, it will also be at right angles to the plane through them; {XI. 4} therefore *FA* is at right angles to the plane through *ED*, *GH*.

But the plane through *ED*, *GH* is the plane of reference; therefore *AF* is at right angles to the plane of reference.

Therefore from the given elevated point *A* the straight line *AF* has been drawn perpendicular to the plane of reference. Q. E. F.

PROPOSITION 12.

To set up a straight line at right angles to a given plane from a given point in it.

Let the plane of reference be the given plane, and *A* the point in it; thus it is required to set up from the point *A* a straight line at right angles to the plane of reference.

Let any elevated point *B* be conceived, from *B* let *BC* be drawn perpendicular to the plane of reference, {XI. 11} and through the point *A* let *AD* be drawn parallel to *BC*. {I. 31}

Then, since *AD*, *CB* are two parallel straight lines, while one of them, *BC*, is at right angles to the plane of reference, therefore the remaining one, *AD*, is also at right angles to the plane of reference. {XI. 8}

Therefore *AD* has been set up at right angles to the given plane from the point *A* in it.

PROPOSITION 13.

From the same point two straight lines cannot be set up at right angles to the same plane on the same side.

For, if possible, from the same point *A* let the two straight lines *AB*, *AC* be set up at right angles to the plane of reference and on the same side, and let a plane be drawn through *BA*, *AC*; it will then make, as section through *A* in the plane of reference, a straight line. {XI. 3}

Let it make *DAE*; therefore the straight lines *AB*, *AC*, *DAE* are in one plane.

And, since *CA* is at right angles to the plane of reference, it will also make right angles with all the straight lines which meet it and are in the plane of

reference. {XI. Def. 3}

But *DAE* meets it and is in the plane of reference; therefore the angle *CAE* is right.

For the same reason the angle *BAE* is also right; therefore the angle *CAE* is equal to the angle *BAE*.

And they are in one plane: which is impossible.

Therefore etc. Q. E. D.

PROPOSITION 14.

Planes to which the same straight line is at right angles will be parallel.

For let any straight line *AB* be at right angles to each of the planes *CD*, *EF*; I say that the planes are parallel.

For, if not, they will meet when produced.

Let them meet; they will then make, as common section, a straight line. {XI. 3}

Let them make *GH*; let a point *K* be taken at random on *GH*, and let *AK*, *BK* be joined.

Now, since *AB* is at right angles to the plane *EF*, therefore *AB* is also at right angles to *BK* which is a straight line in the plane *EF* produced; {XI. Def. 3} therefore the angle *ABK* is right.

For the same reason the angle *BAK* is also right.

Thus, in the triangle *ABK*, the two angles *ABK*, *BAK* are equal to two right angles: which is impossible. {I. 17}

Therefore the planes *CD*, *EF* will not meet when produced; therefore the planes *CD*, *EF* are parallel. {XI. Def. 8}

Therefore planes to which the same straight line is at right angles are parallel. Q. E. D.

PROPOSITION 15.

If two straight lines meeting one another be parallel to two straight lines meeting one another, not being in the same plane, the planes through them are parallel.

For let the two straight lines *AB*, *BC* meeting one another be parallel to the two straight lines *DE*, *EF* meeting one another, not being in the same plane; I say that the planes produced through *AB*, *BC* and *DE*, *EF* will not meet one another.

For let *BG* be drawn from the point *B* perpendicular to the plane through *DE*, *EF* {XI. 11}, and let it meet the plane at the point *G*; through *G* let *GH* be drawn parallel to *ED*, and *GK* parallel to *EF*. {I. 31}

Now, since *BG* is at right angles to the plane through *DE*, *EF*, therefore it will also make right angles with all the straight lines which meet it and are in the plane through *DE*, *EF*. {XI. Def. 3}

But each of the straight lines *GH*, *GK* meets it and is in the plane through *DE*, *EF*; therefore each of the angles *BGH*, *BGK* is right.

And, since *BA* is parallel to *GH*, {XI. 9} therefore the angles *GBA*, *BGH* are equal to two right angles. {I. 29}

But the angle *BGH* is right; therefore the angle *GBA* is also right; therefore *GB* is at right angles to *BA*.

For the same reason *GB* is also at right angles to *BC*.

Since then the straight line *GB* is set up at right angles to the two straight lines *BA*, *BC* which cut one another, therefore *GB* is also at right angles to the plane through *BA*, *BC*. {XI. 4}

But planes to which the same straight line is at right angles are parallel; {XI. 14} therefore the plane through *AB*, *BC* is parallel to the plane through

DE, *EF*.

Therefore, if two straight lines meeting one another be parallel to two straight lines meeting one another, not in the same plane, the planes through them are parallel. Q. E. D.

PROPOSITION 16.

If two parallel planes be cut by any plane, *their common sections are parallel.*

For let the two parallel planes *AB*, *CD* be cut by the plane *EFGH*, and let *EF*, *GH* be their common sections; I say that *EF* is parallel to *GH*.

For, if not, *EF*, *GH* will, when produced, meet either in the direction of *F*, *H* or of *E*, *G*.

Let them be produced, as in the direction of *F*, *H*, and let them, first, meet at *K*.

Now, since *EFK* is in the plane *AB*, therefore all the points on *EFK* are also in the plane *AB*. {XI. 1}

But *K* is one of the points on the straight line *EFK*; therefore *K* is in the plane *AB*.

For the same reason *K* is also in the plane *CD*; therefore the planes *AB*, *CD* will meet when produced.

But they do not meet, because they are, by hypothesis, parallel; therefore the straight lines *EF*, *GH* will not meet when produced in the direction of *F*, *H*.

Similarly we can prove that neither will the straight lines *EF*, *GH* meet when produced in the direction of *E*, *G*.

But straight lines which do not meet in either direction are parallel. {I. Def. 23}

Therefore *EF* is parallel to *GH*.

Therefore etc. Q. E. D.

PROPOSITION 17.

If two straight lines be cut by parallel planes, *they will be cut in the same ratios.*

For let the two straight lines *AB*, *CD* be cut by the parallel planes *GH*, *KL*, *MN* at the points *A*, *E*, *B* and *C*, *F*, *D*; I say that, as the straight line *AE* is to *EB*, so is *CF* to *FD*.

For let *AC*, *BD*, *AD* be joined, let *AD* meet the plane *KL* at the point *O*, and let *EO*, *OF* be joined.

Now, since the two parallel planes *KL*, *MN* are cut by the plane *EBDO*, their common sections *EO*, *BD* are parallel. {XI. 16}

For the same reason, since the two parallel planes *GH*, *KL* are cut by the plane *AOFC*, their common sections *AC*, *OF* are parallel. {*id*.}

And, since the straight line *EO* has been drawn parallel to *BD*, one of the sides of the triangle *ABD*, therefore, proportionally, as *AE* is to *EB*, so is *AO* to *OD*. {VI. 2}

Again, since the straight line *OF* has been drawn parallel to *AC*, one of the sides of the triangle *ADC*, proportionally, as *AO* is to *OD*, so is *CF* to *FD*. {*id*.}

But it was also proved that, as *AO* is to *OD*, so is *AE* to *EB*; therefore also, as *AE* is to *EB*, so is *CF* to *FD*. {V. 11}

Therefore etc. Q. E. D.

PROPOSITION 18.

If a straight line be at right angles to any plane, all the planes through it will also be at right angles to the same plane.

For let any straight line *AB* be at right angles to the plane of reference; I say that all the planes through *AB* are also at right angles to the plane of reference.

For let the plane *DE* be drawn through *AB*, let *CE* be the common section of the plane *DE* and the plane of reference, let a point *F* be taken at random on *CE*, and from *F* let *FG* be drawn in the plane *DE* at right angles to *CE*. {I. 11}

Now, since *AB* is at right angles to the plane of reference, *AB* is also at right angles to all the straight lines which meet it and are in the plane of reference; {XI. Def. 3} so that it is also at right angles to *CE*; therefore the angle *ABF* is right.

But the angle *GFB* is also right; therefore *AB* is parallel to *FG*. {I. 28}

But *AB* is at right angles to the plane of reference; therefore *FG* is also at right angles to the plane of reference. {XI. 8}

Now a plane is at right angles to a plane, when the straight lines drawn, in one of the planes, at right angles to the common section of the planes are at right angles to the remaining plane. {XI. Def. 4}

And *FG*, drawn in one of the planes *DE* at right angles to *CE*, the common section of the planes, was proved to be at right angles to the plane of reference; therefore the plane *DE* is at right angles to the plane of reference.

Similarly also it can be proved that all the planes through *AB* are at right angles to the plane of reference.

Therefore etc. Q. E. D.

PROPOSITION 19.

If two planes which cut one another be at right angles to any plane, their common section will also be at right angles to the same plane.

For let the two planes *AB*, *BC* be at right angles to the plane of reference, and let *BD* be their common section; I say that *BD* is at right angles to the plane of reference.

For suppose it is not, and from the point *D* let *DE* be drawn in the plane *AB* at right angles to the straight line *AD*, and *DF* in the plane *BC* at right angles to *CD*.

Now, since the plane *AB* is at right angles to the plane of reference, and *DE* has been drawn in the plane *AB* at right angles to *AD*, their common section, therefore *DE* is at right angles to the plane of reference. {XI. Def. 4}

Similarly we can prove that *DF* is also at right angles to the plane of reference.

Therefore from the same point *D* two straight lines have been set up at right angles to the plane of reference on the same side: which is impossible. {XI. 13}

Therefore no straight line except the common section *DB* of the planes *AB*, *BC* can be set up from the point *D* at right angles to the plane of reference.

Therefore etc. Q. E. D.

PROPOSITION 20.

If a solid angle be contained by three plane angles, any two, taken together in any manner, are greater than the remaining one.

For let the solid angle at *A* be contained by the three plane angles *BAC*, *CAD*, *DAB*; I say that any two of the angles *BAC*, *CAD*, *DAB*, taken together in any manner, are greater than the remaining one.

If now the angles *BAC*, *CAD*, *DAB* are equal to one another, it is manifest that any two are greater than the remaining one.

But, if not, let *BAC* be greater, and on the straight line *AB*, and at the point *A* on it, let the angle *BAE* be constructed, in the plane through *BA*, *AC*, equal to the angle *DAB*; let *AE* be made equal to *AD*, and let *BEC*, drawn across through the point *E*, cut the straight lines *AB*, *AC* at the points *B*, *C*; let *DB*, *DC* be joined.

Now, since *DA* is equal to *AE*, and *AB* is common, two sides are equal to two sides; and the angle *DAB* is equal to the angle *BAE*; therefore the base *DB* is equal to the base *BE*. {I. 4}

And, since the two sides *BD*, *DC* are greater than *BC*, {I. 20} and of these *DB* was proved equal to *BE*, therefore the remainder *DC* is greater than the remainder *EC*.

Now, since *DA* is equal to *AE*, and *AC* is common, and the base *DC* is greater than the base *EC*, therefore the angle *DAC* is greater than the angle *EAC*. {I. 25}

But the angle *DAB* was also proved equal to the angle *BAE*; therefore the angles *DAB*, *DAC* are greater than the angle *BAC*.

Similarly we can prove that the remaining angles also, taken together two and two, are greater than the remaining one.

Therefore etc. Q. E. D.

PROPOSITION 21.

Any solid angle is contained by plane angles less than four right angles.

Let the angle at *A* be a solid angle contained by the plane angles *BAC*, *CAD*, *DAB*; I say that the angles *BAC*, *CAD*, *DAB* are less than four right angles.

For let points *B*, *C*, *D* be taken at random on the straight lines *AB*, *AC*, *AD* respectively, and let *BC*, *CD*, *DB* be joined.

Now, since the solid angle at *B* is contained by the three plane angles *CBA*, *ABD*, *CBD*, any two are greater than the remaining one; {XI. 20} therefore the angles *CBA*, *ABD* are greater than the angle *CBD*.

For the same reason the angles *BCA*, *ACD* are also greater than the angle *BCD*, and the angles *CDA*, *ADB* are greater than the angle *CDB*; therefore the six angles *CBA*, *ABD*, *BCA*, *ACD*, *CDA*, *ADB* are greater than the three angles *CBD*, *BCD*, *CDB*.

But the three angles *CBD*, *BDC*, *BCD* are equal to two right angles; {I. 32} therefore the six angles *CBA*, *ABD*, *BCA*, *ACD*, *CDA*, *ADB* are greater than two right angles.

And, since the three angles of each of the triangles *ABC*, *ACD*, *ADB* are equal to two right angles, therefore the nine angles of the three triangles, the angles *CBA*, *ACB*, *BAC*, *ACD*, *CDA*, *CAD*, *ADB*, *DBA*, *BAD* are equal to six right angles; and of them the six angles *ABC*, *BCA*, *ACD*, *CDA*, *ADB*, *DBA* are greater than two right angles; therefore the remaining three angles *BAC*, *CAD*, *DAB* containing the solid angle are less than four right angles.

Therefore etc. Q. E. D.

PROPOSITION 22.

If there be three plane angles of which two, taken together in any manner, are greater than the remaining one, and they are contained by equal straight lines, it is possible to construct a triangle out of the straight lines joining the extremities of the equal straight lines.

Let there be three plane angles *ABC*, *DEF*, *GHK*, of which two, taken together in any manner, are greater than the remaining one, namely

the angles ABC, DEF greater than the angle GHK, the angles DEF, GHK greater than the angle ABC,

and, further, the angles *GHK*, *ABC* greater than the angle *DEF*; let the straight lines *AB*, *BC*, *DE*, *EF*, *GH*, *HK* be equal, and let *AC*, *DF*, *GK* be joined; I say that it is possible to construct a triangle out of straight lines equal to *AC*, *DF*, *GK*, that is, that any two of the straight lines *AC*, *DF*, *GK* are greater than the remaining one.

Now, if the angles *ABC*, *DEF*, *GHK* are equal to one another, it is manifest that, *AC*, *DF*, *GK* being equal also, it is possible to construct a triangle out of straight lines equal to *AC*, *DF*, *GK*.

But, if not, let them be unequal, and on the straight line *HK*, and at the point *H* on it, let the angle *KHL* be constructed equal to the angle *ABC*; let *HL* be made equal to one of the straight lines *AB*, *BC*, *DE*, *EF*, *GH*, *HK*, and let *KL*, *GL* be joined.

Now, since the two sides *AB*, *BC* are equal to the two sides *KH*, *HL*, and the angle at *B* is equal to the angle *KHL*, therefore the base *AC* is equal to the base *KL*. {I. 4}

And, since the angles *ABC*, *GHK* are greater than the angle *DEF*, while the angle *ABC* is equal to the angle *KHL*, therefore the angle *GHL* is greater than the angle *DEF*.

And, since the two sides *GH*, *HL* are equal to the two sides *DE*, *EF*, and the angle *GHL* is greater than the angle *DEF*, therefore the base *GL* is greater than the base *DF*. {I. 24}

But *GK*, *KL* are greater than *GL*.

Therefore *GK*, *KL* are much greater than *DF*.

But *KL* is equal to *AC*; therefore *AC*, *GK* are greater than the remaining straight line *DF*.

Similarly we can prove that *AC*, *DF* are greater than *GK*, and further *DF*, *GK* are greater than *AC*.

Therefore it is possible to construct a triangle out of straight lines equal to *AC*, *DF*, *GK*. Q. E. D.

PROPOSITION 23.

To construct a solid angle out of three plane angles two of which, taken together in any manner, are greater than the remaining one: thus the three angles must be less than four right angles.

Let the angles *ABC*, *DEF*, *GHK* be the three given plane angles, and let two of these, taken together in any manner, be greater than the remaining one, while, further, the three are less than four right angles; thus it is required to construct a solid angle out of angles equal to the angles *ABC*, *DEF*, *GHK*.

Let *AB*, *BC*, *DE*, *EF*, *GH*, *HK* be cut off equal to one another, and let *AC*, *DF*, *GK* be joined; it is therefore possible to construct a triangle out of straight lines equal to *AC*, *DF*, *GK*. {XI. 22}

Let *LMN* be so constructed that *AC* is equal to *LM*, *DF* to *MN*, and further *GK* to *NL*, let the circle *LMN* be described about the triangle *LMN*, let its centre be taken, and let it be *O*; let *LO*, *MO*, *NO* be joined; I say that *AB* is greater than *LO*.

For, if not, *AB* is either equal to *LO*, or less.

First, let it be equal.

Then, since *AB* is equal to *LO*, while *AB* is equal to *BC*, and *OL* to *OM*, the two sides *AB*, *BC* are equal to the two sides *LO*, *OM* respectively; and, by hypothesis, the base *AC* is equal to the base *LM*; therefore the angle *ABC* is equal to the angle *LOM*. {I. 8}

For the same reason the angle *DEF* is also equal to the angle *MON*, and further the angle *GHK* to the angle *NOL*; therefore the three angles *ABC*, *DEF*, *GHK* are equal to the three angles *LOM*, *MON*, *NOL*.

But the three angles *LOM*, *MON*, *NOL* are equal to four right angles; therefore the angles *ABC*, *DEF*, *GHK* are equal to four right angles.

But they are also, by hypothesis, less than four right angles: which is absurd.

Therefore *AB* is not equal to *LO*.

I say next that neither is *AB* less than *LO*.

For, if possible, let it be so, and let *OP* be made equal to *AB*, and *OQ* equal to *BC*, and let *PQ* be joined.

Then, since *AB* is equal to *BC*, *OP* is also equal to *OQ*, so that the remainder *LP* is equal to *QM*.

Therefore *LM* is parallel to *PQ*, {VI. 2} and *LMO* is equiangular with *PQO*; {I. 29} therefore, as *OL* is to *LM*, so is *OP* to *PQ*; {VI. 4} and alternately, as *LO* is to *OP*, so is *LM* to *PQ*. {V. 16}

But *LO* is greater than *OP*; therefore *LM* is also greater than *PQ*.

But *LM* was made equal to *AC*; therefore *AC* is also greater than *PQ*.

Since, then, the two sides *AB*, *BC* are equal to the two sides *PO*, *OQ*, and the base *AC* is greater than the base *PQ*, therefore the angle *ABC* is greater than the angle *POQ*. {I. 25}

Similarly we can prove that the angle *DEF* is also greater than the angle *MON*, and the angle *GHK* greater than the angle *NOL*.

Therefore the three angles *ABC*, *DEF*, *GHK* are greater than the three angles *LOM*, *MON*, *NOL*.

But, by hypothesis, the angles *ABC*, *DEF*, *GHK* are less than four right angles; therefore the angles *LOM*, *MON*, *NOL* are much less than four right angles.

But they are also equal to four right angles: which is absurd.

Therefore *AB* is not less than *LO*.

And it was proved that neither is it equal; therefore *AB* is greater than *LO*.

Let then *OR* be set up from the point *O* at right angles to the plane of the circle *LMN*, {XI. 12} and let the square on *OR* be equal to that area by which the square on *AB* is greater than the square on *LO*; {Lemma} let *RL*, *RM*, *RN* be joined.

Then, since *RO* is at right angles to the plane of the circle *LMN*, therefore *RO* is also at right angles to each of the straight lines *LO*, *MO*, *NO*.

And, since *LO* is equal to *OM*, while *OR* is common and at right angles, therefore the base *RL* is equal to the base *RM*. {I. 4}

For the same reason *RN* is also equal to each of the straight lines *RL*, *RM*; therefore the three straight lines *RL*, *RM*, *RN* are equal to one another.

Next, since by hypothesis the square on *OR* is equal to that area by which the square on *AB* is greater than the square on *LO*, therefore the square on *AB* is equal to the squares on *LO*, *OR*.

But the square on *LR* is equal to the squares on *LO*, *OR*, for the angle *LOR* is right; {I. 47} therefore the square on *AB* is equal to the square on *RL*; therefore *AB* is equal to *RL*.

But each of the straight lines *BC*, *DE*, *EF*, *GH*, *HK* is equal to *AB*, while each of the straight lines *RM*, *RN* is equal to *RL*; therefore each of the straight lines *AB*, *BC*, *DE*, *EF*, *GH*, *HK* is equal to each of the straight lines *RL*, *RM*, *RN*.

And, since the two sides *LR*, *RM* are equal to the two sides *AB*, *BC*, and the base *LM* is by hypothesis equal to the base *AC*, therefore the angle *LRM* is equal to the angle *ABC*. {I. 8}

For the same reason the angle *MRN* is also equal to the angle *DEF*, and the angle *LRN* to the angle *GHK*.

Therefore, out of the three plane angles *LRM*, *MRN*, *LRN*, which are equal to the three given angles *ABC*, *DEF*, *GHK*, the solid angle at *R* has been constructed, which is contained by the angles *LRM*, *MRN*, *LRN*. Q. E. F.

LEMMA.

But how it is possible to take the square on *OR* equal to that area by which the square on *AB* is greater than the square on *LO*, we can show as follows.

Let the straight lines *AB*, *LO* be set out, and let *AB* be the greater; let the semicircle *ABC* be described on *AB*, and into the semicircle *ABC* let *AC* be fitted equal to the straight line *LO*, not being greater than the diameter *AB*; {IV. 1} let *CB* be joined

Since then the angle *ACB* is an angle in the semicircle *ACB*, therefore the angle *ACB* is right. {III. 31}

Therefore the square on *AB* is equal to the squares on *AC*, *CB*. {I. 47}

Hence the square on *AB* is greater than the square on *AC* by the square on *CB*.

But *AC* is equal to *LO*.

Therefore the square on *AB* is greater than the square on *LO* by the square on *CB*.

If then we cut off *OR* equal to *BC*, the square on *AB* will be greater than the square on *LO* by the square on *OR*. Q. E. F.

PROPOSITION 24.

If a solid be contained by parallel planes, the opposite planes in it are equal and parallelogrammic.

For let the solid *CDHG* be contained by the parallel planes *AC*, *GF*, *AH*, *DF*, *BF*, *AE*; I say that the opposite planes in it are equal and parallelogrammic.

For, since the two parallel planes *BG*, *CE* are cut by the plane *AC*, their common sections are parallel. {XI. 16}

Therefore *AB* is parallel to *DC*.

Again, since the two parallel planes *BF*, *AE* are cut by the plane *AC*, their common sections are parallel. {XI. 16}

Therefore *BC* is parallel to *AD*.

But *AB* was also proved parallel to *DC*; therefore *AC* is a parallelogram.

Similarly we can prove that each of the planes *DF*, *FG*, *GB*, *BF*, *AE* is a parallelogram.

Let *AH*, *DF* be joined.

Then, since *AB* is parallel to *DC*, and *BH* to *CF*, the two straight lines *AB*, *BH* which meet one another are parallel to the two straight lines *DC*, *CF* which meet one another, not in the same plane; therefore they will contain equal angles; {XI. 10} therefore the angle *ABH* is equal to the angle *DCF*.

And, since the two sides *AB*, *BH* are equal to the two sides *DC*, *CF*, {I. 34} and the angle *ABH* is equal to the angle *DCF*, therefore the base *AH* is equal to the base *DF*, and the triangle *ABH* is equal to the triangle *DCF*. {I. 4}

And the parallelogram *BG* is double of the triangle *ABH*, and the parallelogram *CE* double of the triangle *DCF*; {I. 34} therefore the parallelogram *BG* is equal to the parallelogram *CE*.

Similarly we can prove that *AC* is also equal to *GF*, and *AE* to *BF*.

Therefore etc. Q. E. D.

PROPOSITION 25.

If a parallelepipedal solid be cut by a plane which is parallel to the opposite planes, *then*, *as the base is to the base*, *so will the solid be to the solid.*

For let the parallelepipedal solid *ABCD* be cut by the plane *FG* which is parallel to the opposite planes *RA*, *DH*; I say that, as the base *AEFV* is to the base *EHCF*, so is the solid *ABFU* to the solid *EGCD*.

For let *AH* be produced in each direction, let any number of straight lines whatever, *AK*, *KL*, be made equal to *AE*, and any number whatever, *HM*, *MN*, equal to *EH*; and let the parallelograms *LP*, *KV*, *HW*, *MS* and the solids *LQ*, *KR*, *DM*, *MT* be completed.

Then, since the straight lines *LK*, *KA*, *AE* are equal to one another, the parallelograms *LP*, *KV*, *AF* are also equal to one another, *KO*, *KB*, *AG* are equal to one another, and further *LX*, *KQ*, *AR* are equal to one another, for they are opposite. {XI. 24}

For the same reason the parallelograms *EC*, *HW*, *MS* are also equal to one another, *HG*, *HI*, *IN* are equal to one another, and further *DH*, *MY*, *NT* are equal to one another.

Therefore in the solids *LQ*, *KR*, *AU* three planes are equal to three planes.

But the three planes are equal to the three opposite; therefore the three solids *LQ*, *KR*, *AU* are equal to one another.

For the same reason the three solids *ED*, *DM*, *MT* are also equal to one another.

Therefore, whatever multiple the base *LF* is of the base *AF*, the same multiple also is the solid *LU* of the solid *AU*.

For the same reason, whatever multiple the base *NF* is of the base *FH*, the same multiple also is the solid *NU* of the solid *HU*.

And, if the base *LF* is equal to the base *NF*, the solid *LU* is also equal to the solid *NU*; if the base *LF* exceeds the base *NF*, the solid *LU* also exceeds the solid *NU*; and, if one falls short, the other falls short.

Therefore, there being four magnitudes, the two bases *AF*, *FH*, and the two solids *AU*, *UH*, equimultiples have been taken of the base *AF* and the solid *AU*, namely the base *LF* and the solid *LU*, and equimultiples of the base *HF* and the solid *HU*, namely the base *NF* and the solid *NU*, and it has been proved that, if the base *LF* exceeds the base *FN*, the solid *LU* also exceeds the solid *NU*, if the bases are equal, the solids are equal, and if the base falls short, the solid falls short.

Therefore, as the base *AF* is to the base *FH*, so is the solid *AU* to the solid *UH*. {V. Def. 5} Q. E. D.

PROPOSITION 26.

On a given straight line, and at a given point on it, *to construct a solid angle equal to a given solid angle.*

Let *AB* be the given straight line, *A* the given point on it, and the angle at *D*, contained by the angles *EDC*, *EDF*, *FDC*, the given solid angle; thus it is required to construct on the straight line *AB*, and at the point *A* on it, a solid angle equal to the solid angle at *D*.

For let a point *F* be taken at random on *DF*, let *FG* be drawn from *F* perpendicular to the plane through *ED*, *DC*, and let it meet the plane at *G*, {XI. 11} let *DG* be joined, let there be constructed on the straight line *AB* and at the point *A* on it the angle *BAL* equal to the angle *EDC*, and the angle *BAK* equal to the angle *EDG*, {I. 23} let *AK* be made equal to *DG*, let *KH* be set up from the point *K* at right angles to the plane through *BA*, *AL*, {XI. 12} let *KH* be made equal to *GF*, and let *HA* be joined; I say that the solid angle at *A*, contained by the angles *BAL*, *BAH*, *HAL* is equal to the solid angle at *D* contained by the angles *EDC*, *EDF*, *FDC*.

For let *AB*, *DE* be cut off equal to one another, and let *HB*, *KB*, *FE*, *GE* be joined.

Then, since *FG* is at right angles to the plane of reference, it will also make right angles with all the straight lines which meet it and are in the plane of reference; {XI. Def. 3} therefore each of the angles *FGD*, *FGE* is right.

For the same reason each of the angles *HKA*, *HKB* is also right.

And, since the two sides *KA*, *AB* are equal to the two sides *GD*, *DE* respectively, and they contain equal angles, therefore the base *KB* is equal to the base *GE*. {I. 4}

But *KH* is also equal to *GF*, and they contain right angles; therefore *HB* is also equal to *FE*. {I. 4}

Again, since the two sides *AK*, *KH* are equal to the two sides *DG*, *GF*, and they contain right angles, therefore the base *AH* is equal to the base *FD*. {I. 4}

But *AB* is also equal to *DE*; therefore the two sides *HA*, *AB* are equal to the two sides *DF*, *DE*.

And the base *HB* is equal to the base *FE*; therefore the angle *BAH* is equal to the angle *EDF*. {I. 8}

For the same reason the angle *HAL* is also equal to the angle *FDC*.

And the angle *BAL* is also equal to the angle *EDC*.

Therefore on the straight line *AB*, and at the point *A* on it, a solid angle has been constructed equal to the given solid angle at *D*. Q. E. F.

PROPOSITION 27.

On a given straight line to describe a parallelepipedal solid similar and similarly situated to a given parallelepipedal solid.

Let *AB* be the given straight line and *CD* the given parallelepipedal solid; thus it is required to describe on the given straight line *AB* a parallelepipedal solid similar and similarly situated to the given parallelepipedal solid *CD*.

For on the straight line *AB* and at the point *A* on it let the solid angle, contained by the angles *BAH*, *HAK*, *KAB*, be constructed equal to the solid angle at *C*, so that the angle *BAH* is equal to the angle *ECF*, the angle *BAK* equal to the angle *ECG*, and the angle *KAH* to the angle *GCF*; and let it be contrived that, as *EC* is to *CG*, so is *BA* to *AK*, and, as *GC* is to *CF*, so is *KA* to *AH*. {VI. 12}

Therefore also, ex aequali, as *EC* is to *CF*, so is *BA* to *AH*. {V. 22}

Let the parallelogram *HB* and the solid *AL* be completed.

Now since, as *EC* is to *CG*, so is *BA* to *AK*, and the sides about the equal angles *ECG*, *BAK* are thus proportional, therefore the parallelogram *GE* is similar to the parallelogram *KB*.

For the same reason the parallelogram *KH* is also similar to the parallelogram *GF*, and further *FE* to *HB*; therefore three parallelograms of the solid *CD* are similar to three parallelograms of the solid *AL*.

But the former three are both equal and similar to the three opposite parallelograms, and the latter three are both equal and similar to the three opposite parallelograms; therefore the whole solid *CD* is similar to the whole solid *AL*. {XI. Def. 9}

Therefore on the given straight line *AB* there has been described *AL* similar and similarly situated to the given parallelepipedal solid *CD*. Q. E. F.

PROPOSITION 28.

If a parallelepipedal solid be cut by a plane through the diagonals of the opposite planes, *the solid will be bisected by the plane.*

For let the parallelepipedal solid *AB* be cut by the plane *CDEF* through the diagonals *CF*, *DE* of opposite planes; I say that the solid *AB* will be bisected by the plane *CDEF*.

For, since the triangle *CGF* is equal to the triangle *CFB*, {I. 34} and *ADE* to *DEH*, while the parallelogram *CA* is also equal to the parallelogram *EB*, for they are opposite, and *GE* to *CH*, therefore the prism contained by the two triangles *CGF*, *ADE* and the three parallelograms *GE*, *AC*, *CE* is also equal to the prism contained by the two triangles *CFB*, *DEH* and the three parallelograms *CH*, *BE*, *CE*; for they are contained by planes equal both in multitude and in magnitude. {XI. Def. 10}

Hence the whole solid *AB* is bisected by the plane *CDEF*. Q. E. D.

PROPOSITION 29.

Parallelepipedal solids which are on the same base and of the same height, and in which the extremities of the sides which stand up are on the same straight lines, are equal to one another.

Let *CM*, *CN* be parallelepipedal solids on the same base *AB* and of the same height, and let the extremities of their sides which stand up, namely *AG*, *AF*, *LM*, *LN*, *CD*, *CE*, *BH*, *BK*, be on the same straight lines *FN*, *DK*; I say that the solid *CM* is equal to the solid *CN*.

For, since each of the figures *CH*, *CK* is a parallelogram, *CB* is equal to each of the straight lines *DH*, *EK*, {I. 34} hence *DH* is also equal to *EK*.

Let *EH* be subtracted from each; therefore the remainder *DE* is equal to the remainder *HK*.

Hence the triangle *DCE* is also equal to the triangle *HBK*, {I. 8, 4} and the parallelogram *DG* to the parallelogram *HN*. {I. 36}

For the same reason the triangle *AFG* is also equal to the triangle *MLN*.

But the parallelogram *CF* is equal to the parallelogram *BM*, and *CG* to *BN*, for they are opposite; therefore the prism contained by the two triangles *AFG*, *DCE* and the three parallelograms *AD*, *DG*, *CG* is equal to the prism contained by the two triangles *MLN*, *HBK* and the three parallelograms *BM*, *HN*, *BN*.

Let there be added to each the solid of which the parallelogram *AB* is the base and *GEHM* its opposite; therefore the whole parallelepipedal solid *CM* is equal to the whole parallelepipedal solid *CN*.

Therefore etc. Q. E. D.

PROPOSITION 30.

Parallelepipedal solids which are on the same base and of the same height, and in which the extremities of the sides which stand up are not on the same straight lines, are equal to one another.

Let *CM*, *CN* be parallelepipedal solids on the same base *AB* and of the same height, and let the extremities of their sides which stand up, namely *AF*, *AG*, *LM*, *LN*, *CD*, *CE*, *BH*, *BK*, not be on the same straight lines; I say that the solid *CM* is equal to the solid *CN*.

For let *NK*, *DH* be produced and meet one another at *R*, and further let *FM*, *GE* be produced to *P*, *Q*; let *AO*, *LP*, *CQ*, *BR* be joined.

Then the solid *CM*, of which the parallelogram *ACBL* is the base, and *FDHM* its opposite, is equal to the solid *CP*, of which the parallelogram *ACBL* is the base, and *OQRP* its opposite; for they are on the same base *ACBL* and of the same height, and the extremities of their sides which stand up, namely *AF*, *AO*, *LM*, *LP*, *CD*, *CQ*, *BH*, *BR*, are on the same straight lines *FP*, *DR*. {XI. 29}

But the solid *CP*, of which the parallelogram *ACBL* is the base, and *OQRP* its opposite, is equal to the solid *CN*, of which the parallelogram *ACBL* is the base and *GEKN* its opposite; for they are again on the same base *ACBL* and of the same height, and the extremities of their sides which stand up, namely *AG*, *AO*, *CE*, *CQ*, *LN*, *LP*, *BK*, *BR*, are on the same straight lines *GQ*, *NR*.

Hence the solid *CM* is also equal to the solid *CN*.

Therefore etc. Q. E. D.

PROPOSITION 31.

Parallelepipedal solids which are on equal bases and of the same height are equal to one another.

Let the parallelepipedal solids *AE*, *CF*, of the same height, be on equal bases *AB*, *CD*.

I say that the solid *AE* is equal to the solid *CF*.

First, let the sides which stand up, *HK*, *BE*, *AG*, *LM*, *PQ*, *DF*, *CO*, *RS*, be at right angles to the bases *AB*, *CD*; let the straight line *RT* be produced in a straight line with *CR*; on the straight line *RT*, and at the point *R* on it, let the angle *TRU* be constructed equal to the angle *ALB*, {I. 23} let *RT* be made equal to *AL*, and *RU* equal to *LB*, and let the base *RW* and the solid *XU* be completed.

Now, since the two sides *TR*, *RU* are equal to the two sides *AL*, *LB*, and they contain equal angles, therefore the parallelogram *RW* is equal and similar to the parallelogram *HL*.

Since again *AL* is equal to *RT*, and *LM* to *RS*, and they contain right angles, therefore the parallelogram *RX* is equal and similar to the parallelogram *AM*.

For the same reason *LE* is also equal and similar to *SU*; therefore three parallelograms of the solid *AE* are equal and similar to three parallelograms of the solid *XU*.

But the former three are equal and similar to the three opposite, and the latter three to the three opposite; {XI. 24} therefore the whole parallelepipedal solid *AE* is equal to the whole parallelepipedal solid *XU*. {XI. Def. 10}

Let *DR*, *WU* be drawn through and meet one another at *Y*, let *aTb* be drawn through *T* parallel to *DY*, let *PD* be produced to *a*, and let the solids *YX*, *RI* be completed.

Then the solid *XY*, of which the parallelogram *RX* is the base and *Yc* its opposite, is equal to the solid *XU* of which the parallelogram *RX* is the base and *UV* its opposite, for they are on the same base *RX* and of the same height, and the extremities of their sides which stand up, namely *RY*, *RU*, *Tb*, *TW*, *Se*, *Sd*, *Xc*, *XV*, are on the same straight lines *YW*, *eV*. {XI. 29}

But the solid *XU* is equal to *AE*: therefore the solid *XY* is also equal to the solid *AE*.

And, since the parallelogram *RUWT* is equal to the parallelogram *YT* for they are on the same base *RT* and in the same parallels *RT*, *YW*, {I. 35} while *RUWT* is equal to *CD*, since it is also equal to *AB*, therefore the parallelogram *YT* is also equal to *CD*.

But *DT* is another parallelogram; therefore, as the base *CD* is to *DT*, so is *YT* to *DT*. {V. 7}

And, since the parallelepipedal solid *CI* has been cut by the plane *RF* which is parallel to opposite planes, as the base *CD* is to the base *DT*, so is the solid *CF* to the solid *RI*. {XI. 25}

For the same reason, since the parallelepipedal solid *YI* has been cut by the plane *RX* which is parallel to opposite planes, as the base *YT* is to the base

TD, so is the solid *YX* to the solid *RI*. {XI. 25}

But, as the base *CD* is to *DT*, so is *YT* to *DT*; therefore also, as the solid *CF* is to the solid *RI*, so is the solid *YX* to *RI*. {V. 11}

Therefore each of the solids *CF*, *YX* has to *RI* the same ratio; therefore the solid *CF* is equal to the solid *YX*. {V. 9}

But *YX* was proved equal to *AE*; therefore *AE* is also equal to *CF*.

Next, let the sides standing up, *AG*, *HK*, *BE*, *LM*, *CN*, *PQ*, *DF*, *RS*, not be at right angles to the bases *AB*, *CD*; I say again that the solid *AE* is equal to the solid *CF*.

For from the points *K*, *E*, *G*, *M*, *Q*, *F*, *N*, *S* let *KO*, *ET*, *GU*, *MV*, *QW*, *FX*, *NY*, *SI* be drawn perpendicular to the plane of reference, and let them meet the plane at the points *O*, *T*, *U*, *V*, *W*, *X*, *Y*, *I*, and let *OT*, *OU*, *UV*, *TV*, *WX*, *WY*, *YI*, *IX* be joined.

Then the solid *KV* is equal to the solid *QI*, for they are on the equal bases *KM*, *QS* and of the same height, and their sides which stand up are at right angles to their bases. {First part of this Prop.}

But the solid *KV* is equal to the solid *AE*, and *QI* to *CF*; for they are on the same base and of the same height, while the extremities of their sides which stand up are not on the same straight lines. {XI. 30}

Therefore the solid *AE* is also equal to the solid *CF*.

Therefore etc. Q. E. D.

PROPOSITION 32.

Parallelepipedal solids which are of the same height are to one another as their bases.

Let *AB*, *CD* be parallelepipedal solids of the same height; I say that the parallelepipedal solids *AB*, *CD* are to one another as their bases, that is, that,

as the base *AE* is to the base *CF*, so is the solid *AB* to the solid *CD*.

For let *FH* equal to *AE* be applied to *FG*, {I. 45} and on *FH* as base, and with the same height as that of *CD*, let the parallelepipedal solid *GK* be completed.

Then the solid *AB* is equal to the solid *GK*; for they are on equal bases *AE*, *FH* and of the same height. {XI. 31}

And, since the parallelepipedal solid *CK* is cut by the plane *DG* which is parallel to opposite planes, therefore, as the base *CF* is to the base *FH*, so is the solid *CD* to the solid *DH*. {XI. 25}

But the base *FH* is equal to the base *AE*, and the solid *GK* to the solid *AB*; therefore also, as the base *AE* is to the base *CF*, so is the solid *AB* to the solid *CD*.

Therefore etc. Q. E. D.

PROPOSITION 33.

Similar parallelepipedal solids are to one another in the triplicate ratio of their corresponding sides.

Let *AB*, *CD* be similar parallelepipedal solids, and let *AE* be the side corresponding to *CF*; I say that the solid *AB* has to the solid *CD* the ratio triplicate of that which *AE* has to *CF*.

For let *EK*, *EL*, *EM* be produced in a straight line with *AE*, *GE*, *HE*, let *EK* be made equal to *CF*, *EL* equal to *FN*, and further *EM* equal to *FR*, and let the parallelogram *KL* and the solid *KP* be completed.

Now, since the two sides *KE*, *EL* are equal to the two sides *CF*, *FN*, while the angle *KEL* is also equal to the angle *CFN*, inasmuch as the angle *AEG* is also equal to the angle *CFN* because of the similarity of the solids *AB*, *CD*,

therefore the parallelogram *KL* is equal <and similar> to the parallelogram *CN*.

For the same reason the parallelogram *KM* is also equal and similar to *CR*, and further *EP* to *DF*; therefore three parallelograms of the solid *KP* are equal and similar to three parallelograms of the solid *CD*.

But the former three parallelograms are equal and similar to their opposites, and the latter three to their opposites; {XI. 24} therefore the whole solid *KP* is equal and similar to the whole solid *CD*. {XI. Def. 10}

Let the parallelogram *GK* be completed, and on the parallelograms *GK*, *KL* as bases, and with the same height as that of *AB*, let the solids *EO*, *LQ* be completed.

Then since; owing to the similarity of the solids *AB*, *CD*, as *AE* is to *CF*, so is *EG* to *FN*, and *EH* to *FR*, while *CF* is equal to *EK*, *FN* to *EL*, and *FR* to *EM*, therefore, as *AE* is to *EK*, so is *GE* to *EL*, and *HE* to *EM*.

But, as *AE* is to *EK*, so is *AG* to the parallelogram *GK*, as *GE* is to *EL*, so is *GK* to *KL*, and, as *HE* is to *EM*, so is *QE* to *KM*; {VI. 1} therefore also, as the parallelogram *AG* is to *GK*, so is *GK* to *KL*, and *QE* to *KM*.

But, as *AG* is to *GK*, so is the solid *AB* to the solid *EO*, as *GK* is to *KL*, so is the solid *OE* to the solid *QL*, and, as *QE* is to *KM*, so is the solid *QL* to the solid *KP*; {XI. 32} therefore also, as the solid *AB* is to *EO*, so is *EO* to *QL*, and *QL* to *KP*.

But, if four magnitudes be continuously proportional, the first has to the fourth the ratio triplicate of that which it has to the second; {V. Def. 10} therefore the solid *AB* has to *KP* the ratio triplicate of that which *AB* has to *EO*.

But, as *AB* is to *EO*, so is the parallelogram *AG* to *GK*, and the straight line *AE* to *EK* {VI. 1}; hence the solid *AB* has also to *KP* the ratio triplicate of that which *AE* has to *EK*.

But the solid *KP* is equal to the solid *CD*, and the straight line *EK* to *CF*; therefore the solid *AB* has also to the solid *CD* the ratio triplicate of that which the corresponding side of it, *AE*, has to the corresponding side *CF*.

Therefore etc. Q. E. D.

PORISM.

From this it is manifest that, if four straight lines be <continuously> proportional, as the first is to the fourth, so will a parallelepipedal solid on the first be to the similar and similarly described parallelepipedal solid on the second, inasmuch as the first has to the fourth the ratio triplicate of that which it has to the second.

PROPOSITION 34.

In equal parallelepipedal solids the bases are reciprocally proportional to the heights; and those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal.

Let *AB*, *CD* be equal parallelepipedal solids; I say that in the parallelepipedal solids *AB*, *CD* the bases are reciprocally proportional to the heights, that is, as the base *EH* is to the base *NQ*, so is the height of the solid *CD* to the height of the solid *AB*.

First, let the sides which stand up, namely *AG*, *EF*, *LB*, *HK*, *CM*, *NO*, *PD*, *QR*, be at right angles to their bases; I say that, as the base *EH* is to the base *NQ*, so is *CM* to *AG*.

If now the base *EH* is equal to the base *NQ*, while the solid *AB* is also equal to the solid *CD*, *CM* will also be equal to *AG*.

For parallelepipedal solids of the same height are to one another as the bases; {XI. 32} and, as the base *EH* is to *NQ*, so will *CM* be to *AG*, and it is manifest that in the parallelepipedal solids *AB*, *CD* the bases are reciprocally proportional to the heights.

Next, let the base *EH* not be equal to the base *NQ*, but let *EH* be greater.

Now the solid *AB* is equal to the solid *CD*; therefore *CM* is also greater than *AG*.

Let then *CT* be made equal to *AG*, and let the parallelepipedal solid *VC* be completed on *NQ* as base and with *CT* as height.

Now, since the solid *AB* is equal to the solid *CD*, and *CV* is outside them, while equals have to the same the same ratio, {V. 7} therefore, as the solid *AB* is to the solid *CV*, so is the solid *CD* to the solid *CV*.

But, as the solid *AB* is to the solid *CV*, so is the base *EH* to the base *NQ*, for the solids *AB*, *CV* are of equal height; {XI. 32} and, as the solid *CD* is to the solid *CV*, so is the base *MQ* to the base *TQ* {XI. 25} and *CM* to *CT* {VI. 1}; therefore also, as the base *EH* is to the base *NQ*, so is *MC* to *CT*.

But *CT* is equal to *AG*; therefore also, as the base *EH* is to the base *NQ*, so is *MC* to *AG*.

Therefore in the parallelepipedal solids *AB*, *CD* the bases are reciprocally proportional to the heights.

Again, in the parallelepipedal solids *AB*, *CD* let the bases be reciprocally proportional to the heights, that is, as the base *EH* is to the base *NQ*, so let the height of the solid *CD* be to the height of the solid *AB*; I say that the solid *AB* is equal to the solid *CD*.

Let the sides which stand up be again at right angles to the bases.

Now, if the base *EH* is equal to the base *NQ*, and, as the base *EH* is to the base *NQ*, so is the height of the solid *CD* to the height of the solid *AB*, therefore the height of the solid *CD* is also equal to the height of the solid *AB*.

But parallelepipedal solids on equal bases and of the same height are equal to one another; {XI. 31} therefore the solid *AB* is equal to the solid *CD*.

Next, let the base *EH* not be equal to the base *NQ*, but let *EH* be greater; therefore the height of the solid *CD* is also greater than the height of the solid *AB*, that is, *CM* is greater than *AG*.

Let *CT* be again made equal to *AG*, and let the solid *CV* be similarly completed.

Since, as the base *EH* is to the base *NQ*, so is *MC* to *AG*, while *AG* is equal to *CT*, therefore, as the base *EH* is to the base *NQ*, so is *CM* to *CT*.

But, as the base *EH* is to the base *NQ*, so is the solid *AB* to the solid *CV*, for the solids *AB*, *CV* are of equal height; {XI. 32} and, as *CM* is to *CT*, so is the base *MQ* to the base *QT* {VI. 1} and the solid *CD* to the solid *CV*. {XI. 25}

Therefore also, as the solid *AB* is to the solid *CV*, so is the solid *CD* to the solid *CV*; therefore each of the solids *AB*, *CD* has to *CV* the same ratio.

Therefore the solid *AB* is equal to the solid *CD*. {V. 9}

Now let the sides which stand up, *FE*, *BL*, *GA*, *HK*, *ON*, *DP*, *MC*, *RQ*, not be at right angles to their bases; let perpendiculars be drawn from the points *F*, *G*, *B*, *K*, *O*, *M*, *D*, *R* to the planes through *EH*, *NQ*, and let them meet the planes at *S*, *T*, *U*, *V*, *W*, *X*, *Y*, *a*, and let the solids *FV*, *Oa* be completed; I say that, in this case too, if the solids *AB*, *CD* are equal, the bases are reciprocally proportional to the heights, that is, as the base *EH* is to the base *NQ*, so is the height of the solid *CD* to the height of the solid *AB*.

Since the solid *AB* is equal to the solid *CD*, while *AB* is equal to *BT*, for they are on the same base *FK* and of the same height; {XI. 29, 30} and the solid *CD* is equal to *DX*, for they are again on the same base *RO* and of the same height; {*id.*} therefore the solid *BT* is also equal to the solid *DX*.

Therefore, as the base *FK* is to the base *OR*, so is the height of the solid *DX* to the height of the solid *BT*. {Part 1.}

But the base *FK* is equal to the base *EH*, and the base *OR* to the base *NQ*; therefore, as the base *EH* is to the base *NQ*, so is the height of the solid *DX* to the height of the solid *BT*.

But the solids *DX*, *BT* and the solids *DC*, *BA* have the same heights respectively; therefore, as the base *EH* is to the base *NQ*, so is the height of the solid *DC* to the height of the solid *AB*.

Therefore in the parallelepipedal solids *AB*, *CD* the bases are reciprocally proportional to the heights.

Again, in the parallelepipedal solids *AB*, *CD* let the bases be reciprocally proportional to the heights, that is, as the base *EH* is to the base *NQ*, so let the height of the solid *CD* be to the height of the solid *AB*; I say that the solid *AB* is equal to the solid *CD*.

For, with the same construction, since, as the base *EH* is to the base *NQ*, so is the height of the solid *CD* to the height of the solid *AB*, while the base *EH* is equal to the base *FK*, and *NQ* to *OR*, therefore, as the base *FK* is to the base *OR*, so is the height of the solid *CD* to the height of the solid *AB*.

But the solids *AB*, *CD* and *BT*, *DX* have the same heights respectively; therefore, as the base *FK* is to the base *OR*, so is the height of the solid *DX* to the height of the solid *BT*.

Therefore in the parallelepipedal solids *BT*, *DX* the bases are reciprocally proportional to the heights; therefore the solid *BT* is equal to the solid *DX*. {Part 1.}

But *BT* is equal to *BA*, for they are on the same base *FK* and of the same height; {XI. 29, 30} and the solid *DX* is equal to the solid *DC*. {*id.*}

Therefore the solid *AB* is also equal to the solid *CD*. Q. E. D.

PROPOSITION 35.

If there be two equal plane angles, and on their vertices there be set up elevated straight lines containing equal angles with the original straight lines respectively, if on the elevated straight lines points be taken at random and perpendiculars be drawn from them to the planes in which the original angles are, and if from the points so arising in the planes straight lines be joined to the vertices of the original angles, they will contain, with the elevated straight lines, equal angles.

Let the angles *BAC*, *EDF* be two equal rectilineal angles, and from the points *A*, *D* let the elevated straight lines *AG*, *DM* be set up containing, with the original straight lines, equal angles respectively, namely, the angle *MDE* to the angle *GAB* and the angle *MDF* to the angle *GAC*, let points *G*, *M* be taken at random on *AG*, *DM*, let *GL*, *MN* be drawn from the points *G*, *M* perpendicular to the planes through *BA*, *AC* and *ED*, *DF*, and let them meet the planes at *L*, *N*, and let *LA*, *ND* be joined; I say that the angle *GAL* is equal to the angle *MDN*.

Let *AH* be made equal to *DM*, and let *HK* be drawn through the point *H* parallel to *GL*.

But *GL* is perpendicular to the plane through *BA*, *AC*; therefore *HK* is also perpendicular to the plane through. *BA*, *AC*. {XI. 8}

From the points *K*, *N* let *KC*, *NF*, *KB*, *NE* be drawn perpendicular to the straight lines *AC*, *DF*, *AB*, *DE*, and let *HC*, *CB*, *MF*, *FE* be joined.

Since the square on *HA* is equal to the squares on *HK*, *KA*, and the squares on *KC*, *CA* are equal to the square on *KA*, {I. 47} therefore the square on *HA* is also equal to the squares on *HK*, *KC*, *CA*.

But the square on *HC* is equal to the squares on *HK*, *KC*; {I. 47} therefore the square on *HA* is equal to the squares on *HC*, *CA*.

Therefore the angle *HCA* is right. {I. 48}

For the same reason the angle *DFM* is also right.

Therefore the angle *ACH* is equal to the angle *DFM*.

But the angle *HAC* is also equal to the angle *MDF*.

Therefore *MDF*, *HAC* are two triangles which have two angles equal to two angles respectively, and one side equal to one side, namely, that subtending one of the equal angles, that is, *HA* equal to *MD*; therefore they will also have the remaining sides equal to the remaining sides respectively. {I. 26}

Therefore *AC* is equal to *DF*.

Similarly we can prove that *AB* is also equal to *DE*.

Since then *AC* is equal to *DF*, and *AB* to *DE*, the two sides *CA*, *AB* are equal to the two sides *FD*, *DE*.

But the angle *CAB* is also equal to the angle *FDE*; therefore the base *BC* is equal to the base *EF*, the triangle to the triangle, and the remaining angles to the remaining angles; {I. 4} therefore the angle *ACB* is equal to the angle *DFE*.

But the right angle *ACK* is also equal to the right angle *DFN*; therefore the remaining angle *BCK* is also equal to the remaining angle *EFN*.

For the same reason the angle *CBK* is also equal to the angle *FEN*.

Therefore *BCK*, *EFN* are two triangles which have two angles equal to two angles respectively, and one side equal to one side, namely, that adjacent to the equal angles, that is, *BC* equal to *EF*; therefore they will also have the remaining sides equal to the remaining sides. {I. 26}

Therefore *CK* is equal to *FN*.

But *AC* is also equal to *DF*; therefore the two sides *AC*, *CK* are equal to the two sides *DF*, *FN*; and they contain right angles.

Therefore the base *AK* is equal to the base *DN*. {I. 4}

And, since *AH* is equal to *DM*, the square on *AH* is also equal to the square on *DM*.

But the squares on *AK*, *KH* are equal to the square on *AH*, for the angle *AKH* is right; {I. 47} and the squares on *DN*, *NM* are equal to the square on *DM*, for the angle *DNM* is right; {I. 47} therefore the squares on *AK*, *KH* are equal to the squares on *DN*, *NM*; and of these the square on *AK* is equal to the square on *DN*; therefore the remaining square on *KH* is equal to the square on *NM*; therefore *HK* is equal to *MN*.

And, since the two sides *HA*, *AK* are equal to the two sides *MD*, *DN* respectively, and the base *HK* was proved equal to the base *MN*, therefore the angle *HAK* is equal to the angle *MDN*. {I. 8}

Therefore etc.

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From this it is manifest that, if there be two equal plane angles, and if there be set up on them elevated straight lines which are equal and contain equal angles with the original straight lines respectively, the perpendiculars drawn from their extremities to the planes in which are the original angles are equal to one another. Q. E. D.

PROPOSITION 36.

If three straight lines be proportional, the parallelepipedal solid formed out of the three is equal to the parallelepipedal solid on the mean which is equilateral, but equiangular with the aforesaid solid.

Let *A*, *B*, *C* be three straight lines in proportion, so that, as *A* is to *B*, so is *B* to *C*; I say that the solid formed out of *A*, *B*, *C* is equal to the solid on *B*

which is equilateral, but equiangular with the aforesaid solid.

Let there be set out the solid angle at *E* contained by the angles *DEG*, *GEF*, *FED*, let each of the straight lines *DE*, *GE*, *EF* be made equal to *B*, and let the parallelepipedal solid *EK* be completed, let *LM* be made equal to *A*, and on the straight line *LM*, and at the point *L* on it, let there be constructed a solid angle equal to the solid angle at *E*, namely that contained by *NLO*, *OLM*, *MLN*; let *LO* be made equal to *B*, and *LN* equal to *C*.

Now, since, as *A* is to *B*, so is *B* to *C*, while *A* is equal to *LM*, *B* to each of the straight lines *LO*, *ED*, and *C* to *LN*, therefore, as *LM* is to *EF*, so is *DE* to *LN*.

Thus the sides about the equal angles *NLM*, *DEF* are reciprocally proportional; therefore the parallelogram *MN* is equal to the parallelogram *DF*. {VI. 14}

And, since the angles *DEF*, *NLM* are two plane rectilineal angles, and on them the elevated straight lines *LO*, *EG* are set up which are equal to one another and contain equal angles with the original straight lines respectively, therefore the perpendiculars drawn from the points *G*, *O* to the planes through *NL*, *LM* and *DE*, *EF* are equal to one another; {XI. 35, Por.} hence the solids *LH*, *EK* are of the same height.

But parallelepipedal solids on equal bases and of the same height are equal to one another; {XI. 31} therefore the solid *HL* is equal to the solid *EK*.

And *LH* is the solid formed out of *A*, *B*, *C*, and *EK* the solid on *B*; therefore the parallelepipedal solid formed out of *A*, *B*, *C* is equal to the solid on *B* which is equilateral, but equiangular with the aforesaid solid. Q. E. D.

PROPOSITION 37.

If four straight lines be proportional, the parallelepipedal solids on them which are similar and similarly described will also be proportional; and, if the parallelepipedal solids on them which are similar and similarly described be proportional, the straight lines will themselves also be proportional.

Let *AB*, *CD*, *EF*, *GH* be four straight lines in proportion, so that, as *AB* is to *CD*, so is *EF* to *GH*; and let there be described on *AB*, *CD*, *EF*, *GH* the similar and similarly situated parallelepipedal solids *KA*, *LC*, *ME*, *NG*; I say that, as *KA* is to *LC*, so is *ME* to *NG*.

For, since the parallelepipedal solid *KA* is similar to *LC*, therefore *KA* has to *LC* the ratio triplicate of that which *AB* has to *CD*. {XI. 33}

For the same reason *ME* also has to *NG* the ratio triplicate of that which *EF* has to *GH*. {*id.*}

And, as *AB* is to *CD*, so is *EF* to *GH*.

Therefore also, as *AK* is to *LC*, so is *ME* to *NG*.

Next, as the solid *AK* is to the solid *LC*, so let the solid *ME* be to the solid *NG*; I say that, as the straight line *AB* is to *CD*, so is *EF* to *GH*.

For since, again, *KA* has to *LC* the ratio triplicate of that which *AB* has to *CD*, {XI. 33} and *ME* also has to *NG* the ratio triplicate of that which *EF* has to *GH*, {*id.*} and, as *KA* is to *LC*, so is *ME* to *NG*, therefore also, as *AB* is to *CD*, so is *EF* to *GH*.

Therefore etc. Q. E. D.

PROPOSITION 38.

If the sides of the opposite planes of a cube be bisected, and planes be carried through the points of section, the common section of the planes and the diameter of the cube bisect one another.

For let the sides of the opposite planes *CF*, *AH* of the cube *AF* be bisected at the points *K*, *L*, *M*, *N*, *O*, *Q*, *P*, *R*, and through the points of section let the planes *KN*, *OR* be carried; let *US* be the common section of the planes, and *DG* the diameter of the cube *AF*.

I say that *UT* is equal to *TS*, and *DT* to *TG*.

For let *DU*, *UE*, *BS*, *SG* be joined.

Then, since *DO* is parallel to *PE*, the alternate angles *DOU*, *UPE* are equal to one another. {I. 29}

And, since *DO* is equal to *PE*, and *OU* to *UP*, and they contain equal angles, therefore the base *DU* is equal to the base *UE*, the triangle *DOU* is equal to the triangle *PUE*, and the remaining angles are equal to the remaining angles; {I. 4} therefore the angle *OUD* is equal to the angle *PUE*.

For this reason *DUE* is a straight line. {I. 14}

For the same reason, *BSG* is also a straight line, and *BS* is equal to *SG*.

Now, since *CA* is equal and parallel to *DB*, while *CA* is also equal and parallel to *EG*, therefore *DB* is also equal and parallel to *EG*. {XI. 9}

And the straight lines *DE*, *BG* join their extremities; therefore *DE* is parallel to *BG*. {I. 33}

Therefore the angle *EDT* is equal to the angle *BGT*, for they are alternate; {I. 29} and the angle *DTU* is equal to the angle *GTS*. {I. 15}

Therefore *DTU*, *GTS* are two triangles which have two angles equal to two angles, and one side equal to one side, namely that subtending one of the equal angles, that is, *DU* equal to *GS*, for they are the halves of *DE*, *BG*; therefore they will also have the remaining sides equal to the remaining sides. {I. 26}

Therefore *DT* is equal to *TG*, and *UT* to *TS*.

Therefore etc. Q. E. D.

PROPOSITION 39.

If there be two prisms of equal height, and one have a parallelogram as base and the other a triangle, and if the parallelogram be double of the triangle, the prisms will be equal.

Let *ABCDEF*, *GHKLMN* be two prisms of equal height, let one have the parallelogram *AF* as base, and the other the triangle *GHK*, and let the parallelogram *AF* be double of the triangle *GHK*; I say that the prism *ABCDEF* is equal to the prism *GHKLMN*.

For let the solids *AO*, *GP* be completed.

Since the parallelogram *AF* is double of the triangle *GHK*, while the parallelogram *HK* is also double of the triangle *GHK*, {I. 34} therefore the parallelogram *AF* is equal to the parallelogram *HK*.

But parallelepipedal solids which are on equal bases and of the same height are equal to one another; {XI. 31} therefore the solid *AO* is equal to the solid *GP*.

And the prism *ABCDEF* is half of the solid *AO*, and the prism *GHKLMN* is half of the solid *GP*; {XI. 28} therefore the prism *ABCDEF* is equal to the prism *GHKLMN*.

Therefore etc. Q. E. D.

ENDNOTES.

[1] 1. the plane of reference, τὸ ὑποκείμενον ἐπίπεδον, the plane laid down or assumed.

[2] 2. more elevated, μετεωροτέρῳ.

BOOK XII. HISTORICAL NOTE.

The predominant feature of Book XII. is the use of the *method of exhaustion*, which is applied in Propositions 2, 3-5, 10, 11, 12, and (in a slightly different form) in Propositions 16-18. We conclude therefore that for the content of this Book Euclid was greatly indebted to Eudoxus, to whom the discovery of the method of exhaustion is attributed. The evidence for this attribution comes mainly from Archimedes. (1) In the preface to On the Sphere and Cylinder I., after stating the main results obtained by himself regarding the surface of a sphere or a segment thereof, and the volume and surface of a right cylinder with height equal to its diameter as compared with those of a sphere with the same diameter, Archimedes adds: "Having now discovered that the properties mentioned are true of these figures, I cannot feel any hesitation in setting them side by side both with my former investigations and *with those of the theorems of Eudoxus on solids* which are held to be most irrefragably established, namely that *any pyramid is one third part of the prism which has the same base with the pyramid and equal height* {i.e. Eucl. XII. 7}, and that *any cone is one third part of the cylinder which has the same base with the cone and equal height* {i.e. Eucl. XII. 10}. For, though these properties also were naturally inherent in the figures all along, yet they were in fact unknown to all the many able geometers who lived before Eudoxus and had not been observed by any one." (2) In the preface to the treatise known as the Quadrature of the Parabola Archimedes states the "lemma" assumed by him and known as the "Axiom of Archimedes" (see note on X. 1 above) and proceeds: "Earlier geometers (οἱ πρότερον γεωμέτραι) have also used this lemma; for it is by the use of this same lemma that they have shown that *circles are to one another in the duplicate ratio of their diameters* {Eucl. XII. 2}, and that *spheres are to one another in the triplicate ratio of their diameters* {Eucl. XII. 18}, and further that *every pyramid is one third part of*

the prism which has the same base with the pyramid and equal height {Eucl. XII. 7}; also, that *every cone is one third part of the cylinder which has the same base with the cone and equal height* {Eucl. XII. 10} they proved by assuming *a certain lemma similar to that aforesaid*." Thus in the first passage two theorems of Eucl. XII. are definitely attributed to Eudoxus; and, when Archimedes says, in the second passage, that "earlier geometers" proved these two theorems by means of the lemma known as the "Axiom of Archimedes" and of a lemma similar to it respectively, we can hardly suppose him to be alluding to any other proof than that given by Eudoxus. As a matter of fact, the lemma used by Euclid to prove both propositions (XII. 3-5 and 7, and XII. 10) is the theorem of Eucl. X. 1. As regards the connexion between the two "lemmas" see note on X. 1.

We are not, however, to suppose that none of the *results* obtained by the method of exhaustion had been discovered before the time of Eudoxus (fl. about 368-5 B.C.). Two at least are of earlier date, those of Eucl. XII. 2 and XII. 7.

(*a*) Simplicius (Comment. in Aristot. Phys. p. 61, ed. Diels) quotes Eudemus as saying, in his History of Geometry, that Hippocrates of Chios (fl. say 430 B.C.) first laid it down (ἔθετο) that similar segments of circles are in the ratio of the squares on their bases and that he proved this (ἐδείκνυεν) by proving (ἐκ τοῦ δεῖξαι) that the squares on the diameters have the same ratio as the (whole) circles. We know nothing of the method by which Hippocrates proved this proposition; but, having regard to the evidence from Archimedes quoted above, it is not permissible to suppose that the method was the fully developed *method of exhaustion* as we know it.

(*b*) As regards the two theorems about the volume of a pyramid and of a cone respectively, which Eudoxus was the first to prove, we now have authentic evidence in the short treatise by Archimedes discovered by Heiberg

in a MS. at Constantinople in 1906 and published in Hermes the following year (see now Archimedis opera omnia, ed. Heiberg, 2. ed., Vol. II., 1913, pp. 425-507; T. L. Heath, The Method of Archimedes, Cambridge, 1912). The said treatise, complete in all essentials, bears the title Ἀρχιμήδους περὶ τῶν μηχανικῶν θεωρημάτων πρὸς Ἐρατοσθένην ἔφοδος. This "Method" (or "Plan of attack"), addressed to Eratosthenes, is none other than the ἐφόδιον on which, according to Suidas, Theodosius wrote a commentary, and which is several times cited by Heron in his Metrica; its discovery adds a new and important chapter to the history of the integral calculus. In the preface to this work Archimedes alludes to the theorems which he first discovered by means of mechanical considerations, but proved afterwards by geometry, because the investigation by means of mechanics did not constitute a rigid proof; he observes, however, that the mechanical method is of great use for the *discovery* of theorems, and it is much easier to provide the rigid proof when the fact to be proved has once been discovered than it would be if nothing were known to begin with. He goes on: "Hence too, in the case of those theorems the proof of which was first discovered by Eudoxus, namely those relating to the cone and the pyramid, that the cone is one third part of the cylinder, and the pyramid one third part of the prism, having the same base and equal height, no small part of the credit will naturally be assigned to Democritus, who was the first to make the statement (of the fact) regarding the said figure {i.e. property}, though without proving it." Hence the *discovery* of the two theorems must now be attributed to Democritus (fl. towards the end of 5th cent. B.C.). The words "without proving it" (χωρὶς ἀποδείξεως) do not mean that Democritus gave no sort of proof, but only that he did not give a proof on the rigorous lines required later; for the same words are used by Archimedes of his own investigations by means of mechanics, which, however, do constitute a reasoned argument. The

character of Archimedes' mechanical arguments combined with a passage of Plutarch about a particular question in infinitesimals said to have been raised by Democritus may perhaps give a clue to the line of Democritus' argument as regards the pyramid. The essential feature of Archimedes' mechanical arguments in this tract is that he regards an area as the sum of an infinite number of *siraight lines* parallel to one another and terminated by the boundary or boundaries of the closed figure the area of which is to be found, and a volume as the sum of an infinite number of *plane sections* parallel to one another: which is of course the same thing as taking (as we do in the integral calculus) the sum of an infinite number of strips of breadth *dx* (say), when *dx* becomes indefinitely small, or the sum of an infinite number of parallel laminae of depth *dz* (say), when *dz* becomes indefinitely small. To give only one instance, we may take the case of the area of a segment of a parabola cut off by a chord.

Let *CBA* be the parabolic segment, *CE* the tangent at *C* meeting the diameter *EBD* through the middle point of the chord *CA* in *E*, so that

.

Draw *AF* parallel to *ED* meeting *CE* produced in *F*. Produce *CB* to *H* so that , where *K* is the point in which *CH* meets *AF*; and suppose *CH* to be a lever.

Let any diameter *MNPO* be drawn meeting the curve in *P* and *CF*, *CK*, *CA* in *M*, *N*, *O* respectively.

Archimedes then observes that

("for this is proved in a lemma"), whence

,

so that, if a straight line *TG* equal to *PO* be placed with its middle point at *H*, the straight line *MO* with centre of gravity at *N*, and the straight line *TG*

with centre of gravity at *H*, will balance about *K*.

Taking all other parts of diameters like *PO* intercepted between the curve and *CA*, and placing equal straight lines with their centres of gravity at *H*, these straight lines collected at *H* will balance (about *K*) all the lines like *MO* parallel to *FA* intercepted within the triangle *CFA* in the positions in which they severally lie in the figure.

Hence Archimedes infers that an area equal to that of the parabolic segment hung at *H* will balance (about *K*) the triangle *CFA* hung at its centre of gravity, the point *X* (a point on *CK* such that), and therefore that

,

from which it follows that

.

The same sort of argument is used for solids, *plane sections* taking the place of *straight lines*.

Archimedes is careful to state once more that this method of argument does not constitute a *proof*. Thus, at the end of the above proposition about the parabolic segment, he adds: “This property is of course not proved by what has just been said; but it has furnished a sort of *indication* (ἔμφασίν τινα) that the conclusion is true.”

Let us now turn to the passage of Plutarch (De Comm. Not. adv. Stoicos XXXIX 3) about Democritus above referred to. Plutarch speaks of Democritus as having raised the question in natural philosophy (φυσικῶς): “if a cone were cut by a plane parallel to the base {by which is clearly meant a plane indefinitely near to the base}, what must we think of the surfaces of the sections, that they are equal or unequal? For, if they are unequal, they will make the cone irregular, as having many indentations, like steps, and

unevennesses; but, if they are equal, the sections will be equal, and the cone will appear to have the property of the cylinder and to be made up of equal, not unequal circles, which is very absurd." The phrase "*made up* of equal...circles" (ἐξ ἴσων συγκείμενος...κύκλων) shows that Democritus already had the idea of a solid being the sum of an infinite number of parallel planes, or indefinitely thin laminae, indefinitely near together: a most important anticipation of the same thought which led to such fruitful results in Archimedes. If then one may hazard a conjecture as to Democritus' argument with regard to a pyramid, it seems probable that he would notice that, if two pyramids of the same height and equal triangular bases are respectively cut by planes parallel to the base and dividing the heights in the same ratio, the corresponding sections of the two pyramids are equal, whence he would infer that the pyramids are equal as being the sum of the same infinite number of equal plane sections or indefinitely thin laminae. (This would be a particular anticipation of Cavalieri's proposition that the areal or solid contents of two figures are equal if two sections of them taken at the same height, whatever the height may be, always give equal straight lines or equal surfaces respectively.) And Democritus would of course see that the three pyramids into which a prism on the same base and of equal height with the original pyramid is divided (as in Eucl. XII. 7) satisfy this test of equality, so that the pyramid would be one third part of the prism. The extension to a pyramid with a polygonal base would be easy. And Democritus may have stated the proposition for the cone (of course without an absolute proof) as a natural inference from the result of increasing indefinitely the number of sides in a regular polygon forming the base of a pyramid.

PROPOSITIONS.

PROPOSITION 1.

Similar polygons inscribed in circles are to one another as the squares on the diameters.

Let *ABC*, *FGH* be circles, let *ABCDE*, *FGHKL* be similar polygons inscribed in them, and let *BM*, *GN* be diameters of the circles; I say that, as the square on *BM* is to the square on *GN*, so is the polygon *ABCDE* to the polygon *FGHKL*.

For let *BE*, *AM*, *GL*, *FN* be joined.

Now, since the polygon *ABCDE* is similar to the polygon *FGHKL*, the angle *BAE* is equal to the angle *GFL*, and, as *BA* is to *AE*, so is *GF* to *FL*. {VI. Def. I}

Thus *BAE*, *GFL* are two triangles which have one angle equal to one angle, namely the angle *BAE* to the angle *GFL*, and the sides about the equal angles proportional; therefore the triangle *ABE* is equiangular with the triangle *FGL*. {VI. 6}

Therefore the angle *AEB* is equal to the angle *FLG*.

But the angle *AEB* is equal to the angle *AMB*, for they stand on the same circumference; {III. 27} and the angle *FLG* to the angle *FNG*; therefore the angle *AMB* is also equal to the angle *FNG*.

But the right angle *BAM* is also equal to the right angle *GFN*; {III. 31} therefore the remaining angle is equal to the remaining angle. {I. 32}

Therefore the triangle *ABM* is equiangular with the triangle *FGN*.

Therefore, proportionally, as *BM* is to *GN*, so is *BA* to *GF*. {VI. 4}

But the ratio of the square on *BM* to the square on *GN* is duplicate of the ratio of *BM* to *GN*, and the ratio of the polygon *ABCDE* to the polygon *FGHKL* is duplicate of the ratio of *BA* to *GF*; {VI. 20} therefore also, as the square on *BM* is to the square on *GN*, so is the polygon *ABCDE* to the polygon *FGHKL*.

Therefore etc. Q. E. D.

PROPOSITION 2.

Circles are to one another as the squares on the diameters.

Let *ABCD*, *EFGH* be circles, and *BD*, *FH* their diameters; I say that, as the circle *ABCD* is to the circle *EFGH*, so is the square on *BD* to the square on *FH*.

For, if the square on *BD* is not to the square on *FH* as the circle *ABCD* is to the circle *EFGH*, then, as the square on *BD* is to the square on *FH*, so will the circle *ABCD* be either to some less area than the circle *EFGH*, or to a greater.

First, let it be in that ratio to a less area *S*.

Let the square *EFGH* be inscribed in the circle *EFGH*; then the inscribed square is greater than the half of the circle *EFGH*, inasmuch as, if through the points *E*, *F*, *G*, *H* we draw tangents to the circle, the square *EFGH* is half the square circumscribed about the circle, and the circle is less than the circumscribed square; hence the inscribed square *EFGH* is greater than the half of the circle *EFGH*.

Let the circumferences *EF*, *FG*, *GH*, *HE* be bisected at the points *K*, *L*, *M*, *N*, and let *EK*, *KF*, *FL*, *LG*, *GM*, *MH*, *HN*, *NE* be joined; therefore each of the triangles *EKF*, *FLG*, *GMH*, *HNE* is also greater than the half of the segment

of the circle about it, inasmuch as, if through the points *K*, *L*, *M*, *N* we draw tangents to the circle and complete the parallelograms on the straight lines *EF*, *FG*, *GH*, *HE*, each of the triangles *EKF*, *FLG*, *GMH*, *HNE* will be half of the parallelogram about it, while the segment about it is less than the parallelogram; hence each of the triangles *EKF*, *FLG*, *GMH*, *HNE* is greater than the half of the segment of the circle about it.

Thus, by bisecting the remaining circumferences and joining straight lines, and by doing this continually, we shall leave some segments of the circle which will be less than the excess by which the circle *EFGH* exceeds the area *S*.

For it was proved in the first theorem of the tenth book that, if two unequal magnitudes be set out, and if from the greater there be subtracted a magnitude greater than the half, and from that which is left a greater than the half, and if this be done continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Let segments be left such as described, and let the segments of the circle *EFGH* on *EK*, *KF*, *FL*, *LG*, *GM*, *MH*, *HN*, *NE* be less than the excess by which the circle *EFGH* exceeds the area *S*.

Therefore the remainder, the polygon *EKFLGMHN*, is greater than the area *S*.

Let there be inscribed, also, in the circle *ABCD* the polygon *AOBPCQDR* similar to the polygon *EKFLGMHN*; therefore, as the square on *BD* is to the square on *FH*, so is the polygon *AOBPCQDR* to the polygon *EKFLGMHN*. {XII. 1}

But, as the square on *BD* is to the square on *FH*, so also is the circle *ABCD* to the area *S*; therefore also, as the circle *ABCD* is to the area *S*, so is the

polygon *AOBPCQDR* to the polygon *EKFLGMHN*; {V. 11} therefore, alternately, as the circle *ABCD* is to the polygon inscribed in it, so is the area *S* to the polygon *EKFLGMHN*. {V. 16}

But the circle *ABCD* is greater than the polygon inscribed in it; therefore the area *S* is also greater than the polygon *EKFLGMHN*.

But it is also less: which is impossible.

Therefore, as the square on *BD* is to the square on *FH*, so is not the circle *ABCD* to any area less than the circle *EFGH*.

Similarly we can prove that neither is the circle *EFGH* to any area less than the circle *ABCD* as the square on *FH* is to the square on *BD*.

I say next that neither is the circle *ABCD* to any area greater than the circle *EFGH* as the square on *BD* is to the square on *FH*.

For, if possible, let it be in that ratio to a greater area *S*.

Therefore, inversely, as the square on *FH* is to the square on *DB*, so is the area *S* to the circle *ABCD*.

But, as the area *S* is to the circle *ABCD*, so is the circle *EFGH* to some area less than the circle *ABCD*; therefore also, as the square on *FH* is to the square on *BD*, so is the circle *EFGH* to some area less than the circle *ABCD*: {V. 11} which was proved impossible.

Therefore, as the square on *BD* is to the square on *FH*, so is not the circle *ABCD* to any area greater than the circle *EFGH*.

And it was proved that neither is it in that ratio to any area less than the circle *EFGH*; therefore, as the square on *BD* is to the square on *FH*, so is the circle *ABCD* to the circle *EFGH*.

Therefore etc. Q. E. D.

LEMMA.

I say that, the area *S* being greater than the circle *EFGH*, as the area *S* is to the circle *ABCD*, so is the circle *EFGH* to some area less than the circle *ABCD*.

For let it be contrived that, as the area *S* is to the circle *ABCD*, so is the circle *EFGH* to the area *T*.

I say that the area *T* is less than the circle *ABCD*.

For since, as the area *S* is to the circle *ABCD*, so is the circle *EFGH* to the area *T*, therefore, alternately, as the area *S* is to the circle *EFGH*, so is the circle *ABCD* to the area *T*. {V. 16}

But the area *S* is greater than the circle *EFGH*; therefore the circle *ABCD* is also greater than the area *T*.

Hence, as the area *S* is to the circle *ABCD*, so is the circle *EFGH* to some area less than the circle *ABCD*. Q. E. D.

PROPOSITION 3.

Any pyramid which has a triangular base is divided into two pyramids equal and similar to one another, *similar to the whole and having triangular bases*, *and into two equal prisms; and the two prisms are greater than the half of the whole pyramid.*

Let there be a pyramid of which the triangle *ABC* is the base and the point *D* the vertex; I say that the pyramid *ABCD* is divided into two pyramids equal to one another, having triangular bases and similar to the whole pyramid, and into two equal prisms; and the two prisms are greater than the half of the whole pyramid.

For let *AB*, *BC*, *CA*, *AD*, *DB*, *DC* be bisected at the points *E*, *F*, *G*, *H*, *K*, *L*, and let *HE*, *EG*, *GH*, *HK*, *KL*, *LH*, *KF*, *FG* be joined.

Since *AE* is equal to *EB*, and *AH* to *DH*, therefore *EH* is parallel to *DB*. {VI. 2}

For the same reason *HK* is also parallel to *AB*.

Therefore *HEBK* is a parallelogram; therefore *HK* is equal to *EB*. {I. 34}

But *EB* is equal to *EA*; therefore *AE* is also equal to *HK*.

But *AH* is also equal to *HD*; therefore the two sides *EA*, *AH* are equal to the two sides *KH*, *HD* respectively, and the angle *EAH* is equal to the angle *KHD*; therefore the base *EH* is equal to the base *KD*. {I. 4}

Therefore the triangle *AEH* is equal and similar to the triangle *HKD*.

For the same reason the triangle *AHG* is also equal and similar to the triangle *HLD*.

Now, since two straight lines *EH*, *HG* meeting one another are parallel to two straight lines *KD*, *DL* meeting one another, and are not in the same plane, they will contain equal angles. {XI. 10}

Therefore the angle *EHG* is equal to the angle *KDL*.

And, since the two straight lines *EH*, *HG* are equal to the two *KD*, *DL* respectively, and the angle *EHG* is equal to the angle *KDL*, therefore the base *EG* is equal to the base *KL*; {I. 4} therefore the triangle *EHG* is equal and similar to the triangle *KDL*.

For the same reason the triangle *AEG* is also equal and similar to the triangle *HKL*.

Therefore the pyramid of which the triangle *AEG* is the base and the point *H* the vertex is equal and similar to the pyramid of which the triangle *HKL* is the base and the point *D* the vertex. {XI. Def. 10}

And, since *HK* has been drawn parallel to *AB*, one of the sides of the triangle *ADB*, the triangle *ADB* is equiangular to the triangle *DHK*, {I. 29} and they have their sides proportional; therefore the triangle *ADB* is similar to the triangle *DHK*. {VI. Def. 1}

For the same reason the triangle *DBC* is also similar to the triangle *DKL*, and the triangle *ADC* to the triangle *DLH*.

Now, since the two straight lines *BA*, *AC* meeting one another are parallel to the two straight lines *KH*, *HL* meeting one another, not in the same plane, they will contain equal angles. {XI. 10}

Therefore the angle *BAC* is equal to the angle *KHL*.

And, as *BA* is to *AC*, so is *KH* to *HL*; therefore the triangle *ABC* is similar to the triangle *HKL*.

Therefore also the pyramid of which the triangle *ABC* is the base and the point *D* the vertex is similar to the pyramid of which the triangle *HKL* is the base and the point *D* the vertex.

But the pyramid of which the triangle *HKL* is the base and the point *D* the vertex was proved similar to the pyramid of which the triangle *AEG* is the base and the point *H* the vertex.

Therefore each of the pyramids *AEGH*, *HKLD* is similar to the whole pyramid *ABCD*.

Next, since *BF* is equal to *FC*, the parallelogram *EBFG* is double of the triangle *GFC*.

And since, if there be two prisms of equal height, and one have a parallelogram as base, and the other a triangle, and if the parallelogram be double of the triangle, the prisms are equal, {XI. 39} therefore the prism contained by the two triangles *BKF*, *EHG*, and the three parallelograms *EBFG*, *EBKH*, *HKFG* is equal to the prism contained by the two triangles *GFC*, *HKL* and the three parallelograms *KFCL*, *LCGH*, *HKFG*.

And it is manifest that each of the prisms, namely that in which the parallelogram *EBFG* is the base and the straight line *HK* is its opposite, and that in which the triangle *GFC* is the base and the triangle *HKL* its opposite, is greater than each of the pyramids of which the triangles *AEG*, *HKL* are the bases and the points *H*, *D* the vertices, inasmuch as, if we join the straight lines *EF*, *EK*, the prism in which the parallelogram *EBFG* is the base and the straight line *HK* its opposite is greater than the pyramid of which the triangle *EBF* is the base and the point *K* the vertex.

But the pyramid of which the triangle *EBF* is the base and the point *K* the vertex is equal to the pyramid of which the triangle *AEG* is the base and the point *H* the vertex; for they are contained by equal and similar planes.

Hence also the prism in which the parallelogram *EBFG* is the base and the straight line *HK* its opposite is greater than the pyramid of which the triangle *AEG* is the base and the point *H* the vertex.

But the prism in which the parallelogram *EBFG* is the base and the straight line *HK* its opposite is equal to the prism in which the triangle *GFC* is the base and the triangle *HKL* its opposite, and the pyramid of which the triangle *AEG* is the base and the point *H* the vertex is equal to the pyramid of which the triangle *HKL* is the base and the point *D* the vertex.

Therefore the said two prisms are greater than the said two pyramids of which the triangles *AEG*, *HKL* are the bases and the points *H*, *D* the vertices.

Therefore the whole pyramid, of which the triangle *ABC* is the base and the point *D* the vertex, has been divided into two pyramids equal to one another and into two equal prisms, and the two prisms are greater than the half of the whole pyramid. Q. E. D.

PROPOSITION 4.

If there be two pyramids of the same height which have triangular bases, and cach of them be divided into two pyramids equal to one another and similar to the whole, and into two equal prisms, then, as the base of the one pyramid is to the base of the other pyramid, so will all the prisms in the one pyramid be to all the prisms, being equal in multitude, in the other pyramid.

Let there be two pyramids of the same height which have the triangular bases *ABC*, *DEF*, and vertices the points *G*, *H*, and let each of them be divided into two pyramids equal to one another and similar to the whole and into two equal prisms; {XII. 3} I say that, as the base *ABC* is to the base *DEF*, so are all the prisms in the pyramid *ABCG* to all the prisms, being equal in multitude, in the pyramid *DEFH*,

For, since *BO* is equal to *OC*, and *AL* to *LC*, therefore *LO* is parallel to *AB*, and the triangle *ABC* is similar to the triangle *LOC*.

For the same reason the triangle *DEF* is also similar to the triangle *RVF*.

And, since *BC* is double of *CO*, and *EF* of *FV*, therefore, as *BC* is to *CO*, so is *EF* to *FV*.

And on *BC*, *CO* are described the similar and similarly situated rectilineal figures *ABC*, *LOC*, and on *EF*, *FV* the similar and similarly situated figures *DEF*, *RVF*; therefore, as the triangle *ABC* is to the triangle *LOC*, so is the triangle *DEF* to the triangle *RVF*; {VI. 22} therefore, alternately, as the

triangle *ABC* is to the triangle *DEF*, so is the triangle *LOC* to the triangle *RVF*. {V. 16}

But, as the triangle *LOC* is to the triangle *RVF*, so is the prism in which the triangle *LOC* is the base and *PMN* its opposite to the prism in which the triangle *RVF* is the base and *STU* its opposite; {Lemma following} therefore also, as the triangle *ABC* is to the triangle *DEF*, so is the prism in which the triangle *LOC* is the base and *PMN* its opposite to the prism in which the triangle *RVF* is the base and *STU* its opposite.

But, as the said prisms are to one another, so is the prism in which the parallelogram *KBOL* is the base and the straight line *PM* its opposite to the prism in which the parallelogram *QEVR* is the base and the straight line *ST* its opposite. {XI. 39; cf. XII. 3}

Therefore also the two prisms, that in which the parallelogram *KBOL* is the base and *PM* its opposite, and that in which the triangle *LOC* is the base and *PMN* its opposite, are to the prisms in which *QEVR* is the base and the straight line *ST* its opposite and in which the triangle *RVF* is the base and *STU* its opposite in the same ratio {V. 12}

Therefore also, as the base *ABC* is to the base *DEF*, so are the said two prisms to the said two prisms.

And similarly, if the pyramids *PMNG*, *STUH* be divided into two prisms and two pyramids, as the base *PMN* is to the base *STU*, so will the two prisms in the pyramid *PMNG* be to the two prisms in the pyramid *STUH*.

But, as the base *PMN* is to the base *STU*, so is the base *ABC* to the base *DEF*; for the triangles *PMN*, *STU* are equal to the triangles *LOC*, *RVF* respectively.

Therefore also, as the base *ABC* is to the base *DEF*, so are the four prisms to the four prisms.

And similarly also, if we divide the remaining pyramids into two pyramids and into two prisms, then, as the base *ABC* is to the base *DEF*, so will all the prisms in the pyramid *ABCG* be to all the prisms, being equal in multitude, in the pyramid *DEFH*. Q. E. D.

LEMMA.

But that, as the triangle *LOC* is to the triangle *RVF*, so is the prism in which the triangle *LOC* is the base and *PMN* its opposite to the prism in which the triangle *RVF* is the base and *STU* its opposite, we must prove as follows.

For in the same figure let perpendiculars be conceived drawn from *G*, *H* to the planes *ABC*, *DEF*; these are of course equal because, by hypothesis, the pyramids are of equal height.

Now, since the two straight lines *GC* and the perpendicular from *G* are cut by the parallel planes *ABC*, *PMN*, they will be cut in the same ratios. {XI. 17}

And *GC* is bisected by the plane *PMN* at *N*; therefore the perpendicular from *G* to the plane *ABC* will also be bisected by the plane *PMN*.

For the same reason the perpendicular from *H* to the plane *DEF* will also be bisected by the plane *STU*.

And the perpendiculars from *G*, *H* to the planes *ABC*, *DEF* are equal; therefore the perpendiculars from the triangles *PMN*, *STU* to the planes *ABC*, *DEF* are also equal.

Therefore the prisms in which the triangles *LOC*, *RVF* are bases, and *PMN*, *STU* their opposites, are of equal height.

Hence also the parallelepipedal solids described from the said prisms are of equal height and are to one another as their bases; {XI. 32} therefore their halves, namely the said prisms, are to one another as the base *LOC* is to the base *RVF*. Q. E. D.

PROPOSITION 5.

Pyramids which are of the same height and have triangular bases are to one another as the bases.

Let there be pyramids of the same height, of which the triangles *ABC*, *DEF* are the bases and the points *G*, *H* the vertices; I say that, as the base *ABC* is to the base *DEF*, so is the pyramid *ABCG* to the pyramid *DEFH*.

For, if the pyramid *ABCG* is not to the pyramid *DEFH* as the base *ABC* is to the base *DEF*, then, as the base *ABC* is to the base *DEF*, so will the pyramid *ABCG* be either to some solid less than the pyramid *DEFH* or to a greater.

Let it, first, be in that ratio to a less solid *W*, and let the pyramid *DEFH* be divided into two pyramids equal to one another and similar to the whole and into two equal prisms; then the two prisms are greater than the half of the whole pyramid. {XII. 3}

Again, let the pyramids arising from the division be similarly divided, and let this be done continually until there are left over from the pyramid *DEFH* some pyramids which are less than the excess by which the pyramid *DEFH* exceeds the solid *W*. {X. I}

Let such be left, and let them be, for the sake of argument, *DQRS*, *STUH*; therefore the remainders, the prisms in the pyramid *DEFH*, are greater than the solid *W*.

Let the pyramid *ABCG* also be divided similarly, and a similar number of times, with the pyramid *DEFH*; therefore, as the base *ABC* is to the base *DEF*, so are the prisms in the pyramid *ABCG* to the prisms in the pyramid *DEFH*. {XII. 4}

But, as the base *ABC* is to the base *DEF*, so also is the pyramid *ABCG* to the solid *W*; therefore also, as the pyramid *ABCG* is to the solid *W*, so are the prisms in the pyramid *ABCG* to the prisms in the pyramid *DEFH*; {V. II} therefore, alternately, as the pyramid *ABCG* is to the prisms in it, so is the solid *W* to the prisms in the pyramid *DEFH*. {V. 16}

But the pyramid *ABCG* is greater than the prisms in it; therefore the solid *W* is also greater than the prisms in the pyramid *DEFH*.

But it is also less: which is impossible.

Therefore the prism *ABCG* is not to any solid less than the pyramid *DEFH* as the base *ABC* is to the base *DEF*.

Similarly it can be proved that neither is the pyramid *DEFH* to any solid less than the pyramid *ABCG* as the base *DEF* is to the base *ABC*.

I say next that neither is the pyramid *ABCG* to any solid greater than the pyramid *DEFH* as the base *ABC* is to the base *DEF*.

For, if possible, let it be in that ratio to a greater solid *W*; therefore, inversely, as the base *DEF* is to the base *ABC*, so is the solid *W* to the pyramid *ABCG*.

But, as the solid *W* is to the solid *ABCG*, so is the pyramid *DEFH* to some solid less than the pyramid *ABCG*, as was before proved; {XII. 2, Lemma} therefore also, as the base *DEF* is to the base *ABC*, so is the pyramid *DEFH* to some solid less than the pyramid *ABCG*: {V. II} which was proved absurd.

Therefore the pyramid *ABCG* is not to any solid greater than the pyramid *DEFH* as the base *ABC* is to the base *DEF*.

But it was proved that neither is it in that ratio to a less solid.

Therefore, as the base *ABC* is to the base *DEF*, so is the pyramid *ABCG* to the pyramid *DEFH*. Q. E. D.

PROPOSITION 6.

Pyramids which are of the same height and have polygonal bases are to one another as the bases.

Let there be pyramids of the same height of which the polygons *ABCDE*, *FGHKL* are the bases and the points *M*, *N* the vertices; I say that, as the base *ABCDE* is to the base *FGHKL*, so is the pyramid *ABCDEM* to the pyramid *FGHKLN*.

For let *AC*, *AD*, *FH*, *FK* be joined.

Since then *ABCM*, *ACDM* are two pyramids which have triangular bases and equal height, they are to one another as the bases; {XII. 5} therefore, as the base *ABC* is to the base *ACD*, so is the pyramid *ABCM* to the pyramid *ACDM*.

And, componendo, as the base *ABCD* is to the base *ACD*, so is the pyramid *ABCDM* to the pyramid *ACDM*. {V. 18}

But also, as the base *ACD* is to the base *ADE*, so is the pyramid *ACDM* to the pyramid *ADEM*. {XII. 5}

Therefore, ex aequali, as the base *ABCD* is to the base *ADE*, so is the pyramid *ABCDM* to the pyramid *ADEM*. {V. 22}

And again componendo, as the base *ABCDE* is to the base *ADE*, so is the pyramid *ABCDEM* to the pyramid *ADEM*. {V. 18}

Similarly also it can be proved that, as the base *FGHKL* is to the base *FGH*, so is the pyramid *FGHKLN* to the pyramid *FGHN*.

And, since *ADEM*, *FGHN* are two pyramids which have triangular bases and equal height, therefore, as the base *ADE* is to the base *FGH*, so is the pyramid *ADEM* to the pyramid *FGHN*. {XII. 5}

But, as the base *ADE* is to the base *ABCDE*, so was the pyramid *ADEM* to the pyramid *ABCDEM*.

Therefore also, ex aequali, as the base *ABCDE* is to the base *FGH*, so is the pyramid *ABCDEM* to the pyramid *FGHN*. {V. 22}

But further, as the base *FGH* is to the base *FGHKL*, so also was the pyramid *FGHN* to the pyramid *FGHKLN*.

Therefore also, *ex aequali*, as the base *ABCDE* is to the base *FGHKL*, so is the pyramid *ABCDEM* to the pyramid *FGHKLN*. {V. 22} Q. E. D.

PROPOSITION 7.

Any prism which has a triangular base is divided into three pyramids equal to one another which have triangular bases.

Let there be a prism in which the triangle *ABC* is the base and *DEF* its opposite; I say that the prism *ABCDEF* is divided into three pyramids equal to one another, which have triangular bases.

For let *BD*, *EC*, *CD* be joined.

Since *ABED* is a parallelogram, and *BD* is its diameter, therefore the triangle *ABD* is equal to the triangle *EBD*; {I. 34} therefore also the pyramid of which the triangle *ABD* is the base and the point *C* the vertex is equal to the pyramid of which the triangle *DEB* is the base and the point *C* the vertex. {XII. 5}

But the pyramid of which the triangle *DEB* is the base and the point *C* the vertex is the same with the pyramid of which the triangle *EBC* is the base and the point *D* the vertex; for they are contained by the same planes.

Therefore the pyramid of which the triangle *ABD* is the base and the point *C* the vertex is also equal to the pyramid of which the triangle *EBC* is the base and the point *D* the vertex.

Again, since *FCBE* is a parallelogram, and *CE* is its diameter, the triangle *CEF* is equal to the triangle *CBE*. {I. 34}

Therefore also the pyramid of which the triangle *BCE* is the base and the point *D* the vertex is equal to the pyramid of which the triangle *ECF* is the base and the point *D* the vertex. {XII. 5}

But the pyramid of which the triangle *BCE* is the base and the point *D* the vertex was proved equal to the pyramid of which the triangle *ABD* is the base and the point *C* the vertex; therefore also the pyramid of which the triangle *CEF* is the base and the point *D* the vertex is equal to the pyramid of which the triangle *ABD* is the base and the point *C* the vertex; therefore the prism *ABCDEF* has been divided into three pyramids equal to one another which have triangular bases.

And, since the pyramid of which the triangle *ABD* is the base and the point *C* the vertex is the same with the pyramid of which the triangle *CAB* is the base and the point *D* the vertex, for they are contained by the same planes,

while the pyramid of which the triangle *ABD* is the base and the point *C* the vertex was proved to be a third of the prism in which the triangle *ABC* is the base and *DEF* its opposite, therefore also the pyramid of which the triangle *ABC* is the base and the point *D* the vertex is a third of the prism which has the same base, the triangle *ABC*, and *DEF* as its opposite.

PORISM.

From this it is manifest that any pyramid is a third part of the prism which has the same base with it and equal height. Q. E. D.

PROPOSITION 8.

Similar pyramids which have triangular bases are in the triplicate ratio of their corresponding sides.

Let there be similar and similarly situated pyramids of which the triangles *ABC*, *DEF*, are the bases and the points *G*, *H* the vertices; I say that the pyramid *ABCG* has to the pyramid *DEFH* the ratio triplicate of that which *BC* has to *EF*.

For let the parallelepipedal solids *BGML*, *EHQP* be completed.

Now, since the pyramid *ABCG* is similar to the pyramid *DEFH*, therefore the angle *ABC* is equal to the angle *DEF*, the angle *GBC* to the angle *HEF*, and the angle *ABG* to the angle *DEH*; and, as *AB* is to *DE*, so is *BC* to *EF*, and *BG* to *EH*.

And since, as *AB* is to *DE*, so is *BC* to *EF*, and the sides are proportional about equal angles, therefore the parallelogram *BM* is similar to the parallelogram *EQ*.

For the same reason *BN* is also similar to *ER*, and *BK* to *EO*; therefore the three parallelograms *MB*, *BK*, *BN* are similar to the three *EQ*, *EO*, *ER*.

But the three parallelograms *MB*, *BK*, *BN* are equal and similar to their three opposites, and the three *EQ*, *EO*, *ER* are equal and similar to their three opposites. {XI. 24}

Therefore the solids *BGML*, *EHQP* are contained by similar planes equal in multitude.

Therefore the solid *BGML* is similar to the solid *EHQP*.

But similar parallelepipedal solids are in the triplicate ratio of their corresponding sides. {XI. 33}

Therefore the solid *BGML* has to the solid *EHQP* the ratio triplicate of that which the corresponding side *BC* has to the corresponding side *EF*.

But, as the solid *BGML* is to the solid *EHQP*, so is the pyramid *ABCG* to the pyramid *DEFH*, inasmuch as the pyramid is a sixth part of the solid, because the prism which is half of the parallelepipedal solid {XI. 28} is also triple of the pyramid. {XII. 7}

Therefore the pyramid *ABCG* also has to the pyramid *DEFH* the ratio triplicate of that which *BC* has to *EF*. Q. E. D.

PORISM.

From this it is manifest that similar pyramids which have polygonal bases are also to one another in the triplicate ratio of their corresponding sides.

For, if they are divided into the pyramids contained in them which have triangular bases, by virtue of the fact that the similar polygons forming their bases are also divided into similar triangles equal in multitude and corresponding to the wholes {VI. 20}, then, as the one pyramid which has a triangular base in the one complete pyramid is to the one pyramid which has a triangular base in the other complete pyramid, so also will all the pyramids

which have triangular bases contained in the one pyramid be to all the pyramids which have triangular bases contained in the other pyramid {V. 12}, that is, the pyramid itself which has a polygonal base to the pyramid which has a polygonal base.

But the pyramid which has a triangular base is to the pyramid which has a triangular base in the triplicate ratio of the corresponding sides; therefore also the pyramid which has a polygonal base has to the pyramid which has a similar base the ratio triplicate of that which the side has to the side.

PROPOSITION 9.

In equal pyramids which have triangular bases the bases are reciprocally proportional to the heights; and those pyramids in which the bases are reciprocally proportional to the heights are equal.

For let there be equal pyramids which have the triangular bases *ABC*, *DEF* and vertices the points *G*, *H*; I say that in the pyramids *ABCG*, *DEFH* the bases are reciprocally proportional to the heights, that is, as the base *ABC* is to the base *DEF*, so is the height of the pyramid *DEFH* to the height of the pyramid *ABCG*.

For let the parallelepipedal solids *BGML*, *EHQP* be completed.

Now, since the pyramid *ABCG* is equal to the pyramid *DEFH*, and the solid *BGML* is six times the pyramid *ABCG*, and the solid *EHQP* six times the pyramid *DEFH*, therefore the solid *BGML* is equal to the solid *EHQP*.

But in equal parallelepipedal solids the bases are reciprocally proportional to the heights; {XI. 34} therefore, as the base *BM* is to the base *EQ*, so is the height of the solid *EHQP* to the height of the solid *BGML*.

But, as the base *BM* is to *EQ*, so is the triangle *ABC* to the triangle *DEF*. {I. 34}

Therefore also, as the triangle *ABC* is to the triangle *DEF*, so is the height of the solid *EHQP* to the height of the solid *BGML*. {V. 11}

But the height of the solid *EHQP* is the same with the height of the pyramid *DEFH*, and the height of the solid *BGML* is the same with the height of the pyramid *ABCG*, therefore, as the base *ABC* is to the base *DEF*, so is the height of the pyramid *DEFH* to the height of the pyramid *ABCG*.

Therefore in the pyramids *ABCG*, *DEFH* the bases are reciprocally proportional to the heights.

Next, in the pyramids *ABCG*, *DEFH* let the bases be reciprocally proportional to the heights; that is, as the base *ABC* is to the base *DEF*, so let the height of the pyramid *DEFH* be to the height of the pyramid *ABCG*; I say that the pyramid *ABCG* is equal to the pyramid *DEFH*.

For, with the same construction, since, as the base *ABC* is to the base *DEF*, so is the height of the pyramid *DEFH* to the height of the pyramid *ABCG*, while, as the base *ABC* is to the base *DEF*, so is the parallelogram *BM* to the parallelogram *EQ*, therefore also, as the parallelogram *BM* is to the parallelogram *EQ*, so is the height of the pyramid *DEFH* to the height of the pyramid *ABCG*. {V. 11}

But the height of the pyramid *DEFH* is the same with the height of the parallelepiped *EHQP*, and the height of the pyramid *ABCG* is the same with the height of the parallelepiped *BGML*; therefore, as the base *BM* is to the base *EQ*, so is the height of the parallelepiped *EHQP* to the height of the parallelepiped *BGML*.

But those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal; {XI. 34} therefore the parallelepipedal solid *BGML* is equal to the parallelepipedal solid *EHQP*.

And the pyramid *ABCG* is a sixth part of *BGML*, and the pyramid *DEFH* a sixth part of the parallelepiped *EHQP*; therefore the pyramid *ABCG* is equal to the pyramid *DEFH*.

Therefore etc Q. E. D.

PROPOSITION 10.

Any cone is a third part of the cylinder which has the same base with it and equal height.

For let a cone have the same base, namely the circle *ABCD*, with a cylinder and equal height; I say that the cone is a third part of the cylinder, that is, that the cylinder is triple of the cone.

For if the cylinder is not triple of the cone, the cylinder will be either greater than triple or less than triple of the cone.

First let it be greater than triple, and let the square *ABCD* be inscribed in the circle *ABCD*; {IV. 6} then the square *ABCD* is greater than the half of the circle *ABCD*.

From the square *ABCD* let there be set up a prism of equal height with the cylinder.

Then the prism so set up is greater than the half of the cylinder, inasmuch as, if we also circumscribe a square about the circle *ABCD* {IV. 7}, the square inscribed in the circle *ABCD* is half of that circumscribed about it, and the solids set up from them are parallelepipedal prisms of equal height, while parallelepipedal solids which are of the same height are to one another as

their bases; {XI. 32} therefore also the prism set up on the square *ABCD* is half of the prism set up from the square circumscribed about the circle *ABCD*; {cf. XI. 28, or XII. 6 and 7, Por.} and the cylinder is less than the prism set up from the square circumscribed about the circle *ABCD*; therefore the prism set up from the square *ABCD* and of equal height with the cylinder is greater than the half of the cylinder.

Let the circumferences *AB*, *BC*, *CD*, *DA* be bisected at the points *E*, *F*, *G*, *H*, and let *AE*, *EB*, *BF*, *FC*, *CG*, *GD*, *DH*, *HA* be joined; then each of the triangles *AEB*, *BFC*, *CGD*, *DHA* is greater than the half of that segment of the circle *ABCD* which is about it, as we proved before. {XII. 2}

On each of the triangles *AEB*, *BFC*, *CGD*, *DHA* let prisms be set up of equal height with the cylinder; then each of the prisms so set up is greater than the half part of that segment of the cylinder which is about it, inasmuch as, if we draw through the points *E*, *F*, *G*, *H* parallels to *AB*, *BC*, *CD*, *DA*, complete the parallelograms on *AB*, *BC*, *CD*, *DA*, and set up from them parallelepipedal solids of equal height with the cylinder, the prisms on the triangles *AEB*, *BFC*, *CGD*, *DHA* are halves of the several solids set up; and the segments of the cylinder are less than the parallelepipedal solids set up; hence also the prisms on the triangles *AEB*, *BFC*, *CGD*, *DHA* are greater than the half of the segments of the cylinder about them.

Thus, bisecting the circumferences that are left, joining straight lines, setting up on each of the triangles prisms of equal height with the cylinder, and doing this continually, we shall leave some segments of the cylinder which will be less than the excess by which the cylinder exceeds the triple of the cone. {X. 1}

Let such segments be left, and let them be *AE*, *EB*, *BF*, *FC*, *CG*, *GD*, *DH*, *HA*; therefore the remainder, the prism of which the polygon *AEBFCGDH* is

the base and the height is the same as that of the cylinder, is greater than triple of the cone.

But the prism of which the polygon *AEBFCGDH* is the base and the height the same as that of the cylinder is triple of the pyramid of which the polygon *AEBFCGDH* is the base and the vertex is the same as that of the cone; {XII. 7, Por.} therefore also the pyramid of which the polygon *AEBFCGDH* is the base and the vertex is the same as that of the cone is greater than the cone which has the circle *ABCD* as base.

But it is also less, for it is enclosed by it: which is impossible.

Therefore the cylinder is not greater than triple of the cone.

I say next that neither is the cylinder less than triple of the cone,

For, if possible, let the cylinder be less than triple of the cone, therefore, inversely, the cone is greater than a third part of the cylinder.

Let the square *ABCD* be inscribed in the circle *ABCD*; therefore the square *ABCD* is greater than the half of the circle *ABCD*.

Now let there be set up from the square *ABCD* a pyramid having the same vertex with the cone; therefore the pyramid so set up is greater than the half part of the cone, seeing that, as we proved before, if we circumscribe a square about the circle, the square *ABCD* will be half of the square circumscribed about the circle, and if we set up from the squares parallelepipedal solids of equal height with the cone, which are also called prisms, the solid set up from the square *ABCD* will be half of that set up from the square circumscribed about the circle; for they are to one another as their bases. {XI. 32}

Hence also the thirds of them are in that ratio; therefore also the pyramid of which the square *ABCD* is the base is half of the pyramid set up from the square circumscribed about the circle.

And the pyramid set up from the square about the circle is greater than the cone, for it encloses it.

Therefore the pyramid of which the square *ABCD* is the base and the vertex is the same with that of the cone is greater than the half of the cone.

Let the circumferences *AB*, *BC*, *CD*, *DA* be bisected at the points *E*, *F*, *G*, *H*, and let *AE*, *EB*, *BF*, *FC*, *CG*, *GD*, *DH*, *HA* be joined; therefore also each of the triangles *AEB*, *BFC*, *CGD*, *DHA* is greater than the half part of that segment of the circle *ABCD* which is about it.

Now, on each of the triangles *AEB*, *BFC*, *CGD*, *DHA* let pyramids be set up which have the same vertex as the cone; therefore also each of the pyramids so set up is, in the same manner, greater than the half part of that segment of the cone which is about it.

Thus, by bisecting the circumferences that are left, joining straight lines, setting up on each of the triangles a pyramid which has the same vertex as the cone, and doing this continually, we shall leave some segments of the cone which will be less than the excess by which the cone exceeds the third part of the cylinder. {X. 1}

Let such be left, and let them be the segments on *AE*, *EB*, *BF*, *FC*, *CG*, *GD*, *DH*, *HA*; therefore the remainder, the pyramid of which the polygon *AEBFCGDH* is the base and the vertex the same with that of the cone, is greater than a third part of the cylinder.

But the pyramid of which the polygon *AEBFCGDH* is the base and the vertex the same with that of the cone is a third part of the prism of which the polygon *AEBFCGDH* is the base and the height is the same with that of the cylinder; therefore the prism of which the polygon *AEBFCGDH* is the base

and the height is the same with that of the cylinder is greater than the cylinder of which the circle *ABCD* is the base.

But it is also less, for it is enclosed by it: which is impossible.

Therefore the cylinder is not less than triple of the cone.

But it was proved that neither is it greater than triple; therefore the cylinder is triple of the cone; hence the cone is a third part of the cylinder.

Therefore etc. Q. E. D.

PROPOSITION 11.

Cones and cylinders which are of the same height are to one another as their bases.

Let there be cones and cylinders of the same height, let the circles *ABCD*, *EFGH* be their bases, *KL*, *MN* their axes and *AC*, *EG* the diameters of their bases; I say that, as the circle *ABCD* is to the circle *EFGH*, so is the cone *AL* to the cone *EN*.

For, if not, then, as the circle *ABCD* is to the circle *EFGH*, so will the cone *AL* be either to some solid less than the cone *EN* or to a greater.

First, let it be in that ratio to a less solid *O*, and let the solid *X* be equal to that by which the solid *O* is less than the cone *EN*; therefore the cone *EN* is equal to the solids *O*, *X*.

Let the square *EFGH* be inscribed in the circle *EFGH*; therefore the square is greater than the half of the circle.

Let there be set up from the square *EFGH* a pyramid of equal height with the cone; therefore the pyramid so set up is greater than the half of the cone, inasmuch as, if we circumscribe a square about the circle, and set up from it a

pyramid of equal height with the cone, the inscribed pyramid is half of the circumscribed pyramid, for they are to one another as their bases, {XII. 6} while the cone is less than the circumscribed pyramid.

Let the circumferences *EF*, *FG*, *GH*, *HE* be bisected at the points *P*, *Q*, *R*, *S*, and let *HP*, *PE*, *EQ*, *QF*, *FR*, *RG*, *GS*, *SH* be joined.

Therefore each of the triangles *HPE*, *EQF*, *FRG*, *GSH* is greater than the half of that segment of the circle which is about it.

On each of the triangles *HPE*, *EQF*, *FRG*, *GSH* let there be set up a pyramid of equal height with the cone; therefore, also, each of the pyramids so set up is greater than the half of that segment of the cone which is about it.

Thus, bisecting the circumferences which are left, joining straight lines, setting up on each of the triangles pyramids of equal height with the cone, and doing this continually, we shall leave some segments of the cone which will be less than the solid *X*. {X. 1}

Let such be left, and let them be the segments on *HP*, *PE*, *EQ*, *QF*, *FR*, *RG*, *GS*, *SH*; therefore the remainder, the pyramid of which the polygon *HPEQFRGS* is the base and the height the same with that of the cone, is greater than the solid *O*.

Let there also be inscribed in the circle *ABCD* the polygon *DTAUBVCW* similar and similarly situated to the polygon *HPEQFRGS*, and on it let a pyramid be set up of equal height with the cone *AL*.

Since then, as the square on *AC* is to the square on *EG*, so is the polygon *DTAUBVCW* to the polygon *HPEQFRGS*, {XII. 1} while, as the square on *AC* is to the square on *EG*, so is the circle *ABCD* to the circle *EFGH*, {XII. 2} therefore also, as the circle *ABCD* is to the circle *EFGH*, so is the polygon *DTAUBVCW* to the polygon *HPEQFRGS*.

But, as the circle *ABCD* is to the circle *EFGH*, so is the cone *AL* to the solid *O*, and, as the polygon *DTAUBVCW* is to the polygon *HPEQFRGS*, so is the pyramid of which the polygon *DTAUBVCW* is the base and the point *L* the vertex to the pyramid of which the polygon *HPEQFRGS* is the base and the point *N* the vertex. {XII. 6}

Therefore also, as the cone *AL* is to the solid *O*, so is the pyramid of which the polygon *DTAUBVCW* is the base and the point *L* the vertex to the pyramid of which the polygon *HPEQFRGS* is the base and the point *N* the vertex; {V. 11} therefore, alternately, as the cone *AL* is to the pyramid in it, so is the solid *O* to the pyramid in the cone *EN*. {V. 16}

But the cone *AL* is greater than the pyramid in it; therefore the solid *O* is also greater than the pyramid in the cone *EN*.

But it is also less: which is absurd.

Therefore the cone *AL* is not to any solid less than the cone *EN* as the circle *ABCD* is to the circle *EFGH*.

Similarly we can prove that neither is the cone *EN* to any solid less than the cone *AL* as the circle *EFGH* is to the circle *ABCD*.

I say next that neither is the cone *AL* to any solid greater than the cone *EN* as the circle *ABCD* is to the circle *EFGH*.

For, if possible, let it be in that ratio to a greater solid *O*; therefore, inversely, as the circle *EFGH* is to the circle *ABCD*, so is the solid *O* to the cone *AL*.

But, as the solid *O* is to the cone *AL*, so is the cone *EN* to some solid less than the cone *AL*; therefore also, as the circle *EFGH* is to the circle *ABCD*, so is the cone *EN* to some solid less than the cone *AL*: which was proved impossible.

Therefore the cone *AL* is not to any solid greater than the cone *EN* as the circle *ABCD* is to the circle *EFGH*.

But it was proved that neither is it in this ratio to a less solid; therefore, as the circle *ABCD* is to the circle *EFGH*, so is the cone *AL* to the cone *EN*.

But, as the cone is to the cone, so is the cylinder to the cylinder, for each is triple of each; {XII. 10}

Therefore also, as the circle *ABCD* is to the circle *EFGH*, so are the cylinders on them which are of equal height.

Therefore etc. Q. E. D.

PROPOSITION 12.

Similar cones and cylinders are to one another in the triplicate ratio of the diameters in their bases.

Let there be similar cones and cylinders, let the circles *ABCD*, *EFGH* be their bases, *BD*, *FH* the diameters of the bases, and *KL*, *MN* the axes of the cones and cylinders; I say that the cone of which the circle *ABCD* is the base and the point *L* the vertex has to the cone of which the circle *EFGH* is the base and the point *N* the vertex the ratio triplicate of that which *BD* has to *FH*.

For, if the cone *ABCDL* has not to the cone *EFGHN* the ratio triplicate of that which *BD* has to *FH*, the cone *ABCDL* will have that triplicate ratio either to some solid less than the cone *EFGHN* or to a greater.

First, let it have that triplicate ratio to a less solid *O*.

Let the square *EFGH* be inscribed in the circle *EFGH*; {IV. 6} therefore the square *EFGH* is greater than the half of the circle *EFGH*.

Now let there be set up on the square *EFGH* a pyramid having the same vertex with the cone; therefore the pyramid so set up is greater than the half part of the cone.

Let the circumferences *EF*, *FG*, *GH*, *HE* be bisected at the points *P*, *Q*, *R*, *S*, and let *EP*, *PF*, *FQ*, *QG*, *GR*, *RH*, *HS*, *SE* be joined.

Therefore each of the triangles *EPF*, *FQG*, *GRH*, *HSE* is also greater than the half part of that segment of the circle *EFGH* which is about it.

Now on each of the triangles *EPF*, *FQG*, *GRH*, *HSE* let a pyramid be set up having the same vertex with the cone; therefore each of the pyramids so set up is also greater than the half part of that segment of the cone which is about it.

Thus, bisecting the circumferences so left, joining straight lines, setting up on each of the triangles pyramids having the same vertex with the cone, and doing this continually, we shall leave some segments of the cone which will be less than the excess by which the cone *EFGHN* exceeds the solid *O*. {X. 1}

Let such be left, and let them be the segments on *EP*, *PF*, *FQ*, *QG*, *GR*, *RH*, *HS*, *SE*; therefore the remainder, the pyramid of which the polygon *EPFQGRHS* is the base and the point *N* the vertex, is greater than the solid *O*.

Let there be also inscribed in the circle *ABCD* the polygon *ATBUCVDW* similar and similarly situated to the polygon *EPFQGRHS*, and let there be set up on the polygon *ATBUCVDW* a pyramid having the same vertex with the cone; of the triangles containing the pyramid of which the polygon *ATBUCVDW* is the base and the point *L* the vertex let *LBT* be one, and of the triangles containing the pyramid of which the polygon *EPFQGRHS* is the base and the point *N* the vertex let *NFP* be one; and let *KT*, *MP* be joined.

Now, since the cone *ABCDL* is similar to the cone *EFGHN*, therefore, as *BD* is to *FH*, so is the axis *KL* to the axis *MN*. {XI. Def. 24}

But, as *BD* is to *FH*, so is *BK* to *FM*; therefore also, as *BK* is to *FM*, so is *KL* to *MN*.

And, alternately, as *BK* is to *KL*, so is *FM* to *MN*. {V. 16}

And the sides are proportional about equal angles, namely the angles *BKL*, *FMN*; therefore the triangle *BKL* is similar to the triangle *FMN*. {VI. 6}

Again, since, as *BK* is to *KT*, so is *FM* to *MP*, and they are about equal angles, namely the angles *BKT*, *FMP*, inasmuch as, whatever part the angle *BKT* is of the four right angles at the centre *K*, the same part also is the angle *FMP* of the four right angles at the centre *M*; since then the sides are proportional about equal angles, therefore the triangle *BKT* is similar to the triangle *FMP*. {VI. 6}

Again, since it was proved that, as *BK* is to *KL*, so is *FM* to *MN*, while *BK* is equal to *KT*, and *FM* to *PM*, therefore, as *TK* is to *KL*, so is *PM* to *MN*; and the sides are proportional about equal angles, namely the angles *TKL*, *PMN*, for they are right; therefore the triangle *LKT* is similar to the triangle *NMP*. {VI. 6}

And since, owing to the similarity of the triangles *LKB*, *NMF*, as *LB* is to *BK*, so is *NF* to *FM*, and, owing to the similarity of the triangles *BKT*, *FMP*, as *KB* is to *BT*, so is *MF* to *FP*, therefore, *ex aequali*, as *LB* is to *BT*, so is *NF* to *FP*. {V. 22}

Again since, owing to the similarity of the triangles *LTK*, *NPM*, as *LT* is to *TK*, so is *NP* to *PM*, and, owing to the similarity of the triangles *TKB*, *PMF*, as *KT* is to *TB*, so is *MP* to *PF*; therefore, *ex aequali*, as *LT* is to *TB*, so is *NP* to *PF*. {V. 22}

But it was also proved that, as *TB* is to *BL*, so is *PF* to *FN*.

Therefore, ex aequali, as *TL* is to *LB*, so is *PN* to *NF*. {V. 22}

Therefore in the triangles *LTB*, *NPF* the sides are proportional; therefore the triangles *LTB*, *NPF* are equiangular; {VI. 5} hence they are also similar. {VI. Def. I}

Therefore the pyramid of which the triangle *BKT* is the base and the point *L* the vertex is also similar to the pyramid of which the triangle *FMP* is the base and the point *N* the vertex, for they are contained by similar planes equal in multitude. {XI. Def. 9}

But similar pyramids which have triangular bases are to one another in the triplicate ratio of their corresponding sides. {XII. 8}

Therefore the pyramid *BKTL* has to the pyramid *FMPN* the ratio triplicate of that which *BK* has to *FM*.

Similarly, by joining straight lines from *A*, *W*, *D*, *V*, *C*, *U* to *K*, and from *E*, *S*, *H*, *R*, *G*, *Q* to *M*, and setting up on each of the triangles pyramids which have the same vertex with the cones, we can prove that each of the similarly arranged pyramids will also have to each similarly arranged pyramid the ratio triplicate of that which the corresponding side *BK* has to the corresponding side *FM*, that is, which *BD* has to *FH*.

And, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents; {V. 12} therefore also, as the pyramid *BKTL* is to the pyramid *FMPN*, so is the whole pyramid of which the polygon *ATBUCVDW* is the base and the point *L* the vertex to the whole pyramid of which the polygon *EPFQGRHS* is the base and the point *N* the vertex; hence also the pyramid of which *ATBUCVDW* is the base and the point *L* the vertex

has to the pyramid of which the polygon *EPFQGRHS* is the base and the point *N* the vertex the ratio triplicate of that which *BD* has to *FH*.

But, by hypothesis, the cone of which the circle *ABCD* is the base and the point *L* the vertex has also to the solid *O* the ratio triplicate of that which *BD* has to *FH*; therefore, as the cone of which the circle *ABCD* is the base and the point *L* the vertex is to the solid *O*, so is the pyramid of which the polygon *ATBUCVDW* is the base and *L* the vertex to the pyramid of which the polygon *EPFQGRHS* is the base and the point *N* the vertex; therefore, alternately, as the cone of which the circle *ABCD* is the base and *L* the vertex is to the pyramid contained in it of which the polygon *ATBUCVDW* is the base and *L* the vertex, so is the solid *O* to the pyramid of which the polygon *EPFQGRHS* is the base and *N* the vertex. {V. 16}

But the said cone is greater than the pyramid in it; for it encloses it.

Therefore the solid *O* is also greater than the pyramid of which the polygon *EPFQGRHS* is the base and *N* the vertex.

But it is also less: which is impossible.

Therefore the cone of which the circle *ABCD* is the base and *L* the vertex has not to any solid less than the cone of which the circle *EFGH* is the base and the point *N* the vertex the ratio triplicate of that which *BD* has to *FH*:

Similarly we can prove that neither has the cone *EFGHN* to any solid less than the cone *ABCDL* the ratio triplicate of that which *FH* has to *BD*.

I say next that neither has the cone *ABCDL* to any solid greater than the cone *EFGHN* the ratio triplicate of that which *BD* has to *FH*.

For, if possible, let it have that ratio to a greater solid *O*.

Therefore, inversely, the solid *O* has to the cone *ABCDL* the ratio triplicate of that which *FH* has to *BD*.

But, as the solid *O* is to the cone *ABCDL*, so is the cone *EFGHN* to some solid less than the cone *ABCDL*.

Therefore the cone *EFGHN* also has to some solid less than the cone *ABCDL* the ratio triplicate of that which *FH* has to *BD*: which was proved impossible.

Therefore the cone *ABCDL* has not to any solid greater than the cone *EFGHN* the ratio triplicate of that which *BD* has to *FH*.

But it was proved that neither has it this ratio to a less solid than the cone *EFGHN*.

Therefore the cone *ABCDL* has to the cone *EFGHN* the ratio triplicate of that which *BD* has to *FH*.

But, as the cone is to the cone, so is the cylinder to the cylinder, for the cylinder which is on the same base as the cone and of equal height with it is triple of the cone; {XII. 10} therefore the cylinder also has to the cylinder the ratio triplicate of that which *BD* has to *FH*.

Therefore etc. Q. E. D.

PROPOSITION 13.

If a cylinder be cut by a plane which is parallel to its opposite planes, then, as the cylinder is to the cylinder, so will the axis be to the axis.

For let the cylinder *AD* be cut by the plane *GH* which is parallel to the opposite planes *AB*, *CD*, and let the plane *GH* meet the axis at the point *K*; I say that, as the cylinder *BG* is to the cylinder *GD*, so is the axis *EK* to the axis *KF*.

For let the axis *EF* be produced in both directions to the points *L*, *M*, and let there be set out any number whatever of axes *EN*, *NL* equal to the axis *EK*, and any number whatever *FO*, *OM* equal to *FK*; and let the cylinder *PW* on the axis *LM* be conceived of which the circles *PQ*, *VW* are the bases.

Let planes be carried through the points *N*, *O* parallel to *AB*, *CD* and to the bases of the cylinder *PW*, and let them produce the circles *RS*, *TU* about the centres *N*, *O*.

Then, since the axes *LN*, *NE*, *EK* are equal to one another, therefore the cylinders *QR*, *RB*, *BG* are to one another as their bases. {XII. 11}

But the bases are equal; therefore the cylinders *QR*, *RB*, *BG* are also equal to one another.

Since then the axes *LN*, *NE*, *EK* are equal to one another, and the cylinders *QR*, *RB*, *BG* are also equal to one another, and the multitude of the former is equal to the multitude of the latter, therefore, whatever multiple the axis *KL* is of the axis *EK*, the same multiple also will the cylinder *QG* be of the cylinder *GB*.

For the same reason, whatever multiple the axis *MK* is of the axis *KF*, the same multiple also is the cylinder *WG* of the cylinder *GD*.

And, if the axis *KL* is equal to the axis *KM*, the cylinder *QG* will also be equal to the cylinder *GW*, if the axis is greater than the axis, the cylinder will also be greater than the cylinder, and if less, less.

Thus, there being four magnitudes, the axes *EK*, *KF* and the cylinders *BG*, *GD*, there have been taken equimultiples of the axis *EK* and of the cylinder *BG*, namely the axis *LK* and the cylinder *QG*, and equimultiples of the axis *KF* and of the cylinder *GD*, namely the axis *KM* and the cylinder *GW*; and it

has been proved that, if the axis *KL* is in excess of the axis *KM*, the cylinder *QG* is also in excess of the cylinder *GW*, if equal, equal, and if less, less.

Therefore, as the axis *EK* is to the axis *KF*, so is the cylinder *BG* to the cylinder *GD*. {V. Def. 5} Q. E. D.

PROPOSITION 14.

Cones and cylinders which are on equal bases are to one another as their heights.

For let *EB*, *FD* be cylinders on equal bases, the circles *AB*, *CD*; I say that, as the cylinder *EB* is to the cylinder *FD*, so is the axis *GH* to the axis *KL*.

For let the axis *KL* be produced to the point *N*, let *LN* be made equal to the axis *GH*, and let the cylinder *CM* be conceived about *LN* as axis.

Since then the cylinders *EB*, *CM* are of the same height, they are to one another as their bases. {XII. 11}

But the bases are equal to one another; therefore the cylinders *EB*, *CM* are also equal.

And, since the cylinder *FM* has been cut by the plane *CD* which is parallel to its opposite planes, therefore, as the cylinder *CM* is to the cylinder *FD*, so is the axis *LN* to the axis *KL*. {XII. 13}

But the cylinder *CM* is equal to the cylinder *EB*, and the axis *LN* to the axis *GH*; therefore, as the cylinder *EB* is to the cylinder *FD*, so is the axis *GH* to the axis *KL*.

But, as the cylinder *EB* is to the cylinder *FD*, so is the cone *ABG* to the cone *CDK*. {XII. 10}

Therefore also, as the axis *GH* is to the axis *KL*, so is the cone *ABG* to the cone *CDK* and the cylinder *EB* to the cylinder *FD*. Q. E. D.

PROPOSITION 15.

In equal cones and cylinders the bases are reciprocally proportional to the heights; and those cones and cylinders in which the bases are reciprocally proportional to the heights are equal.

Let there be equal cones and cylinders of which the circles *ABCD*, *EFGH* are the bases; let *AC*, *EG* be the diameters of the bases, and *KL*, *MN* the axes, which are also the heights of the cones or cylinders; let the cylinders *AO*, *EP* be completed.

I say that in the cylinders *AO*, *EP* the bases are reciprocally proportional to the heights, that is, as the base *ABCD* is to the base *EFGH*, so is the height *MN* to the height *KL*.

For the height *LK* is either equal to the height *MN* or not equal.

First, let it be equal.

Now the cylinder *AO* is also equal to the cylinder *EP*.

But cones and cylinders which are of the same height are to one another as their bases; {XII. 11} therefore the base *ABCD* is also equal to the base *EFGH*.

Hence also, reciprocally, as the base *ABCD* is to the base *EFGH*, so is the height *MN* to the height *KL*.

Next, let the height *LK* not be equal to *MN*, but let *MN* be greater; from the height *MN* let *QN* be cut off equal to *KL*, through the point *Q* let the cylinder *EP* be cut by the plane *TUS* parallel to the planes of the circles *EFGH*, *RP*,

and let the cylinder *ES* be conceived erected from the circle *EFGH* as base and with height *NQ*.

Now, since the cylinder *AO* is equal to the cylinder *EP*, therefore, as the cylinder *AO* is to the cylinder *ES*, so is the cylinder *EP* to the cylinder *ES*. {V. 7}

But, as the cylinder *AO* is to the cylinder *ES*, so is the base *ABCD* to the base *EFGH*, for the cylinders *AO*, *ES* are of the same height; {XII. 11} and, as the cylinder *EP* is to the cylinder *ES*, so is the height *MN* to the height *QN*, for the cylinder *EP* has been cut by a plane which is parallel to its opposite planes. {XII. 13}

Therefore also, as the base *ABCD* is to the base *EFGH*, so is the height *MN* to the height *QN*. {V. 11}

But the height *QN* is equal to the height *KL*; therefore, as the base *ABCD* is to the base *EFGH*, so is the height *MN* to the height *KL*.

Therefore in the cylinders *AO*, *EP* the bases are reciprocally proportional to the heights.

Next, in the cylinders *AO*, *EP* let the bases be reciprocally proportional to the heights, that is, as the base *ABCD* is to the base *EFGH*, so let the height *MN* be to the height *KL*; I say that the cylinder *AO* is equal to the cylinder *EP*.

For, with the same construction, since, as the base *ABCD* is to the base *EFGH*, so is the height *MN* to the height *KL*, while the height *KL* is equal to the height *QN*, therefore, as the base *ABCD* is to the base *EFGH*, so is the height *MN* to the height *QN*

But, as the base *ABCD* is to the base *EFGH*, so is the cylinder *AO* to the cylinder *ES*, for they are of the same height; {XII. 11} and, as the height *MN*

is to *QN*, so is the cylinder *EP* to the cylinder *ES*; {XII. 13} therefore, as the cylinder *AO* is to the cylinder *ES*, so is the cylinder *EP* to the cylinder *ES*. {V. 11}

Therefore the cylinder *AO* is equal to the cylinder *EP*. {V. 9}

And the same is true for the cones also. Q. E. D.

PROPOSITION 16.

Given two circles about the same centre, to inscribe in the greater circle an equilateral polygon with an even number of sides which does not touch the lesser circle.

Let *ABCD*, *EFGH* be the two given circles about the same centre *K*; thus it is required to inscribe in the greater circle *ABCD* an equilateral polygon with an even number of sides which does not touch the circle *EFGH*.

For let the straight line *BKD* be drawn through the centre *K*, and from the point *G* let *GA* be drawn at right angles to the straight line *BD* and carried through to *C*; therefore *AC* touches the circle *EFGH*. {III. 16, Por.}

Then, bisecting the circumference *BAD*, bisecting the half of it, and doing this continually, we shall leave a circumference less than *AD*. {X. 1}

Let such be left, and let it be *LD*; from *L* let *LM* be drawn perpendicular to *BD* and carried through to *N*, and let *LD*, *DN* be joined; therefore *LD* is equal to *DN*. {III. 3, I. 4}

Now, since *LN* is parallel to *AC*, and *AC* touches the circle *EFGH*, therefore *LN* does not touch the circle *EFGH*; therefore *LD*, *DN* are far from touching the circle *EFGH*.

If then we fit into the circle *ABCD* straight lines equal to the straight line *LD* and placed continuously, there will be inscribed in the circle *ABCD* an

equilateral polygon with an even number of sides which does not touch the lesser circle *EFGH*. Q. E. F.

PROPOSITION 17.

Given two spheres about the same centre, to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface.

Let two spheres be conceived about the same centre *A*; thus it is required to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface.

Let the spheres be cut by any plane through the centre; then the sections will be circles, inasmuch as the sphere was produced by the diameter remaining fixed and the semicircle being carried round it; {XI. Def. 14} hence, in whatever position we conceive the semicircle to be, the plane carried through it will produce a circle on the circumference of the sphere.

And it is manifest that this circle is the greatest possible, inasmuch as the diameter of the sphere, which is of course the diameter both of the semicircle and of the circle, is greater than all the straight lines drawn across in the circle or the sphere.

Let then *BCDE* be the circle in the greater sphere, and *FGH* the circle in the lesser sphere; let two diameters in them, *BD*, *CE*, be drawn at right angles to one another; then, given the two circles *BCDE*, *FGH* about the same centre, let there be inscribed in the greater circle *BCDE* an equilateral polygon with an even number of sides which does not touch the lesser circle *FGH*, let *BK*, *KL*, *LM ME* be its sides in the quadrant *BE*. let *KA* be joined and carried through to *N*, let *AO* be set up from the point *A* at right angles to the plane of the circle *BCDE*, and let it meet the surface of the sphere at *O*, and through *AO* and each of the straight lines *BD*, *KN* let planes be carried;

they will then make greatest circles on the surface of the sphere, for the reason stated.

Let them make such, and in them let *BOD*, *KON* be the semicircles on *BD*, *KN*.

Now, since *OA* is at right angles to the plane of the circle *BCDE*, therefore all the planes through *OA* are also at right angles to the plane of the circle *BCDE*; {XI. 18} hence the semicircles *BOD*, *KON* are also at right angles to the plane of the circle *BCDE*.

And, since the semicircles *BED*, *BOD*, *KON* are equal, for they are on the equal diameters *BD*, *KN*, therefore the quadrants *BE*, *BO*, *KO* are also equal to one another.

Therefore there are as many straight lines in the quadrants *BO*, *KO* equal to the straight lines *BK*, *KL*, *LM*, *ME* as there are sides of the polygon in the quadrant *BE*.

Let them be inscribed, and let them be *BP*, *PQ*, *QR*, *RO* and *KS*, *ST*, *TU*, *UO*, let *SP*, *TQ*, *UR* be joined, and from *P*, *S* let perpendiculars be drawn to the plane of the circle *BCDE*; {XI. 11} these will fall on *BD*, *KN*, the common sections of the planes, inasmuch as the planes of *BOD*, *KON* are also at right angles to the plane of the circle *BCDE*. {cf. XI. Def. 4}

Let them so fall, and let them be *PV*, *SW*, and let *WV* be joined.

Now since, in the equal semicircles *BOD*, *KON*, equal straight lines *BP*, *KS* have been cut off, and the perpendiculars *PV*, *SW* have been drawn, therefore *PV* is equal to *SW*, and *BV* to *KW*. {III. 27, I. 26}

But the whole *BA* is also equal to the whole *KA*; therefore the remainder *VA* is also equal to the remainder *WA*; therefore, as *BV* is to *VA*, so is *KW* to *WA*; therefore *WV* is parallel to *KB*. {VI. 2}

And, since each of the straight lines *PV*, *SW* is at right angles to the plane of the circle *BCDE*, therefore *PV* is parallel to *SW*. {XI. 6}

But it was also proved equal to it; therefore *WV*, *SP* are also equal and parallel. {I. 33}

And, since *WV* is parallel to *SP*, while *WV* is parallel to *KB*, therefore *SP* is also parallel to *KB*. {XI. 9}

And *BP*, *KS* join their extremities; therefore the quadrilateral *KBPS* is in one plane, inasmuch as, if two straight lines be parallel, and points be taken at random on each of them, the straight line joining the points is in the same plane with the parallels. {XI. 7}

For the same reason each of the quadrilaterals *SPQT*, *TQRU* is also in one plane.

But the triangle *URO* is also in one plane. {XI. 2}

If then we conceive straight lines joined from the points *P*, *S*, *Q*, *T*, *R*, *U* to *A*, there will be constructed a certain polyhedral solid figure between the circumferences *BO*, *KO*, consisting of pyramids of which the quadrilaterals *KBPS*, *SPQT*, *TQRU* and the triangle *URO* are the bases and the point *A* the vertex.

And, if we make the same construction in the case of each of the sides *KL*, *LM*, *ME* as in the case of *BK*, and further in the case of the remaining three quadrants, there will be constructed a certain polyhedral figure inscribed in the sphere and contained by pyramids, of which the said quadrilaterals and the triangle *URO*, and the others corresponding to them, are the bases and the point *A* the vertex.

I say that the said polyhedron will not touch the lesser sphere at the surface on which the circle *FGH* is.

Let *AX* be drawn from the point *A* perpendicular to the plane of the quadrilateral *KBPS*, and let it meet the plane at the point *X*; {XI. 11} let *XB*, *XK* be joined.

Then, since *AX* is at right angles to the plane of the quadrilateral *KBPS*, therefore it is also at right angles to all the straight lines which meet it and are in the plane of the quadrilateral. {XI. Def. 3}

Therefore *AX* is at right angles to each of the straight lines *BX*, *XK*.

And, since *AB* is equal to *AK*, the square on *AB* is also equal to the square on *AK*.

And the squares on *AX*, *XB* are equal to the square on *AB*, for the angle at *X* is right; {I. 47} and the squares on *AX*, *XK* are equal to the square on *AK*. {*id.*}

Therefore the squares on *AX*, *XB* are equal to the squares on *AX*, *XK*.

Let the square on *AX* be subtracted from each; therefore the remainder, the square on *BX*, is equal to the remainder, the square on *XK*; therefore *BX* is equal to *XK*.

Similarly we can prove that the straight lines joined from *X* to *P*, *S* are equal to each of the straight lines *BX*, *XK*.

Therefore the circle described with centre *X* and distance one of the straight lines *XB*, *XK* will pass through *P*, *S* also, and *KBPS* will be a quadrilateral in a circle.

Now, since *KB* is greater than *WV*, while *WV* is equal to *SP*, therefore *KB* is greater than *SP*.

But *KB* is equal to each of the straight lines *KS*, *BP*; therefore each of the straight lines *KS*, *BP* is greater than *SP*.

And, since *KBPS* is a quadrilateral in a circle, and *KB*, *BP*, *KS* are equal, and *PS* less, and *BX* is the radius of the circle, therefore the square on *KB* is greater than double of the square on *BX*.

Let *KZ* be drawn from *K* perpendicular to *BV*.

Then, since *BD* is less than double of *DZ*, and, as *BD* is to *DZ*, so is the rectangle *DB*, *BZ* to the rectangle *DZ*, *ZB*, if a square be described upon *BZ* and the parallelogram on *ZD* be completed, then the rectangle *DB*, *BZ* is also less than double of the rectangle *DZ*, *ZB*.

And, if *KD* be joined, the rectangle *DB*, *BZ* is equal to the square on *BK*, and the rectangle *DZ*, *ZB* equal to the square on *KZ*; {III. 31, VI. 8 and Por.} therefore the square on *KB* is less than double of the square on *KZ*.

But the square on *KB* is greater than double of the square on *BX*; therefore the square on *KZ* is greater than the square on *BX*.

And, since *BA* is equal to *KA*, the square on *BA* is equal to the square on *AK*.

And the squares on *BX*, *XA* are equal to the square on *BA*, and the squares on *KZ*, *ZA* equal to the square on *KA*; {I. 47} therefore the squares on *BX*, *XA* are equal to the squares on *KZ*, *ZA*, and of these the square on *KZ* is greater than the square on *BX*; therefore the remainder, the square on *ZA*, is less than the square on *XA*.

Therefore *AX* is greater than *AZ*; therefore *AX* is much greater than *AG*.

And *AX* is the perpendicular on one base of the polyhedron, and *AG* on the surface of the lesser sphere; hence the polyhedron will not touch the lesser sphere on its surface.

Therefore, given two spheres about the same centre, a polyhedral solid has been inscribed in the greater sphere which does not touch the lesser sphere at its surface. Q. E. F.

PORISM.

But if in another sphere also a polyhedral solid be inscribed similar to the solid in the sphere *BCDE*, the polyhedral solid in the sphere *BCDE* has to the polyhedral solid in the other sphere the ratio triplicate of that which the diameter of the sphere *BCDE* has to the diameter of the other sphere.

For, the solids being divided into their pyramids similar in multitude and arrangement, the pyramids will be similar.

But similar pyramids are to one another in the triplicate ratio of their corresponding sides; {XII. 8, Por.} therefore the pyramid of which the quadrilateral *KBPS* is the base, and the point *A* the vertex, has to the similarly arranged pyramid in the other sphere the ratio triplicate of that which the corresponding side has to the corresponding side, that is, of that which the radius *AB* of the sphere about *A* as centre has to the radius of the other sphere.

Similarly also each pyramid of those in the sphere about *A* as centre has to each similarly arranged pyramid of those in the other sphere the ratio triplicate of that which *AB* has to the radius of the other sphere.

And, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents; {V. 12} hence the whole polyhedral solid in the sphere about *A* as centre has to the whole polyhedral solid in the other sphere the ratio triplicate of that which *AB* has to the radius of the other sphere, that is, of that which the diameter *BD* has to the diameter of the other sphere. Q. E. D.

PROPOSITION 18.

Spheres are to one another in the triplicate ratio of their respective diameters.

Let the spheres *ABC*, *DEF* be conceived, and let *BC*, *EF* be their diameters; I say that the sphere *ABC* has to the sphere *DEF* the ratio triplicate of that which *BC* has to *EF*.

For, if the sphere *ABC* has not to the sphere *DEF* the ratio triplicate of that which *BC* has to *EF*, then the sphere *ABC* will have either to some less sphere than the sphere *DEF*, or to a greater, the ratio triplicate of that which *BC* has to *EF*.

First, let it have that ratio to a less sphere *GHK*, let *DEF* be conceived about the same centre with *GHK*, let there be inscribed in the greater sphere *DEF* a polyhedral solid which does not touch the lesser sphere *GHK* at its surface, {XII. 17} and let there also be inscribed in the sphere *ABC* a polyhedral solid similar to the polyhedral solid in the sphere *DEF*; therefore the polyhedral solid in *ABC* has to the polyhedral solid in *DEF* the ratio triplicate of that which *BC* has to *EF*. {XII. 17, Por.}

But the sphere *ABC* also has to the sphere *GHK* the ratio triplicate of that which *BC* has to *EF*; therefore, as the sphere *ABC* is to the sphere *GHK*, so is the polyhedral solid in the sphere *ABC* to the polyhedral solid in the sphere *DEF*; and, alternately, as the sphere *ABC* is to the polyhedron in it, so is the sphere *GHK* to the polyhedral solid in the sphere *DEF*. {V. 16}

But the sphere *ABC* is greater than the polyhedron in it; therefore the sphere *GHK* is also greater than the polyhedron in the sphere *DEF*.

But it is also less, for it is enclosed by it.

Therefore the sphere *ABC* has not to a less sphere than the sphere *DEF* the ratio triplicate of that which the diameter *BC* has to *EF*.

Similarly we can prove that neither has the sphere *DEF* to a less sphere than the sphere *ABC* the ratio triplicate of that which *EF* has to *BC*.

I say next that neither has the sphere *ABC* to any greater sphere than the sphere *DEF* the ratio triplicate of that which *BC* has to *EF*.

For, if possible, let it have that ratio to a greater, *LMN*; therefore, inversely, the sphere *LMN* has to the sphere *ABC* the ratio triplicate of that which the diameter *EF* has to the diameter *BC*.

But, inasmuch as *LMN* is greater than *DEF*, therefore, as the sphere *LMN* is to the sphere *ABC*, so is the sphere *DEF* to some less sphere than the sphere *ABC*, as was before proved. {XII. 2, Lemma}

Therefore the sphere *DEF* also has to some less sphere than the sphere *ABC* the ratio triplicate of that which *EF* has to *BC*: which was proved impossible.

Therefore the sphere *ABC* has not to any sphere greater than the sphere *DEF* the ratio triplicate of that which *BC* has to *EF*.

But it was proved that neither has it that ratio to a less sphere.

Therefore the sphere *ABC* has to the sphere *DEF* the ratio triplicate of that which *BC* has to *EF*. Q. E. D.

BOOK XIII. HISTORICAL NOTE.

I have already given, in the note to IV. 10, the evidence upon which the construction of the five regular solids is attributed to the Pythagoreans. Some of them, the cube, the tetrahedron (which is nothing but a pyramid), and the octahedron (which is only a double pyramid with a square base), cannot but have been known to the Egyptians. And it appears that dodecahedra have been found, of bronze or other material, which may belong to periods earlier than Pythagoras' time by some centuries (for references see Cantor's Geschichte der Mathematik I{3}, pp. 175-6).

It is true that the author of the scholium No. I to Eucl. XIII. says that the Book is about "the five so-called Platonic figures, which however do not belong to Plato, three of the aforesaid five figures being due to the Pythagoreans, namely the cube, the pyramid and the dodecahedron, while the octahedron and the icosahedron are due to Theaetetus." This statement (taken probably from Geminus) may perhaps rest on the fact that Theaetetus was the first to write at any length about the two last-mentioned solids. We are told indeed by Suidas (S. V. Θεαίτητος) that Theaetetus "first wrote on the 'five solids' as they are called." This no doubt means that Theaetetus was the first to write a complete and systematic treatise on all the regular solids; it does not exclude the possibility that Hippasus or others had already written on the dodecahedron. The fact that Theaetetus wrote upon the regular solids agrees very well with the evidence which we possess of his contributions to the theory of irrationals, the connexion between which and the investigation of the regular solids is seen in Euclid's Book XIII.

Theaetetus flourished about 380 B.C., and his work on the regular solids was soon followed by another, that of Aristaeus, an elder contemporary of Euclid, who also wrote an important book on Solid Loci, i.e. on conics treated as loci. This Aristaeus (known as "the elder") wrote in the period

about 320 B.C. We hear of his Comparison of the five regular solids from Hypsicles (2nd cent. B.C.), the writer of the short book commonly included in the editions of the Elements as Book XIV. Hypsicles gives in this Book some six propositions supplementing Eucl. XIII.; and he introduces the second of the propositions (Heiberg's Euclid, Vol. v. p. 6) as follows:

"*The same circle circumscribes both the pentagon of the dodecahedron and the triangle of the icosahedron when both are inscribed in the same sphere*. This is proved by Aristaeus in the book entitled Comparison of the five figures."

Hypsicles proceeds (pp. 7 sqq.) to give a proof of this theorem. Allman pointed out (Greek Geometry from Thales to Euclid, 1889, pp. 201-2) that this proof depends on eight theorems, six of which appear in Euclid's Book XIII. (in Propositions 8, 10, 12, 15, 16 with Por., 17); two other propositions not mentioned by Allman are also used, namely XIII. 4 and 9. This seems, as Allman says, to confirm the inference of Bretschneider (p. 171) that, as Aristaeus' work was the newest and latest in which, before Euclid's time, this subject was treated, we have in Eucl. XIII. at least a partial recapitulation of the contents of the treatise of Aristaeus.

After Euclid, Apollonius wrote on the comparison of the dodecahedron and the icosahedron inscribed in one and the same sphere. This we also learn from Hypsicles, who says in the next words following those about Aristaeus above quoted: "But it is proved by Apollonius in the second edition of his Comparison of the dodecahedron with the icosahedron that, as the surface of the dodecahedron is to the surface of the icosahedron {inscribed in the same sphere}, so is the dodecahedron itself {i.e. its volume} to the icosahedron, because the perpendicular is the same from the centre of the sphere to the pentagon of the dodecahedron and to the triangle of the icosahedron."

PROPOSITIONS.

PROPOSITION 1.

If a straight line be cut in extreme and mean ratio, the square on the greater segment added to the half of the whole is five times the square on the half.

For let the straight line *AB* be cut in extreme and mean ratio at the point *C*, and let *AC* be the greater segment; let the straight line *AD* be produced in a straight line with *CA*, and let *AD* be made half of *AB*; I say that the square on *CD* is five times the square on *AD*.

For let the squares *AE*, *DF* be described on *AB*, *DC*, and let the figure in *DF* be drawn; let *FC* be carried through to *G*.

Now, since *AB* has been cut in extreme and mean ratio at *C*, therefore the rectangle *AB*, *BC* is equal to the square on *AC*. {VI. Def. 3, VI. 17}

And *CE* is the rectangle *AB*, *BC*, and *FH* the square on *AC*; therefore *CE* is equal to *FH*.

And, since *BA* is double of *AD*, while *BA* is equal to *KA*, and *AD* to *AH*, therefore *KA* is also double of *AH*.

But, as *KA* is to *AH*, so is *CK* to *CH*; {VI. 1} therefore *CK* is double of *CH*.

But *LH*, *HC* are also double of *CH*.

Therefore *KC* is equal to *LH*, *HC*.

But *CE* was also proved equal to *HF*; therefore the whole square *AE* is equal to the gnomon *MNO*.

And, since *BA* is double of *AD*, the square on *BA* is quadruple of the square on *AD*, that is, *AE* is quadruple of *DH*.

But *AE* is equal to the gnomon *MNO*; therefore the gnomon *MNO* is also quadruple of *AP*; therefore the whole *DF* is five times *AP*.

And *DF* is the square on *DC*, and *AP* the square on *DA*; therefore the square on *CD* is five times the square on *DA*.

Therefore etc. Q. E. D.

PROPOSITION 2.

If the square on a straight line be five times the square on a segment of it, then, when the double of the said segment is cut in extreme and mean ratio, the greater segment is the remaining part of the original straight line.

For let the square on the straight line *AB* be five times the square on the segment *AC* of it, and let *CD* be double of *AC*; I say that, when *CD* is cut in extreme and mean ratio, the greater segment is *CB*.

Let the squares *AF*, *CG* be described on *AB*, *CD* respectively, let the figure in *AF* be drawn, and let *BE* be drawn through.

Now, since the square on *BA* is five times the square on *AC*, *AF* is five times *AH*.

Therefore the gnomon *MNO* is quadruple of *AH*.

And, since *DC* is double of *CA*, therefore the square on *DC* is quadruple of the square on *CA*, that is, *CG* is quadruple of *AH*.

But the gnomon *MNO* was also proved quadruple of *AH*; therefore the gnomon *MNO* is equal to *CG*.

And, since *DC* is double of *CA*, while *DC* is equal to *CK*, and *AC* to *CH*, therefore *KB* is also double of *BH*. {VI. 1}

But *LH*, *HB* are also double of *HB*; therefore *KB* is equal to *LH*, *HB*.

But the whole gnomon *MNO* was also proved equal to the whole *CG*; therefore the remainder *HF* is equal to *BG*.

And *BG* is the rectangle *CD*, *DB*, for *CD* is equal to *DG*; and *HF* is the square on *CB*; therefore the rectangle *CD*, *DB* is equal to the square on *CB*.

Therefore, as *DC* is to *CB*, so is *CB* to *BD*.

But *DC* is greater than *CB*; therefore *CB* is also greater than *BD*.

Therefore, when the straight line *CD* is cut in extreme and mean ratio, *CB* is the greater segment.

Therefore etc. Q. E. D.

LEMMA.

That the double of *AC* is greater than *BC* is to be proved thus.

If not, let *BC* be, if possible, double of *CA*.

Therefore the square on *BC* is quadruple of the square on *CA*; therefore the squares on *BC*, *CA* are five times the square on *CA*.

But, by hypothesis, the square on *BA* is also five times the square on *CA*; therefore the square on *BA* is equal to the squares on *BC*, *CA*: which is impossible. {II. 4}

Therefore *CB* is not double of *AC*.

Similarly we can prove that neither is a straight line less than *CB* double of *CA*; for the absurdity is much greater.

Therefore the double of *AC* is greater than *CB*. Q. E. D.

PROPOSITION 3.

If a straight line be cut in extreme and mean ratio, the square on the lesser segment added to the half of the greater segment is five times the square on the half of the greater segment.

For let any straight line *AB* be cut in extreme and mean ratio at the point *C*, let *AC* be the greater segment, and let *AC* be bisected at *D*; I say that the square on *BD* is five times the square on *DC*.

For let the square *AE* be described on *AB*, and let the figure be drawn double.

Since *AC* is double of *DC*, therefore the square on *AC* is quadruple of the square on *DC*, that is, *RS* is quadruple of *FG*.

And, since the rectangle *AB*, *BC* is equal to the square on *AC*, and *CE* is the rectangle *AB*, *BC*, therefore *CE* is equal to *RS*.

But *RS* is quadruple of *FG*; therefore *CE* is also quadruple of *FG*.

Again, since *AD* is equal to *DC*, *HK* is also equal to *KF*.

Hence the square *GF* is also equal to the square *HL*.

Therefore *GK* is equal to *KL*, that is, *MN* to *NE*; hence *MF* is also equal to *FE*.

But *MF* is equal to *CG*; therefore *CG* is also equal to *FE*.

Let *CN* be added to each; therefore the gnomon *OPQ* is equal to *CE*.

But *CE* was proved quadruple of *GF*; therefore the gnomon *OPQ* is also quadruple of the square *FG*.

Therefore the gnomon *OPQ* and the square *FG* are five times *FG*.

But the gnomon *OPQ* and the square *FG* are the square *DN*.

And *DN* is the square on *DB*, and *GF* the square on *DC*.

Therefore the square on *DB* is five times the square on *DC*. Q. E. D.

PROPOSITION 4.

If a straight line be cut in extreme and mean ratio, the square on the whole and the square on the lesser segment together are triple of the square on the greater segment.

Let *AB* be a straight line, let it be cut in extreme and mean ratio at *C*, and let *AC* be the greater segment; I say that the squares on *AB*, *BC* are triple of the square on *CA*.

For let the square *ADEB* be described on *AB*, and let the figure be drawn.

Since then *AB* has been cut in extreme and mean ratio at *C*, and *AC* is the greater segment, therefore the rectangle *AB*, *BC* is equal to the square on *AC*. {VI. Def. 3, VI. 17}

And *AK* is the rectangle *AB*, *BC*, and *HG* the square on *AC*; therefore *AK* is equal to *HG*.

And, since *AF* is equal to *FE*, let *CK* be added to each; therefore the whole *AK* is equal to the whole *CE*; therefore *AK*, *CE* are double of *AK*.

But *AK*, *CE* are the gnomon *LMN* and the square *CK*; therefore the gnomon *LMN* and the square *CK* are double of *AK*.

But, further, *AK* was also proved equal to *HG*; therefore the gnomon *LMN* and the squares *CK*, *HG* are triple of the square *HG*.

And the gnomon *LMN* and the squares *CK*, *HG* are the whole square *AE* and *CK*, which are the squares on *AB*, *BC*, while *HG* is the square on *AC*.

Therefore the squares on *AB*, *BC* are triple of the square on *AC*. Q. E. D.

PROPOSITION 5.

If a straight line be cut in extreme and mean ratio, and there be added to it a straight line equal to the greater segment, the whole straight line has been cut in extreme and mean ratio, and the original straight line is the greater segment.

For let the straight line *AB* be cut in extreme and mean ratio at the point *C*, let *AC* be the greater segment, and let *AD* be equal to *AC*.

I say that the straight line *DB* has been cut in extreme and mean ratio at *A*, and the original straight line *AB* is the greater segment.

For let the square *AE* be described on *AB*, and let the figure be drawn.

Since *AB* has been cut in extreme and mean ratio at *C*, therefore the rectangle *AB*, *BC* is equal to the square on *AC*. {VI. Def. 3, VI. 17}

And *CE* is the rectangle *AB*, *BC*, and *CH* the square on *AC*; therefore *CE* is equal to *HC*.

But *HE* is equal to *CE*, and *DH* is equal to *HC*; therefore *DH* is also equal to *HE*.

Therefore the whole *DK* is equal to the whole *AE*.

And *DK* is the rectangle *BD*, *DA*, for *AD* is equal to *DL*; and *AE* is the square on *AB*; therefore the rectangle *BD*, *DA* is equal to the square on *AB*.

Therefore, as *DB* is to *BA*, so is *BA* to *AD*. {VI. 17}

And *DB* is greater than *BA*; therefore *BA* is also greater than *AD*. {V. 14}

Therefore *DB* has been cut in extreme and mean ratio at *A*, and *AB* is the greater segment. Q. E. D.

PROPOSITION 6.

If a rational straight line be cut in extreme and mean ratio, each of the segments is the irrational straight line called apotome.

Let *AB* be a rational straight line, let it be cut in extreme and mean ratio at *C*, and let *AC* be the greater segment; I say that each of the straight lines *AC*, *CB* is the irrational straight line called apotome.

For let *BA* be produced, and let *AD* be made half of *BA*.

Since then the straight line *AB* has been cut in extreme and mean ratio, and to the greater segment *AC* is added *AD* which is half of *AB*, therefore the square on *CD* is five times the square on *DA*. {XIII. 1}

Therefore the square on *CD* has to the square on *DA* the ratio which a number has to a number; therefore the square on *CD* is commensurable with the square on *DA*. {X. 6}

But the square on *DA* is rational, for *DA* is rational, being half of *AB* which is rational; therefore the square on *CD* is also rational; {X. Def. 4} therefore *CD* is also rational.

And, since the square on *CD* has not to the square on *DA* the ratio which a square number has to a square number, therefore *CD* is incommensurable in length with *DA*; {X. 9} therefore *CD*, *DA* are rational straight lines commensurable in square only; therefore *AC* is an apotome. {X. 73}

Again, since *AB* has been cut in extreme and mean ratio, and *AC* is the greater segment, therefore the rectangle *AB*, *BC* is equal to the square on *AC*. {VI. Def. 3, VI. 17}

Therefore the square on the apotome *AC*, if applied to the rational straight line *AB*, produces *BC* as breadth.

But the square on an apotome, if applied to a rational straight line, produces as breadth a first apotome; {X. 97} therefore *CB* is a first apotome.

And *CA* was also proved to be an apotome.

Therefore etc. Q. E. D.

PROPOSITION 7.

If three angles of an equilateral pentagon, taken either in order or not in order, *be equal*, *the pentagon will be equiangular.*

For in the equilateral pentagon *ABCDE* let, first, three angles taken in order, those at *A*, *B*, *C*, be equal to one another; I say that the pentagon *ABCDE* is equiangular.

For let *AC*, *BE*, *FD* be joined.

Now, since the two sides *CB*, *BA* are equal to the two sides *BA*, *AE* respectively, and the angle *CBA* is equal to the angle *BAE*, therefore the base *AC* is equal to the base *BE*, the triangle *ABC* is equal to the triangle *ABE*, and

the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend, {I. 4} that is, the angle *BCA* to the angle *BEA*, and the angle *ABE* to the angle *CAB*; hence the side *AF* is also equal to the side *BF*. {I. 6}

But the whole *AC* was also proved equal to the whole *BE*; therefore the remainder *FC* is also equal to the remainder *FE*.

But *CD* is also equal to *DE*.

Therefore the two sides *FC*, *CD* are equal to the two sides *FE*, *ED*; and the base *FD* is common to them; therefore the angle *FCD* is equal to the angle *FED*. {I. 8}

But the angle *BCA* was also proved equal to the angle *AEB*; therefore the whole angle *BCD* is also equal to the whole angle *AED*.

But, by hypothesis, the angle *BCD* is equal to the angles at *A*, *B*; therefore the angle *AED* is also equal to the angles at *A*, *B*.

Similarly we can prove that the angle *CDE* is also equal to the angles at *A*, *B*, *C*; therefore the pentagon *ABCDE* is equiangular.

Next, let the given equal angles not be angles taken in order, but let the angles at the points *A*, *C*, *D* be equal; I say that in this case too the pentagon *ABCDE* is equiangular.

For let *BD* be joined.

Then, since the two sides *BA*, *AE* are equal to the two sides *BC*, *CD*, and they contain equal angles, therefore the base *BE* is equal to the base *BD*, the triangle *ABE* is equal to the triangle *BCD*, and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend; {I. 4} therefore the angle *AEB* is equal to the angle *CDB*.

But the angle *BED* is also equal to the angle *BDE*, since the side *BE* is also equal to the side *BD*. {I. 5}

Therefore the whole angle *AED* is equal to the whole angle *CDE*.

But the angle *CDE* is, by hypothesis, equal to the angles at *A*, *C*; therefore the angle *AED* is also equal to the angles at *A*, *C*.

For the same reason the angle *ABC* is also equal to the angles at *A*, *C*, *D*.

Therefore the pentagon *ABCDE* is equiangular. Q. E. D.

PROPOSITION 8.

If in an equilateral and equiangular pentagon straight lines subtend two angles taken in order, they cut one another in extreme and mean ratio, and their greater segments are equal to the side of the pentagon.

For in the equilateral and equiangular pentagon *ABCDE* let the straight lines *AC*, *BE*, cutting one another at the point *H*, subtend two angles taken in order, the angles at *A*, *B*; I say that each of them has been cut in extreme and mean ratio at the point *H*, and their greater segments are equal to the side of the pentagon.

For let the circle *ABCDE* be circumscribed about the pentagon *ABCDE*. {IV. 14}

Then, since the two straight lines *EA*, *AB* are equal to the two *AB*, *BC*, and they contain equal angles, therefore the base *BE* is equal to the base *AC*, the triangle *ABE* is equal to the triangle *ABC*, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend. {I. 4}

Therefore the angle *BAC* is equal to the angle *ABE*; therefore the angle *AHE* is double of the angle *BAH*. {I. 32}

But the angle *EAC* is also double of the angle *BAC*, inasmuch as the circumference *EDC* is also double of the circumference *CB*; {III. 28, VI. 33} therefore the angle *HAE* is equal to the angle *AHE*; hence the straight line *HE* is also equal to *EA*, that is, to *AB*. {I. 6}

And, since the straight line *BA* is equal to *AE*, the angle *ABE* is also equal to the angle *AEB*. {I. 5}

But the angle *ABE* was proved equal to the angle *BAH*; therefore the angle *BEA* is also equal to the angle *BAH*.

And the angle *ABE* is common to the two triangles *ABE* and *ABH*; therefore the remaining angle *BAE* is equal to the remaining angle *AHB*; {I. 32} therefore the triangle *ABE* is equiangular with the triangle *ABH*; therefore, proportionally, as *EB* is to *BA*, so is *AB* to *BH*. {VI. 4}

But *BA* is equal to *EH*; therefore, as *BE* is to *EH*, so is *EH* to *HB*.

And *BE* is greater than *EH*; therefore *EH* is also greater than *HB*. {V. 14}

Therefore *BE* has been cut in extreme and mean ratio at *H*, and the greater segment *HE* is equal to the side of the pentagon.

Similarly we can prove that *AC* has also been cut in extreme and mean ratio at *H*, and its greater segment *CH* is equal to the side of the pentagon. Q. E. D.

PROPOSITION 9.

If the side of the hexagon and that of the decagon inscribed in the same circle be added together, the whole straight line has been cut in extreme and mean ratio, and its greater segment is the side of the hexagon.

Let *ABC* be a circle; of the figures inscribed in the circle *ABC* let *BC* be the side of a decagon, *CD* that of a hexagon, and let them be in a straight line; I say that the whole straight line *BD* has been cut in extreme and mean ratio, and *CD* is its greater segment.

For let the centre of the circle, the point *E*, be taken, let *EB*, *EC*, *ED* be joined, and let *BE* be carried through to *A*.

Since *BC* is the side of an equilateral decagon, therefore the circumference *ACB* is five times the circumference *BC*; therefore the circumference *AC* is quadruple of *CB*.

But, as the circumference *AC* is to *CB*, so is the angle *AEC* to the angle *CEB*; {VI. 33} therefore the angle *AEC* is quadruple of the angle *CEB*.

And, since the angle *EBC* is equal to the angle *ECB*, {I. 5} therefore the angle *AEC* is double of the angle *ECB*. {I. 32}

And, since the straight line *EC* is equal to *CD*, for each of them is equal to the side of the hexagon inscribed in the circle *ABC*, {IV. 15, Por.} the angle *CED* is also equal to the angle *CDE*; {I. 5} therefore the angle *ECB* is double of the angle *EDC*. {I. 32}

But the angle *AEC* was proved double of the angle *ECB*; therefore the angle *AEC* is quadruple of the angle *EDC*.

But the angle *AEC* was also proved quadruple of the angle *BEC*; therefore the angle *EDC* is equal to the angle *BEC*.

But the angle *EBD* is common to the two triangles *BEC* and *BED*; therefore the remaining angle *BED* is also equal to the remaining angle *ECB*; {I. 32} therefore the triangle *EBD* is equiangular with the triangle *EBC*.

Therefore, proportionally, as *DB* is to *BE*, so is *EB* to *BC*. {VI. 4}

But *EB* is equal to *CD*.

Therefore, as *BD* is to *DC*, so is *DC* to *CB*.

And *BD* is greater than *DC*; therefore *DC* is also greater than *CB*.

Therefore the straight line *BD* has been cut in extreme and mean ratio, and *DC* is its greater segment. Q. E. D.

PROPOSITION 10.

If an equilateral pentagon be inscribed in a circle, the square on the side of the pentagon is equal to the squares on the side of the hexagon and on that of the decagon inscribed in the same circle.

Let *ABCDE* be a circle, and let the equilateral pentagon *ABCDE* be inscribed in the circle *ABCDE*.

I say that the square on the side of the pentagon *ABCDE* is equal to the squares on the side of the hexagon and on that of the decagon inscribed in the circle *ABCDE*.

For let the centre of the circle, the point *F*, be taken, let *AF* be joined and carried through to the point *G*, let *FB* be joined, let *FH* be drawn from *F* perpendicular to *AB* and be carried through to *K*, let *AK*, *KB* be joined, let *FL* be again drawn from *F* perpendicular to *AK*, and be carried through to *M*, and let *KN* be joined.

Since the circumference *ABCG* is equal to the circumference *AEDG*, and in them *ABC* is equal to *AED*, therefore the remainder, the circumference *CG*, is equal to the remainder *GD*.

But *CD* belongs to a pentagon; therefore *CG* belongs to a decagon.

And, since *FA* is equal to *FB*, and *FH* is perpendicular, therefore the angle *AFK* is also equal to the angle *KFB*. {I. 5, I. 26}

Hence the circumference *AK* is also equal to *KB*; {III. 26} therefore the circumference *AB* is double of the circumference *BK*; therefore the straight line *AK* is a side of a decagon.

For the same reason *AK* is also double of *KM*.

Now, since the circumference *AB* is double of the circumference *BK*, while the circumference *CD* is equal to the circumference *AB*, therefore the circumference *CD* is also double of the circumference *BK*.

But the circumference *CD* is also double of *CG*; therefore the circumference *CG* is equal to the circumference *BK*.

But *BK* is double of *KM*, since *KA* is so also; therefore *CG* is also double of *KM*.

But, further, the circumference *CB* is also double of the circumference *BK*, for the circumference *CB* is equal to *BA*.

Therefore the whole circumference *GB* is also double of *BM*; hence the angle *GFB* is also double of the angle *BFM*. {VI. 33}

But the angle *GFB* is also double of the angle *FAB*, for the angle *FAB* is equal to the angle *ABF*.

Therefore the angle *BFN* is also equal to the angle *FAB*.

But the angle *ABF* is common to the two triangles *ABF* and *BFN*; therefore the remaining angle *AFB* is equal to the remaining angle *BNF*; {I. 32} therefore the triangle *ABF* is equiangular with the triangle *BFN*.

Therefore, proportionally, as the straight line *AB* is to *BF*, so is *FB* to *BN*; {VI. 4} therefore the rectangle *AB*, *BN* is equal to the square on *BF*. {VI. 17}

Again, since *AL* is equal to *LK*, while *LN* is common and at right angles, therefore the base *KN* is equal to the base *AN*; {I. 4} therefore the angle *LKN* is also equal to the angle *LAN*.

But the angle *LAN* is equal to the angle *KBN*; therefore the angle *LKN* is also equal to the angle *KBN*.

And the angle at *A* is common to the two triangles *AKB* and *AKN*.

Therefore the remaining angle *AKB* is equal to the remaining angle *KNA*; {I. 32} therefore the triangle *KBA* is equiangular with the triangle *KNA*.

Therefore, proportionally, as the straight line *BA* is to *AK*, so is *KA* to *AN*; {VI. 4} therefore the rectangle *BA*, *AN* is equal to the square on *AK*. {VI. 17}

But the rectangle *AB*, *BN* was also proved equal to the square on *BF*; therefore the rectangle *AB*, *BN* together with the rectangle *BA*, *AN*, that is, the square on *BA* {II. 2}, is equal to the square on *BF* together with the square on *AK*.

And *BA* is a side of the pentagon, *BF* of the hexagon {IV. 15, Por.}, and *AK* of the decagon.

Therefore etc. Q. E. D.

PROPOSITION 11.

If in a circle which has its diameter rational an equilateral pentagon be inscribed, the side of the pentagon is the irrational straight line called minor.

For in the circle *ABCDE* which has its diameter rational let the equilateral pentagon *ABCDE* be inscribed; I say that the side of the pentagon is the irrational straight line called minor.

For let the centre of the circle, the point *F*, be taken, let *AF*, *FB* be joined and carried through to the points, *G*, *H*, let *AC* be joined, and let *FK* be made a fourth part of *AF*.

Now *AF* is rational; therefore *FK* is also rational.

But *BF* is also rational; therefore the whole *BK* is rational.

And, since the circumference *ACG* is equal to the circumference *ADG*, and in them *ABC* is equal to *AED*, therefore the remainder *CG* is equal to the remainder *GD*.

And, if we join *AD*, we conclude that the angles at *L* are right, and *CD* is double of *CL*.

For the same reason the angles at *M* are also right, and *AC* is double of *CM*.

Since then the angle *ALC* is equal to the angle *AMF*, and the angle *LAC* is common to the two triangles *ACL* and *AMF*, therefore the remaining angle *ACL* is equal to the remaining angle *MFA*; {I. 32} therefore the triangle *ACL* is equiangular with the triangle *AMF*; therefore, proportionally, as *LC* is to *CA*, so is *MF* to *FA*.

And the doubles of the antecedents may be taken; therefore, as the double of *LC* is to *CA*, so is the double of *MF* to *FA*.

But, as the double of *MF* is to *FA*, so is *MF* to the half of *FA*; therefore also, as the double of *LC* is to *CA*, so is *MF* to the half of *FA*.

And the halves of the consequents may be taken; therefore, as the double of *LC* is to the half of *CA*, so is *MF* to the fourth of *FA*.

And *DC* is double of *LC*, *CM* is half of *CA*, and *FK* a fourth part of *FA*; therefore, as *DC* is to *CM*, so is *MF* to *FK*.

Componendo also, as the sum of *DC*, *CM* is to *CM*, so is *MK* to *KF*; {V. 18} therefore also, as the square on the sum of *DC*, *CM* is to the square on *CM*, so is the square on *MK* to the square on *KF*.

And since, when the straight line subtending two sides of the pentagon, as *AC*, is cut in extreme and mean ratio, the greater segment is equal to the side of the pentagon, that is, to *DC*, {XIII. 8} while the square on the greater segment added to the half of the whole is five times the square on the half of the whole, {XIII. 1} and *CM* is half of the whole *AC*, therefore the square on *DC*, *CM* taken as one straight line is five times the square on *CM*.

But it was proved that, as the square on *DC*, *CM* taken as one straight line is to the square on *CM*, so is the square on *MK* to the square on *KF*; therefore the square on *MK* is five times the square on *KF*.

But the square on *KF* is rational, for the diameter is rational; therefore the square on *MK* is also rational; therefore *MK* is rational

And, since *BF* is quadruple of *FK*, therefore *BK* is five times *KF*; therefore the square on *BK* is twenty-five times the square on *KF*.

But the square on *MK* is five times the square on *KF*; therefore the square on *BK* is five times the square on *KM*; therefore the square on *BK* has not to the square on *KM* the ratio which a square number has to a square number; therefore *BK* is incommensurable in length with *KM*. {X. 9}

And each of them is rational.

Therefore *BK*, *KM* are rational straight lines commensurable in square only.

But, if from a rational straight line there be subtracted a rational straight line which is commensurable with the whole in square only, the remainder is irrational, namely an apotome; therefore *MB* is an apotome and *MK* the annex to it. {X. 73}

I say next that *MB* is also a fourth apotome.

Let the square on *N* be equal to that by which the square on *BK* is greater than the square on *KM*; therefore the square on *BK* is greater than the square on *KM* by the square on *N*.

And, since *KF* is commensurable with *FB*, componendo also, *KB* is commensurable with *FB*. {X. 15}

But *BF* is commensurable with *BH*; therefore *BK* is also commensurable with *BH*. {X. 12}

And, since the square on *BK* is five times the square on *KM*, therefore the square on *BK* has to the square on *KM* the ratio which 5 has to 1.

Therefore, convertendo, the square on *BK* has to the square on *N* the ratio which 5 has to 4 {V. 19, Por.}, and this is not the ratio which a square number has to a square number; therefore *BK* is incommensurable with *N*; {X. 9} therefore the square on *BK* is greater than the square on *KM* by the square on a straight line incommensurable with *BK*.

Since then the square on the whole *BK* is greater than the square on the annex *KM* by the square on a straight line incommensurable with *BK*, and the whole *BK* is commensurable with the rational straight line, *BH*, set out, therefore *MB* is a fourth apotome. {X. Deff. III. 4}

But the rectangle contained by a rational straight line and a fourth apotome is irrational, and its square root is irrational, and is called minor. {X. 94}

But the square on *AB* is equal to the rectangle *HB*, *BM*, because, when *AH* is joined, the triangle *ABH* is equiangular with the triangle *ABM*, and, as *HB* is to *BA*, so is *AB* to *BM*.

Therefore the side *AB* of the pentagon is the irrational straight line called minor. Q. E. D.

PROPOSITION 12.

If an equilateral triangle be inscribed in a circle, *the square on the side of the triangle is triple of the square on the radius of the circle.*

Let *ABC* be a circle, and let the equilateral triangle *ABC* be inscribed in it; I say that the square on one side of the triangle *ABC* is triple of the square on the radius of the circle.

For let the centre *D* of the circle *ABC* be taken, let *AD* be joined and carried through to *E*, and let *BE* be joined.

Then, since the triangle *ABC* is equilateral, therefore the circumference *BEC* is a third part of the circumference of the circle *ABC*.

Therefore the circumference *BE* is a sixth part of the circumference of the circle; therefore the straight line *BE* belongs to a hexagon; therefore it is equal to the radius *DE*. {IV. 15, Por.}

And, since *AE* is double of *DE*, the square on *AE* is quadruple of the square on *ED*, that is, of the square on *BE*.

But the square on *AE* is equal to the squares on *AB*, *BE*; {III. 31, I. 47} therefore the squares on *AB*, *BE* are quadruple of the square on *BE*.

Therefore, separando, the square on *AB* is triple of the square on *BE*.

But *BE* is equal to *DE*; therefore the square on *AB* is triple of the square on *DE*.

Therefore the square on the side of the triangle is triple of the square on the radius. Q. E. D.

PROPOSITION 13.

To construct a pyramid, to comprehend it in a given sphere, and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid.

Let the diameter *AB* of the given sphere be set out, and let it be cut at the point *C* so that *AC* is double of *CB*; let the semicircle *ADB* be described on *AB*, let *CD* be drawn from the point *C* at right angles to *AB*, and let *DA* be joined; let the circle *EFG* which has its radius equal to *DC* be set out, let the equilateral triangle *EFG* be inscribed in the circle *EFG*, {IV. 2} let the centre of the circle, the point *H*, be taken, {III. 1} let *EH*, *HF*, *HG* be joined; from the point *H* let *HK* be set up at right angles to the plane of the circle *EFG*, {XI. 12} let *HK* equal to the straight line *AC* be cut off from *HK*, and let *KE*, *KF*, *KG* be joined.

Now, since *KH* is at right angles to the plane of the circle *EFG*, therefore it will also make right angles with all the straight lines which meet it and are in the plane of the circle *EFG*. {XI. Def. 3}

But each of the straight lines *HE*, *HF*, *HG* meets it: therefore *HK* is at right angles to each of the straight lines *HE*, *HF*, *HG*.

And, since *AC* is equal to *HK*, and *CD* to *HE*, and they contain right angles, therefore the base *DA* is equal to the base *KE*. {I. 4}

For the same reason each of the straight lines *KF*, *KG* is also equal to *DA*; therefore the three straight lines *KE*, *KF*, *KG* are equal to one another.

And, since *AC* is double of *CB*, therefore *AB* is triple of *BC*.

But, as *AB* is to *BC*, so is the square on *AD* to the square on *DC*, as will be proved afterwards.

Therefore the square on *AD* is triple of the square on *DC*.

But the square on *FE* is also triple of the square on *EH*, {XIII. 12} and *DC* is equal to *EH*; therefore *DA* is also equal to *EF*.

But *DA* was proved equal to each of the straight lines *KE*, *KF*, *KG*; therefore each of the straight lines *EF*, *FG*, *GE* is also equal to each of the straight lines *KE*, *KF*, *KG*; therefore the four triangles *EFG*, *KEF*, *KFG*, *KEG* are equilateral.

Therefore a pyramid has been constructed out of four equilateral triangles, the triangle *EFG* being its base and the point *K* its vertex.

It is next required to comprehend it in the given sphere and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid.

For let the straight line *HL* be produced in a straight line with *KH*, and let *HL* be made equal to *CB*.

Now, since, as *AC* is to *CD*, so is *CD* to *CB*, {VI. 8, Por.} while *AC* is equal to *KH*, *CD* to *HE*, and *CB* to *HL*, therefore, as *KH* is to *HE*, so is *EH* to *HL*; therefore the rectangle *KH*, *HL* is equal to the square on *EH*. {VI. 17}

And each of the angles *KHE*. *EHL* is right; therefore the semicircle described on *KL* will pass through *E* also. {cf. VI. 8, III. 31.}

If then, *KL* remaining fixed, the semicircle be carried round and restored to the same position from which it began to be moved, it will also pass through the points *F*, *G*, since, if *FL*, *LG* be joined, the angles at *F*, *G* similarly become right angles; and the pyramid will be comprehended in the given sphere.

For *KL*, the diameter of the sphere, is equal to the diameter *AB* of the given sphere, inasmuch as *KH* was made equal to *AC*, and *HL* to *CB*.

I say next that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid

For, since *AC* is double of *CB*, therefore *AB* is triple of *BC*; and, convertendo, *BA* is one and a half times *AC*.

But, as *BA* is to *AC*, so is the square on *BA* to the square on *AD*.

Therefore the square on *BA* is also one and a half times the square on *AD*.

And *BA* is the diameter of the given sphere, and *AD* is equal to the side of the pyramid.

Therefore the square on the diameter of the sphere is one and a half times the square on the side of the pyramid. Q. E. D.

LEMMA.

It is to be proved that, as *AB* is to *BC*, so is the square on *AD* to the square on *DC*.

For let the figure of the semicircle be set out, let *DB* be joined, let the square *EC* be described on *AC*, and let the parallelogram *FB* be completed.

Since then, because the triangle *DAB* is equiangular with the triangle *DAC*, as *BA* is to *AD*, so is *DA* to *AC*, {VI. 8, VI. 4} therefore the rectangle *BA*, *AC* is equal to the square on *AD*. {VI. 17}

And since, as *AB* is to *BC*, so is *EB* to *BF*, {VI. 1} and *EB* is the rectangle *BA*, *AC*, for *EA* is equal to *AC*, and *BF* is the rectangle *AC*, *CB*, therefore, as *AB* is to *BC*, so is the rectangle *BA*, *AC* to the rectangle *AC*, *CB*.

And the rectangle *BA*, *AC* is equal to the square on *AD*, and the rectangle *AC*, *CB* to the square on *DC*, for the perpendicular *DC* is a mean proportional between the segments *AC*, *CB* of the base, because the angle *ADB* is right. {VI. 8, Por.}

Therefore, as *AB* is to *BC*, so is the square on *AD* to the square on *DC*. Q. E. D.

PROPOSITION 14.

To construct an octahedron and comprehend it in a sphere, *as in the preceding case*; *and to prove that the square on the diameter of the sphere is double of the square on the side of the octahedron.*

Let the diameter *AB* of the given sphere be set out, and let it be bisected at *C*; let the semicircle *ADB* be described on *AB*, let *CD* be drawn from *C* at right angles to *AB*, let *DB* be joined; let the square *EFGH*, having each of its sides equal to *DB*, be set out, let *HF*, *EG* be joined, from the point *K* let the straight line *KL* be set up at right angles to the plane of the square *EFGH* {XI. 12}, and let it be carried through to the other side of the plane, as *KM*; from the straight lines *KL*, *KM* let *KL*, *KM* be respectively cut off equal to one of the straight lines *EK*, *FK*, *GK*, *HK*, and let *LE*, *LF*, *LG*, *LH*, *ME*, *MF*, *MG*, *MH* be joined.

Then, since *KE* is equal to *KH*, and the angle *EKH* is right, therefore the square on *HE* is double of the square on *EK*. {I. 47}

Again, since *LK* is equal to *KE*, and the angle *LKE* is right, therefore the square on *EL* is double of the square on *EK*. {*id.*}

But the square on *HE* was also proved double of the square on *EK*; therefore the square on *LE* is equal to the square on *EH*; therefore *LE* is equal to *EH*.

For the same reason *LH* is also equal to *HE*; therefore the triangle *LEH* is equilateral.

Similarly we can prove that each of the remaining triangles of which the sides of the square *EFGH* are the bases, and the points *L*, *M* the vertices, is

equilateral; therefore an octahedron has been constructed which is contained by eight equilateral triangles.

It is next required to comprehend it in the given sphere, and to prove that the square on the diameter of the sphere is double of the square on the side of the octahedron.

For, since the three straight lines *LK*, *KM*, *KE* are equal to one another, therefore the semicircle described on *LM* will also pass through *E*.

And for the same reason, if, *LM* remaining fixed, the semicircle be carried round and restored to the same position from which it began to be moved, it will also pass through the points *F*, *G*, *H*, and the octahedron will have been comprehended in a sphere.

I say next that it is also comprehended in the given sphere.

For, since *LK* is equal to *KM*, while *KE* is common, and they contain right angles, therefore the base *LE* is equal to the base *EM*. {I. 4}

And, since the angle *LEM* is right, for it is in a semicircle, {III. 31} therefore the square on *LM* is double of the square on *LE*. {I. 47}

Again, since *AC* is equal to *CB*, *AB* is double of *BC*.

But, as *AB* is to *BC*, so is the square on *AB* to the square on *BD*; therefore the square on *AB* is double of the square on *BD*.

But the square on *LM* was also proved double of the square on *LE*.

And the square on *DB* is equal to the square on *LE*, for *EH* was made equal to *DB*.

Therefore the square on *AB* is also equal to the square on *LM*; therefore *AB* is equal to *LM*.

And *AB* is the diameter of the given sphere; therefore *LM* is equal to the diameter of the given sphere.

Therefore the octahedron has been comprehended in the given sphere, and it has been demonstrated at the same time that the square on the diameter of

the sphere is double of the square on the side of the octahedron. Q. E. D.

PROPOSITION 15.

To construct a cube and comprehend it in a sphere, like the pyramid; and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube.

Let the diameter *AB* of the given sphere be set out, and let it be cut at *C* so that *AC* is double of *CB*; let the semicircle *ADB* be described on *AB*, let *CD* be drawn from *C* at right angles to *AB*, and let *DB* be joined; let the square *EFGH* having its side equal to *DB* be set out, from *E*, *F*, *G*, *H* let *EK*, *FL*, *GM*, *HN* be drawn at right angles to the plane of the square *EFGH*, from *EK*, *FL*, *GM*, *HN* let *EK*, *FL*, *GM*, *HN* respectively be cut off equal to one of the straight lines *EF*, *FG*, *GH*, *HE*, and let *KL*, *LM*, *MN*, *NK* be joined; therefore the cube *FN* has been constructed which is contained by six equal squares.

It is then required to comprehend it in the given sphere, and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube.

For let *KG*, *EG* be joined.

Then, since the angle *KEG* is right, because *KE* is also at right angles to the plane *EG* and of course to the straight line *EG* also, {XI. Def. 3} therefore the semicircle described on *KG* will also pass through the point *E*.

Again, since *GF* is at right angles to each of the straight lines *FL*, *FE*, *GF* is also at right angles to the plane *FK*; hence also, if we join *FK*, *GF* will be at right angles to *FK*; and for this reason again the semicircle described on *GK* will also pass through *F*.

Similarly it will also pass through the remaining angular points of the cube.

If then, *KG* remaining fixed, the semicircle be carried round and restored to the same position from which it began to be moved, the cube will be comprehended in a sphere.

I say next that it is also comprehended in the given sphere.

For, since *GF* is equal to *FE*, and the angle at *F* is right, therefore the square on *EG* is double of the square on *EF*.

But *EF* is equal to *EK*; therefore the square on *EG* is double of the square on *EK*; hence the squares on *GE*, *EK*, that is the square on *GK* {I. 47}, is triple of the square on *EK*.

And, since *AB* is triple of *BC*, while, as *AB* is to *BC*, so is the square on *AB* to the square on *BD*, therefore the square on *AB* is triple of the square on *BD*.

But the square on *GK* was also proved triple of the square on *KE*.

And *KE* was made equal to *DB*; therefore *KG* is also equal to *AB*.

And *AB* is the diameter of the given sphere; therefore *KG* is also equal to the diameter of the given sphere.

Therefore the cube has been comprehended in the given sphere; and it has been demonstrated at the same time that the square on the diameter of the sphere is triple of the square on the side of the cube. Q. E. D.

PROPOSITION 16.

To construct an icosahedron and comprehend it in a sphere, like the aforesaid figures; and to prove that the side of the icosahedron is the irrational straight line called minor.

Let the diameter *AB* of the given sphere be set out, and let it be cut at *C* so that *AC* is quadruple of *CB*, let the semicircle *ADB* be described on *AB*, let the straight line *CD* be drawn from *C* at right angles to *AB*, and let *DB* be joined; let the circle *EFGHK* be set out and let its radius be equal to *DB*, let

the equilateral and equiangular pentagon *EFGHK* be inscribed in the circle *EFGHK*, let the circumferences *EF*, *FG*, *GH*, *HK*, *KE* be bisected at the points *L*, *M*, *N*, *O*, *P*, and let *LM*, *MN*, *NO*, *OP*, *PL*, *EP* be joined.

Therefore the pentagon *LMNOP* is also equilateral, and the straight line *EP* belongs to a decagon.

Now from the points *E*, *F*, *G*, *H*, *K* let the straight lines *EQ*, *FR*, *GS*, *HT*, *KU* be set up at right angles to the plane of the circle, and let them be equal to the radius of the circle *EFGHK*, let *QR*, *RS*, *ST*, *TU*, *UQ*, *QL*, *LR*, *RM*, *MS*, *SN*, *NT*, *TO*, *OU*, *UP*, *PQ* be joined.

Now, since each of the straight lines *EQ*, *KU* is at right angles to the same plane, therefore *EQ* is parallel to *KU*. {XI. 6}

But it is also equal to it; and the straight lines joining those extremities of equal and parallel straight lines which are in the same direction are equal and parallel. {I. 33}

Therefore *QU* is equal and parallel to *EK*.

But *EK* belongs to an equilateral pentagon; therefore *QU* also belongs to the equilateral pentagon inscribed in the circle *EFGHK*.

For the same reason each of the straight lines *QR*, *RS*, *ST*, *TU* also belongs to the equilateral pentagon inscribed in the circle *EFGHK*; therefore the pentagon *QRSTU* is equilateral.

And, since *QE* belongs to a hexagon, and *EP* to a decagon, and the angle *QEP* is right, therefore *QP* belongs to a pentagon; for the square on the side of the pentagon is equal to the square on the side of the hexagon and the square on the side of the decagon inscribed in the same circle. {XIII. 10}

For the same reason *PU* is also a side of a pentagon.

But *QU* also belongs to a pentagon; therefore the triangle *QPU* is equilateral.

For the same reason each of the triangles *QLR*, *RMS*, *SNT*, *TOU* is also equilateral.

And, since each of the straight lines *QL*, *QP* was proved to belong to a pentagon, and *LP* also belongs to a pentagon, therefore the triangle *QLP* is equilateral.

For the same reason each of the triangles *LRM*, *MSN*, *NTO*, *OUP* is also equilateral.

Let the centre of the circle *EFGHK* the point *V*, be taken; from *V* let *VZ* be set up at right angles to the plane of the circle, let it be produced in the other direction, as *VX*, let there be cut off *VW*, the side of a hexagon, and each of the straight lines *VX*, *WZ*, being sides of a decagon, and let *QZ*, *QW*, *UZ*, *EV*, *LV*, *LX*, *XM* be joined.

Now, since each of the straight lines *VW*, *QE* is at right angles to the plane of the circle, therefore *VW* is parallel to *QE*. {XI. 6}

But they are also equal; therefore *EV*, *QW* are also equal and parallel. {I. 33}

But *EV* belongs to a hexagon; therefore *QW* also belongs to a hexagon.

And, since *QW* belongs to a hexagon, and *WZ* to a decagon, and the angle *QWZ* is right, therefore *QZ* belongs to a pentagon. {XIII. 10}

For the same reason *UZ* also belongs to a pentagon, inasmuch as, if we join *VK*, *WU*, they will be equal and opposite, and *VK*, being a radius, belongs to a hexagon; {IV. 15, Por.} therefore *WU* also belongs to a hexagon.

But *WZ* belongs to a decagon, and the angle *UWZ* is right; therefore *UZ* belongs to a pentagon. {XIII. 10}

But *QU* also belongs to a pentagon; therefore the triangle *QUZ* is equilateral.

For the same reason each of the remaining triangles of which the straight lines *QR*, *RS*, *ST*, *TU* are the bases, and the point *Z* the vertex, is also

equilateral.

Again, since *VL* belongs to a hexagon, and *VX* to a decagon, and the angle *LVX* is right, therefore *LX* belongs to a pentagon. {XIII. 10}

For the same reason, if we join *MV*, which belongs to a hexagon, *MX* is also inferred to belong to a pentagon.

But *LM* also belongs to a pentagon; therefore the triangle *LMX* is equilateral.

Similarly it can be proved that each of the remaining triangles of which *MN*, *NO*, *OP*, *PL* are the bases, and the point *X* the vertex, is also equilateral.

Therefore an icosahedron has been constructed which is contained by twenty equilateral triangles.

It is next required to comprehend it in the given sphere, and to prove that the side of the icosahedron is the irrational straight line called minor.

For, since *VW* belongs to a hexagon, and *WZ* to a decagon, therefore *VZ* has been cut in extreme and mean ratio at *W*, and *VW* is its greater segment; {XIII. 9} therefore, as *ZV* is to *VW*, so is *VW* to *WZ*.

But *VW* is equal to *VE*, and *WZ* to *VX*; therefore, as *ZV* is to *VE*, so is *EV* to *VX*.

And the angles *ZVE*, *EVX* are right; therefore, if we join the straight line *EZ*, the angle *XEZ* will be right because of the similarity of the triangles *XEZ*, *VEZ*.

For the same reason, since, as *ZV* is to *VW*, so is *VW* to *WZ*, and *ZV* is equal to *XW*, and *VW* to *WQ*, therefore, as *XW* is to *WQ*, so is *QW* to *WZ*.

And for this reason again, if we join *QX*, the angle at *Q* will be right; {VI. 8} therefore the semicircle described on *XZ* will also pass through *Q*. {III. 31}

And if, *XZ* remaining fixed, the semicircle be carried round and restored to the same position from which it began to be moved, it will also pass through

Q and the remaining angular points of the icosahedron, and the icosahedron will have been comprehended in a sphere.

I say next that it is also comprehended in the given sphere.

For let *VW* be bisected at *A'*.

Then, since the straight line *VZ* has been cut in extreme and mean ratio at *W*, and *ZW* is its lesser segment, therefore the square on *ZW* added to the half of the greater segment, that is *WA'*, is five times the square on the half of the greater segment; {XIII. 3} therefore the square on *ZA'* is five times the square on .

And *ZX* is double of *ZA'*, and *VW* double of ; therefore the square on *ZX* is five times the square on *WV*.

And, since *AC* is quadruple of *CB*, therefore *AB* is five times *BC*.

But, as *AB* is to *BC*, so is the square on *AB* to the square on *BD*; {VI. 8, V. Def. 9} therefore the square on *AB* is five times the square on *BD*.

But the square on *ZX* was also proved to be five times the square on *VW*.

And *DB* is equal to *VW*, for each of them is equal to the radius of the circle *EFGHK*; therefore *AB* is also equal to *XZ*.

And *AB* is the diameter of the given sphere; therefore *XZ* is also equal to the diameter of the given sphere.

Therefore the icosahedron has been comprehended in the given sphere

I say next that the side of the icosahedron is the irrational straight line called minor.

For, since the diameter of the sphere is rational, and the square on it is five times the square on the radius of the circle *EFGHK*, therefore the radius of the circle *EFGHK* is also rational; hence its diameter is also rational.

But, if an equilateral pentagon be inscribed in a circle which has its diameter rational, the side of the pentagon is the irrational straight line called minor. {XIII. 11}

And the side of the pentagon *EFGHK* is the side of the icosahedron.

Therefore the side of the icosahedron is the irrational straight line called minor.

PORISM.

From this it is manifest that the square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron has been described, and that the diameter of the sphere is composed of the side of the hexagon and two of the sides of the decagon inscribed in the same circle. Q. E. D.

PROPOSITION 17.

To construct a dodecahedron and comprehend it in a sphere, like the aforesaid figures, and to prove that the side of the dodecahedron is the irrational straight line called apotome.

Let *ABCD*, *CBEF*, two planes of the aforesaid cube at right angles to one another, be set out, let the sides *AB*, *BC*, *CD*, *DA*, *EF*, *EB*, *FC* be bisected at *G*, *H*, *K*, *L*, *M*, *N*, *O* respectively, let *GK*, *HL*, *MH*, *NO* be joined, let the straight lines *NP*, *PO*, *HQ* be cut in extreme and mean ratio at the points *R*, *S*, *T* respectively, and let *RP*, *PS*, *TQ* be their greater segments; from the points *R*, *S*, *T* let *RU*, *SV*, *TW* be set up at right angles to the planes of the cube towards the outside of the cube, let them be made equal to *RP*, *PS*, *TQ*, and let *UB*, *BW*, *WC*, *CV*, *VU* be joined.

I say that the pentagon *UBWCV* is equilateral, and in one plane, and is further equiangular.

For let *RB*, *SB*, *VB* be joined.

Then, since the straight line *NP* has been cut in extreme and mean ratio at *R*, and *RP* is the greater segment, therefore the squares on *PN*, *NR* are triple of the square on *RP*. {XIII. 4}

But *PN* is equal to *NB*, and *PR* to *RU*; therefore the squares on *BN*, *NR* are triple of the square on *RU*.

But the square on *BR* is equal to the squares on *BN*, *NR*; {I. 47} therefore the square on *BR* is triple of the square on *RU*; hence the squares on *BR*, *RU* are quadruple of the square on *RU*.

But the square on *BU* is equal to the squares on *BR*, *RU*; therefore the square on *BU* is quadruple of the square on *RU*; therefore *BU* is double of *RU*.

But *VU* is also double of *UR*, inasmuch as *SR* is also double of *PR*, that is, of *RU*; therefore *BU* is equal to *UV*.

Similarly it can be proved that each of the straight lines *BW*, *WC*, *CV* is also equal to each of the straight lines *BU*, *UV*.

Therefore the pentagon *BUVCW* is equilateral.

I say next that it is also in one plane.

For let *PX* be drawn from *P* parallel to each of the straight lines *RU*, *SV* and towards the outside of the cube, and let *XH*, *HW* be joined; I say that *XHW* is a straight line.

For, since *HQ* has been cut in extreme and mean ratio at *T*, and *QT* is its greater segment, therefore, as *HQ* is to *QT*, so is *QT* to *TH*.

But *HQ* is equal to *HP*, and *QT* to each of the straight lines *TW*, *PX*; therefore, as *HP* is to *PX*, so is *WT* to *TH*.

And *HP* is parallel to *TW*, for each of them is at right angles to the plane *BD*; {XI. 6} and *TH* is parallel to *PX*, for each of them is at right angles to the plane *BF*. {*id.*}

But if two triangles, as *XPH*, *HTW*, which have two sides proportional to two sides be placed together at one angle so that their corresponding sides are also parallel, the remaining straight lines will be in a straight line; {VI. 32} therefore *XH* is in a straight line with *HW*.

But every straight line is in one plane; {XI. 1} therefore the pentagon *UBWCV* is in one plane.

I say next that it is also equiangular.

For, since the straight line *NP* has been cut in extreme and mean ratio at *R*, and *PR* is the greater segment, while *PR* is equal to *PS*, therefore *NS* has also been cut in extreme and mean ratio at *P*, and *NP* is the greater segment; {XIII. 5} therefore the squares on *NS*, *SP* are triple of the square on *NP*. {XIII. 4}

But *NP* is equal to *NB*, and *PS* to *SV*; therefore the squares on *NS*, *SV* are triple of the square on *NB*; hence the squares on *VS*, *SN*, *NB* are quadruple of the square on *NB*.

But the square on *SB* is equal to the squares on *SN*, *NB*; therefore the squares on *BS*, *SV*, that is, the square on *BV* — for the angle *VSB* is right — is quadruple of the square on *NB*; therefore *VB* is double of *BN*.

But *BC* is also double of *BN*; therefore *BV* is equal to *BC*.

And, since the two sides *BU*, *UV* are equal to the two sides *BW*, *WC*, and the base *BV* is equal to the base *BC*, therefore the angle *BUV* is equal to the angle *BWC*. {I. 8}

Similarly we can prove that the angle *UVC* is also equal to the angle *BWC*; therefore the three angles *BWC*, *BUV*, *UVC* are equal to one another.

But if in an equilateral pentagon three angles are equal to one another, the pentagon will be equiangular, {XIII. 7} therefore the pentagon *BUVCW* is equiangular.

And it was also proved equilateral; therefore the pentagon *BUVCW* is equilateral and equiangular, and it is on one side *BC* of the cube.

Therefore, if we make the same construction in the case of each of the twelve sides of the cube, a solid figure will have been constructed which is contained by twelve equilateral and equiangular pentagons, and which is called a dodecahedron.

It is then required to comprehend it in the given sphere, and to prove that the side of the dodecahedron is the irrational straight line called apotome.

For let *XP* be produced, and let the produced straight line be *XZ*; therefore *PZ* meets the diameter of the cube, and they bisect one another, for this has been proved in the last theorem but one of the eleventh book. {XI. 38}

Let them cut at *Z*; therefore *Z* is the centre of the sphere which comprehends the cube, and *ZP* is half of the side of the cube.

Let *UZ* be joined.

Now, since the straight line *NS* has been cut in extreme and mean ratio at *P*, and *NP* is its greater segment, therefore the squares on *NS*, *SP* are triple of the square on *NP*. {XIII. 4}

But *NS* is equal to *XZ*, inasmuch as *NP* is also equal to *PZ*, and *XP* to *PS*.

But further *PS* is also equal to *XU*, since it is also equal to *RP*; therefore the squares on *ZX*, *XU* are triple of the square on *NP*.

But the square on *UZ* is equal to the squares on *ZX*, *XU*; therefore the square on *UZ* is triple of the square on *NP*.

But the square on the radius of the sphere which comprehends the cube is also triple of the square on the half of the side of the cube, for it has previously been shown how to construct a cube and comprehend it in a sphere, and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube. {XIII. 15}

But, if whole is so related to whole, so is half to half also; and *NP* is half of the side of the cube; therefore *UZ* is equal to the radius of the sphere which comprehends the cube.

And *Z* is the centre of the sphere which comprehends the cube; therefore the point *U* is on the surface of the sphere.

Similarly we can prove that each of the remaining angles of the dodecahedron is also on the surface of the sphere; therefore the dodecahedron has been comprehended in the given sphere.

I say next that the side of the dodecahedron is the irrational straight line called apotome.

For since, when *NP* has been cut in extreme and mean ratio, *RP* is the greater segment, and, when *PO* has been cut in extreme and mean ratio, *PS* is the greater segment, therefore, when the whole *NO* is cut in extreme and mean ratio, *RS* is the greater segment.

{Thus, since, as *NP* is to *PR*, so is *PR* to *RN*, the same is true of the doubles also, for parts have the same ratio as their equimultiples; {V. 15} therefore as *NO* is to *RS*, so is *RS* to the sum of *NR*, *SO*.

But *NO* is greater than *RS*; therefore *RS* is also greater than the sum of *NR*, *SO*; therefore *NO* has been cut in extreme and mean ratio, and *RS* is its greater segment.}

But *RS* is equal to *UV*; therefore, when *NO* is cut in extreme and mean ratio, *UV* is the greater segment.

And, since the diameter of the sphere is rational, and the square on it is triple of the square on the side of the cube, therefore *NO*, being a side of the cube, is rational.

{But if a rational line be cut in extreme and mean ratio, each of the segments is an irrational apotome.}

Therefore *UV*, being a side of the dodecahedron, is an irrational apotome. {XIII. 6}

PORISM.

From this it is manifest that, when the side of the cube is cut in extreme and mean ratio, the greater segment is the side of the dodecahedron. Q. E. D.

PROPOSITION 18.

To set out the sides of the five figures and to compare them with one another.

Let *AB*, the diameter of the given sphere, be set out, and let it be cut at *C* so that *AC* is equal to *CB*, and at *D* so that *AD* is double of *DB*; let the semicircle *AEB* be described on *AB*, from *C*, *D* let *CE*, *DF* be drawn at right angles to *AB*, and let *AF*, *FB*, *EB* be joined.

Then, since *AD* is double of *DB*, therefore *AB* is triple of *BD*.

Convertendo, therefore, *BA* is one and a half times *AD*.

But, as *BA* is to *AD*, so is the square on *BA* to the square on *AF*, {V. Def. 9, VI. 8} for the triangle *AFB* is equiangular with the triangle *AFD*; therefore the square on *BA* is one and a half times the square on *AF*.

But the square on the diameter of the sphere is also one and a half times the square on the side of the pyramid. {XIII. 13}

And *AB* is the diameter of the sphere; therefore *AF* is equal to the side of the pyramid.

Again, since *AD* is double of *DB*, therefore *AB* is triple of *BD*.

But, as *AB* is to *BD*, so is the square on *AB* to the square on *BF*; {VI. 8, V. Def. 9} therefore the square on *AB* is triple of the square on *BF*.

But the square on the diameter of the sphere is also triple of the square on the side of the cube. {XIII. 15}

And *AB* is the diameter of the sphere; therefore *BF* is the side of the cube.

And, since *AC* is equal to *CB*, therefore *AB* is double of *BC*.

But, as *AB* is to *BC*, so is the square on *AB* to the square on *BE*; therefore the square on *AB* is double of the square on *BE*.

But the square on the diameter of the sphere is also double of the square on the side of the octahedron. {XIII. 14}

And *AB* is the diameter of the given sphere; therefore *BE* is the side of the octahedron.

Next, let *AG* be drawn from the point *A* at right angles to the straight line *AB*, let *AG* be made equal to *AB*, let *GC* be joined, and from *H* let *HK* be drawn perpendicular to *AB*.

Then, since *GA* is double of *AC*, for *GA* is equal to *AB*, and, as *GA* is to *AC*, so is *HK* to *KC*, therefore *HK* is also double of *KC*.

Therefore the square on *HK* is quadruple of the square on *KC*; therefore the squares on *HK*, *KC*, that is, the square on *HC*, is five times the square on *KC*.

But *HC* is equal to *CB*; therefore the square on *BC* is five times the square on *CK*.

And, since *AB* is double of *CB*, and, in them, *AD* is double of *DB*, therefore the remainder *BD* is double of the remainder *DC*.

Therefore *BC* is triple of *CD*; therefore the square on *BC* is nine times the square on *CD*.

But the square on *BC* is five times the square on *CK*; therefore the square on *CK* is greater than the square on *CD*; therefore *CK* is greater than *CD*.

Let *CL* be made equal to *CK*, from *L* let *LM* be drawn at right angles to *AB*, and let *MB* be joined.

Now, since the square on *BC* is five times the square on *CK*, and *AB* is double of *BC*, and *KL* double of *CK*, therefore the square on *AB* is five times the square on *KL*.

But the square on the diameter of the sphere is also five times the square on the radius of the circle from which the icosahedron has been described. {XIII. 16, Por.}

And *AB* is the diameter of the sphere; therefore *KL* is the radius of the circle from which the icosahedron has been described; therefore *KL* is a side of the hexagon in the said circle. {IV. 15, Por.}

And, since the diameter of the sphere is made up of the side of the hexagon and two of the sides of the decagon inscribed in the same circle, {XIII. 16, Por.} and *AB* is the diameter of the sphere, while *KL* is a side of the hexagon, and *AK* is equal to *LB*, therefore each of the straight lines *AK*, *LB* is a side of the decagon inscribed in the circle from which the icosahedron has been described.

And, since *LB* belongs to a decagon, and *ML* to a hexagon, for *ML* is equal to *KL*, since it is also equal to *HK*, being the same distance from the centre, and each of the straight lines *HK*, *KL* is double of *KC*, therefore *MB* belongs to a pentagon. {XIII. 10}

But the side of the pentagon is the side of the icosahedron; {XIII. 16} therefore *MB* belongs to the icosahedron.

Now, since *FB* is a side of the cube, let it be cut in extreme and mean ratio at *N*, and let *NB* be the greater segment; therefore *NB* is a side of the dodecahedron. {XIII. 17, Por.}

And, since the square on the diameter of the sphere was proved to be one and a half times the square on the side *AF* of the pyramid, double of the square on the side *BE* of the octahedron and triple of the side *FB* of the cube, therefore, of parts of which the square on the diameter of the sphere contains

six, the square on the side of the pyramid contains four, the square on the side of the octahedron three, and the square on the side of the cube two.

Therefore the square on the side of the pyramid is fourthirds of the square on the side of the octahedron, and double of the square on the side of the cube; and the square on the side of the octahedron is one and a half times the square on the side of the cube.

The said sides, therefore, of the three figures, I mean the pyramid, the octahedron and the cube, are to one another in rational ratios.

But the remaining two, I mean the side of the icosahedron and the side of the dodecahedron, are not in rational ratios either to one another or to the aforesaid sides; for they are irrational, the one being minor {XIII. 16} and the other an apotome {XIII. 17}.

That the side *MB* of the icosahedron is greater than the side *NB* of the dodecahedron we can prove thus.

For, since the triangle *FDB* is equiangular with the triangle *FAB*, {VI. 8} proportionally, as *DB* is to *BF*, so is *BF* to *BA*. {VI. 4}

And, since the three straight lines are proportional, as the first is to the third, so is the square on the first to the square on the second; {V. Def. 9, VI. 20, Por.} therefore, as *DB* is to *BA*, so is the square on *DB* to the square on *BF*; therefore, inversely, as *AB* is to *BD*, so is the square on *FB* to the square on *BD*.

But *AB* is triple of *BD*; therefore the square on *FB* is triple of the square on *BD*.

But the square on *AD* is also quadruple of the square on *DB*, for *AD* is double of *DB*; therefore the square on *AD* is greater than the square on *FB*; therefore *AD* is greater than *FB*; therefore *AL* is by far greater than *FB*.

And, when *AL* is cut in extreme and mean ratio, *KL* is the greater segment, inasmuch as *LK* belongs to a hexagon, and *KA* to a decagon; {XIII. 9} and,

when *FB* is cut in extreme and mean ratio, *NB* is the greater segment; therefore *KL* is greater than *NB*.

But *KL* is equal to *LM*; therefore *LM* is greater than *NB*.

Therefore *MB*, which is a side of the icosahedron, is by far greater than *NB* which is a side of the dodecahedron. Q. E. D.

I say next that *no other figure*, *besides the said five figures*, *can be constructed which is contained by equilateral and equiangular figures equal to one another.*

For a solid angle cannot be constructed with two triangles, or indeed planes.

With three triangles the angle of the pyramid is constructed, with four the angle of the octahedron, and with five the angle of the icosahedron; but a solid angle cannot be formed by six equilateral and equiangular triangles placed together at one point, for, the angle of the equilateral triangle being two-thirds of a right angle, the six will be equal to four right angles: which is impossible, for any solid angle is contained by angles less than four right angles. {XI. 21}

For the same reason, neither can a solid angle be constructed by more than six plane angles.

By three squares the angle of the cube is contained, but by four it is impossible for a solid angle to be contained, for they will again be four right angles.

By three equilateral and equiangular pentagons the angle of the dodecahedron is contained; but by four such it is impossible for any solid angle to be contained, for, the angle of the equilateral pentagon being a right angle and a fifth, the four angles will be greater than four right angles: which is impossible.

Neither again will a solid angle be contained by other polygonal figures by reason of the same absurdity.

Therefore etc. Q. E. D.

LEMMA.

But that *the angle of the equilateral and equiangular pentagon is a right angle and a fifth* we must prove thus.

Let *ABCDE* be an equilateral and equiangular pentagon, let the circle *ABCDE* be circumscribed about it, let its centre *F* be taken, and let *FA*, *FB*, *FC*, *FD*, *FE* be joined.

Therefore they bisect the angles of the pentagon at *A*, *B*, *C*, *D*, *E*.

And, since the angles at *F* are equal to four right angles and are equal, therefore one of them, as the angle *AFB*, is one right angle less a fifth; therefore the remaining angles *FAB*, *ABF* consist of one right angle and a fifth.

But the angle *FAB* is equal to the angle *FBC*; therefore the whole angle *ABC* of the pentagon consists of one right angle and a fifth. Q. E. D.

Data

Translated by Robert Simson. 1830

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PREFACE.

EUCLID'S DATA is the first in order of the books written by the antient Geometers to facilitate and promote the method of Resolution or Analysis. In the general, a thing is said to be given which is either actually exhibited, or can be found out, that is, which is either known by Hypothesis, or that can be demonstrated to be known; and the Propositions in the Book of Euclid's Data shew what things can be found out or known from those that by Hypothesis are already known; so that in the Analysis or Investigation of a Problem, from the things that are laid down to be known or given, by the help of these Propositions other things are demonstrated to be given, and from these other things are again shewn to be given, and so on, until that which was proposed to be found out in the Problem is demonstrated to be given, and when this is done the Problem is solved, and its Composition is made and derived from the Compositions of the Data which were made use of in the Analysis. And thus the Data of Euclid are of the most general and necessary use in the solution of Problems of every kind.

Euclid is reckoned to be the Author of the Book of the Data both by the antient and modern Geometers; and there seems to be no doubt of his having written a Book on this subject, but which in the course of so many ages has been much vitiated by unskilful Editors in several places, both in the order of the Propositions, and in the Definitions and Demonstrations themselves. To correct the errors which are now found in it, and bring it nearer to the accuracy with which it was, no doubt, at first written by Euclid, is the design of this Edition, that so it may be rendered more useful to Geometers, at least to beginners who desire to learn the investigatory method of the antients. And for their sakes the Composition of most of the Data are subjoined to their Demonstrations, that the Compositions of Problems solved by help of the Data may be the more easily made.

Marinus the Philosopher's preface which in the Greek Edition is prefixed to the Data is here left out, as being of no use to understand them. at the end of it he says that Euclid has not used the synthetical, but the analytical method in delivering them; in which he is quite mistaken; for in the Analysis of a Theorem the thing to be demonstrated is assumed in the Analysis; but in the Demonstrations of the Data, the thing to be demonstrated, which is that something or other is given, is never once assumed in the Demonstration, from which it is manifest that every one of then is demonstrated synthetically; tho' indeed if a Proposition of the Data be turned into a Problem, for example the 84th or 85th in the former Editions, which here are the 85th and 86th, the Demonstration of the Proposition becomes the Analysis of the Problem.

Wherein this Edition differs from the Greek, and the reasons of the alterations from it will be shewn in the Notes at the end of the Data.

DEFINITIONS.

I.

SPACES, lines and angles are said to be given in magnitude, when equals to them can be found.

II.

A ratio is said to be given, when a ratio of a given magnitude to a given magnitude which is the same ratio with it can be found.

III

Rectilineal figures are said to be given in species, which have each of their angles given, and the ratios of their sides given.

IV.

Points, lines and spaces are said to be given in position, which have always the same situation, and which are either actually exhibited, or can be found.

A.

An angle is said to be given in position, which is contained by straight lines given in position.

V.

A circle is said to be given in magnitude, when a straight line from its center to the circumference is given in magnitude.

VI.

A circle is said to be given in position and magnitude, the center of which is given in position, and a straight line from it to the circumference is given in magnitude.

VII.

Segments of circles are said to be given in magnitude, when the angles in them, and their bases are given in magnitude.

VIII.

Segments of circles are said to be given in position and magnitude, when the angles in them are given in magnitude, and their bases are given both in position and magnitude.

IX.

A magnitude is said to be greater than another by a given magnitude, when this given magnitude being taken from it, the remainder is equal to the other magnitude.

X.

A magnitude is said to be less than another by a given magnitude, when this given magnitude being added to it, the whole is equal to the other magnitude.

PROPOSITIONS.

PROPOSITION I.

THE ratios of given magnitudes to one another is given.

Let A, B be two given magnitudes, the ratio of A to B is given. Because A is a given magnitude, there may be found one equal to it; let this be C. and because B is given, one equal to it may be found; let it be D. and since A is equal to C, and B to D; therefore A is to B, as C to D; and consequently the ratio of A to B is given, because the ratio of the given magnitudes C, D which is the same with it has been found.

PROP. II.

2.

IF a given magnitude has a given ratio to another magnitude, “and if unto the two magnitudes by which the given ratio is exhibited, and the given magnitude, a fourth proportional can be found;” the other magnitude is given.

Let the given magnitude A have a given ratio to the magnitude B; if a fourth proportional can be found to the three magnitudes above-named, B is given in magnitude.

Because A is given, a magnitude may be found equal to it; let this be C. and because the ratio of A to B is given, a ratio which is the same with it may be found; let this be the ratio of the given magnitude E to the given magnitude F. unto the magnitudes E, F, C find a fourth proportional D, which, by the Hypothesis, can be done. wherefore because A is to B, as E to F; and as E to F,

so is C to D; A is to B, as C to D. but A is equal to C, therefore B is equal to D. the magnitude B is therefore given, because a magnitude D equal to it has been found.

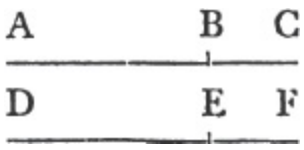
The limitation within the inverted commas is not in the Greek text, but is now necessarily added; and the same must be understood in all the Propositions of the Book which depend upon this second Proposition, where it is not expressly mentioned. See the Note upon it.

PROP. III.

3.

IF any given magnitudes be added together, their sum shall be given.

Let any given magnitudes AB, BC be added together, their sum AC is given.



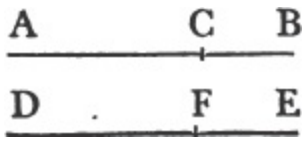
Because AB is given, a magnitude equal to it may be found; let this be DE. and because BC is given, one equal to it may be found; let this be EF. wherefore because AB is equal to DE, and BC equal to EF; the whole AC is equal to the whole DF. AC is therefore given, because DF has been found, which is equal to it.

PROP. IV.

4.

IF a given magnitude be taken from a given magnitude; the remaining magnitude shall be given.

From the given magnitude AB let the given magnitude AC be taken; the remaining magnitude CB is given.



Because AB is given, a magnitude equal to it may be found; let this be DE. and because AC is given, one equal to it may be found; let this be DF. wherefore because AB is equal to DE, and AC to DF; the remainder CB is equal to the remainder FE. CB is therefore given, because FE which is equal to it has been found.

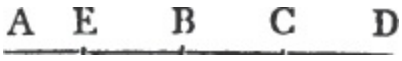
12.

PROP. V.

IF of three magnitudes, the first together with the second be given, and also the second together with the third; either the first is equal to the third, or one of them is greater than the other by a given magnitude.

Let AB, BC, CD be three magnitudes, of which AB together with BC, that is AC, is given; and also BC together with CD, that is BD, is given. either AB is equal to CD, or one of them is greater than the other by a given magnitude.

Because AC, BD are each of them given, they are either equal to one another, or not equal. first, let them be equal, and because AC is equal to BD, take away the common part BC; therefore the remainder AB is equal to the remainder CD.



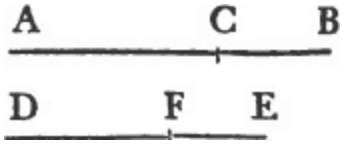
But if they be unequal, let AC be greater than BD, and make CE equal to BD. therefore CE is given, because BD is given, and the whole AC is given, therefore AE the remainder is given. and because EC is equal to BD, by taking BC from both, the remainder EB is equal to the remainder CD. and AE is given, wherefore AB exceeds EB, that is CD by the given magnitude AE.

PROP. VI

5.

IF a magnitude has a given ratio to a part of it; it shall also have a given ratio to the remaining part of it.

Let the magnitude AB have a given ratio to AC a part of it; it has also a given ratio to the remainder BC.



Because the ratio of AB to AC is given, a ratio may be found which is the same to it. let this be the ratio of DE a given magnitude to the given

magnitude DF. and because DE, DF are given, the remainder FE is given, and because AB is to AC, as DE to DF, by conversion AB is to BC, as DE to EF. therefore the ratio of AB to BC is given, because the ratio of the given magnitudes DE, EF which is the same with it has been found.

COR. From this it follows, that the parts AC, CB have a given ratio to one another. because as AB to BC, so is DE to EF; by division, AC is to CB, as DF to FE; and DF, FE are given; therefore the ratio of AC to CB is given.

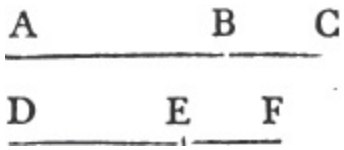
PROP. VII.

6.

See. N

IF two magnitudes which have a given ratio to one another, be added together; the whole magnitude shall have to each of them a given ratio.

Let the magnitudes AB, BC which have a given ratio to one another, be added together; the whole AC has to each of the magnitudes AB, BC a given ratio.

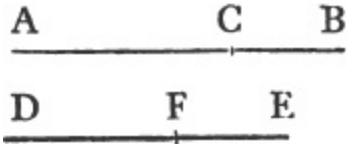


Because the ratio of AB to BC is given, a ratio maybe found which is the same with it; let this be the ratio of the given magnitudes DE, EF. and because DE, DF are given, the whole DF is given. and because as AB to BC, so is DE to EF; by composition, AC is to CB, as DF to FE; and by conversion, AC is to AB, as DF to DE. wherefore because AC is to each of the magnitudes AB, BC, as DF to each of the others DE, EF; the ratio of AC to each of the magnitudes AB, BC is given.

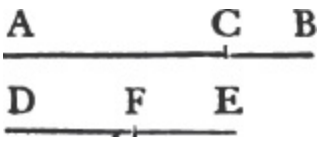
PROP. VIII.

7.

IF a given magnitude be divided into two parts which have a given ratio to one another, and if a fourth proportional can be found to the sum of the two magnitudes by which the given ratio is exhibited, one of them, and the given magnitude; each of the parts is given.



Let the given magnitude AB be divided into the parts AC, CB which have a given ratio to one another; if a fourth proportional can be found to the above-named magnitudes; AC and CB are each of them given. Because the ratio of AC to CB is given, the ratio of AB to BC is given; therefore a ratio which is:



the same with it can be found, let this be the ratio of the given magnitudes DE, EF. and because the given magnitude AB has to BC the given ratio of DE to EF, if unto DE, EF, AB a fourth proportional can be found, this which is BC is given; and because AB is given the other part AC is given.

In the same manner, and with the like limitation, If the difference AC of two magnitudes AB, BC which have a given ratio be given; each of the magnitudes AB, BC is given.

PROP. IX.

8.

MAGNITUDES which have given ratios to the same magnitude, have also a given ratio to one another.

Let A, C have each of them a given ratio to B; A has a given ratio to C.

Because the ratio of A to B is given, a ratio which is the same to it may be found; let this be the ratio of the given magnitudes D, E. and because the ratio of B to C is given, a ratio which is the same with it may be found; let this be the ratio of the given magnitudes F, G. to F, G, E find a fourth proportional H, if it can be done; and because as A is to B, so is D to E; and as B to C, so is (F to G, and so is) E to H; ex aequali, as A to C, so is D to H. therefore the ratio of A to C is given, because the ratio of the given magnitudes D and H, which is the same with it, has been found. but if a fourth proportional to F, G, E cannot be found, then it can only be said that the ratio of A to C is compounded of the ratios of A to B, and B to C, that is of the given ratios of D to E, and F to G.

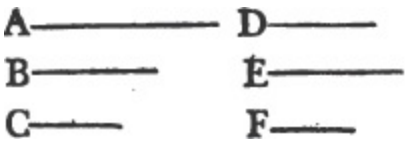
PROP. X.

9.

IF two or more magnitudes have given ratios to one another, and if they have given ratios, tho' they be not the same, to some other magnitudes; these other magnitudes shall also have given ratios to one another.

Let two or more magnitudes A, B, C have given ratios to one another; and let them have given ratios, tho' they be not the same, to some other

magnitudes D, E, F. the magnitudes D, E, F have given ratios to one another.

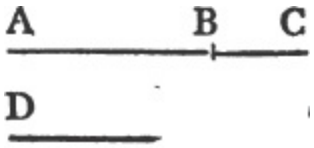


Because the ratio of A to B is given, and likewise the ratio of A to D; therefore the ratio of D to B is given; but the ratio of B to E is given, therefore the ratio of D to E is given, and because the ratio of B to C is given, and also the ratio of B to E; the ratio of E to C is given. and the ratio of C to F is given; wherefore the ratio of E to F is given. D, E, F have therefore given ratios to one another.

PROP. XI.

22.

IF two magnitudes have each of them a given ratio to another magnitude; both of them together shall have a given ratio to that other.



Let the magnitudes AB, BC have a given ratio to the magnitude D; AC has a given ratio to the same D.

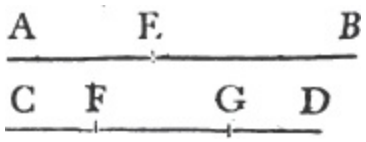
Because AB, BC have each of them a given ratio to D, the ratio of AB to BC is given. and by composition, the ratio of AC to CB is given. but the ratio of BC to D is given; therefore the ratio of AC to D is given.

PROP. XII.

23.

IF the whole have to the whole a given ratio, and the parts have to the parts given, but not the same, ratios, every one of them, whole or part, shall have to every one a given ratio.

Let the whole AB have a given ratio to the whole CD, and the parts AE, EB have given, but not the same, ratios to the parts CF, FD; every one shall have to every one, whole or part, a given ratio.

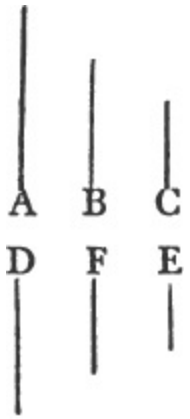


Because the ratio of AE to CF is given, as AE to CF, so make AB to CG; the ratio therefore of AB to CG is given; wherefore the ratio of the remainder EB to the remainder FG is given, because it is the same with the ratio of AB to CG. and the ratio of EB to FD is given, wherefore the ratio of FD to FG is given; and by conversion, the ratio of FD to DG is given. and because AB has to each of the magnitudes CD, CG a given ratio, the ratio of CD to CG is given; and therefore the ratio of CD to DG is given. but the ratio of GD to DF is given, wherefore the ratio of CD to DF is given, and consequently the ratio of CF to FD is given; but the ratio of CF to AE is given, as also the ratio of FD to EB; wherefore the ratio of AE to EB is given; as also the ratio of AB to each of them. the ratio therefore of every one to every one is given.

PROP. XIII.

24.

IF the first of three proportional straight lines has a given ratio to the third, the first shall also have a given ratio to the second.



Let A, B, C be three proportional straight lines, that is as A to B, so is B to C; if A has to C a given ratio, A shall also have to B a given ratio.

Because the ratio of A to C is given, a ratio which is the same with it may be found; let this be the ratio of the given straight lines D, E; and between D and E find a mean proportional F;

therefore the rectangle contained by D and E is equal to the square of F, and the rectangle D, E is given because its sides D, E are given; wherefore the square of F, and the straight line F is given. and because as A is to C, so is D to E; but as A to C, so is the square of A to the square of B; and as D to E, so is the square of D to the square of F; therefore the square of A is to the square of B, as the square of D to the square of F. as therefore the straight line A to the straight line B, so is the straight line D to the straight line F. therefore the ratio of A to B is given, because the ratio of the given straight lines D, F which is the same with it has been found.

PROP. XIV.

A.

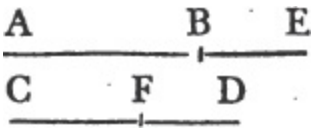
IF a magnitude together with a given magnitude has a given ratio to another magnitude; the excess of this other magnitude above a given magnitude has a

given ratio to the first magnitude. and if the excess of a magnitude above a given magnitude has a given ratio to another magnitude; this other magnitude together with a given magnitude has a given ratio to the first magnitude.

Let the magnitude AB together with the given magnitude BE, that is AE, have a given ratio to the magnitude CD; the excess of CD above a given magnitude has a given ratio to AB.

Because the ratio of AE to CD is given, as AE to CD, so make BE to FD; therefore the ratio of BE to FD is given, and BE is given, wherefore FD is given. and because as AE to CD, so is BE to FD, the remainder AB is to the remainder CF, as AE to CD. but the ratio of AE to CD is given, therefore the ratio of AB to CF is given; that is, CF the excess of CD above the given magnitude FD has a given ratio to AB.

Next, Let the excess of the magnitude AB above the given magnitude BE, that is, let AE have a given ratio to the magnitude CD; CD together with a given magnitude has a given ratio to AB.



Because the ratio of AE to CD is given, as AE to CD, so make BE to FD; therefore the ratio of BE to FD is given, and BE is given, wherefore FD is given. and because as AE to CD, so is BE to FD, AB is to CF, as AE to CD. but the ratio of AE to CD is given, therefore the ratio of AB to CF is given; that is CF which is equal to CD together with the given magnitude DF has a given ratio to AB.

PROP. XV.

B.

IF a magnitude together with that to which another magnitude has a given ratio, be given; this other is given together with that to which the first magnitude has a given ratio. Let AB, CD be two magnitudes of which AB together with BE to which CD has a given ratio, is given; CD is given together with that magnitude to which AB has a given ratio.

Because the ratio of CD to BE is given, as BE to CD, so make AE to FD; therefore the ratio of AE to FD is given, and AE is given; wherefore FD is given. and because as BE to CD, so is AE to FD; AB is to FC, as BE to CD. and the ratio of BE to CD is given, wherefore the ratio of AB to FC is given. and FD is given, that is CD together with FC to which AB has a given ratio is given.

PROP. XVI.

10.

IF the excess of a magnitude above a given magnitude, has a given ratio to another magnitude; the excess of both together above a given magnitude shall have to that other a given ratio. and if the excess of two magnitudes together above a given magnitude. has to one of them a given ratio; either the excess of the other above a given magnitude has to that one a given ratio; or the other is given together with the magnitude to which that one has a given ratio.

Let the excess of the magnitude AB above a given magnitude, have a given ratio to the magnitude BC; the excess of AC, both of them together, above a given magnitude, has a given ratio to BC.

Let AD be the given magnitude the excess of AB above which, viz. DB, has a given ratio to BC. and because DB has a given ratio to BC, the ratio of

DC to CB is given, and AD is given; therefore DC, the excess of AC above the given magnitude AD, has a given ratio to BC.

Next, let the excess of two magnitudes AB, BC together above a given magnitude have to one of them BC a given ratio; either the excess of the other of them AB above a given magnitude shall have to BC a given ratio; or AB is given together with the magnitude to which BC has a given ratio.

Let AD be the given magnitude, and first let it be less than AB; and because DC the excess of AC above AD has a given ratio to BC, DB has a given ratio to BC; that is DB, the excess of AB above the given magnitude AD, has a given ratio to BC.

But let the given magnitude be greater than AB, and make AE equal to it; and because EC, the excess of AC above AE, has to BC a given ratio, BC has a given ratio to BE; and because AE is given, AB together with BE to which BC has a given ratio, is given.

PROP. XVII.

11.

See. N

IF the excess of a magnitude above a given magnitude has a given ratio to another magnitude; the excess of the same first magnitude above a given magnitude, shall have a given ratio to both the magnitudes together. and if the excess of either of two magnitudes above a given magnitude has a given ratio to both magnitudes together; the excess of the same above a given magnitude shall have a given ratio to the other.

Let the excess of the magnitude AB above a given magnitude have a given ratio to the magnitude BC; the excess of AB above given magnitude has a

given ratio to AC.

Let AD be the given magnitude; and because DB, the excess of AB above AD, has a given ratio to BC; the ratio of DC to DB a given. make the ratio of AD to DE the same with this ratio; therefore the ratio of AD to DE is given. and AD is given, wherefore DE, and the remainder AE are given. and because as DC to DB, so is AD to DE, AC is to EB, as DC to DB; and the ratio of DC to DB is given, wherefore the ratio of AC to EB is given. and because the ratio of EB to AC is given, and that AE is given, therefore EB the excess of AB above the given magnitude AE, has a given ratio to AC.

Next, let the excess of AB above a given magnitude have a given ratio to AB and BC together, that is to AC; the excess of AB above a given magnitude has a given ratio to BC.

Let AE be the given magnitude; and because EB the excess of AB above AE has to AC a given ratio, as AC to EB, so make AD to DE; therefore the ratio of AD to DE is given, as also the ratio of AD to AE. and AE is given, wherefore AD is given. and because as the whole, AC, to the whole, EB, so is AD to DE; the remainder DC is to the remainder DB, as AC to and the ratio of AC to EB is given, wherefore the ratio of DC to DB is given, as also the ratio of DB to BC. and AD is given, therefore DB, the excess of AB above the given magnitude AD, has a given ratio to BC.

PROP. XVIII.

14.

IF to each of two magnitudes, which have a given ratio to one another, a given magnitude be added; the wholes shall either have a given ratio to one another, or the excess of one of them above a given magnitude shall have a given ratio to the other.

Let the two magnitudes AB, CD have a given ratio to one another, and to AB let the given magnitude BE be added, and the given magnitude DF to CD. the wholes AE, CF either have a given ratio to one another, or the excess of one of them above a given magnitude has a given ratio to the other.

Because BE, DF are each of them given, their ratio is given.

and if this ratio be the same with the ratio of AB to CD, the ratio of AE to CF, which is the same with the given ratio of AB to CD, shall be given.

But if the ratio of BE to DF be not the same with the ratio of AB to CD; either it is greater than the ratio of AB to CD, or, by inversion, the ratio of DF to BE is greater than the ratio of CD to AB. first, let the ratio of BE to DF be greater than the ratio of AB to CD; and as AB to CD, so make BG to DF; therefore the ratio of BG to DF is given; and DF is given, therefore BG is given. and because BE has a greater ratio to DF than (AB to CD, that is than) BG to DF, BE is greater than BG. and because as AB to CD, so is BG to DF, therefore AG is to CF, as AB to CD. but the ratio of AB to CD is given, wherefore the ratio of AG to CF is given; and because BE, BG are each of them given, GE is given. therefore AG, the excess of AE above the given magnitude GE has a given ratio to CF. the other case is demonstrated in the same manner.

PROP. XIX.

15.

IF from each of two magnitudes, which have a given ratio to one another, a given magnitude be taken; the remainders shall either have a given ratio to one another, or the excess of one of them above a given magnitude, shall have a given ratio to the other.

Let the magnitudes AB, CD have a given ratio to one another, and from AB let the given magnitude AE be taken, and from CD, the given magnitude CF. the remainders EB, FD shall either have a given ratio to one another, or the excess of one of them above a given magnitude shall have a given ratio to the other.

Because AE, CF are each of them given, their ratio is given; and if this ratio be the same with the ratio of AB to CD, the ratio of the remainder EB to the remainder FD, which is the same with the given ratio of AB to CD, shall be given.

But if the ratio of AB to CD be not the same with the ratio of AE to CF, either it is greater than the ratio of AE to CF, or, by inversion, the ratio of CD to AB is greater than the ratio of CF to AE. first, let the ratio of AB to CD be greater than the radio of AE to CF, and as AB to CD, so make AG to CF; therefore the ratio of AG to CF is given, and CF is given, wherefore AG is given. and because the ratio of AB to CD, that is the ratio of AG to CF, is greater than the ratio of AE to CF; AG is greater than AE. and AG, AE are given, therefore the remainder EG is given. and as AB to CD, so is AG to CF, and so is the remainder GB to the remainder FD; and the ratio of AB to CD is given, wherefore the ratio of GB to FD is therefore GB, the excess of EB above the given magnitude DG, has a given ratio to FD. in the same manner the other case is demonstrated.

PROP. XX.

16.

IF to one of two magnitudes which have a given ratio to one another, a given magnitude be added, and from the other a given magnitude be taken; the

excess of the sum above a given magnitude shall have a given ratio to the remainder.

Let the two magnitudes AB, CD have a given ratio to one another, and to AB let the given magnitude EA be added, and from CD let the given magnitude CF be taken; the excess of the sum EB above a given magnitude has a given ratio to the remainder FD.

Because the ratio of AB to CD is given, make as AB to CD, so AG to CF. therefore the ratio of AG to CF is given, and CF is given, wherefore AG is given; and EA is given, therefore the whole EG is given. and because as AB to CD, so is AG to CF, and so is the remainder GB to the remainder FD; the ratio of GB to FD is given. and EG is given,

therefore GB, the excess of the sum EB above the given magnitude EG, has a given ratio to the remainder FD.

PROP. XXI.

C.

IF two magnitudes have a given ratio to one another, if a given magnitude be added to one of them, and the other be taken from a given magnitude; the sum together with the magnitude to which the remainder has a given ratio, is given. and the remainder is given together with the magnitude to which the sum has a given ratio.

Let the two magnitudes AB, CD have a given ratio to one another; and to AB let the given magnitude BE be added, and let CD be taken from the given magnitude FD. the sum AE is given together with the magnitude to which the remainder FC has a given ratio.

Because the ratio of AB to CD is given, make as AB to CD, so GB to FD. therefore the ratio of GB to FD is given, and FD is given, wherefore GB is given; and BE is given, the whole GE is therefore given. and because as AB to CD, so is GB to FD, and so is GA to FC; the ratio of GA to FC is given. and AE together with GA is given, because GE is given; therefore the sum AE together with GA to which the remainder FC has a given ratio, is given. the second part is manifest from Pro.

PROP. XXII.

See. N

IF two magnitudes have a given ratio to one another, if from one of them a given magnitude be taken, and the other be taken from a given magnitude; each of the remainders is given together with the magnitude to which the other remainder has a given ratio.

Let the two magnitudes AB, CD have a given ratio to one another, and from AB let the given magnitude AE be taken, and let CD be taken from the given magnitude CF; the remainder EB is given together with the magnitude to which the other remainder DF has a given ratio.

Because the ratio of AB to CD is given, make as AB to CD, so AG to CF. the ratio of AG to CF is therefore given, and CF is given, wherefore AG is given; and AE is given, and therefore the remainder EG is given. and because as AB to CD, so is AG to CE, and so is the remainder BG to the remainder DF; the ratio of BG to DF is given. and EB together with BG is given, because EG is given. therefore the remainder EB together with BG to which DF the other remainder has a given ratio is given. the second part is plain from Pro.

PROP. XXIII.

20.

IF from two given magnitudes there be taken magnitudes which have a given radio to one another, the remainders shall either have a given ratio to one another, or the excess of one of them above a given magnitude shall have a given ratio to the other.

Let AB, CD be two given magnitudes, and from them let the magnitudes AE, CF which have a given ratio to one another be taken; the remainders EB, FD either have a given ratio to one another, or the excess or one or them above a given magnitude has a given ratio to the other.

Because AB, CD are each of them given, the ratio of AB to CD is given. and if this ratio be the same with the ratio of AE to CF, then the remainder EB has the same given ratio to the remainder FD.

But if the ratio of AB to CD be not the same with the ratio of AE to CF, it is either greater than it, or, by inversion, the ratio of CD to AB is greater than the ratio CF to AE. first, let the ratio of AB to CD be greater than the ratio of AE to CF; and as AE to CF, so make AG to CD, therefore the ratio of AG to CD is given, because the ratio of AE to CF is given; and CD is given, wherefore AG is given; and because the ratio of AB to CD is greater than the ratio of (AE to CF, that is, than the ratio of) AG to CD; AB is greater than AG. and AB, AG are given, therefore the remainder BG is given. and because as AE to CF, so is AG to CD, and so is EG to FD; the ratio of EG to FD is given. and GB is given, therefore EG the excess of EB above the given magnitude GB, has a given ratio to FD. the other case is shewn in the same way.

PROP. XXIV.

13.

IF there be three magnitudes, the first of which has a given ratio to the second, and the excess of the second above a given magnitude has a given ratio to the third; the excess of the first above a given magnitude shall also have a given ratio to the third.

Let AB, CD, E be three magnitudes, of which AB has a given ratio to CD; and the excess of CD above a given magnitude has a given ratio to E. the excess of AB above a given magnitude has a given ratio to E.

Let CF be the given magnitude the excess of CD above which, viz. FD has a given ratio to E. and because the ratio of AB to CD is given, as AB to CD so make AG to CF; therefore the ratio of AG to CF is given; and CF is given, wherefore AG is given. and because as AB to CD, so is AG to CF, and so is. GB to FD; the ratio of GB to FD is given. and the ratio of FD to E is given, wherefore the ratio of GB to E is given. and AG is given, therefore GB the excess of AB above the given magnitude AG has a given ratio to E.

COR. 1. And if the first has a given ratio to the second, and the excess of the first above a given magnitude has a given ratio to the third; the excess of the second above a given magnitude shall have a given ratio to the third. for if the second be called the first, and the first the second, this Corollary will be the same with the Proposition.

COR. 2. Also if the first has a given ratio to the second, the excess of the third above a given magnitude has also a given ratio to the second, the same excess shall have a given ratio to the first; as is evident from the 9th Dat.

PROP. XXV.

IF there be three magnitudes, the excess of the first where of above a given magnitude has a given ratio to the second; and the excess of the third above a given magnitude has a given ratio to the same second. the first shall either have a given ratio to the third, or the excess of one of them above a given magnitude shall have a given ratio to the other.

Let AB, C, DE, be three magnitudes, and let the excesses of each of the two AB, DE above given magnitudes have given ratios to C; AB, DE either have a given ratio to one another, or the excess of one of them above a given magnitude has a given ratio to the other.

Let FB the excess of AB above the given magnitude AF have a given ratio to C; and let GE the excess of DE above the given magnitude DG have a given ratio to C; and because FB, GE have each of them a given ratio to C, they have a given ratio to one another. but to FB, GE the given magnitudes AF, DG are added; therefore the whole magnitudes AB, DE have either a given ratio to one another, or the excess of one of them above a given magnitude has a given ratio to the other.

PROP. XXVI.

18.

IF there be three magnitudes the excesses of one of which above given magnitudes have given ratios to the other two magnitudes; these two shall either have a given ratio to one another, or the excess of one of them above a given magnitude shall have a given ratio to the other.

Let AB, CD, EF be three magnitudes, and let GD the excess of one of them CD above the given magnitude CG have a given ratio to AB; and also let KD the excess of the same CD above the given magnitude CK have a

given ratio to EF. either AB has a given ratio to EF, or the excess of one of them above a given magnitude has a given ratio to the other.

Because GD has a given ratio to AB, as GD to AB, so make CG to HA; therefore the ratio of CG to HA is given; and CG is given, wherefore HA is given. and because as GD to AB, so is CG to HA, and so is CD to HB; the ratio of CD to HB is given. also because KD has a given ratio to EF, as KD to EF, so make CK to LE; therefore the ratio of CK to LE is given; and CK is given, wherefore LE is given. and because as KD to EF, so is CK to LE, and so is CD to LF; the ratio of CD to LF is given. but the ratio of CD to HB is given, wherefore the ratio of HB to LF is given. and from HB, LF the given magnitudes HA, LE being taken, the remainders AB, EF shall either have a given ratio to one another, or the excess of one of them above a given magnitude has a given ratio to the other.

“Another Demonstration.

Let AB, C, DE be three magnitudes, and let the excesses of one of them C above given magnitudes have given ratios to AB and DE. either AB, DE have a given ratio to one another, or the excess of one of them above a given magnitude has a given ratio to the other.

Because the excess of C above a given magnitude has a given ratio to AB, therefore AB together with a given magnitude has a given ratio to C. let this given magnitude be AF, wherefore FB has a given ratio to C. also, because the excess of C above a given magnitude has a given ratio to DE, therefore DE together with a given magnitude has a given ratio to C. let this given magnitude be DG, wherefore GE has a given ratio to C. and FB has a given ratio to C, therefore the ratio of FB to GE is given. and from FB, GE the given magnitudes AF, DG being taken, the remainders AB, DE either have a

given ratio to one another, or the excess of one of them above a given magnitude has a given ratio to the other.”

PROP. XXVII.

19.

IF there be three magnitudes the excess of the first of which above a given magnitude has a given ratio to the second; and the excess of the second above a given magnitude has also a given ratio to the third. the excess of the first above a given magnitude shall have a given ratio to the third.

Let AB, CD, E be three magnitudes the excess of the first of which AB above the given magnitude AG, viz. GB has a given ratio to CD; and FD the excess of CD above the given magnitude CF, has a given ratio to E. the excess of AB above a given magnitude has a given ratio to E.

Because the ratio of GB to CD is given, as GB to CD, so make GH to CF; therefore the ratio of GH to CF is given; and CF is given, wherefore GH is given; and AG is given, wherefore the whole AH is given. and because as GB to CD, so is GH to CF, and so is the remainder HB to the remainder FB; the ratio of HB to FD is given. and the ratio of FD to E is given. wherefore the ratio of HB to E is given. and AH is given; therefore HB the excess of AB above the given magnitude AH has a given ratio to E.

“Otherwise.

Let AB, C, D be three magnitudes, the excess EB of the first of which AB above the given magnitude AE has a given ratio to C, and the excess of C above a given magnitude has a given ratio to D. the excess of AB above a given magnitude has a given ratio to D.

Because EB has a given ratio to C, and the excess of C above a given magnitude has a given ratio to D; therefore the excess of EB above a given magnitude has a given ratio to D. let this given magnitude be EF, therefore FB the excess of EB above EF has a given ratio to D. and AF is given, because AE, EF are given.

therefore FB the excess of AB above the given magnitude AF has a given ratio to D.”

PROP. XXVIII.

25.

IF two lines given in position cut one another, the point or points in which they cut one another are given.

Let two lines AB, CD given in position cut one another in the point E; the point E is given.

Because the lines AB, CD are given in position, they have always the same situation, and therefore the point, or points, in which they cut one another have always the same situation. and because the lines AB, CD can be found, the point, or points, in which they cut one another, are likewise found; and therefore are given in position.

PROP. XXIX.

26.

IF the extremities of a straight line be given in position; the straight line is given in position and magnitude.

Because the extremities of the straight line are given, they can be found; let these be the points A, B, between which a straight line AB can be drawn; this has an invariable position, because between two given points there can be drawn but one straight line. and when the straight line AB is drawn, its magnitude is at the same time exhibited, or given. therefore the straight line AB is given in position and magnitude.

PROP. XXX.

27.

IF one of the extremities of a straight line given in position and magnitude be given; the other extremity shall also be given.

Let the point A be given, to wit one of the extremities of straight line given in magnitude, and which lies in the straight line AC given in position; the other extremity is also given.

Because the straight line is given in magnitude, one equal to can be found; let this be the straight line D. from the grace straight line AC cut off AB equal to the lesser D. therefore the other extremity B of the straight line AB is found. and the point B has always the same situation, because any other point in AC, upon the same side of A, cuts off between it and the point A greater or less straight line than AB, that is than D. therefore the point B is given. and it is plain another such point can be found in AC produced upon the other side of the point A.

PROP. XXXI.

28.

IF a straight line be drawn through a given point parallel to a straight line given in position; that straight line is given in position.

Let A be a given point, and BC a straight line given in position; the straight line drawn thro' A parallel to BC is given in position.

Thro' A draw the straight line DAE parallel to BC; the straight line DAE has always the same position, because no other straight line can be drawn through A parallel to BC. therefore the straight line DAE which has been found is given in position,

.

PROP. XXXII.

IF a straight line be drawn to a given point in a given straight line, and makes a given angle with it. that straight line is given in position.

Let AB be a straight line given in position, and C a given point in it, the straight line drawn to C which makes a given angle with CB, is given in position.

Because the angle is given, one equal to it can be found; let this be the angle at D. at the given point C in the given straight line AB make the angle ECB equal to the angle at D. therefore the straight line EC has always the same situation, because any other straight line FC drawn to the point C makes with CB a greater or less angle than the angle ECB or the angle at D. therefore the straight line EC which has been found is given in position.

It is to be observed that there are two straight lines EC, GC upon one side of AB that make equal angles with it, and which make equal angles with it when produced to the other side.

PROP. XXXIII.

IF a straight line be drawn from a given point, to a straight line given in position, and makes a given angle with it; that straight line is given in position.

From the given point A let the straight line AD be drawn to the straight line BC given in position, and make with it a given angle ADC; AD is given in position.

Thro' the point A draw the straight line EAF parallel to BC; and because thro' the given point A the straight line EAF is drawn parallel to BC which is given in position, EAF is therefore given in position. and because the straight line AD meets the parallels BC, EF, the angle EAD is equal to the angle ADC; and ADC is:

given, wherefore also the angle EAD is given. therefore because the straight line DA is drawn to the given point A in the straight line EF given in position, and makes with it a given angle EAD; AD is given in position.

PROP. XXXIV.

31.

IF from a given point to a straight line given in position, a straight line be drawn which is given in magnitude; the same is also given in position.

Let A be a given point, and BC a straight line given in position; a straight line given in magnitude drawn from the position A to BC is given in position.

Because the straight line is given in magnitude, one equal to it can be found; let this be the straight line D. from the point A draw AE perpendicular to BC; and because AE is the shortest of all the straight lines which can be drawn from the point A to BC, the straight line D, to which one equal is to be drawn from the point A to BC, cannot be less than AE. If therefore D be

equal to AE, AE is the straight line given in magnitude drawn from the point A to BC. and it is evident that AE is given in position because it is drawn from the given point A to BC which is given position, and makes with BC the given angle AEC.

But if the straight line D be not equal to AE, it must be greater than it. produce AE, and make AF equal to D; and from the center A, at the distance AF describe the circle GFH, and join AC, AH. because the circle GFH is given in position, and the straight line BC is also given in position; therefore their intersection G is given; and the point A is given; wherefore AG is given in position, that is, the straight line AG given in magnitude (for it is equal to D) and drawn from the given point A to the straight line BC given in position, is also given in position. and in like manner AH is given in position. therefore, in this case there are two straight lines AG,

AH of the same given magnitude which can be drawn from a given point A to a straight line BC given in position.

PROP. XXXV.

32.

IF a straight line be drawn between two parallel straight lines given in position, and makes given angles with them; the straight line is given in magnitude.

Let the straight line EF be drawn between the parallels AB, CD which are given in position, and make the given angles BEF, EFD; EF is given in magnitude.

In CD take the given point G, and thro' G draw GH parallel to EF. and because CD meets the parallels GH, EF, the angle EFD is equal to the angle HGD. and EFD is a given angle, wherefore the angle HGD is given and

because HG is drawn to the given point G in the straight line CD given in position, and makes a given angle HGD; the straight line HG is given in position and AB is given in position, therefore the point H is given; and the point G is also given, wherefore GH is given in magnitude and EF is equal to it; therefore EF is given in magnitude.

PROP. XXXVI.

33.

IF a straight line given in magnitude be drawn between two parallel straight lines given in position; it shall make given angles with the parallels.

Let the straight line EF given in magnitude be drawn between the parallel straight lines AB, CD which are given in position; the angles AEF, EFC shall be given.

Because EF is given in magnitude, a straight line equal to it can be found; let this be G. in AB take a given point H, and from it draw HK perpendicular to CD. therefore the straight line G, that is EF, cannot be less than HK.

and if G be equal to HK, EF also is equal to it; wherefore EF is at right angles to CD, for if it be not, EF would be greater than HK, which is absurd. therefore the angle EFD is a right and consequently a given angle.

But if the straight line G be not equal to HK, it must be greater than it. produce HK, and take HL equal to G; and from the center H, at the distance HL describe the circle MLN, and join HM, HN. and because the circle MLN, and the straight line CD are given in position, the points M, N are given; and the point H is given, wherefore the straight lines HM, HN are given in position. and CD is given in position, therefore the angles HMN, HNM are

given in position. of the straight lines HM, HN let HN be that which is not parallel to EF, for EF cannot be parallel to both of them; and draw EO parallel to HN. EO therefore is equal to HN, that is to G; and EF is equal to G, wherefore EO is equal to EF, and the angle EFO to the angle EOF, that is to the given angle HNM. and because the angle HNM which is equal to the angle EFO or EFD has been found, therefore the angle EFD, that is the angle AEF, is given in magnitude, and consequently the angle EFC.

PROP. XXXVII.

E.

IF a straight line given in magnitude be drawn from a point to a straight line given in position, in a given angle; the straight line drawn thro' that point parallel to the straight line given in position, is given in position.

Let the straight line AD given in magnitude be drawn from the point A to the straight line BC given in position, in the given angle ADC; the straight line EAF drawn through A parallel to BC is given in position.

In BC take a given point G, and draw GH parallel to AD. and because HG is drawn to a given point G in the straight line BC given in position, in a given:

angle HGC, for it is equal to the given angle ADC; HG is given in position; but it is given also in magnitude, because it is equal to AD which is given in magnitude, therefore because G one of the extremities of the straight line GH given in position and magnitude is given, the other extremity H is given. and the straight line EAF which is drawn through the given point H parallel to BC given in position, is therefore given in position.

PROP. XXXVIII.

34.

IF a straight line be drawn from a given point to two parallel straight lines given in position; the ratio of the segments between the given point and the parallels shall be given.

Let the straight line EFG be drawn from the given point E to the parallels AB, CD; the ratio of EF to EG is given.

From the point E draw EHK perpendicular to CD. and because from a given point E the straight line EK is drawn to CD which is given in position, in a given angle EKC; EK is given in position.

and AB, CD are given in position; therefore the points are H, K are given. and the point E is given, wherefore EH, EK are given in magnitude, and the ratio of them is therefore given. but as EH to EK, so is EF to EG, because AB, CD are parallels. therefore the ratio of EF to EG is given.

PROP. XXXIX.

35. 36.

IF the ratio of the segments of a straight line between a given point in it and two parallel straight lines be given; if one of the parallels be given in position, the other is also given in position.

From the given point A let the straight line AED be drawn to the two parallel straight lines FG, BC, and let the ratio of the segments AE, AD be given; if one of the parallels BC be given in position, the other FG is also given in position.

From the point A draw AH perpendicular to BC, and let it meet FG in K. and because AH is drawn from the given point A to the straight line BC given

in position, and makes a given angle AHD; AH is given in position. and BC is likewise given in position, therefore the point H is given. the point A is also given, wherefore AH is given in magnitude. and, because FG, BC are parallels, as AE to AD, so is AK to AH; and the ratio of AE to AD is given, wherefore the ratio of AK to AH is given; but AH is given in magnitude, therefore AK is given in magnitude; and it is also given in position, and the point A is given; wherefore the point K is given. and because the straight line FG is drawn thro' the given point K parallel to BC which is given in position, therefore FG is given in position.

PROP. XL.

37. 38.

IF the ratio of the segments of a straight line into which it is cut by three parallel straight lines, be given; If two of the parallels are given in position, the third also is given in position.

Let AB, CD, HK be three parallel straight lines, of which AB, CD are given in position; and let the ratio of the segments GE,

GF into which the straight line GEF is cut by the three parallels, be given; the third parallel HK is given in position.

In AB take a given point L, and draw LM perpendicular to CD, meeting HK in N. because LM is drawn from the given point L to CD which is given in position, and makes a given angle LMD; LM is given in position. and CD is given in position, wherefore the point M is given; and the point L is given, LM is therefore given in magnitude. and because the ratio of GS, to

GF is given, and as GE to GF, so is NL to NM; the ratio of NL to NM is given; and therefore the ratio of ML to LN is given. but LM is given in magnitude, wherefore LN is given magnitude; and it is also given in position, and the point L is given; wherefore the point N is given, and because the straight line HK is drawn thro' the given point N parallel to CD which is given in position, therefore HK is given in position.

PROP. XLI.

F.

IF a straight line meets three parallel straight lines which are given in position; the segments into which they cut it, have a given ratio.

Let the parallel straight lines AB, CD, EF given in position be cut by the straight line GHK; the ratio of GH to HK is given.

In AB take a given point L, and draw LM perpendicular to CD, meeting EF in N; therefore LM is given in position; and CD, EF are given in position, wherefore the points M, N are given. and the point L is given, therefore the straight lines LM, MN are given in magnitude; and the ratio of LM to MN is therefore given, but as LM to MN, so is GH to HK; wherefore the ratio of GH to HK is given.

PROP. XLII.

39.

See. N.

IF each of the sides of a triangle be given in magnitude; the triangle is given in species.

Let each of the sides of the triangle ABC be given in magnitude; the triangle ABC is given in species.

Make a triangle DEF the sides of which are equal, each to each, to the given straight lines AB, BC, CA; which can be done, because any two of them must be greater than the third; and let DE be equal to AB, EF to BC, and FD to CA. and because the two sides ED, DF are equal to the two BA, AC, each to each, and the base EF equal to the base BC; the angle EDF is equal to the angle BAC. therefore because the angle EDF, which is equal to the angle BAC, has been found, the angle BAC is given, in like manner the angles at B, C are given, and because the sides AB, BC, CA are given, their ratios to one another are given. therefore the triangle ABC is given in species.

PROP. XLIII.

IF each of the angles of a triangle be given in magnitude; the triangle is given in species.

Let each of the angles of the triangle ABC be given in magnitude; the triangle ABC is given in species.

Take a straight line DE given in position and magnitude, and at the points D, E make the angle EDF equal to the angle BAC, and the angle DEF equal to ABC; therefore the other angles EFD, BCA are equal. and each of the angles at the points A, B, C is given, wherefore

each of those at the points D, E, F is given. and because the straight line FD is drawn to the given point D in DE which is given in position, making the given angle EDF; therefore DF is given in position in like manner EF also is given in position; wherefore the point F is given, and the points D, E are given; therefore each of the straight lines DE, EF, FD is given in magnitude,

wherefore the triangle DEF is given in species; and it is similar to the triangle ABC; which therefore is given in species.

PROP. XLIV.

41.

IF one of the angles of a triangle be given, and if the sides about it have a given ratio to one another; the triangle is given in species.

Let the triangle ABC have one of its angles BAC given, and let the sides BA, AC about it have a given ratio to one another; the triangle ABC is given in species.

Take a straight line DE given in position and magnitude, and at the point D in the given straight line DE make the angle EDF equal to the given angle BAC; wherefore the angle EDF is given. and because the straight line FD is drawn to the given point D in ED which is given in position, making the given angle EDF; therefore FD is given in position and because the ratio of BA to AC is given, make the ratio of ED to DF the same with it, and join EF. and because the ratio of ED to DF is given, and ED is given, therefore DF is given in magnitude; and it is given also in position, and the point D is given, wherefore the point F is given. and the points D, E are given, wherefore DE, EF, FD are given in magnitude; and the triangle DEF is therefore given in species. and because the triangles ABC, DEF have one angle BAC equal to one angle EDF, and the sides about these angles proportionals; the triangles are similar. but the triangle DEF is given in species, and therefore also the triangle ABC.

PROP. XLV.

42.

IF the sides of a triangle have to one another given ratios; the triangle is given in species.

Let the sides of the triangle ABC have given ratios to one another. the triangle ABC is given in species.

Take a straight line D given in magnitude; and because the ratio of AB to LC is given, make the ratio of D to E the same with it; and D is. given, therefore E is given. and because the ratio of BC to CA is given, to this make the ratio of E to F the same; and E is given, and therefore F. and because as AB to BC, so is D to E, by composition AB and BC together are to BC, as D and E to E; but as BC to CA, so is E to F; therefore, ex aequali, as AB and BC are to CA, so are D and E to F. and AB and BC are greater than CA, therefore D and E are greater then F. in the same another any two of the there D, E, F are greater the then the GHK when sides are equal to D, E, F, so that GH be equal to D, HK to E, and KG to F. and because D, E, F are, each of them, given, therefore GH, HK, KG are each of them given in magnitude; therefore the triangle GHK is given in species. but as AB to BC, so is (D to E, that is) GH to HK; and as BC to CA, so is (E to F, that is) HK to KG; therefore, ex acquali, as AB to AC, so is GH to GK. wherefore the triangle ABC is equiangular and similar to the triangle GHK. and the triangle GHK is given in species; therefore also the triangle ABC is given in species.

COR. If a triangle is required to be made the sides of which shall have the same ratios which three given straight lines D, E, F have to one another; it is necessary that every two of them be greater than the third.

PROP. XLVI.

43.

IF the sides of a right angled triangle about one of the acute angles have a given ratio to one another; the triangle is given in species.

Let the sides AB, BC about the acute angle ABC of the triangle ABC which has a right angle at A, have a given ratio to one another; the triangle ABC is given in species.

Take a straight line DE given in position and magnitude; and because the ratio of AB to BC is given, make as AB to BC, so DE to EF; and because DE has a given ratio to EF, and DE is given, therefore EF is given. and because as AB to BC, so is DE to EF, and AB is less than BC, therefore DE is less than EF. from the point D draw DG at right angles to DE, and from the center E at the distance EF describe a circle which shall meet DG in two points, let G be either of them, and join EG; therefore the circumference of the circle is given in position. and the straight line DG is given in position, because it is drawn to the given point D in DE given in position, in a given angle, therefore the point G is given, and the points D, E are given, wherefore DE, EG, GD are given in magnitude, and the triangle DEG in species. and because the triangles ABC, DEG have the angle BAC equal to the angle EDG, and the sides about the angles ABC, DEG proportionals, and each of the other angles BCA, EGD less than a right angle; the triangle ABC is equiangular and similar to the triangle DEG. but DEG is given in species, therefore the triangle ABC is given in species. and in the same manner, the triangle made by drawing a straight line from E to the other point in which the circle meets DG is given in species.

PROP. XLVII.

44.

IF a triangle has one of its angles which is not a right angle given, and if the sides about another angle have a given ratio to one another; the triangle is given in species.

Let the triangle ABC have one of its angles ABC a given, but not a right angle, and let the sides BA, AC about another angle BAC have a given ratio to one another; the triangle ABC is given in species.

First, Let the given ratio be the ratio of equality, that is, let the sides BA, AC and consequently the angles ABC, ACB be equal. and because the angle ABC is given, the angle ACB, and also the remaining angle BAC is given. therefore the triangle ABC is given in species. and it is evident that in this case the given angle ABC must be acute.

Next, Let the given ratio be the ratio of a less to a greater, that is, let the side AB adjacent to the given angle be less than the side AC. take a straight line DE given in position and magnitude, and make the angle DEF equal to the given angle ABC; therefore EF is given in position. and because the ratio of BA to AC is given, as BA to AC, so make ED to DG; and because the ratio of ED to DG is given, and ED is given, the straight line DG is given. and BA is less than AC, therefore ED is less than DG. from the center D, at the distance DG describe the circle GF meeting EF in F, and join DF. and because the circle is given in position, as also the straight line EF, the point F is given. and the points D, E are given, wherefore the straight lines DE, EF, FD are given in magnitude, and the triangle DEF in species. and because BA is less than AC, the angle ACB is less than the angle ABC, and therefore ACB is less than a right angle. in the same manner, because ED is less than DG or DF, the angle DFE is less than a right angle, and because the triangles

ABC, DEF have the angle ABC equal to the angle DEF, and the sides about the angles BAC, EDF proportionals, and each of the other angles ACB, DFE less than a right angle; the triangles ABC, DEF are similar. and DEF is given in species, wherefore the triangle ABC is also given in species.

Thirdly, Let the given ratio be the ratio of a greater to a less, that is, let the side AB adjacent to the given angle be greater than AC. and, as in the last case, take a straight line DE given in position and magnitude, and make the angle DEF equal to the given angle ABC; therefore EF is given in position. also draw DG perpendicular to EF; therefore if the ratio of BA to AC be the same with the ratio of ED to the perpendicular DG, the triangles ABC, DEG are similar, because the angles ABC, DEG are equal, and DGE is a right angle. therefore the angle ACB is a right angle, and the triangle ABC is given in species.

But if, in this last case, the given ratio of BA to AC be not the same with the ratio of ED to DG, that is, with the ratio of BA to the perpendicular AM drawn from A to BC; the ratio of BA to AC must be less than the ratio of BA to AM, because AC is greater than AM. make as BA to AC, so ED to DH; therefore the ratio of ED to DH is less than the ratio of (BA to AM, that is than the ratio of) ED to DG; and consequently DH is greater P than DG; and because BA is greater than AC, ED is greater than DH. from the center D, at the distance DH, describe the circle KHF which necessarily meets the straight line EF in two points, because DH is greater than DG, and less than DE. let the circle meet EF in the points F, K which are given, as was shewn in the preceding case; and, DF, DK being joined, the triangles DEF, DEK are given in species, as was there shewn. from the center A, at the distance AC describe a circle meeting BC again in L. and if the

angle ACB be less than a right angle, ALB must be greater than a right angle; and on the contrary. in the same manner, if the angle DFE be less than a right angle, DKE must be greater than one; and on the contrary. let each of the angles ACB, DFE be either less or greater than a right angle; and because in the triangles ABC, DEF the angles ABC, DEF are equal, and the sides BA, AC, and ED, DF about two of the other angles proportionals, the triangle ABC is similar to the triangle DEF. in the same manner, the triangle ABL is similar to DEK. and the triangles DEF, DEK are given in species, therefore also the triangles ABC, ABL are given in species. and from this it is evident, that, in this third case, there are always two triangles of a different species to which the things mentioned as given in the Proposition can agree.

PROP. XLVIII.

45.

IF a triangle has one angle given, and if both the sides together about that angle have a given ratio to the remaining side; the triangle is given in species.

Let the triangle ABC have the angle BAC given, and let the sides BA, AC together about that angle have a given ratio to BC; the triangle ABC is given in species.

Bisect the angle BAC by the straight line AD; therefore the angle BAD is given. and because as BA to AC, so is BD to DC, by permutation, as AB to BD, so is AC to CD; and as BA and AC together to BC, so is AB to BD. but the ratio of BA and AC together to BC is given, wherefore the ratio of AB to BD is given; and the angle BAD is given, therefore the triangle ABD is given in species. and the angle ABD is therefore given.; the angle BAC is also given, wherefore the triangle ABC is given in species.

A triangle which shall have the things that are mentioned in the Proposition to be given, can be found in the following manner. let EFG be the given angle, and let the ratio of H to K be the

given ratio which the two sides about the angle EFG must have to the third side of the triangle, therefore because two sides of a triangle are greater than the third side, the ratio of H to K must be the ratio of a greater to a less. bisect the angle EFG by the straight line FL, and by the 47th Proposition find a triangle of which EFL is one of the angles, and in which the ratio of the sides about the angle opposite to FL is the same with the ratio of H to K; to do which, take FE given in position and magnitude, and draw EL perpendicular to FL. then, if the ratio of H to K be the same with the ratio of FE to EL, produce EL and let it meet FG in P; the triangle FEP is that which was to be found. for it has the given angle EFG, and because this angle is bisected by FL, the sides EF, FP together are to EP, as FE to EL, that is as H to K.

But if the ratio of H to K be not the same with the ratio of FE to EL, it must be less than it, as was shewn in Pro. and in this case there are two triangles each of which has the given angle EFL, and the ratio of the sides about the angle opposite to FL the same with the ratio of H to K. by Pro. find these triangles EFM, EFN each of which has the angle EFL for one of its angles, and the ratio of the side FE to EM or EN the same with the ratio of H to K; and let the angle EMF be greater, and ENF less than a right angle. and because H is greater than K, EF is greater than EN, and therefore the angle EFN, that is the angle NFG, is less than the angle ENF. to each of these add the angles NEF, EFN; therefore the angles NEF, EFG are less than the angles NEF, EFN, FNE, that is than two right angles; therefore the straight lines EN, FG must meet together when produced; let them meet in O, and produce EM

to G. each of the triangles EFG, EFO has the things mentioned to be given in the Proposition. for each of them has the given angle EFG, and because this angle is bisected by the straight line FMN, the sides EF, FG together have to EG the third side the ratio of FE to EM, that is of H to K. in like manner, the sides EF, FO together have to EO the ratio which H has to K.

PROP. XLIX.

46.

IF a triangle has one angle given, and if the sides about another angle, both together, have a given ratio to the third side; the triangle is given in species.

Let the triangle ABC have one angle ABC given, and let the two sides BA, AC about another angle BAC have a given ratio to BC; the triangle ABC is given in species.

Suppose the angle BAC to be bisected by the straight line AD; BA and AC together are to BC, as AB to BD, as was shewn in the preceding Proposition. but the ratio of BA and AC together to BC is given, therefore also the ratio of AB to BD is given. and the angle ABD is given, wherefore the triangle ABD is given in species; and consequently the angle BAD, and its double the angle BAC are given; and the angle ABC is given. therefore the triangle ABC is given in species.

A triangle which shall have the things mentioned in the Proposition to be given, may be thus found. Let EFG be the given angle, and the ratio of H to K the given ratio; and by Pro. find the triangle EFL which has the angle EFG for one of its angles, and the ratio of the sides EF, FL about this angle the same with the ratio of H to K; and make the angle LEM equal to the angle

FEL. and because the ratio of H to K is the ratio which two sides of a triangle have to the third, H must be greater than K; and because EF is to FL, as H to K, therefore EF is greater than FL, and the angle FEL, that is LEM is therefore less than the angle ELF. wherefore the angles LFE, FEM are less than two right angles, as was shewn in the foregoing Proposition, and the straight lines FL, EM must meet if produced; let them meet in G. EFG is the triangle which was to be found; for EFG is one of its angles, and because the angle FEG is bisected by EL, the two sides FE, EG together have to the third side FG the ratio of EF to FL, that is the given ratio of H to K.

PROP. L.

76.

IF from the vertex of a triangle given in species, a straight line be drawn to the base in a given angle; it shall have a given ratio to the base.

From the vertex A of the triangle ABC which is given in species, let AD be drawn to the base BC in a given angle ADB; the ratio of AD to BC is given.

Because the triangle ABC is given in species, the angle ABD is given, and the angle ADB is given; therefore the triangle ABD is given in species; wherefore the ratio of AD to AB is given, and the ratio of AB to BC is given; and therefore the ratio of AD to BC is given.

PROP. LI.

47.

RECTILINEAL figures given in species, are divided into triangles which are given in species.

Let the rectilineal figure ABCDE be given in species; ABCDE may be divided into triangles given in species.

Join BE, BD, and because ABCDE is given in species, the angle BAE is given, and the ratio of BA to BE is given; wherefore the triangle BAE is given in species, and the angle AEB is therefore given. but the whole angle AED is given, and therefore the remaining angle BED is given. and the ratio of AE to EB is given, as also the ratio of AE to ED; therefore the ratio of BE to ED is given. and the angle BED is given, wherefore the triangle BED is given in species. in the same manner the triangle BDC is given in species. therefore rectilineal figures which are given in species are divided into triangles given in species.

PROP. LII.

48.

IF two triangles given in species be described upon the same straight line; they shall have a given ratio to one another.

Let the triangles ABC, ABD given in species be described upon the same straight line AB; the ratio of the triangle ABC to the triangle ABD is given.

Thro’ the point C draw CE parallel to AB, and let it meet DA produced in E, and join BE. because the triangle ABC is given is species, the angle BAC, that is the angle ACE, is given; and because the triangle ABD is given in species, the angle DAB, that is the angle AEC is given. therefore the triangle ACE is given in species; wherefore the ratio of EA to AC is given, and the ratio of CA to AB is given, as also the ratio of BA to AD; therefore the ratio

of EA to AD is given. and the triangle ACB is equal to the triangle AEB, and as the triangle AEB, or ACB, is to the triangle ADB, so is the straight line EA to AD. but the ratio of EA to AD is given, therefore the ratio of the triangle ACB to the triangle ADB is given.

PROBLEM.

To find the ratio of two triangles ABC, ABD given in species and which are described upon the same straight line AB.

Take a straight line FG given in position and magnitude, and because the angles of the triangles ABC, ABD are given, at the points F, G of the straight line FG make the angles GFH, GFK equal to the angles BAC, BAD; and the angles FGH, FGK equal to the angles ABC, ABD, each to each. therefore the triangle ABC, ABD are equiangular to the triangles FGH, FGK, each to each. through the point II draw HL parallel to FG meeting KF produced in L. and because the angles BAC, BAD are equal to the angles GFH, GFK, each to each; therefore the angles ACE, AEC are equal to FHL, FLH, each to each, and the triangle AEC equiangular to the triangle FLH. therefore as EA to AC,

so is LF to FH; and as CA to AB, so HF to FG; and as BA to AD, so GF to FK; wherefore, ex aequali, as EA to AD, so is LF to FK. but, as was shewn, the triangle ABC is to the triangle ABD, as the straight line EA to AD, that is as LF to FK. the ratio therefore of LF to FK has been found which is the same with the ratio of the triangle ABC to the triangle ABD.

PROP. LIII.

49.

IF two rectilineal figures given in species be described upon the same straight line; they shall have a given ratio to one another.

Let any two rectilineal figures ABCDE, ABFG which are given in species, be described upon the same straight line AB; the ratio of them to one another is given.

Join AC, AD, AF; each of the triangles AED, ADC, ACB, AGF, ABF is given in species. and because the triangles ADE, ADC given in species are described upon the same straight line AD, the ratio of EAD to DAC is given; and, by composition, the ratio of EACD to DAC is given. and the ratio of DAC to CAB is given, because they are described upon the same straight line AC; therefore the ratio of EACD to ACB is given; and, by composition, the ratio of ABCDE to ABC is given. in the same manner, the ratio of ABFG to ABF is given. but the ratio of the triangle ABC to the triangle ABF is given; wherefore because the ratio of ABCDE to ABC is given, as also the ratio of ABC to ABF, and the ratio of ABF to ABFG; the ratio of the rectilineal ABCDE to the rectilineal ABFG is given.

PROBLEM.

To find the ratio of two rectilineal figures given in species, and described upon the same straight line.

Let ABCDE, ABFG be two rectilineal figures given In species, and described upon the same straight line AB, and join AC, AD, AF. take a straight line HK given in position and magnitude, and by the 52. Dat. find the ratio of the triangle ADE to the triangle ADC, and make the ratio of HK to KL the same with it. find also the ratio of the triangle ACD to the triangle ACB, and make the ratio of KL to LM the same. also, find the ratio of the triangle ABC to the triangle ABF, and make the ratio of LM to MN the same. and lastly, find the ratio of the triangle AFB to the triangle AFG, and make the ratio of MN to NO the same. then the ratio of ABCDE to ABFG is the same with the ratio of HM to MO.

Because the triangle EAD is to the triangle DAC, as the straight line HK to KL; and as triangle DAC to CAB, so is the straight line KL to LM; therefore by using composition as often as the number of triangles requires, the rectilineal ABCDE is to the triangle ABC, as the straight line HM to ML. in like manner, because triangle GAF is to FAB, as ON to NM, by composition, the rectilineal ABFG is to the triangle ABF, as MO to MN; and, by inversion, as ABF to ABFG, so is NM to MO. and the triangle ABC is to ABF, as LM to MN. wherefore because as ABCDE to ABC, so is HM to ML; and as ABC to ABF, so is LM to MN; and as ABF to ABFG, so is MN to MO; ex aequali, as the rectilineal ABCDE to ABFG, so is the straight line HM to MO.

PROP. LIV.

50.

IF two straight lines have a given ratio to one another; the similar rectilineal figures described upon them similarly, shall have a given ratio to one another.

Let the straight lines AB, CD have a given ratio to one another, and let the similar and similarly placed rectilineal figures E, F be described upon them; the ratio of E to F is given.

To AB, CD let G be a third proportional; therefore as AB to CD, so is CD to G. and the ratio of AB to CD is given, wherefore the ratio of CD to G is given; and consequently the ratio of AB to G is also given. but as AB to G, so is the figure E to the figure F. therefore the ratio of E to F is given.

PROBLEM.

To find the ratio of two similar rectilineal figures E, F similarly described upon straight lines AB, CD which have a given ratio to one another. let G be

a third proportional to AB, CD.

Take a straight line H given in magnitude; and because the ratio of AB to CD is given, make the ratio of H to K the same with it; and because H is given, K is given. as H is to K, so make K to L; then the ratio of E to F is the same with the ratio of H to L. for AB is to CD, as H to K, wherefore CD is to G, as K to L; and, ex aequali, as AB to G, so is H to L. but the figure E is to the figure F, as AB to G, that is as H to L.

PROP. LV.

IF two straight lines have a given ratio to one another; the rectilineal figures given in species described upon them, shall have to one another a given ratio.

Let AB, CD be two straight lines which have a given ratio to one another; the rectilineal figures E, F given in species and described upon them, have a given ratio to one another.

Upon the straight line AB describe the figure AG similar and similarly placed to the figure F; and because F is given in species, AG is also given in species. therefore since the figures E, AG which are given in species, are described upon the same straight line AB, the ratio of E to AG is given. and because the ratio of AB to CD is given, and upon them are described the similar and similarly placed rectilineal figures AG, F, the ratio of AG to F is given. and the ratio of AG to E is given; therefore the ratio of E to F is given.

PROBLEM.

To find the ratio of two rectilineal figures E, F given in species, and described upon the straight lines AB, CD which have a given ratio to one another.

Take a straight line H given in magnitude; and because the rectilineal figures E, AG given in species are described upon the same straight line AB,

find their ratio by the 53. Dat. and make the ratio of H to K the same; K is therefore given. and because the similar rectilineal figures AG, F are described upon the straight lines AB, CD which have a given ratio, find their ratio by the 54. Dat. and make the ratio of K to L the same. the figure E has to F the same ratio which H has to L. for, by the construction, as E is to AG, so is H to K; and as AG to F, so is K to L; therefore, ex aequali, as E to F, so is H to L.

PROP. LVI.

52.

IF a rectilineal figure given in species be described upon a straight line given in magnitude; the figure is given in magnitude.

Let the rectilineal figure ABCDE given in species be described upon the straight line AB given in magnitude; the figure ABCDE is given in magnitude.

Upon AB let the square AF be described; therefore AF is given in species and magnitude. and because the rectilineal figures ABCDE, AF given in species are described upon the same straight line AB, the ratio of ABCDE to AF is given. but the square AF is given in magnitude, therefore also the figure ABCDE is given in magnitude.

PROP.

To find the magnitude of a rectilineal figure given in species described upon a straight line given in magnitude.

Take the straight line GH equal to the given straight line AB, and by the 53. Dat. find the ratio which the square AF upon AB has to the figure

ABCDE; and make the ratio of GH to HK the same; and upon GH describe the square GL, and complete the parallelogram LHKM the figure ABCDE is equal to LHKM because AF is to ABCDE, as the straight line GH to HK, that is, as the figure GL to HM; and AF is equal to GL, therefore ABCDE is equal to HM.

PROP. LVII.

IF two rectilineal figures are given in species, and if a side of one of them has a given ratio to a side of the other; the ratios of the remaining sides to the remaining sides shall be given.

Let AC, DF be two rectilineal figures given in species, and let be ratio of the side AB to the side DE be given; the ratios of be remaining sides to the remaining sides are also given.

Because the ratio of AB to DE is given, as also the ratios of AB to BC, and of DE to EF; the ratio of BC to EF is given in the same manner, the ratios of the other sides to the other sides are given.

The ratio which BC has to EF may be found thus; take a straight line G given in magnitude, and because the ratio of BC to BA is given, make the ratio of G to H the same; and because the ratio of AB to DE given, make the ratio of H to K be same; and make the ratio of K to L the same with the given ratio of DE to EF. since therefore as BC to BA, so is G to H; and s BA to DE, so is H to K; and as DE to EF, so is K to L; ex equali, BC is to EF, as G to L. therefore the ratio of G to L as been found which is the same with the ratio of BC to EF.

PROP. LVIII.

IF two similar rectilineal figures have a given ratio to one another; their homologous sides have also a given ratio to one another.

Let the two similar rectilineal figures A, B have a given ratio to one another; their homologous sides have also a given ratio.

Let the side CD be homologous to EF, and to CD, EF let the straight line G be a third proportional, as therefore CD to G, D is the figure A to B; and the ratio of A to B is given, therefore the ratio of CD to G is given; and CD, EF, G are proportionals, wherefore the ratio of CD to EF is given.

The ratio of CD to EF may be round thus; take a straight line H given in magnitude; and because the ratio of the figure A to B is given, make the ratio of H to K the same with it. and, as the 13. Dat. directs to be done, find a mean proportional L between H and K; the ratio of CD to EF is the same with that of H to L. let G be a third proportional to CD, EF; therefore as CD to G, so is (A to B, and so is) H to K. and as CD to EF, so is H to L, as is shewn in the 13. Dat.

PROP. LIX.

54.

IF two rectilineal figures given in species have a given ratio to one another; their sides shall likewise have given ratios to one another.

Let the two rectilineal figures A, B given in species have a given ratio to one another; their sides shall also have given ratios to one another.

If the figure A be similar to B, their homologous sides shall have a given ratio to one another, by the preceding Proposition; and because the figures are given in species, the sides of each of them have given ratios to one another. therefore each side of one of them has to each side of the other a given ratio.

But if the figure A be not similar to B, let CD, EF be any two of their sides; and upon EF conceive the figure EG to be described similar and similarly placed to the figure A, so that CD, EF be homologous sides; therefore EG is given in species. and the figure B is given in species, wherefore the ratio of B to EG is given; and the ratio of A to B is given, therefore the ratio of the figure A to EG is given. and A is similar to EG, therefore the ratio of the side CB to EF is given; and consequently the ratios of the remaining sides to the remaining sides are given.

The ratio of CD to EF may be found thus; take a straight H given in magnitude, and because the ratio of the figure A to B given, make the ratio of H to K the same with it. and by the Dat. find the ratio of the figure B to EG, and make the ratio of H to L the same; between H and L find a mean proportional M; ratio of CD to EF is the same with the ratio of H to M. because, the figure A is to B, as H to K; and as B to EG, so is K to L; ex aequali, as A to EG, so is H to L. and the figures A, EG are so b. 9. D and M is a mean proportional between H and L; therefore, as C is shewn in the preceding Proposition, CO is to EF, as H to M.

PROP. LX.

IF a rectilineal figure be given in species and magnitude, the sides of it shall be given in magnitude.

Let the rectilineal figure A be given in species and magnitude; sides are given in magnitude.

Take a straight line BC given in position and magnitude; and upon BC describe the figure D similar, and similarly placed, to the figure A, and let EF be the side of the figure A homologous to BC the side of D; therefore the

figure D is given in species. and because upon the given straight line. C the figure D given in species is described, D is given in magnitude. and the figure A is given in magnitude, therefore the ratio of A to D is given. and the figure A is similar to D; therefore the ratio of he side EF to the homologous side BC is given. and BC is given, wherefore EF is given. and the ratio of EF to EG is given, therefore EG is given. and, in the same manner, each of the other sides of the figure A can be shewn to be given.

PROBLEM.

To describe a rectilineal figure A similar to a given figure D, and equal to another given figure H. It is Pro. B. 6. Elem.

Because each of the figures D, H is given, their ratio is given, which may be found by making upon the given straight line BC the parallelogram BK equal to D, and upon its side CK making the parallelogram KL equal to H in the angle KCL equal to the angle MBC. therefore the ratio of D to H, that is of BK to KL is the same with the ratio of BC to CL. and because the figures D, A are similar, and that the ratio of D to A, or H, is the same with the ratio of BC to CL; by the 58. Dat. the ratio of the homologous sides BC, EF is the same with the ratio of BC to the mean proportional between BC and CL. find EF the mean proportional; then EF is the side of the figure to be described, homologous to BC the side of D, and the figure itself can be described by the 18th Prop. B. 6. which, by the construction, is similar to D. and because D is to A, as BC to CL, that is as the figure BK to KL; and that D is equal to BK, therefore A is equal to KL, that is to H.

PROP. LXI.

IF a parallelogram given in magnitude has one of its sides and one of its angles given in magnitude; the other side also is given.

Let the parallelogram ABCD given in magnitude, have the side AB and the angle BAC given in magnitude; the other side AC is given.

Take a straight line EF given in position and magnitude; and because the parallelogram AD is given in magnitude, a rectilineal figure equal to it can be found. and a parallelogram equal to this figure can be applied to the given straight line EF in an angle equal to the given angle BAC. let this be the parallelogram EFGH having the angle FEG equal to the angle BAC. and because the parallelograms AD, EH are equal, and have the angles at A and E equal; the sides about them are reciprocally proportional. therefore is AB to EF, so is EG to AC; and AB, EF, EG are given, therefore also AC is given. Whence the way of finding AC is manifest.

PROP. LXII.

H.

IF a parallelogram has a given angle, the rectangle contained by the sides about that angle has a given ratio to the parallelogram.

Let the parallelogram ABCD have the given angle ABC; the rectangle AB, BC has a given ratio to the parallelogram AC.

From the point A draw AE perpendicular to BC. because the angle ABC is given, as also the angle AEB; the triangle ABE is given in species; therefore the ratio of BA to AE is given. but as BA to AE, so is the rectangle AB, BC to the rectangle AE, BC; therefore the ratio of the rectangle AB, BC to AE, BC, that is to the parallelogram AC is given.

And it is evident how the ratio of the rectangle to the parallelogram may be found, by making the angle FGH equal to the given angle ABC, and drawing, from any point F in one of its sides, FK perpendicular to the other

GH; for GF is to FK, as BA to AE, that is, as the rectangle AB, BC to the parallelogram AC.

66.

COR. And if a triangle ABC has a given angle ABC, the rectangle AB, BC contained by the sides about that angle, shall have a given ratio to the triangle ABC.

Complete the parallelogram ABCD; therefore, by this Proposition, the rectangle AB, BC has a given ratio to the parallelogram AC; and AC has a given ratio to its half the triangle ABC. therefore the rectangle AB, BC has a given ratio to the triangle ABC.

And the ratio of the rectangle to the triangle is found thus; make the triangle FGK as was shewn in the Proposition; the ratio of GF to the half of the perpendicular FK is the same with the ratio of the rectangle AB, BC to the triangle ABC. because, as was shewn, GF is to FK, as AB, BC to the parallelogram AC; and FK is to its half, as AC is to its half which is the triangle ABC; therefore, ex aequali, GF is to the half of FK, as AB, BC rectangle is to the triangle ABC.

PROP. LXIII.

56.

IF two parallelograms be equiangular, as a side of the first to a side of the second, so is the other side of the second to the straight line to which the other side of the first has the same ratio which the first parallelogram has to the second. and consequently if the ratio of the first parallelogram to the second be given, the ratio of the other side of the first to that straight line is

given; and if the ratio of the other side of the first to that straight line be given, the ratio of the first parallelogram to the second is given.

Let AC, DF be two equiangular parallelograms, as BC a side of the first is to EF a side of the second, so is DE the other side of the second to the straight line to which AB the other side of the first has the same ratio which AC has to DF.

Produce the straight line AB, and make as BC to EF, so DE to BG, and complete the parallelogram BGHC. therefore because BC, or GH is to EF, as DE to BG, the sides about the equal angles BGH, DEF are reciprocally proportional; wherefore the parallelogram BH is equal to DF. and AB is to BG, as the parallelogram AC is to BH, that is to DF. as therefore BC is to EF, so is DE to BG which is the straight line to which AB the same ratio that AC has to DF.

And if the ratio of the parallelogram AC to DF be given, then the ratio of the straight line AB to BG is given; and if the ratio of AB to the straight line BG be given, the ratio of the parallelogram AC to DF is given.

PROP. LXIV.

74. 73.

IF two parallelograms have unequal, but given, angles, and if as a side of the first to a side of the second, so the other side of the second be made to a certain straight line; if the ratio of the first parallelogram to the second be given, the ratio of the other side of the first to that straight line shall be given. and if the ratio of the other side of the first to that straight line be given, the ratio of the first parallelogram to the second shall be given.

Let ABCD, EFGH be two parallelograms which have the unequal, but given, angles ABC, EFG; and as BC to FG, so make EF to the straight line

M. if the ratio of the parallelogram AC to EG be given, the ratio of AB to M is given.

At the point B of the straight line BC make the angle CBK equal to the angle EFG, and complete the parallelogram KBCL. and because the ratio of AC to EG is given, and that AC is equal to the parallelogram KC, therefore the ratio of KC to EG is given; and KC, EG are equiangular; therefore as BC to FG, so is EF to the straight line to which KB has a given ratio, viz. the same which the parallelogram KC has to EG. but as BC to FG, so is EF the straight line M; therefore KB has a given ratio to M

and the ratio of AB to BK is given, because the triangle ABK is given in species. therefore the ratio of AB to M is given.

And if the ratio of AB to M be given, the ratio of the parallelogram AC to EG is given. for since the ratio of KB to BA is given, as also the ratio of AB to M, the ratio of KB to M is given. and because the parallelograms KC, EG are equiangular, as BC to FG, so is EF to the straight line to which KB has the same ratio which the parallelogram KC has to EG. but as BC to FG, so is EF to M. therefore KB is to M, as the parallelogram KC is to EG. and the ratio of KB to M is given, therefore the ratio of the parallelogram KC, that is of AC to EG is given.

75.

COR. And if two triangles ABC, EFG have two equal angles, or two unequal, but given, angles ABC, EFG, and if as BC a side of the first to FG a side of the second, so the other side of the second EF be made to a straight line M; if the ratio of the triangles be given, the ratio of the other side of the first to the straight line M is given.

Complete the parallelograms ABCD, EFGH; and because the ratio of the triangle ABC to the triangle EFG is given, the ratio of the parallelogram AC to EG is given, because the parallelograms are double of the triangles. and because BC is to FG, as EF to M, the ratio of AB to M is given by the 63. Dat. if the angles ABC, EFG are equal; but if they be unequal, but given angles, the ratio of AB to M is given by this Proposition.

And if the ratio of AB to M be given, the ratio of the parallelogram AC to EG is given by the same Propositions; and therefore the ratio of the triangle ABC to EFG is given.

PROP. LXV.

68.

IF two equiangular parallelograms have a given ratio to one another, and if one side has to one side a given ratio; the other side shall also have to the other side a given ratio.

Let the two equiangular parallelograms AB, CD have a given ratio to one another, and let the side EB have a given ratio to the side FD; the other side AE has also a given ratio to the other side CF.

Because the two equiangular parallelograms AB, CD have a given ratio to one another; as EB a side of the first is to FD a side of the second, so is FC the other side of the second to the straight line to which AE the other side of the first has the same given ratio which the first parallelogram AB has to the other CD. let this straight line be EG; therefore the ratio of AE to EG is given. and EB is to FD, as FC to EG, therefore the ratio of FC to EG is given, because the ratio of EB to FD is given. and because the ratio of AE to EG, as also the ratio of FC to EG is given; the ratio of AE to CF is given.

The ratio of AE to CF may be found thus; take a straight line H given in magnitude, and because the ratio of the parallelogram AB to CD is given, make the ratio of H to K the same with it. and because the ratio of FD to EB is given, make the ratio of K to L the same, the ratio of AE to CF is the same with the ratio of H to L. make as EB to FD, so FC to EG, therefore, by inversion, as FD to EB, so is EG to FC. and as AE to EG, so is (the parallelogram AB to CD, and so is) H to K; but as EG to FC, so is (FD to EB, and so is) K to L; therefore, ex aequali, as AE to FC, so is H to L.

PROP. LXVI.

69.

IF two parallelograms have unequal, but given, angles, and a given ratio to one another; if one side has to one side a given ratio, the other side has also a given ratio to the other side.

Let the two parallelograms ABCD, EFGH which have the given unequal angles ABC, EFG, have a given ratio to one another, and let the ratio of BC to FG be given; the ratio also of AB to EF is given.

At the point B of the straight line BC make the angle CBK equal to the given angle EFG, and complete the parallelogram BKLC. and because each of the angles BAK, AKB is given, the triangle ABK is given in species; therefore the ratio of AB to BK is given. and because, by the hypothesis, the ratio of the parallelogram AC

to EG is given, and that AC is equal to BL; therefore the ratio of BL to EG is given. and because BL is equiangular to EG, and by the hypothesis, the ratio of BC to FG is given; therefore the ratio of KB to EF is given. and the ratio of KB to BA is given; the ratio therefore of AB to EF is given.

The ratio of AB to EF may be found thus; take the straight line MN given in position and magnitude; and make the angle NMO equal to the given angle BAK, and the angle MNO equal to the given angle EFG or AKB. and because the parallelogram BL is equiangular to EG, and has a given ratio to it, and that the ratio of BC to FG is given; find by the 65. Dat. the ratio of KB to EF; and make the ratio of NO to OP the same with it. then the ratio of AB to EF is the same with the ratio of MO to OP. for since the triangle ABK is equiangular to MON, as AB to BK, so is MO to ON; and as KB to EF, so is NO to OP; therefore, ex aequali, as AB to EF, so is MO to OP.

PROP. LXVII.

70.

IF the sides of two equiangular parallelograms have given ratios to one another; the parallelograms shall have a given ratio to one another.

Let ABCD, EFGH be two equiangular parallelograms, and let the ratio of AB to EF, as also the ratio of BC to FG be given; the ratio of the parallelogram AC to EG is given.

Take a straight line K given in magnitude, and because the ratio of AB to EF is given, make ratio of K to L the same with it; therefore L is given and because the ratio of BC to FG is given, make the ratio of L to M the same. therefore M is given; and K is given, wherefore the ratio of K to M is given. but the parallelogram AC is to the parallelogram EG, as the straight line K to the straight line M, as is demonstrated in the 23. Prop. of B. 6. Elem. therefore the ratio of AC to EG is given.

From this it is plain how the ratio of two equiangular parallelograms may be found when the ratios of their sides are given.

PROP. LXVIII.

70.

IF the sides of two parallelograms which have unequal, but given, angles, have given ratios to one another; the parallelograms shall have a given ratio to one another.

Let two parallelograms ABCD, EFGH which have the given unequal angles ABC, EFG have the ratios of their sides, viz. of AB to EF, and of BC to FG given; the ratio of the parallelogram AC to EG is given.

At the point B of the straight line BC make the angle CBK equal to the given angle EFG, and complete the parallelogram KBCL. and because each of the angles BAK, BKA is given, the triangle ABK is given in species. therefore the ratio of AB to BK is given; and the ratio of AB to EF is given, wherefore the ratio of BK to EF is given. and the ratio of BC to FG is given; and the angle KDC is equal to the angle EFG; therefore the ratio of the parallelogram KC to EG is given. but KC is equal to AC; therefore the ratio of AC to EG is given.

The ratio of the parallelogram AC to EG may be found thus; take the straight line MN given in position and magnitude, and make the angle MNO equal to the given angle KAB, and the angle NMO equal to the given angle AKB or FEH. and because the ratio of AB to EF is given, make the ratio of NO to P the same; also make the ratio of P to Q the same with the given ratio of BC to FG. the parallelogram AC is to EG, as MO to Q.

Because the angle KAB is equal to the angle MNO, and the angle AKB equal to the angle NMO; the triangle AKB is equiangular to NMO. therefore as KB to BA, so is MO to ON; and as BA to EF, so is NO to P; wherefore, ex aequali, as KB to EF, so is MO to P. and BC is to FG, as P to Q, and the

parallelograms KC, EG are equiangular; therefore, as was shewn in Pro. the parallelogram KC, that is AC, is to EG, as MO to Q.

COR. 1. If two triangles ABC, DEF have two equal angles, 71. or two unequal, but given angles ABC, DEF, and if the ratios of the sides about these angles, viz. the ratios of AB to DE, and of BC to EF be given; the triangles shall have a given ratio to one another.

Complete the parallelograms BG, EH; the ratio of BG to EH is given; and therefore the triangles which are the halves of them have a given ratio to one another.

COR. 2. If the safes BC, EF of two triangles ABC, DEF have a given ratio to one another, and if also the straight lines AG, DH which are drawn to the bases from the opposite angles, either in equal angles, or unequal, but given, angles AGC, DHF have a given ratio to one another; the triangles shall have a given ratio to one another.

Draw BK, EL parallel to AG, DH, and complete the parallelograms KC, LF. and because the angles AGC, DHF, or their equals the angles KBC, LEF are either equal, or unequal, but given; and that the ratio of AG to DH, that is of KB to LE is given, as also the ratio of BC to EF; therefore the ratio of the parallelogram KC to LF is given. wherefore also the ratio of the triangle ABC to DEF is given.

PROP. LXIX.

61.

IF a parallelogram which has a given angle be applied to one side of a rectilineal figure given in species; if the figure have a given ratio to the

parallelogram, the parallelogram is given in species.

Let ABCD be a rectilineal figure given in species, and to one side of it AB let the parallelogram ABEF having the given angle ABE be applied; if the figure ABCD has a given ratio to the parallelogram BF, the parallelogram BF is given in species.

Thro' the point A draw AG parallel to BC, and thro' the point C draw CG parallel to AB, and produce G A, CB to the points H,

K. because the angle ABC is given, and the ratio of AB to BC is given, the figure ABCD being given in species; therefore the parallelogram BG is given in species. and because upon the same straight line AB the two rectilineal figures BD, BG given in species are described, the ratio of BD to BG is given. and, by hypothesis, the ratio of BD to the parallelogram BF is given; wherefore the ratio of BF, that is of the parallelogram BH, to BG is given, and therefore the ratio of the straight line KB to BC is given. and the ratio of BC to BA is given, wherefore the ratio of KB to BA is given. and because the angle ABC is given, the adjacent angle ABK is given; and the angle ABE is given, therefore the remaining angle KBE is given. the angle EKB is also given, because it is equal to the angle ABK; therefore the triangle BKE is given in species, and consequently the ratio of EB to BK is given. and the ratio of KB to BA is given, wherefore the ratio of EB to BA is given. and the angle ABE is given, therefore the parallelogram BF is given in species.

A parallelogram similar to BF may be found thus; take a straight line LM given in position and magnitude; and because the angles ABK, ABE are given, make the angle NLM equal to ABK, and the angle NLO equal to ABE and because the ratio of BF to BD is given, make the ratio of LM to P the same with it; and because the ratio of the figure BD to BG is given, find this ratio by the 53. Dat. and make the ratio of P to Q the same. also, because the

ratio of CB to BA is given, make the ratio of Q to R the same. and take LN equal to R, thro' the point M draw OM parallel to LN, and complete the parallelogram NLOS; then this is similar to the parallelogram BF.

Because the angle ABK is equal to NLM, and the angle ABE to NLO; the angle KBE is equal to MLO. and the angles BKE, LMO are equal, because the angle ABK is equal to NLM. therefore the triangles BKE, LMO are equiangular to one another, wherefore as BE to BK, so is LO to LM. and because as the figure BF to BD, so is the straight line LM to P; and as BD to BG, so is P to Q; ex aequali, as BF, that is BH, to BG, so is LM to Q. but BH is to BG, as KB to BC; as therefore KB to BC, so is LM to Q. and because BE is to BK, as LO to LM; and as BK to BC, so is LM to Q; and as BC to BA, so Q was made to R; therefore, ex aequali, as BE to BA, so is LO to R, that is to LN. and the angles ABE, NLO are equal; therefore the parallelogram BF is similar to LS.

PROP. LXX.

62. 78.

IF two straight lines have a given ratio to one another, and upon one of them be described a rectilineal figure given in species, and upon the other a parallelogram having a given angle; if the figure have a given ratio to the parallelogram, the parallelogram is given in species.

Let the two straight lines AB, CD have a given ratio to one another, and upon AB let the figure AEB given in species be described, and upon CD the parallelogram DF having the given angle FCD; if the ratio of AEB to DF be given, the parallelogram DF is given in species.

Upon the straight line AB conceive the parallelogram AG to be described similar and similarly placed to FD. and because the ratio of AB to CD is

given, and upon them are described the similar rectilineal figures AG, FD; the ratio of AG to FD is given. and the ratio of FD to AEB is given; therefore the ratio of AEB to AG is given; and the angle ABG is given, because it is equal to the angle FCD. because therefore the parallelogram AG which has a given angle ABG is applied to a side AB of the figure AEB given in species, and the ratio of AEB to AG is given, the parallelogram AG is given in species. but FD is similar to AG; therefore FD is given in species.

A parallelogram similar to FD may be found thus; take a straight line H given in magnitude; and because the ratio of the figure AEB to FD is given, make the ratio of II to K the same with it. also because the ratio of the straight line CD to AB is given, find by the 54. Dat. the ratio which the figure FD described upon CD has to the figure AG described upon AB similar to FD; and make the ratio of K to L the same with this ratio, and because the ratios of H to K, and of K to L are given, the ratio of H to L is given

because therefore as AEB to FD, so is H to K; and as FD to AG, so is K to L; ex aequali, as AEB to AG, so is H to L; therefore the ratio of AEB to AG is given. and the figure AEB is given in species, and to its side AB the parallelogram AG is applied in the given angle ABG, therefore by the 69. Dat. a parallelogram may be found similar to AG. let this be the parallelogram MN; MN also is similar to FD. for, by the construction, MN is similar to AG, and AG is similar to FD; therefore the parallelogram FD is similar to MN.

PROP. LXXI.

81.

IF the extremes of three proportional straight lines have given ratios to the extremes of other three proportional straight lines; the means shall also have a

given ratio to one another. and if one extreme has a given ratio to one extreme, and the mean to the mean; likewise the other extreme shall have to the other a given ratio.

Let A, B, C be three proportional straight lines, and D, E, F three other; and let the ratios of A to D, and of C to F be given. then the ratio of B to E is also given.

Because the ratio of A to D, as also of C to F is given, the ratio of the rectangle A, C to the rectangle D, F is given. but the square of B is equal to the rectangle A, C; and the square of E to the rectangle D, F. therefore the ratio of the square of B to the square of E is given; wherefore also the ratio of the straight line B to E is given.

Next, let the ratio of A to D, and of B to E be given; then the ratio of C to F is also given.

Because the ratio of B to E is given, the ratio of the square of B to the square of E is given. therefore the ratio of the rectangle A, C to the rectangle D, F is given. and the ratio of the side A to the side D is given; therefore the ratio of the other side C to the other F is given.

COR. And if the extremes of four proportionals have to the extremes of four other proportionals given ratios, and one of the means a given ratio to one of the means; the other mean shall have a given ratio to the other mean. as may be shewn in the same manner as in the foregoing Proposition.

PROP. LXXII.

82.

IF four straight lines be proportionals; as the first is to the straight line to which the second has a given ratio; so is the third to the straight line to which

the fourth has a given ratio.

Let A, B, C, D be four proportional straight lines, viz. as A to B, so C to D; as A is to the straight line to which B has a given ratio, so is C to the straight line to which D has a given ratio.

Let E be the straight line to which B has a given ratio, and as B to E, so make D to F. the ratio of B to E is given, and therefore the ratio of D to F. and because as A to B, so is C to D; and as B to E, so D to F; therefore, ex aequali, as A to E, so is C to F. and E is the straight line to which B has a given ratio, and F that to which D has a given ratio; therefore as A is to the straight line to which B has a given ratio, so is C to that to which D has a given ratio.

PROP. LXXIII.

83.

IF four straight lines be proportionals; as the first is to the straight line to which the second has a given ratio, so is the straight line to which the third has a given ratio to the fourth.

Let the straight line A be to B, as C to D; as A to the straight line to which B has a given ratio, so is the straight line to which C has a given ratio to D.

Let E be the straight line to which B has a given ratio, and as B to E, so make F to C; because the ratio of B to E is given, the ratio of C to F is given. and because A is to B, as C to D; and as B to E, so F to C; therefore, ex aequali in proportione perturbata, A is to E, as F to D; that is A is to E to which B has a given ratio, as F, to which C has a given ratio, is to D.

PROP. LXXIV.

IF a triangle has a given obtuse angle; the excess of the square of the side which subtends the obtuse angle above the squares of the sides which contain it, shall have a given ratio to the triangle.

Let the triangle ABC have a given obtuse angle ABC; and produce the straight line CB, and from the point A draw AD perpendicular to BC. the excess of the square of AC above the squares of AB, BC, that is the double of the rectangle contained by DB, BC, has a given ratio to the triangle ABC.

Because the angle ABC is given, the angle ABD is also given; and the angle ADB is given, wherefore the triangle ABD is given in species; and therefore the ratio of AD to DB is given. and as AD to DB, so is the rectangle AD, BC to the rectangle DB, BC; wherefore the ratio of the rectangle AD, BC to the rectangle DB, BC is given, as also the ratio of twice the rectangle DB, BC to the rectangle AD, BC. but the ratio or the rectangle AD, BC to the triangle ABC is given, because it is double of the triangle; therefore the ratio of twice the rectangle DB, BC to the triangle ABC is given. and twice the rectangle DB, BC is the excess of the square of AC above the squares of AB, BC. therefore this excess has a given ratio to the triangle ABC.

And the ratio of this excess to the triangle ABC may be found thus; take a straight line EF given in position and magnitude; and because the angle ABC is given, at the point F of the straight line EF make the angle EFG equal to the angle ABC; produce GF, and draw EH perpendicular to FG. then the ratio of the excess of the square of AC above the squares of AB, BC to the triangle ABC is the same with the ratio of quadruple the straight line HF to HE.

Because the angle ABD is equal to the angle EFH, and the angle ADB to EHF, each being a right angle; the triangle ADB is equiangular to EHF. therefore as BD to DA, so FH to HE; and as quadruple of BD to DA, so is quadruple of FH to HE. but as twice BD is to DA, so is twice the rectangle

DB, BC to the rectangle AD, BC; and as DA to the half of it, so is the rectangle AD, BC to its half the triangle ABC; therefore, ex ae quali, as twice BD is to the half of DA, that is, as quadruple of BD is to DA, that is, as quadruple of FH to HE, so is twice the rectangle DB, BC to the triangle ABC.

PROP. LXXV.

65.

IF a triangle has a given acute angle; the space by which the square of the side subtending the acute angle is less than the squares of the sides which contain it, shall have a given ratio to the triangle.

Let the triangle ABC have a given acute angle ABC, and draw AD perpendicular to BC; the space by which the square of AC is less than the squares of AB, BC, that is the double of the rectangle contained by CB, BD, has a given ratio to the triangle ABC.

Because the angles ABD, ADB are each of them given, the triangle ABD is given in species; and therefore the ratio of BD to DA is given. and as BD to DA, so is the rectangle CB, BD to the rectangle CB, AD; therefore the ratio of these rectangles is given, as also the ratio of twice the rectangle CB, BD to the rectangle CB, AD. but the rectangle CB, AD has a given ratio to its half the triangle ABC, therefore the ratio of twice the rectangle CB, BD to the triangle ABC is given. and twice the rectangle CB, BD is the space b which the square of AC is less than the squares of AB, BC; therefore the ratio of this space to the triangle ABC is given. and the ratio may be found as in the preceding Proportion.

LEMMA.

IF from the vertex A of an Isosceles triangle ABC, any straight line AD be drawn to the base BC; the square of the side AB is equal to the rectangle BD, DC of the segments of the base together with the square of AD. but if AD be drawn to the base produced, the square of AD is equal to the rectangle BD, DC together with the square of AB.

CAS. 1. Bisect the base BC in E, and join AE which will be perpendicular to BC; wherefore the square of AB is equal to the squares of AE, EB. but the square of EB is equal to the rectangle BD, DC together with the square of DE. therefore the square of AB is equal to the squares of AE, ED, that is to the square of AD, together with the rectangle BD, DC. the other case is shewn in the same way by 6. 2. Elem.

PROP. LXXVI.

67.

IF a triangle have a given angle, the excess of the square of the straight line which is equal to the two sides that contain the given angle, above the square of the third side, shall have a given ratio to the triangle.

Let the triangle ABC have the given angle BAC, the excess of the straight line which is equal to BA, AC together above the square of BC, shall have a given ratio to the triangle ABC.

Produce BA, and take AD equal to AC, join DC and produce it to E, and thro' the point B draw BE parallel to AC; join AE, and draw AF perpendicular to DC. and because AD is equal to AC, BD is equal to BE; and BC is drawn from the vertex B of the Isosceles triangle DBE, therefore, by the Lemma, the square of BD, that is of BA and AC together, is equal to the

rectangle DC, CE together with the square of BC; and therefore the square of BA, AC together, that is of BD is greater than the square of BC by the rectangle DC, CE; and this rectangle has a given ratio to the triangle ABC. because the angle BAC is given, the adjacent angle CAD is given; and each of the angles ADC, DCA is given, for each of them is the half of the given angle BAC; therefore the triangle ADC is given in species; and AF is drawn from its vertex to the base in a given angle, wherefore the ratio of AF to the base CD is given. and as CD to AF, so is the rectangle DC, CE to the rectangle AF, CE; and the ratio of the rectangle AF, CE to its half the triangle ACE is given; therefore the ratio of the rectangle DC, CE to the triangle ACE, that is to the triangle ABC is given. and the rectangle DC, CE is the excess of the square of BA, AC together above the square of BC; therefore the ratio of this excess to the triangle ABC is given.

The ratio which the rectangle DC, CE has to the triangle ABC is found thus. take the straight line GH given in position and magnitude, and at the point G in GH make the angle HGK equal to the given angle CAD, and take GK equal to GH, join KH, and draw GL perpendicular to it. then the ratio of HK to the half of GL is the same with the ratio of the rectangle DC, CE to the triangle ABC. because the angles HGK, DAC at the vertices of the Isosceles triangles GHK, ADC are equal to one another, these triangles are similar, and because GL, AF are perpendicular to the bases HK, DC, as HK to GL, so is (DC to AF, and so is) the rectangle DC, CE to the rectangle AF, CE; but as GL to its half, so is the rectangle AF, CE to its half which is the triangle ACE, or the triangle ABC; therefore, ex aequali, HK is to the half of the straight line GL, as the rectangle DC, CE is to the triangle ABC.

COR. And if a triangle have a given angle, the space by which the square of the straight line which is the difference of the sides which contain the given angle is less than the square of the third side, shall have a given ratio to

the triangle. this is demonstrated the same way as the preceding Proposition, by help of the second case of the Lemma.

PROP. LXXVII.

I.

IF the perpendicular drawn from a given angle of a triangle to the opposite side, or base, has a given ratio to the base; the triangle is given in species.

Let the triangle ABC have the given angle BAC, and let the perpendicular AD drawn to the base BC, have a given ratio to it; the triangle ABC is given in species.

IF ABC be an Isosceles triangle, it is evident that if any one of its angles be given, the rest are also given; and therefore the triangle

is given in species, without the consideration of the ratio of the perpendicular to the base, which in this case is given by Pro. But when ABC is not an Isosceles triangle, take any straight line EF given in position and magnitude, and upon it describe the segment of a circle EGF containing an angle equal to the given angle BAC; draw GH bisecting EF at right angles, and join EG, GF. then since the angle EGF is equal to the angle BAC, and that EGF is an Isosceles triangle and ABC is not, the angle FEG is not equal to the angle CBA. draw EL making the angle FEL equal to the angle CBA, join FL, and draw LM perpendicular to EF. then because the triangles ELF, BAC are equiangular, as also are the triangles MLE, DAB, as ML to LE, so is DA to AB; and as LE to EF, so is AD to BC; wherefore, ex aequali, as LM to EF, so is AD to BC. and because the ratio of AD to BC is given, therefore the ratio of LM to EF is given; and EF is given, wherefore LM also is given complete the parallelogram LMFK, and because LM is given, FK is given in

magnitude; it is also given in position, and the point F is given, and consequently the point K; and because thro' K the straight line KL is drawn parallel to EF which is given in position, therefore KL is given in position; and the

circumference ELF is given in position, therefore the point L is given. and because the points L, E, F are given, the straight lines LE, EF, FL are given in magnitude; therefore, the triangle LEF is given in species. and the triangle ABC is similar to LEF, wherefore also ABC is given in species.

Because LM is less than GH, the ratio of LM to EF, that is the given ratio of AD to BC must be less than the ratio of GH to EF which the straight line, in a segment of a circle containing an angle equal to the given angle, that bisects the base of the segment at right angles, has unto the base.

COR. 1. If two triangles ABC, LEF have one angle BAC equal to one angle ELF, and if the perpendicular AD be to the base BC, as the perpendicular LM to the base EF; the triangles ABC, LEF are similar.

Describe the circle EGF about the triangle ELF, and draw LN parallel to EF, join EN, NF, and draw NO perpendicular to EF. because the angles ENF, ELF are equal, and that the angle EFN is equal to the alternate angle FNL, that is to the angle FEL in the same segment, therefore the triangle NEF is similar to LEF. and in the segment EGF there can be no other triangle upon the base EF which has the ratio of its perpendicular to that base the same with the ratio of LM or NO to EF, because the perpendicular must be greater or less than LM or NO. but, as has been shewn in the preceding demonstration, a triangle similar to ABC can be described in the segment EGF upon the base EF, and the ratio of its perpendicular to the base is the same, as was there shewn, with the ratio of AD to BC, that is of LM to EF. therefore that triangle must be either LEF, or NEF, which therefore are similar to the triangle ABC.

COR. 2. If a triangle ABC has a given angle BAC, and if the straight line AR drawn from the given angle to the opposite side BC, in a given angle ARC, has a given ratio to BC; the triangle ABC is given in species.

Draw AD perpendicular to BC; therefore the triangle ARD is given in species; wherefore the ratio of AD to AR is given; and the ratio of AR to BC is given, and consequently the ratio of AD to BC is given; and the triangle ABC is therefore given in species.

COR. 3. If two triangles ABC, LEF have one angle BAC equal to one angle ELF, and if straight lines drawn from these angles to the bases, making with them given and equal angles, have the same ratio to the bases, each to each; then the triangles are similar. for, having drawn perpendiculars to the bases from the equal angles, as one perpendicular is to its base, so is the other to its base. wherefore, by Cor. 1. the triangles are similar.

A triangle similar to ABC may be found thus; having described the segment EGF and drawn the straight line GH as was directed in the Proposition, find FK which has to EF the given ratio of AD to BC; and place FK at right angles to EF from the point F. then because, as has been shewn, the ratio of AD to BC, that is of FK to EF, must be less than the ratio of GH to EF; therefore FK is less than GH; and consequently the parallel to EF drawn through the point K must meet the circumference of the segment in two points. let L be either of them, and join EL, LF, and draw LM perpendicular to EF. then because the angle BAC is equal to the angle ELF, and that AD is to BC, as KF, that is LM to EF, the triangle ABC is similar to triangle LEF. by Cor. 1.

PROP. LXXVIII.

IF a triangle have one angle given, and if the ratio of the rectangle of the sides which contain the given angle to the square of the third side be given; the triangle is given in species.

Let the triangle ABC have the given angle BAC, and let the ratio of the rectangle BA, AC to the square of BC be given; the triangle ABC given in species.

From the point A draw AD perpendicular to BC; the rectangle AD, BC has a given ratio to its half the triangle ABC. and because the angle BAC is given, the ratio of the triangle ABC to the rectangle BA, AC is given; and, by the hypothesis, the ratio of the rectangle BA, AC to the square of BC is given. therefore the ratio of the rectangle AD, BC to the square of BC, that is the ratio of the straight line AD to BC is given. wherefore the triangle ABC is given in species.

A triangle similar to ABC may be found thus; take a straight line EF given in position and magnitude, and make the angle FEG equal to the given angle BAC, and draw FH perpendicular to EG, and BK perpendicular to AC; therefore the triangles ABK, EFH are similar. and the rectangle AD, BC, or the rectangle BK, AC, which is equal to it, is to the rectangle BA, AC, as the straight line BK to BA, that is as FH to FE; let the given ratio of the rectangle BA, AC to the square of BC be the same with the ratio of the straight line EF to FL; therefore, ex aequali, the radio of the rectangle AD, BC to the square of BC, that is the ratio of the straight line AD to BC, is the same with the ratio of HF to FL. and because AD is not greater than the straight line MN in the segment of the circle described about the triangle ABC, which bisects BC at right angles; the ratio of AD to BC, that is of HF to FL, must not be greater than the ratio of MN to BC. let it be so, and by the 77. Dat. find a triangle OPQ which has one of its angles POQ equal to the given angle BAC, and the

ratio of the perpendicular OR, drawn from that angle, to the base PQ the same with the ratio of HF to FL. then the triangle ABC is similar to OPQ. because, as has been shewn, the ratio of AD to BC is the same with the ratio of (HF to FL, that is, by the construction, with the ratio of) OR to PQ; and the angle BAC is equal to the angle POQ. therefore the triangle ABC is similar to the triangle POQ.

Otherwise,

Let the triangle ABC have the given angle BAC, and let the ratio of the rectangle BA, AC to the square of BC be given; the triangle ABC is given in species.

Because the angle BAC is given, the excess of the square of both the sides BA, AC together above the square of the third side BC has a given ratio to the triangle ABC. let the figure D be equal to this excess; therefore the ratio of D to the triangle ABC is given; and the ratio of the triangle ABC to the rectangle BA, AC is given, because BAC is a given angle; and the rectangle BA, AC has a given ratio to the square of BC; wherefore the ratio of D to the square of BC is given. and, by composition, the ratio of the space D together with the square of BC to the square of BC is given. but D together with the square of BC is equal to the square of both BA and AC together; therefore the ratio of the square of BA, AC together to the square of BC is given; and the ratio of BA, AC together to BC is therefore given. and the angle BAC is given, wherefore the triangle ABC is given in species.

The composition of this which depends upon those of the 76. and 48. Propositions is more complex than the preceding composition which depends upon that of Pro. which is easy.

PROP. LXXIX.

K.

IF a triangle have a given angle, and if the straight line drawn from that angle to the base, making a given angle with it, divides the base into segments which have a given ratio to one another; the triangle is given in species.

Let the triangle ABC have the given angle BAC, and let the straight line AD drawn to the base BC making the given angle ADB, divide BC into the segments BD, DC which have a given ratio to one another; the triangle ABC is given in species.

Describe the circle BAC about the triangle, and from its center E draw EA, EB, EC, ED. because the angle BAC is given, the angle BEC at the center, which is the double of it, is given. and the ratio of BE to EC is given, because they are equal to one another; therefore the triangle BEC is given in species, and the ratio of EB to BC is given. also the ratio of CB to BD is given, because the ratio of BD to DC is given; therefore the ratio of EB to BD is given. and the angle EBC is given, wherefore the triangle EBD is given in species, and the ratio of EB, that is of EA to ED is therefore given. and the angle EDA is given, because each of the angles BDE, BDA is given. therefore the triangle AED is given in species, and the angle AED given; also the angle DEC is given, because each of the angles BED, BEC is given; therefore the angle AEC is given. and the ratio of EA to EC, which are equal, is given; and the triangle AEC is therefore given in species, and the angle ECA given. and the angle ECB is given, wherefore the angle ACB is given. and the angle BAC is also given; therefore the triangle ABC is given in species.

A triangle similar to ABC may be found, by taking a straight line given in position and magnitude, and dividing it in the given ratio which the segments

BD, DC are required to have to one another; then if upon that straight line a segment of a circle be described containing an angle equal to the given angle BAC, and a straight line be drawn from the point of division in an angle equal to the given angle ADB, and from the point where it meets the circumference, straight lines be drawn to the extremity of the first line, these together with the first line shall contain a triangle similar to ABC, as may easily be shewn.

The Demonstration may be also made in the manner of that of the 77. Prop. and that of the 77. may be made in the manner of this.

PROP. LXXX.

L.

IF the sides about an angle of a triangle have a given ratio to one another, and if the perpendicular drawn from that angle to the base has a given ratio to the base; the triangle is given in species.

Let the sides BA, AC, about the angle BAC of the triangle ABC have a given ratio to one another, and let the perpendicular AD have a given ratio to the base BC; the triangle ABC is given in species.

First, let the sides AB, AC be equal to one another, therefore the perpendicular AD bisects the base BC. and the ratio of AD to BC, and therefore to its half DB is given; and the angle ADB is given. wherefore the triangle* ABD and consequently the triangle ABC is given in species.

But let the sides be unequal, and BA be greater than AC; and make the angle CAE equal to the angle ABC. because the angle AEB is common to the triangles AEB, CEA, they are similar; therefore as AB to BE, so is CA to AE,

and, by permutation, as BA to AC, so is BE to EA, and so is EA to EC. and the ratio of BA to AC is given, therefore the ratio of BE to EA, and the ratio of E A to EC, as also the ratio of BE to EC is given; wherefore the ratio of EB to BC is given. and the ratio of AD to BC is given by the Hypothesis, therefore the ratio of AD to BE is given; and the ratio of BE to EA was shewn to be given; wherefore the ratio of AD to AE is given. and ADE is a right angle, therefore the triangle ADE is given in species, and the angle AEB given; the ratio of BE to EA is likewise given, therefore the triangle ABE is given in species, and consequently the angle EAB, as also the angle ABE, that is the angle CAE is given; therefore the angle BAC is given, and the angle ABC being also given, the triangle ABC is given in species.

How to find a triangle which shall have the things which are mentioned to be given in the Proposition, is evident in the first case. and to find it the more easily in the other case, it is to be observed that if the straight line EF equal to EA be placed in EB towards B, the point F divides the base BC into the segments BF, FC which have to one another the ratio of the sides BA, AC. because BE, EA, or EF, and EC were shewn to be proportionals, therefore* BF is to FC, as BE to EF, or EA, that is as BA to AC. and AE cannot be less than the altitude of the triangle ABC, but it may be equal to it; which if it be, the triangle, in this case, as also the ratio of the sides, may be thus found, having given the ratio of the perpendicular to the base. take the straight line GH given in position and magnitude, for the base of the triangle to be found; and let the given ratio of the perpendicular to the base be that of the straight line K to GH, that is, let K be equal to the perpendicular; and suppose GLH to be the triangle which is to be found. therefore having made the angle HLM equal to LGH, it is required that LM be perpendicular to GM and equal to K. and because GM. ML, MH are proportionals, as was shewn of BE, EA, EC, the rectangle GMH is equal to the square of ML. add the common square of

NH, (having bisected GH in N) and the square of NM is equal to the squares of the given straight lines NH and ML, or K. therefore the square of NM, and its side NM, is given, as also the point M, viz. by taking the straight line NM the square of which is equal to the squares of NH, ML. draw ML equal to K, at right angles to GM. and because ML is given in position and magnitude, therefore the point L is given; join LG, LH, then the triangle LGH is that which was to be found. for the square of NM is equal to the squares of NH and ML, and taking away the common square of NH, the rectangle GMH is equal to the square of ML; therefore as GM to ML, so is ML to MH, and the triangle LGM is therefore equiangular to HLM, and the angle HLM equal to the angle LGM, and the straight line LM, drawn from the vertex of the triangle making the angle HLM equal to LGH, is perpendicular to the base and equal to the given straight line K, as was required. and the ratio of the sides GL, LH is the same with the ratio of GM to ML, that is with the ratio of the straight line which is made up of GN the half of the given base and of NM the square of which is equal to the squares of GN and K, to the straight line K.

And whether this ratio of GM to ML is greater or less than the ratio of the sides of any other triangle upon the base GH, and of which the altitude is equal to the straight line K, that is, the vertex of which is in the parallel to GH drawn thro' the point L, may be thus found. Let OGH be any such triangle, and draw OP making the angle HOP equal to the angle OGH; therefore, as before, GP, PO, PH are proportionals. and PO cannot be equal to LM, because the rectangle GPH, would be equal to the rectangle GMH, which is impossible, for the point P cannot fall upon M, because O would then fall on L; nor can PO be less than LM, therefore it is greater; and consequently the rectangle GPH is greater than the rectangle GMH, and the straight line GP

greater than GM. therefore the ratio of GM to MH is greater than the ratio of GP to PH, and the ratio of the square of GM to the square of ML is therefore greater than the ratio of the square of GP to the square of PO, and the ratio of the straight line GM to ML, greater than the ratio of GP to PO. but as GM to ML, so is GL to LH; and as GP to PO, so is GO to OH; therefore the ratio of GL to LH is greater than the ratio of GO to OH; wherefore the ratio of GL to LH is the greatest of all others; and consequently the given ratio of the greater side to the less must not be greater than this ratio.

But if the ratio of the sides be not the same with this greatest ratio of GM to ML, it must necessarily be less than it. Let any less ratio be given, and the same things being supposed, viz. that GH is the base, and K equal to the altitude of the triangle, it may be found as follows. Divide GH in the point Q, so that the ratio of GQ to QH may be the same with the given radio of the sides; and as GQ to QH, so make GP to PQ, and so will PQ be to PH; wherefore the square of GP is to the square of PQ, as the straight line GP to PH. and because GM, ML, MH are proportionals, the square of GM is to the square of ML, as the straight line GM to MH. but the ratio of GQ to QH, that is the ratio of GP to PQ, is less than the ratio of GM to ML; and therefore the ratio of the square of GP to the square of PQ is less than the ratio of the square of GM to that of ML; and consequently the ratio of the straight line GP to PH is less than the ratio of GM to MH, and, by division, the ratio of GH to HP is less than that of GH to HM; wherefore k the straight line HP is greater than HM, and the rectangle GPH, that is the square of PQ. greater than the rectangle GMH, that is than the square of ML, and the straight line PQ is therefore greater than ML. draw LR parallel to GP, and from P draw PR at right angles to GP. because PQ is greater than ML, or PR, the circle described from the center P, at the distance PQ, must necessarily cut LR in two points; let these be O, S, and join OG, OH; SG, SH; each of the triangles

OGH, SGH have the things mentioned to be given in the Proposition. join OP, SP; and because as GP to PQ, or PO, so is PO to PH, the triangle OGP is equiangular to HOP; as, therefore, OG to GP, so is HO to OP, and, by permutation, as GO to OH, so is GP to PO, or PQ, and so is GQ to QH. therefore the triangle OGH has the ratio of its sides GO, OH the same with the given ratio of GQ to QH; and the perpendicular has to the base the given ratio of K to GH, because the perpendicular is equal to LM, or K. the like may be shewn in the same way of the triangle SGH.

This construction by which the triangle OGH is found, is shorter than that which would be deduced from the Demonstration of the Datum; by reason that the base GH is given in position and magnitude, which was not supposed in the Demonstration. the same thing is to be observed in the next Proposition.

PROP. LXXXI.

IF the sides about an angle of a triangle be unequal and have a given ratio to one another, and if the perpendicular from that angle to the base divides it into segments that have a given ratio to one another; the triangle is given in species.

Let ABC be a triangle the sides of which about the angle BAC are unequal, and have a given ratio to one another, and let the perpendicular AD to the base BC divide it into the segments BD, DC which have a given ratio to one another; the triangle ABC is given in species.

Let AB be greater than AC, and make the angle CAE equal to the angle ABC; and because the angle AEB is common to the triangles ABE, CAE, they are equiangular to one another. therefore as AB to BE, so is CA to AE, and, by permutation, as AB to AC, so BE to EA, and so is EA to EC. but the

ratio of BA to AC is given, therefore the ratio of BE to EA, as also the ratio of EA to EC is given; wherefore the ratio of BE to EC, as also the ratio of EC to CB is given. and the ratio of BC to CD is given, because the ratio of BD to DC is given; therefore the ratio of EC to CD is given, and consequently the ratio of DE to EC. and the ratio of EC to EA was shewn to be given, therefore the ratio of DE to EA is given. and ADE is a right angle, wherefore the triangle ADE is given in species, and the angle AED given. and the ratio of CE to EA is given, therefore the triangle AEC is given in species, and consequently the angle ACE is given, as also the adjacent angle ACB. in the same manner, because the ratio of BE to EA is given, the triangle BEA is given in species, and the angle ABE is therefore given. and the angle ACB is given; wherefore the triangle ABC is given in species.

But the ratio of the greater side BA to the other AC must be less than the ratio of the greater segment BD to DC. because the square of BA is to the square of AC, as the squares of BD, DA to the squares of DC, DA; and the squares of BD, DA have to the squares of DC, DA a less ratio than the square of BD has to the square of DC†, because the square of BD is greater than the square of DC; therefore the square of BA has to the square of AC a less ratio than the square of BD has to that of DC. and consequently the ratio of BA to AC is less than the ratio of BD to DC.

This being premised, a triangle which shall have the things mentioned to be given in the Proposition, and to which the triangle ABC is similar, may be found thus. take a straight line GH given in position and magnitude, and divide it in K so that the ratio of GK to KH may be the same with the given ratio of BA to AC; divide also GH in L so that the ratio of GL to LH may be the same with the given ratio of BD to DC, and draw LM at right angles to GH. and because the ratio of the sides of a triangle is less than the ratio of the segments of the base, as has been shewn, the ratio of GK to KH is less than

the ratio of GL to LH, wherefore the point L must fall betwixt K and H. also make as GK to KH, so GN to NK, and so shall NK be to NH. and from the center N, at the distance NK describe a circle, and let its circumference meet LM in O, and join OG, OH; then OGH is the triangle which was to be described. because GN is to NK, or NO, as NO to NH, the triangle OGN is equiangular to HON; therefore as OG to GN, so is HO to ON, and, by permutation, as GO to OH, so is GN to NO, or NK, that is as GK to KH, that is in the given ratio of the sides. and, by the construction, GL, LH have to one another the given ratio of the segments of the base.

PROP. LXXXII.

60.

IF a parallelogram given in species and magnitude be encreased, or diminished by a gnomon given in magnitude; the sides of the gnomon are given in magnitude.

First, let the parallelogram AB given in species and magnitude be encreased by the given gnomon ECBDFG; each of the straight lines CE, DF is given.

Because AB is given in species and magnitude, and that the gnomon ECBDFG is given, therefore the whole space AG is given in magnitude. but AG is also given in species, because it is similar to AB; therefore the sides of AG are given. each of the straight lines AE, AF is therefore given; and each of the straight lines CA, AD is given, therefore each of the remainders EC, DF is given.

Next, let the parallelogram AG given in species and magnitude be diminished by the given gnomon ECBDFG; each of the straight lines CE, DF

is given.

Because the parallelogram AG is given, as also its gnomon ECBDFG; the remaining space AB is given in magnitude. but it is also given in species; because it is similar to AG; therefore its sides CA, AD are given. and each of the straight lines EA, AF is given; therefore EC, DF are each of them given.

The gnomon and its sides CE, DF may be found thus in the first case. let H be the given space to which the gnomon must be made equal, and find a parallelogram similar to AB and equal to the figures AB and H together, and place its sides AE, AF from the point A, upon the straight lines AC, AD, and complete the parallelogram AG, which is about the same diameter with AB. because therefore AG is equal to both AB and H, take away the common part AB, the remaining gnomon ECBDFG is equal to the remaining figure H. therefore a gnomon equal to H, and its sides CE, DF are found. and in like manner they may be found in the other case, in which the given figure H must be less than the figure FE from which it is to be taken.

PROP. LXXXIII.

58.

IF a parallelogram equal to a given space be applied to a given straight line, deficient by a parallelogram given in species; the sides of the defect are given.

Let the parallelogram AC equal to a given space be applied to the given straight line AB, deficient by the parallelogram BDCL given in species; each of the straight lines CD, DB are given.

Bisect AB in E; therefore EB is given in magnitude. upon EB describe the parallelogram EF similar to DL and similarly placed; therefore EF is given in

species, and is about the same diameter with DL; let BCG be the diameter, and construct the figure. therefore because the figure EF given in species is described upon the given straight line EB, EF is given in magnitude. and the gnomon ELH is equal to the given figure AC, therefore since EF is diminished by the given gnomon ELH, the sides EK, FH of the gnomon are given. but EK is equal to DC, and FH to DB; wherefore CD, DB are each of them given.

This Demonstration is the Analysis of the Problem in the 28. Prop. of Book 6. the construction and Demonstration of which Proposition is the Composition of the Analysis. and because the given space AC or its equal the gnomon ELH is to be taken from the figure EF described upon the half of AB similar to BC, therefore AC must not be greater than EF, as is shewn in the 27. Prop. B. 6.

PROP. LXXXIV.

59.

IF a parallelogram equal to a given space be applied to a given straight line, exceeding by a parallelogram given in species; the sides of the excess are given.

Let the parallelogram AC equal to a given space be applied to the given straight line AB, exceeding by the parallelogram BDCL given in species; each of the straight lines CD, DB are given.

Bisect AB in E; therefore EB is given in magnitude. upon EB describe the parallelogram EF similar to LD, and similarly placed; therefore EF is given in species, and is about the same diameter

with LD. let CBG be the diameter, and construct the figure. therefore because the figure EF given in species is described upon the given straight line EB, EF is given in magnitude. and the gnomon ELH is and equal to the given figure AC; wherefore since EF is encreased by the given gnomon ELH, its sides EK, FH are given. but EK is equal to CD, and FH to BD; therefore CD, DB are each of them given.

This Demonstration is the Analysis of the Problem in the 29. Prop. Book 6. the construction and Demonstration of which is the Composition of the Analysis.

COR. If a parallelogram given in species be applied to a given straight line, exceeding by a parallelogram equal to a given space; the sides of the parallelogram are given.

Let the parallelogram ADCE given in species be applied to the given straight line AB exceeding by the parallelogram BDCG equal to a given space; the sides AD, DC of the parallelogram are given.

Draw the diameter DE of the parallelogram AC, and construct the figure. because the parallelogram AK is equal to BC which is given, therefore AK is given. and BK is similar to AC, therefore BK is given in species. and since the parallelogram AK given in magnitude is applied to the given straight line AB, exceeding by the parallelogram BK given in species, therefore, by this Proposition, BD, DK the sides of the excess are given. and the straight line AB is given, therefore the whole AD, as also DC to which it has a given ratio is given.

PROB.

To apply a parallelogram similar to a given one to a given straight line AB, exceeding by a parallelogram equal to a given space.

To the given straight line AB apply the parallelogram AK equal to the given space, exceeding by the parallelogram BK similar to the one given. draw DF the diameter of BK, and thro' the point A draw AE parallel to BF meeting DF produced in E, and complete the parallelogram AC.

The parallelogram BC is equal to AK, that is to the given space; and the parallelogram AC is similar to BK. therefore the parallelogram AC is applied to the straight line AB similar to the one given and exceeding by the parallelogram BC which is equal to the given space.

PROP. LXXXV.

34.

IF two straight lines contain a parallelogram given in magnitude, in a given angle; if the difference of the straight lines be given, they shall each of them be given.

Let AB, BC contain the parallelogram AC given in magnitude, in the given angle ABC, and let the excess of BC above AB be given; each of the straight lines AB, BC is given.

Let DC be the given excess of BC above BA, therefore the remainder BD is equal to BA. complete the parallelogram AD, and because AB is equal to BD, the ratio of AB to BD is given. and the angle ABD is given, therefore the parallelogram AD is given in species. and because the given parallelogram AC is applied to the given straight line DC, exceeding by the parallelogram AD given in species, the sides of the excess are given; therefore BD is given. and DC is given, wherefore the whole BC is given. and AB is given, therefore AB, BC are each of them given.

PROP. LXXXVI.

85.

IF two straight lines contain a parallelogram given in magnitude, in a given angle; if both of them together be given, they shall each of them be given.

Let the two straight lines AB, BC contain the parallelogram AC given in magnitude, in the given angle ABC, and let AB, BC together be given; each of the straight lines AB, BC is given.

Produce CB and make BD equal to BA, and complete the parallelogram ABDE. because DB is equal to BA, and the angle ABD given, because the adjacent angle ABC is given; the parallelogram AD is given in species. and because AB, BC together are given, and AB is equal to BD; therefore DC is given. and because the given parallelogram AC is applied to the given straight line DC, deficient by the parallelogram AD given in species, the sides AB, BD of the defect are given. and DC is given, wherefore the remainder BC is given; and each of the straight lines AB, BC is therefore given.

PROP. LXXXVII.

87.

IF two straight lines contain a parallelogram given in magnitude, in a given angle; if the excess of the square of the greater above the square of the lesser be given, each of the straight lines shall be given.

Let the two straight lines AB, BC contain the given parallelogram AC in the given angle ABC; if the excess of the square of BC above the square of BA be given; AB and BC are each of them given.

Let the given excess of the square of BC above the square of BA be the rectangle CB, BD; take this from the square of BC, the remainder, which is the rectangle BC, CD is equal to the square of AB. and because the angle ABC of the parallelogram AC is given, the ratio of the rectangle of the sides AB, BC to the parallelogram AC is given; and AC is given, therefore the rectangle AB, BC is given; and the rectangle CB, BD is given; therefore the ratio of the rectangle CB, BD to the rectangle AB, BC, that is the ratio of the straight line DB to BA is given; therefore the ratio of the square of DB to the square of BA is given. and the square of BA is equal to the rectangle BC, CD; wherefore the ratio of the rectangle BC, CD to the square of BD is given, as also the ratio of four times the rectangle BC, CD to the square of BD; and, by composition, the ratio of four times the rectangle BC, CD together with the square of BD to the square of BD is given. but four times the rectangle BC, CD together with the square of BD is equal to the square of the straight lines BC, CD taken together; therefore the ratio of the square of BC, CD together to the square of BD is given; wherefore the ratio of the straight line BC together with CD to BD is given. and, by composition, the ratio of BC together with CD and DB, that is the ratio of twice BC to BD is given; therefore the ratio of BC to BD is given, as also the ratio of the square of BC to the rectangle CB, BD. but the rectangle CB, BD is given, being the given excess of the squares of BC, BA; therefore the square of BC, and the straight line BC is given, and the ratio of BC to BD, as also of BD to BA has been shewn to be given; therefore the ratio of BC to BA is given; and BC is given, wherefore BA is given.

The preceding Demonstration is the Analysis of this Problem, viz.

A parallelogram AC which has a given angle ABC being given in magnitude, and the excess of the square of BC one of its sides above the square of the other BA being given; to find the sides. and

The Composition is as follows,

Let EFG be the given angle to which the angle ABC is required to be equal, and from any point E in EE draw EG perpendicular to FG; let the rectangle EG, GH be the given space to which the parallelogram AC is to be made equal; and the rectangle HG, GL be the given excess of the squares of BC, BA.

Take, in the straight line GE, GK equal to FE, and make GM double of GK; join ML, and in GL produced take LN equal to LM. bisect GN in O, and between GH, GO find a mean proportional BC. as OG to GL, so make CB to BD; and make the angle CBA equal to GFE, and as LG to GK, so make DB to BA; and complete the parallelogram AC. AC is equal to the rectangle EG, GH, and the excess of the squares of CB, BA is equal to the rectangle HG, GL.

Because as CB to BD, so is OG to GL, the square of CB is to the rectangle CB, BD, as the rectangle HG, GO to the rectangle HG, GL. and the square of CB is equal to the rectangle HG, GO, because GO, BC, GH are proportionals; therefore the rectangle CB, BD is equal to HG, GL. and because as CB. to BD, so is OG to GL, twice CB is to BD, as twice OG, that is GN, to GL; and, by division, as BC together with CD is to BD, so is NL, that is LM, to LG. therefore the square of BC together with CD is to the square of BD, as the square of ML to the square of LG. but the square of BC and CD together is equal to four times the rectangle BC, CD together with the square of BD; therefore four times the rectangle BC, CD together with the square of BD is to the square of BD, as the square of ML to the square of LG. and, by division, four times the rectangle BC, CD is to the square of BD, as the square of MG to the square of GL; wherefore the rectangle BC, CD is to the square of BD, as (the square of KG the half of MG to the square of GL, that

is as) the square of AB to the square of BD, because as LG to GK, so DB was made to BA. therefore the rectangle BC, CD is equal to the square of AB; to each of these add the rectangle CB, BD, and the square of BC becomes equal to the square of AB together with the rectangle CB, BD. therefore this rectangle, that is the given rectangle HG, GL is the excess of the squares of BC, AB. from the point A draw AP perpendicular to BC, and because the angle ABP is equal to the angle EFG, the triangle ABP is equiangular to EFG. and DB was made to BA, as LG to GK, therefore as the rectangle CB, BD to CB, BA, so is the rectangle HG, GL to HG, GK; and as the rectangle CB, BA to

AP, BC, so is (the straight line BA to AP, and so is FE or GK to EG, and so is) the rectangle HG, GK to HG, GE; therefore, ex aequali, as the rectangle CB, BD to AP, BC, so is the rectangle HG, GL to EG, GH. and the rectangle CB, BD is equal to HG, GL, therefore the rectangle AP, BC, that is the parallelogram AC is equal to the given rectangle EG, GH.

PROP. LXXXVIII.

N.

IF two straight lines contain a parallelogram given in magnitude, in a given angle; if the sum of the squares of its sides be given, the sides shall each of them be given.

Let the two straight lines AB, BC contain the parallelogram ABCD given in magnitude in the given angle ABC, and let the sum of the squares of AB, BC be given; AB, BC are each of them given.

First, let ABC be a right angle; and because twice the rectangle contained by two equal straight lines is equal to both their squares; but if two straight

lines are unequal, twice the rectangle contained by them is less than the sum of their squares, as is evident from the 7. Prop. B. 2. Elem. therefore twice the given space, to which space the Rectangle of which the sides are to be found, is equal, must not be greater than the given sum of the squares of the sides. and if twice that space be equal to the given sum of the squares, the sides of the rectangle must necessarily be equal to one another. therefore in this case describe a square ABCD equal to the given rectangle, and its sides AB, BC are those which were to be found. for the rectangle AC is equal to the given space, and the sum of the squares of its sides AB, BC is equal to twice the rectangle AC, that is, by the hypothesis, to the given space to which the sum of the squares was required to be equal.

But if twice the given rectangle be not equal to the given sum of the squares of the sides, it must be less than it, as has been shewn. Let ABCD be the rectangle, join AC and draw BE perpendicular to it, and complete the rectangle AEBF, and describe the circle ABC about the triangle ABC; AC is its diameter. and because the triangle ABC is similar to AEB, as AC to CB, so is AB to BE; therefore the rectangle AC, BE is equal to AB, BC; and the rectangle AB, BC is given, wherefore AC, BE is given. and because the sum of the squares of AB, BC is given, the square of AC which is equal to that sum is given; and AC itself is therefore given in magnitude. let AC be likewise given in position, and the point A; therefore AF is given in position. and the rectangle AC, BE is given, as has been shewn, and AC is given, wherefore BE is given in magnitude, as also AF which is equal to it; and AF is also given in position, and the point A is given; wherefore the point is given, and the straight line FB in position. and the circumference ABC is given in position, wherefore the point B is given. and the points A, C are

given; therefore the straight lines AB, BC are given in position and magnitude.

The sides AB, BC of the rectangle may be found thus; let the rectangle GH, GK be the given space to which the rectangle AB, BC is equal; and let GH, GL be the given rectangle to which the sum of the squares of AB, BC is equal. find a square equal to the rectangle GH, GL, and let its side AC be given in position; upon AC as a diameter describe the semicircle ABC, and as AC to GH, so make GK to AF, and from the point A place AF at right angles to AC. therefore the rectangle CA, AF is equal to GH, GK; and, by the hypothesis, twice the rectangle GH, GK is less

than GH, GL, that is than the square of AC; wherefore twice the rectangle CA, AF is less than the square of AC, and the rectangle CA, AF itself less than half the square of AC, that is than the rectangle contained by the diameter AC and its half; wherefore AF is less than the semidiameter of the circle, and consequently the straight line drawn through the point F parallel to AC must meet the circumference in two points. let B be either of them, and join AB, BC and complete the rectangle ABCD; ABCD is the rectangle which was to be found. draw BE perpendicular to AC; therefore BE is equal to AF, and because the angle ABC in a semicircle is a right angle, the rectangle AB, BC is equal to AC, BE, that is to the rectangle CA, AF which is equal to the given rectangle GH, GK. and the squares of AB, BC are together equal to the square of AC, that is to the given rectangle GH, GL.

But if the given angle ABC of the parallelogram AC be not a right angle, in this case because ABC is a given angle, the ratio of the rectangle contained by the sides AB, BC to the parallelogram AC is given; and AC is given, therefore the rectangle AB, BC is given. and the sum of the squares of AB, BC is given; therefore the sides AB, BC are given by the preceding case.

The sides AB, BC and the parallelogram AC may be found thus. let EFG be the given angle of the parallelogram, and from any point E in FE draw EG perpendicular to FG. and let the rectangle EG, FH be the given space to which the parallelogram is to be made equal, and let EF, FK be the given rectangle to which the sum of the squares of the sides is to be equal. and, by the preceding case, find the sides of a rectangle which is equal to the given rectangle EF, FH, and the squares of the sides of which are together equal to the given rectangle EF, FK. therefore, as was shewn in that case, twice the rectangle EF, FH must not be greater than the rectangle EF, FK; let it be so, and let AB, BC be the sides of the rectangle joined in the angle ABC equal to the given angle EFG; and complete the parallelogram ABCD, which will be that which was to be found. draw AL perpendicular to BC, and because the angle ABL is equal to EFG, the triangle ABL is equiangular to EFG. and the parallelogram AC, that is the rectangle AL, BC is to the rectangle AB, BC as (the straight line AL to AB, that is as EG to EF, that is as) the rectangle EG, FH to EF, FH. and, by the construction, the rectangle AB, BC is equal to EF, FH, therefore the rectangle AL, BC, or, its equal, the parallelogram AC is equal to the given rectangle EG, FH. and the squares of AB, BC are together equal, by construction, to the given rectangle EF, FK.

PROP. LXXXIX.

86.

IF two straight lines contain a given parallelogram in a given angle, and if the excess of the square of one of them above a given space has a given ratio to the square of the other; each of the straight lines shall be given.

Let the two straight lines AB, BC contain the given parallelogram AC in the given angle ABC, and let the excess of the square of BC above a given space have a given ratio to the square of AB; each of the straight lines AB, BC is given.

Because the excess of the square of BC above a given space has a given ratio to the square of BA, let the rectangle CB, BD be the given space; take this from the square of BC, the remainder, to wit, the rectangle BC, CD has a given ratio to the square of BA. draw AE perpendicular to BC, and let the square of BF be equal to the rectangle BC, CD. then because the angle ABC, as also BEA is given, the triangle ABE is given in species, and the ratio of AE to AB given. and because the ratio of the rectangle BC, CD, that is of the square of BF to the square of BA is given, the ratio of the straight line BF to BA is given. and the ratio of AE to AB is given, wherefore the ratio of AE to BF is given, as also the ratio of the rectangle AE, BC, that is of the parallelogram AC to the rectangle FB, BC; and AC is given, wherefore the rectangle FB, BC is given. and the excess of the square of BC above the square of BF, that is above the rectangle BC, CD is given, for it is equal to the given rectangle CB, BD. therefore because the rectangle contained by the straight lines FB, BC is given, and also the excess of the square of BC above the square of BF; FB, BC are each of them given. and the ratio of FB to BA is given; therefore AB, BC are given.

The Composition is as follows.

Let GHK be the given angle to which the angle of the parallelogram is to be made equal, and from any point G in HG draw GK perpendicular to HK; let GK, HL be the rectangle to which the parallelogram is to be made equal, and let LH, HM be the rectangle equal to the given space which is to be taken

from the square of one of the sides; and let the ratio of the remainder to the square of the other side be the same with the ratio of the square of the given straight line NH to the square of the given straight line HG.

By help of the 87. Dat. find two straight lines BC, BF which contain a rectangle equal to the given rectangle NH, HL, and such that the excess of the square of BC above the square of BF be equal to the given rectangle LH, HM; and join CB, BF in the angle FBC equal to the given angle GHK. and as NH to HG, so make FB to BA, and complete the parallelogram AC, and draw AE perpendicular to BC. then AC is equal to the rectangle GK, HL; and if from the square of BC, the given rectangle LH, HM be taken, the remainder shall have to the square of BA the same ratio which the square of NH has to the square of HG.

Because, by the construction, the square of BC is equal to the square of BF together with the rectangle LH, HM; if from the square of BC there be taken the rectangle LH, HM, there remains the square of BF which has to the square of BA the same ratio which the square of NH has to the square of HG, because as NH to HG, so FB was made to BA; but as HG to GK, so is BA to AE, because the triangle GHK is equiangular to ABE; therefore, ex aequali, as NH to GK, so is FB to AE. wherefore the rectangle NH, HL is to the rectangle GK, HL, as the rectangle FB, BC to AE, BC. but, by the construction, the rectangle NH, HL is equal to FB, BC; therefore the rectangle GK, HL is equal to the rectangle AE, BC, that is to the parallelogram AC.

The Analysis of this Problem might have been made as in the 86. Prop. in the Greek, and the composition of it may be made as that which is in Pro. of this Edition.

PROP. XC.

O.

IF two straight lines contain a given parallelogram in a given angle, and if the square of one of them be given together with the space which has a given ratio to the square of the other; each of the straight lines shall be given.

Let the two straight lines AB, BC contain the given parallelogram AC in the given angle ABC, and let the square of BC be given together with the space which has a given ratio to the square of AB; AB, BC are each of them given.

Let the square of BD be the space which has the given ratio to the square of AB; therefore, by the hypothesis, the square of BC together with the square of BD is given. from the point A draw AE perpendicular to BC, and because the angles ABE, BEA are given, the triangle ABE is given in species; therefore the ratio of BA to AE is given. and because the ratio of the square of BD to the square of BA is given, the ratio of the straight line BD to BA is given; and the ratio of BA to AE is given, therefore the ratio of AE to BD is given, as also the ratio of the rectangle AE, BC, that is of the parallelogram AC to the rectangle DB, BC. and AC is given, therefore the rectangle DB, BC is given; and the square of BC together with the square of BD is given, therefore because the rectangle contained by the two straight lines DB, BC is given, and the sum of their squares is given; the straight lines DB, BC are each of them given. and the ratio of DB to BA is given; therefore AB, BC are given.

The Composition is as follows.

Let FGH be the given angle to which the angle of the parallelogram is to be made equal, and from any point F in GF draw FH perpendicular to GH; and let the rectangle FH, GK be that to which the parallelogram is to be made equal; and let the rectangle KG, GL be the space to which the square of one

of the sides of the parallelogram together with the space which has a given ratio to the square of the other side, is to be made equal; and let this given ratio be the same which the square of the given straight line MG has to the square of GF.

By the 83. Dat. find two straight lines DB, BC which contain a rectangle equal to the given rectangle MG, GK, and such that the sum of their squares is equal to the given rectangle KG, GL therefore, by the determination of the Problem in that Proposition, twice the rectangle MG, GK must not be greater than the rectangle KG, GL. let it be so, and join the straight lines DB, BC in the angle DBC equal to the given angle FGH. and as MG to

GF, so make DB to BA, and complete the parallelogram AC. AC is equal to the rectangle FH, GK; and the square of BC together with the square of BD which by the construction has to the square of BA the given ratio which the square of MG has to the square of GF, is equal, by the construction, to the given rectangle KG, GL. Draw AE perpendicular to BC.

Because as DB to BA, so is MG to GF; and as BA to AE, so GF to FH; ex aequali, as DB to AE, so is MG to FH. therefore as the rectangle DB, BC to AE, BC, so is the rectangle MG, GK to FH, GK. and the rectangle DB, BC is equal to the rectangle MG, GK; therefore the rectangle AE, BC, that is the parallelogram AC, is equal to the rectangle FH, GK.

PROP. XCI.

88.

IF a straight line drawn within a circle given in magnitude cuts off a segment which contains a given angle; the straight line is given in magnitude.

In the circle ABC given in magnitude, let the straight line AC be drawn cutting off the segment AEC which contains the given angle AEC; the straight line AC is given in magnitude.

Take D the center of the circle, join AD and produce it to

E, and join EC. the angle ACE being a right angle is given; and the angle AEC is given; therefore the triangle ACE is given in species, and the ratio of EA to AC is therefore given. and EA is given in magnitude, because the circle is given in magnitude; AC is therefore given in magnitude.

PROP. XCII.

89.

IF a straight line given in magnitude be drawn within a circle given in magnitude; it shall cut off a segment containing a given angle.

Let the straight line AC given in magnitude be drawn within the circle ABC given in magnitude; it shall cut off a segment containing a given angle.

Take D the center of the circle, join AD and produce it to E, and join EC. and because each of the straight lines EA, AC is given, their ratio is given; and the angle ACE is a right angle, therefore the triangle ACE is given in species, and consequently the angle AEC is given.

PROP. XCIII. 90.

IF from any point in the circumference of a circle given in position two straight lines be drawn meeting the circumference and containing a given angle; if the point in which one of them meets the circumference again be given, the point in which the other meets it is also given.

From any point A in the circumference of a circle ABC given in position, let AB, AC be drawn to the circumference making the given angle BAC; if the point B be given, the point C is also given.

Take D the center of the circle, and join BD, DC. and because each of the points B, D is given, BD is given in position. and because the angle BAC is given, the angle BDC is given. therefore because the straight line DC is drawn to the given point D in the straight line BD given in position in the given angle BDC, DC is given in position. and the circumference ABC is given in position, therefore the point C is given.

PROP. XCIV.

91.

IF from a given point a straight line be drawn touching a circle given in position; the straight line is given in position and magnitude.

Let the straight line AB be drawn from the given point A touching the circle BC given in position; AB is given in position and magnitude.

Take D the center of the circle, and join DA, DB. because each of the points D, A is given, the straight line AD is given in position and magnitude. and DBA is a right angle, wherefore DA is a diameter of the circle DBA described about the triangle DBA; and that circle is therefore given in position. and the circle BC is given in position, therefore the point B is given. the point A is also given; therefore the straight line AB is given in position and magnitude.

PROP. XCV.

92.

IF a straight line be drawn from a given point without a circle given in position; the rectangle contained by the segments betwixt the point and the circumference of the circle is given.

Let the straight line ABC be drawn from the given point A without the circle BCD given in position, cutting it in B, C; the rectangle BA, AC is given.

From the point A draw AD touching the circle; therefore AD is given in position and magnitude. and because AD is given, the square of AD is given which is equal to the rectangle BA, AC. therefore the rectangle BA, AC is given.

PROP. XCVI.

93.

IF a straight line be drawn thro' a given point within a circle given in position, the rectangle contained by the segments betwixt the point and the circumference of the circle is given.

Let the straight line BAC be drawn thro' the given point A within the circle BCE given in position; the rectangle BA, AC is given.

Take D the center of the circle, join AD and produce it to the points E, F. because the points A, D are given, the straight line AD is given in position; and the circle BEC is given in position; therefore the points E, F are given. and the point A is given, therefore EA, AF are each of them given; and the rectangle EA, AF is therefore given; and it is equal to the rectangle BA, AC which consequently is given.

PROP. XCVII.

94.

IF a straight line be drawn within a circle given in magnitude cutting off a segment containing a given angle; if the angle in the segment be bisected by a straight line produced till it meets the circumference, the straight lines which contain the given angle shall both of them together have a given ratio to the straight line which bisects the angle. and the rectangle contained by both these lines together which contain the given angle, and the part of the bisecting line cut off below the base of the segment, shall be given.

Let the straight line BC be drawn within the circle ABC given in magnitude cutting off a segment containing the given angle BAC. and let the angle BAC be bisected by the straight line AD; BA together with AC has a given ratio to AD; and the rectangle contained by BA and AC together, and the straight line ED cut off from AD below BC the base of the segment, is given.

Join BD; and because BC is drawn within the circle ABC given in magnitude cutting off the segment BAC containing the given angle BAC; BC is given in magnitude. by the same reason BD is given; therefore the ratio of BC to BD is given. and because the angle BAC is bisected by AD, as BA to AC, so is BE to EC; and, by permutation, as AB to BE, so is AC to CE; wherefore as BA and AC together to BC, so is AC to CE. and because the angle BAE is equal to EAC, and the angle ACE to ADB; the triangle ACE is equiangular to the triangle ADB; therefore as AC to CE, so is AD to DB. but as AC to CE, so is BA together with AC to BC; as therefore BA and AC to BC, so is AD to DB; and, by permutation, as BA and AC to AD, so is BC to

BD. and the ratio of BC to BD is given, therefore the ratio of BA together with AC to AD is given.

Also the rectangle contained by BA and AC together, and DE is given.

Because the triangle BDE is equiangular to the triangle ACE, as BD to DE, so is AC to CE; and as AC to CE, so is BA and AC to BC; therefore as BA and AC to BC, so is BD to DE. wherefore the rectangle contained by BA and AC together, and DE is equal to the rectangle CB, BD. but CB, BD is given; therefore the rectangle contained by BA and AC together, and DE is given.

Otherwise.

Produce CA and make AF equal to AB, and join BF. and because the angle BAC is double of each of the angles BFA, BAD, the angle BFC is equal to BAD; and the angle BCA is equal to BDA, therefore the triangle FCB is equiangular to ADB. as therefore FC to CB, so is AD to DB, and, by permutation, as FC, that is BA and AC together to AD, so is CB to BD. and the ratio of CB to BD is given, therefore the ratio of BA and AC to AD is given.

And because the angle BFC is equal to the angle DAC, that is to the angle DBC, and the angle ACB equal to the angle ADB; the triangle FCB is equiangular to BDE, as therefore FC to CB, so is BD to DE; therefore the rectangle contained by FC, that is BA and AC together, and DE is equal to the rectangle CB, BD which is given, and therefore the rectangle contained by BA, AC together, and DE is given.

PROP. XCVIII.

P.

IF a straight line be drawn within a circle given in magnitude cutting off a segment containing a given angle; it the angle adjacent to the angle in the segment be bisected by a straight line produced till it meet the circumference again and the base of the segment; the excess of the straight lines which contain the given angle shall have a given ratio to the segment of the bisecting line which is within the circle; and the rectangle contained by the same excess and the segment of the bisecting line betwixt the base produced and the point where it again meets the circumference, shall be given.

Let the straight line BC be drawn within the circle ABC given in magnitude cutting off a segment containing the given angle BAC, and let the angle CAF adjacent to BAC be bisected by the straight line DAE meeting the circumference again in D, and BC the base of the segment produced in E; the excess of BA, AC has a given ratio to AD; and the rectangle which is contained by the same excess and the straight line ED, is given.

Join BD, and thro' B draw BG parallel to DE meeting AC produced in G. and because BC cuts off from the circle ABC given in magnitude the segment BAC containing a given angle, BC is therefore given in magnitude. by the same reason BD is given, because the angle BAD is equal to the given angle EAF; therefore the ratio of BC to BD is given. and because the angle CAE is equal to EAF, of which CAE is equal to the alternate angle AGB, and EAF to the interior and opposite angle ABG; therefore the angle AGB is equal to ABG, and the straight line AB equal to AG; so that GC is the excess of BA, AC. and because the angle BGC is equal to GAE, that is to EAF, or the angle BAD; and that the angle BCG is equal to the opposite interior angle BDA of the quadrilateral BCAD in the circle; therefore the triangle BGC is equiangular to BDA therefore as CC to CB, so is AD to DB, and, by permutation, as GC, which is the excess of BA, AC to AD, so is CB to BD.

and the ratio of CB to BD is given; therefore the ratio of the excess of BA, AC to AD is given.

And because the angle GBC is equal to the alternate angle DEB, and the angle BCG equal to BDE; the triangle BCG is equiangular to BDE. therefore as GC to CB, so is BD to DE, and consequently the rectangle GC, DE is equal to the rectangle CB, BD which is given, because its sides CB, BD are given. therefore the rectangle contained by the excess of BA, AC and the straight line DE is given.

PROP. XCIX.

95.

IF from a given point in the diameter of a circle given in position, or in the diameter produced, a straight line be drawn to any point in the circumference, and from that point a straight line be drawn at right angles to the first, and from the point in which this meets the circumference again, a straight line be drawn parallel to the first; the point in which this parallel meets the diameter is given; and the rectangle contained by the two parallels is given.

In BC the diameter of the circle ABC given in position, or in BC produced, let the given point D be taken, and from D let a straight line DA be drawn to any point A in the circumference, and let AE be drawn at right angles to DA, and from the point E where it meets the circumference again let EF be drawn parallel to DA meeting BC in F; the point F is given, as also the rectangle AD, EF.

Produce EF to the circumference in G, and join AG. because GEA is a right angle, the straight line AG is the diameter of the circle ABC; and BC is also a diameter of it; therefore the point H where they meet is the center of

the circle, and consequently H is given. and the point D is given, wherefore DH is given in magnitude. and because AD is parallel to FG, and GH equal to HA; DH is equal to HF, and AD equal to GF. and DH is

given, therefore HF is given in magnitude; and it is also given in position, and the point H is given, therefore the point F is given.

And because the straight line EFG is drawn from a given point F without or within the circle ABC given in position, therefore the rectangle EF, FG is given. and GF is equal to AD, wherefore the rectangle AD, EF is given.

PROP. C.

Q

IF from a given point in a straight line given in position, a straight line be drawn to any point in the circumference of a circle given in position; and from this point a straight line be drawn making with the first an angle equal to the difference of a right angle and the angle contained by the straight line given in position, and the straight line which joins the given point and the center of the circle; and from the point in which the second line meets the circumference again, a third straight line be drawn making with the second an angle equal to that which the first makes with the second. the point in which this third line meets the straight line given in position is given; as also the rectangle contained by the first straight line and the segment of the third betwixt the circumference and the straight line given in position, is given.

Let the straight line CD be drawn from the given point C in the straight line AB given in position, to the circumference of the circle DEF given in position of which G is the center; join CG, and from the point D let DF be drawn making the angle CDF equal to the difference of a right angle and the angle BCG, and from the point F let FE be drawn making the angle DFE

equal to CDF, meeting AB in H. the point H is given; as also the rectangle CD, FH.

Let CD, FH meet one another in the point K from which draw KL perpendicular to DF; and let DC meet the circumference again in M, and let FH meet the same in E, and join MG, GF, CH.

Because the angles MDF, DFE are equal to one another, the circumferences MF, DE are equal; and adding or taking away the common part ME, the circumference DM is equal to EF; therefore the straight line DM is equal to the straight line EF, and the angle GMD to the angle GFE; and the angles GMC, GFH are equal to one another, because they are either the same with the angles GMD, GFE, or adjacent to them. and because the angles KDL, LKD are together equal to a right angle, that is, by the hypothesis, to the angles KDL, GCB; the angle GCB or GCH is equal to the angle (LKD, that is to the angle) LKF or GKH. therefore the points C, K, H, G are in the circumference of a circle; and the angle GCK is therefore equal to the angle GHF; and the angle CMC is equal to GFH, and the straight line GM to GF; therefore CG is equal to GH, and CM to HF. and because CG is equal to GH, the angle GCH is equal to GHC; but the angle GCH is given, therefore GHC is given; and consequently the angle CGH is given. and CG is given in position, and the point G; therefore GH is given in position; and CB is also given in position, wherefore the point H is given.

And because HF is equal to CM, the rectangle DC, FH is equal to DC, CM. but DC, CM is given, because the point C is given; therefore the rectangle DC, FH is given.

FINIS

NOTES.

DEFINITION II.

THIS is made more explicit than in the Greek text, to prevent a mistake which the Author of the second Demonstration of the 24th Proposition in the Greek Edition has fallen into, of thinking that a ratio is given to which another ratio is shewn to be equal, tho' this other be not exhibited in given magnitude. See the Notes on that Proposition which is the 13 th in this Edition. besides by this Definition, as it is now given, some Propositions are demonstrated, which in the Greek are not so well done by help of Pro.

DEF. IV.

In the Greek text Def. 4. is thus "Points, lines, spaces and angles are said to be given in position which have always the same situation." but this is imperfect and useless, because there are innumerable cases in which things may be given according to this Definition, and yet their position cannot be found. for instance, let the triangle ABC be given in position, and let it be proposed to draw a straight line BD from the angle at B to the opposite side AC which shall cut off the angle DBC which shall be the seventh part of the angle ABC. suppose this is done, therefore the straight line BD is invariable in its position, that is, has always the same situation; for any other straight line drawn from the point B on either side of BD cuts off an angle greater or lesser than the seventh part of the angle ABC; therefore, according to this Definition, the straight line BD is given in position, as also the point D in which it meets the straight line AC which is given in position. but from the things here given, neither the straight line BD nor the point D can be found by the help of Euclid's Elements only, by which every thing in his Data is

supposed may be found. this Definition is therefore of no use. we have amended it by adding “and which are either actually exhibited or can be found;” for nothing is to be reckoned given, which cannot be found, or is not actually exhibited.

The Definition of an angle given by position is taken out of the 4th, and given more distinctly by itself in the Definition marked A.

DEF. XI. XII. XIII. XIV. XV.

The 11th and 12th are omitted because they cannot be given in English so as to have any tolerable sense, and therefore wherever the terms defined occur, the words which express their meaning are made use of in their place.

The 13. 14. 15. are omitted as being of no use.

It is to be observed in general of the Data in this book, that they are to be understood to be given Geometrically, not always Arithmetically, that is, they cannot always be exhibited in numbers; for instance, if the side of a square be given, the ratio of it to its diameter is given geometrically, but not in numbers; and the diameter is given, but tho’ the number of any equal parts in the side be given, for example to, the number of them in the diameter cannot be given. and the like holds in many other cases.

PROPOSITION I.

In this it is shewn that A is to B, as C to D, from this that A is to C, as B to D, and then by permutation; but it follows directly, without these two steps, from 7. 5.

PROP. II.

The limitation added at the end of this Proposition between the inverted commas is quite necessary, because without it the Proposition cannot always be demonstrated. for the Author having said “because A is given, a magnitude equal to it can be found, let this be C; and because the ratio of A

to B is given, a ratio which is the same to it can be found" adds, "let it be found, and let it be the ratio of C to Δ." Now from the second Definition nothing more follows than that some ratio, suppose the ratio of E to Z, can be found, which is the same with the ratio of A to B; and when the Author supposes that the ratio of C to Δ, which is

also the same with the ratio of A to B, can be found, he necessarily supposes that to the three magnitudes E, Z, C a fourth proportional Δ may be found; but this cannot always be done by the Elements of Euclid; from which it is plain Euclid must have understood the Proposition under the limitation which is now added to his text. An example will make this clear; let A be a given angle, and B another angle to which A has a given ratio, for instance, the ratio of the given straight line E to the given one Z, then, having found an angle C equal to A, how can the angle Δ be found to which C has the same ratio that E has to Z? certainly no way, until it be shewn how to find an angle to which a given angle has a given ratio, which cannot be done by Euclid's Elements, nor probably by any Geometry known in his time. Therefore in all the Propositions of this book which depend upon this second, the above-mentioned limitation must be understood, tho' it be not explicitly mentioned.

PROP. V.

The order of the Propositions in the Greek text between Pro. and Pro. is now changed into another which is more natural, by placing those which are more simple before those which are more complex; and by placing together those which are of the same kind, some of which were mixed among others of a different kind. thus Pro. in the Greek is now made the 5. and those which were the 22. and 23. are made the 11. and 12. as they are more simple than the Propositions concerning magnitudes the excess of one of which above a

given magnitude has a given ratio to the other, after which these two were placed; and the 24. in the Greek text is, for the same reason, made the 13.

PROP. VI. VII.

These are universally true, tho' in the Greek text they are demonstrated by Pro. which has a limitation; they are therefore now shewn without it.

PROP. XII.

In the 23. Prop. in the Greek text, which here is the 12. the words "" are wrong translated by Claud. Hardy in his Edition of Euclid's Data printed at Paris Ann., 1625, which was the first Edition of the Greek text; and Dr. Gregory follows him in translating them by the words "etsi non easdem, "as if the Greek had been as in Pro. of the Greek text. Euclid's meaning is that the ratios mentioned in the Proposition must not be the same; for if they were, the Proposition would not be true. whatever ratio the whole has to the whole, if the ratios of the parts of the first to the parts of the other be the same with this ratio, one part of the first may be double, triple, &c. of the other part of it, or have any other ratio to it, and consequently cannot have a given ratio to it. wherefore these words must be rendered by "non autem easdem, "but not the same ratios, as Zambertus has translated them in his Edition.

PROP. XIII.

Some very ignorant Editor has given a second Demonstration of this Proposition in the Greek text, which has been as ignorantly kept in it by Claud. Hardy and Dr. Gregory, and has been retained in the translations of Zambertus and others; Carolus Renaldinus gives it only. the author or it has thought that a ratio was given if another ratio could be shewn to be the same

to it, tho' this last ratio be not found. but this is altogether absurd, because from it would be deduced that the ratio of the sides of any two squares is given, and the ratio of the diameters of any two circles, &c. and it is to be observed that the moderns frequently take given ratios, and ratios that are always the same for one and the same thing, and Sir Isaac Newton has fallen into this mistake in the 17th Lemma of his Principia, Ed. 1713. and in other places. but this should be carefully avoided, as it may lead into other errors.

PROP. XIV. XV.

Euclid in this book has several Propositions concerning magnitudes, the excess of one of which above a given magnitude has a given ratio to the other; but he has given none concerning magnitudes whereof one together with a given magnitude has a given ratio to the other; tho' these last occur as frequently in the solution or Problems as the first. the reason of which is, that the last may be all demonstrated by help of the first; for if a magnitude together with a given magnitude has a given ratio to another magnitude; the excess of this other above a given magnitude shall have a given ratio to the first, and on the contrary; as we have demonstrated in Pro. and for a like reason Pro. has been added to the Data. one example will make the thing clear; suppose it were to be demonstrated, That if a magnitude A together with a given magnitude has a given ratio to another magnitude B, that the two magnitudes A and B, together with a given magnitude have a given ratio to that other magnitude B; which is the same Proposition with respect to the last kind of magnitudes above-mentioned, that the first part of Pro. in this Edition is in respect of the first kind. this is shewn thus; from the hypothesis, and by the first part of Pro. the excess of B above a given magnitude has unto A a given ratio; and therefore, by the first part of Pro. the excess of B above a given magnitude has unto B and A together a given ratio; and by the second

part of Pro. A and B together with a given magnitude has unto B a given ratio; which is the thing that was to be demonstrated. in like manner the other Propositions concerning the last kind of magnitudes may be shewn.

PROP. XVI. XVII.

In the third part of Pro. in the Greek text, which is the 16. in this Edition, after the ratio of EC to CB has been shewn to be given; from this, by inversion and conversion, the ratio of BC to BE is demonstrated to be given; but, without these two steps, the conclusion should have been made only by citing the 6. Proposition. and in like manner, in the first part of Pro. in the Greek, which in this Edition is the 17. from the ratio of DB to BC being given, the ratio of DC to DB is shewn to be given, by inversion and Composition, instead of citing Pro, and the same fault occurs in the second part of the same Pro.

PROP. XXI. XXII.

These are now added, as being wanting to complete the subject treated of in the four preceding Propositions.

PROP. XXIII.

This which is Pro. in the Greek text, was separated from Pro. 15. 16. in that text, after which it should have been immediately placed, as being of the same kind. it is now put into its proper place. but Pro. in the Greek is left out, as being the same with Pro. in that text, which is here Pro.

PROP. XXIV.

This, which is Pro. in the Greek, is now put into its proper place, having been disjoined from the three following it in this Edition, which are of the same kind.

PROP. XXVIII.

This which in the Greek text is Pro. and several of the following Propositions, are there deduced from Def. 4. which is not sufficient, as has been mentioned in the Note on that Definition; they are therefore now shewn more explicitly.

PROP. XXXIV. XXXVI.

Each of these has a Determination, which is now added, which occasions a change in their Demonstrations.

PROP. XXXVII. XXXIX. XL. XLI.

The 35. and 36. Propositions in the Greek text are joined into one, which makes the 39. in this Edition, because the same Enuntiation and Demonstration serves both. and for the same reason Pro. 38. in the Greek are joined into one which here is the 40.

Pro. is added to the Data, as it frequently occurs in the solution of Problems. and Pro. is added to complete the rest.

PROP. XLII.

This is Pro. in the Greek text, where the whole construction of Pro. of Book 1. of the Elements is put without need into the Demonstration, but is now only cited.

PROP. XLV.

This is Pro. in the Greek, where the three straight lines made use of in the construction are said, but not shewn, to be such that any two of them is greater than the third, which is now done.

PROP. XLVII.

This is Pro. in the Greek text, but the Demonstration of it is changed into another wherein the several cases of it are shewn, which, tho' necessary, is not done in the Greek.

PROP. XLVIII.

There are two cases in this Proposition, arising from the two cases of the 3d part of Pro. on which the 48. depends. and in the Composition these two cases are explicitly given.

PROP. LII.

The Construction and Demonstration of this which is Pro. in the Greek, are made something shorter than in that text.

PROP. LIII.

Pro. in the Greek text is omitted, being only a case of Pro. in that text, which is Pro. in this Edition.

PROP. LVIII.

This is not in the Greek text, but its Demonstration is contained in that of the first part of Pro. in that text; which Proposition is concerning figures that are

given in species; this 58. is true of similar figures, tho' they be not given in species, and as it frequently occurs, it was necessary to add it.

PROP. LIX. LXI.

This is the 54. in the Greek; and the 77. in the Greek, being the very same with it, is left out. and a shorter Demonstration is given of Pro.

PROP. LXII.

This which is most frequently useful is not in the Greek, and is necessary to Pro. 88. in this Edition, as also, tho' not mentioned, to Pro. 87. in the former Editions. Pro. in the Greek text is made a Corollary to it.

PROP. LXIV.

This contains both Pro. and 73. in the Greek text; the first case of the 74. is a repetition of Pro. from which it is separated in that text by many Propositions; and as there is no order in these Propositions, as they stand in the Greek, they are now put into the order which seemed most convenient and natural.

The Demonstration of the first part of Pro. in the Greek is grossly vitiated. Dr. Gregory says that the sentences he has inclosed betwixt two stars are superfluous and ought to be cancelled; but he has not observed that what follows them is absurd, being to prove that the ratio [see his figure] of AT to TK is given, which by the Hypothesis at the beginning of the Proposition is expressly given; so that the whole of this part was to be altered, which is done in this Pro.

PROP. LXVII. LXVIII.

Pro. in the Greek text is divided into these two, for the sake of distinctness; and the Demonstration of the 67. is rendered shorter than that of the first part of Pro. in the Greek by means of Pro. of Book 6. of the Elements.

PROP. LXX.

This is Pro. in the Greek text; Pro. in that text is only a particular case of it, and is therefore omitted.

Dr. Gregory in the Demonstration of Pro. cites the 49. Prop. Dat. to prove that the ratio of the figure AEB to the parallelogram AH is given, whereas this was shewn a few lines before; and besides the 49. Prop. is not applicable to these two figures, because AH is not given in species, but is, by the step for which the citation is brought, proved to be given in species.

PROP. LXXIII.

Pro. in the Greek text is neither well enuntiated nor demonstrated. the 73. which in this Edition is put in place of it, is really the same, as will appear by considering [see Dr. Gregory's Edition] that A, B, D, E in the Greek text are four proportionals, and that the Proposition is to shew that Δ, which has a given ratio to E, is to D, as B is to the straight line to which A has a given ratio; or, by inversion, that is to Δ, as the straight line to which A has a given ratio is to B; that is, if the proportionals be placed in this order, viz. D, E, A, B, that the first P is to Δ to which the second E has a given ratio, as the straight line to which the third A has a given ratio is to the fourth B; which is the Enuntiation of this 73. and was thus changed that it might be made like to that at Pro. in this Edition, which is the 82. in the Greek text. and the Demonstration of Pro. is the same with that of Pro. only making use of Pro. instead of Pro. of Book 5. of the Elements.

PROP. LXXVII.

This is put in place of Pro. in the Greek text which is not a Datum, but a Theorem premised as a Lemma to Pro. in that text, and Pro. is made Cor. 1. to Pro. in this Edition. Cl. Hardy in his Edition of the Data takes notice, that, in Pro. of the Greek text, the parallel KL in the figure of Pro. in this Edition must meet the circumference, but does not demonstrate it, which is done here at the end of Cor. 3. of Pro. in the construction for finding a triangle similar to ABC.

PROP. LXXVIII.

The Demonstration of this which is Pro. in the Greek is rendered a good deal shorter by help of Pro.

PROP. LXXIX. LXXX. LXXXI.

These are added to Euclid's Data, as Propositions which are often useful in the solution of Problems.

PROP. LXXXII.

This which is Pro. in the Greek text is placed before the 83. and 84. which in the Greek are the 58. and 59. because the Demonstration of these two in this Edition is deduced from that of Pro. from which they naturally follow.

PROP. LXXXVIII. XC.

Dr. Gregory in his preface to Euclid's Works which he published at Oxford in 1703, after having told that he had supplied the defects of the Greek text of the Data in innumerable places from several Manuscripts, and corrected Cl.

Hardy's translation by Mr. Bernard's, adds, that the 86. Theorem " or Proposition, " seemed to be remarkably vitiated, but which could not be restored by help of the Manuscripts; then he gives three different translations of it in Latin, according to which he thinks it may be read; the two first have no distinct meaning, and the third which he says is the best, tho' it contains a true Proposition which is the 90. in this Edition, has no connexion in the least with the Greek text. and it is strange that Dr. Gregory did not observe, that if Pro. was changed into this, the Demonstration of the 86. must be cancelled, and another put in its place. but, the truth is, both the Enuntiation and the Demonstration of Pro. are quite entire and right, only Pro. which is more simple, ought to have been placed before it; and the deficiency which the Doctor justly observes to be in this part of Euclid's Data, and which no doubt is owing to the carelessness and ignorance of the Greek Editors should have been supplied, not by changing Pro. which is both entire and necessary, but by adding the two Propositions which are the 88. and 90. in this Edition.

PROP. XCVIII. C.

These were communicated to me by two excellent Geometers, the first of them by the Right Honourable the Earl Stanhope, and the other by Dr. Matthew Stewart; to which I have added the Demonstrations.

Tho' the order of the Propositions has been in many places changed from that in former Editions, yet this will be of little disadvantage, as the antient Geometers never cite the Data, and the Moderns very rarely.

AS that part of the Composition of a Problem which is its Construction may not be so readily deduced from the Analysis by beginners; for their sake the following Example is given in which the derivation of the several parts of the Construction from the Analysis is particularly shewn, that they may be assisted to do the like in other Problems.

PROBLEM.

Having given the magnitude of a parallelogram, the angle of which ABC is given, and also the excess of the square of its side BC above the square of the side AB; To find its sides and describe it.

The Analysis of this is the same with the Demonstration of the 87. Prop. of the Data. and the Construction that is given of the Problem at the end of that Proposition, is thus derived from the Analysis.

Let EFG be equal to the given angle ABC, and because in the Analysis it is said that the ratio of the rectangle AB, BC to the parallelogram

AC is given by the 62. Prop. Dat. therefore from a point in FE, the perpendicular EG is drawn to FG, as the ratio of FE to EG is the ratio of the rectangle AB, BC to the parallelogram AC by what is shewn at the end of Pro. Next the magnitude of AC is exhibited by making the rectangle EG, GH equal to it, and the given excess of the square of BC above the square of BA, to which excess the rectangle CB, BD is equal, is exhibited by the rectangle HG, GL. then in the Analysis the rectangle AB, BC is said to be given, and this is equal to the rectangle FE, GH, because the rectangle AB, BC is to the parallelogram AC, as (FE to EG, that is as the rectangle) FE, GH to EG, GH; and the parallelogram AC is equal to the rectangle EG, GH, therefore the rectangle AB, BC is equal to FE, GH. and consequently the ratio of the rectangle CB, BD, that is of the rectangle HG, GL, to AB, BC, that is of the straight line DB to BA, is the same with the ratio (of the rectangle GL, GH to FE, GH, that is) of the straight line GL to FE, which ratio of DB to BA is the next thing said to be given in the Analysis. from this it is plain that the square of FE is to the square of GL, as the square of BA which is equal to the rectangle BC, CD is to the square of BD, the ratio of which spaces is the next thing said to be given. and from this it follows that four times the square of

FE is to the square of GL, as four times the rectangle BC, CD is to the square of BD; and, by Composition, four times the square of FE together with the square of GL is to the square of GL, as four times the rectangle BC, CD together with the square of BD, is to the square of BD, that is [8. 6.] as the square of the straight lines BC, CD taken together is to the square of BD, which ratio is the next thing said to be given in the Analysis. and because four times the square of FE and the square of GL are to be added together, therefore in the perpendicular EG there is taken KG equal to FE, and MG equal to the double of it, because thereby the squares of MG, GL, that is, joining ML, the square of ML is equal to four times the square of FE and to the square of GL. and because the square of ML is to the square of GL, as the square of the straight line made up of BC and CD is to the square of BD, therefore [22. 6.] ML is to LG, as BC together with CD is to BD, and, by Composition, ML and LG together, that is, producing GL to N, so that ML be equal to LN, the straight line NG is to GL, as twice BC is to BD; and by taking GO equal to the half of NG, GO is to GL, as BC to BD the ratio of which is said to be given in the Analysis. and from this it follows, that the rectangle HG, GO is to HG, GL, as the square of BC is to the rectangle CB, BD which is equal to the rectangle HG, GL, and therefore the square of BC is equal to the rectangle HG, GO, and BC is consequently found by taking a mean proportional betwixt HG and GO, as is said in the Construction. and because it was shewn that GO is to GL, as BC to BD, and that now the three first are found, the fourth BD is found by 12. 6. it was likewise shewn that LG is to FE, or GK, as DB to BA, and the three first are now found, and there by the fourth BA. make the angle ABC equal to EFG, and complete the parallelogram of which the sides are AB, BC, and the construction is finished; the rest of the Composition contains the Demonstration.

AS the Propositions from the 13. to the 28. may be thought by beginners to be less useful than the rest, because they cannot so readily see how they are to be made use of in the solution of Problems; on this account the two following Problems are added, to shew that they are equally useful with the other Propositions, and from which it may easily be judged that many other Problems depend upon these Propositions.

PROBLEM 1.

TO find three straight lines such, that the ratio of the first to the second is given; and if a given straight line be taken from the second, the ratio of the remainder to the third is given; also the rectangle contained by the first and third is given.

Let AB be the first straight line, CD the second, and EF the third. and because the ratio of AB to CD is given, and that if a given straight line be taken from CD, the ratio of the remainder EF is given; therefore the excess of the first AB above a given straight line has a given ratio to the third EF. Let BH be that given straight line, therefore AH the excess of AB above it has a given ratio to EF; and consequently the rectangle BA, AH has a given ratio to the rectangle AB, EF, which last rectangle is given by the Hypothesis; therefore the rectangle BA, AH is given, and BH the excess of its sides is given; wherefore the sides AB, AH are given. and because the ratios of AB to CD, and of AH to EF are given; CD and EF are given.

The Composition.

Let the given ratio of KL to KM be that which AB is required to have to CD; and let DG be the given straight line which is to be taken from CD, and let the given ratio of KM to KN be that which the remainder must have to EF; also, let the given rectangle NK, KO be that to which the rectangle AB, EF is

required to be equal. find the given straight line BH which is to be taken from AB, which is done, as plainly appears from Pro. Dat.

by making as KM to KL, so GD to HB. to the given straight line BH apply a rectangle equal to LK, KO exceeding by a square, and let BA, AH be its sides. then is AB the first of the straight lines required to be found. and by making as LK to KM, so AB to DC, DC will be the second. and lastly, make as KM to KN, so CG to EF, and EF is the third.

For as AB to CD, so is HB to GD, each of these ratios being the same with the ratio of LK to KM; therefore AH is to CG, as (AB to CD, that is, as) LK to KM; and as CG to EF, so is KM to KN; wherefore, ex aequali, as AH to EF, so is LK to KN. and as the rectangle BA, AH to the rectangle BA, EF, so is the rectangle LK, KO to the rectangle KN, KO. and, by the Construction, the rectangle BA, AH is equal to LK, KO, therefore the rectangle AB, EF is equal to the given rectangle NK, KO. and AB has to CD the given ratio of KL to KM; and from CD the given straight line GD being taken, the remainder CG has to EF the given ratio of KM to KN. Q. E. D.

PROB. II.

TO find three straight lines such, that the ratio of the first to the second is given; and if a given straight line be taken from the second, the ratio of the remainder to the third is given; also the sum of the squares of the first and third is given.

Let AB be the first straight line, BC the second, and BD the third. and because the ratio of AB to BC is given, and that if a given straight line be taken from BC, the ratio of the remainder to BD is given; therefore the excess of the first AB above a given straight line has a given ratio to the third BD. let AE be that given straight line, therefore the remainder EB has a given ratio to BD. let BD be placed at right angles to EB, and join DE, then the triangle EBD is given in species; wherefore the angle BED is given. let AE

which is given in magnitude be given also in position, and the straight line ED will be given in position. join AD, and because the sum of the squares of AB, BD, that is, the square of AD is given, therefore the straight line AD is given in magnitude; and it is also given in position, because from the given point A it is drawn to the straight line ED given in position. therefore the point D in which the two straight lines AD, ED given in position cut one another is given. and the straight line DB which is at right angles to AB is given in position, and AB is given in position, therefore the point B is given. and the points A, D are given, wherefore the straight lines AB, BD are given. and the ratio of AB to BC is given, and therefore BC is given.

The Composition.

Let the given ratio of FG to GH be that which AB is required to have to BC, and let HK be the given straight line which is to be taken from BC, and let the ratio which the remainder. is required to have to BD be the given ratio of HG to GL, and place GL at right angles to FH, and join LF, LH. Next, as HG is to

GF, so make HK to AE; produce AE to N so that AN be the straight line to the square of which the sum of the squares of AB, BD is required to be equal; and make the angle NED equal to the angle GFL. from the center A at the distance AN describe a circle, and let its circumference meet ED in D, and draw DB perpendicular to AN, and DM making the angle BDM equal to the angle GLH. lastly, produce BM to C so that MC be equal to HK. then is AB the first, BC the second and BD the third of the straight lines that were to be found.

For the triangle EBD, FGL, as also DBM, LGH being equiangular, as EB to BD, so is FG to GL; and as DB to BM, so is LG to CH; therefore, ex acquali, as LB to BM, so is (FG to CH, and so is) AE to HK or MC;

wherefore AB is to BC, as AE to HK, that is, as FG to GH, that is, in the given ratio. and from the straight line BC taking MC which is equal to the given straight line HK, the remainder BM has to BD the given ratio of HG to GL. and the form of the squares of AB, BD is equal to the square of AD or AN which is the given space. Q. E. D.

I believe it would be in vain to try to deduce the preceding Construction from an Algebraical Solution of the Problem.

FINIS.

On Divisions of Figures

Translated by Raymond Clare Archibald, 1915

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TOMY OLD TEACHER

AND FRIEND

ALFRED DEANE SMITH

PROFESSOR OF GREEK AND LATIN

AT MOUNT ALLISON UNIVERSITY

FOR FORTY-FOUR YEARS

SCHOLAR OF GREAT ATTAINMENTS

THE WONDER OF ALL WHO KNOW HIM

THESE PAGES ARE AFFECTIONATELY DEDICATED

INTRODUCTORY

Euclid, famed founder of the Alexandrian School of Mathematics, was the author of not less than nine works. Approximately complete texts, all carefully edited, of four of these, (1) the *Elements,* (2) the Data, (3) the *Optics,* (4) the *Phenomena,* are now our possession. In the case of (5) the *Pseudaria,* (6) the *Surface-Loci,* (7) the *Conics,* our fragmentary knowledge, derived wholly from Greek sources, makes conjecture as to their content of the vaguest nature. On (8) the *Porisms,* Pappus gives extended comment. As to (9), the book *On Divisions (of figures),* Proclus alone among Greeks makes explanatory reference. But in an Arabian MS., translated by Woepcke into French over sixty years ago, we have not only the enunciations of all of the propositions but also the proofs of four of them.

Whilst elaborate restorations of the *Porisms* by Simson and Chasles have been published, no previous attempt has been made (the pamphlet of Ofterdinger is not forgotten) to restore the proofs of the book *On Divisions (of figures).* And, except for a short sketch in Heath's monumental edition of Euclid's *Elements,* nothing but passing mention of Euclid's book *On Divisions* has appeared in English.

In this little volume I have attempted:

(1) to give, with necessary commentary, a restoration of Euclid's work based on the Woepcke text and on a thirteenth century geometry of Leonardo Pisano.

(2) to take due account of the various questions which arise in connection with (a) certain MSS. of "Muhammed Bagdedinus," *(b)* the Dee-Commandinus book on divisions of figures.

(3) to indicate the writers prior to 1500 who have dealt with propositions of Euclid's work.

(4) to make a selection from the very extensive bibliography of the subject during the past 400 years.

In the historical survey the MSS. of "Muhammed Bagdedinus" play an important rôle, and many recent historians, for example Heiberg, Cantor,

Hankel, Loria, Suter, and Steinschneider, have contributed to the discussion. As it is necessary for me to correct errors, major and minor, of all of these writers, considerable detail has to be given in the first part of the volume; the brief second part treats of writers on divisions before 1500; the third part contains the restoration proper, with its thirty-six propositions. The Appendix deals with literature since 1500.

A score of the propositions are more or less familiar as isolated problems of modern English texts, and are also to be found in many recent English, German and French books and periodicals. But any approximately accurate restoration of the work as a whole, in Euclidean manner, can hardly fail of appeal to anyone interested in elementary geometry or in Greek mathematics of twenty-two centuries ago.

In the spelling of Arabian names, I have followed Suter.

It is a pleasure to have to acknowledge indebtedness to the two foremost living authorities on Greek Mathematics. I refer to Professor J. L. Heiberg of the University of Copenhagen and to Sir Thomas L. Heath of London. Professor Heiberg most kindly sent me the proof pages of the forthcoming concluding volume of Euclid's *Opera Omnia,* which contained the references to Euclid's book *On Divisions of Figures.* To Sir Thomas my debt is great. On nearly every page that follows there is evidence of the influence of his publications; moreover, he has read this little book in proof and set me right at several points, more especially in connection with discussions in Note 113 and Paragraph 50.

R. C. A.

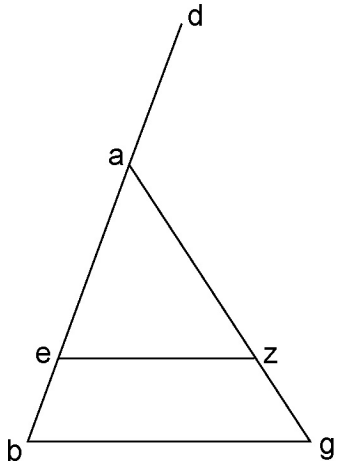
BROWN UNIVERSITY, *June,* 1915.

PROPOSITION 1.

22. “To divide a given triangle into equal parts by a line parallel to its base.” [Leonardo 5, p. 119, ll. 7-9.]

Let *abg* be the given triangle which it is required to bisect by a line parallel to *bg*. Produce *ba* to *d* till *ba = 2ad*. Then in *ba* find a point *e* such that ba: ae = ae: ad.

Through *e* draw *ez* parallel to *bg*; then the triangle *abg* is divided by the line *ez* into two equal parts, of which one is the triangle *aez*, and the other the quadrilateral *ebgz*.



Leonardo then gives three proofs, but as the first and second are practically equivalent, I shall only indicate the second and third.

I. When three lines are proportional, as the first is to the third so is a figure on the first to the similar and similarly situated figure described on the second [VI. 19, “Porism”].

∴ *ba: ad* = figure on *ba:* similar and similarly situated figure on *ae.*

Hence $ba : ad = \triangle abg : \triangle aez$
$= 2 : 1.$

$\therefore \triangle abg = 2\triangle aez.$

II. $ba : ae = ae : ad.$
$\therefore ba \cdot ad = ae^2,$

and since ad is one-half of ba,

$$ba^2 = 2\,ae^2.$$

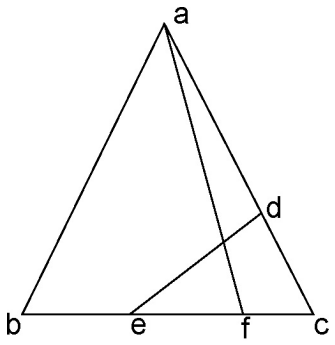
And since bg is parallel to ez,

$ba : ae = ga : az.$
$\therefore ba^2 : ae^2 = ga^2 : az^2.$ [VI. 22]

But $ba^2 = 2\,ae^2.$
$\therefore ga^2 = 2\,az^2.$

Then $ba \cdot ag = 2\,ae \cdot az,$ [VI. 22]
$\therefore \triangle abg = 2\,\triangle aez$ [88].

Then follows a numerical example.



the points *d, e*, respectively. I say that

$$\triangle abc : \triangle dec = ac \cdot cb : dc \cdot ce.$$

Proof: To ac apply the triangle $afc = \triangle dec$. [I. 44]
Since the triangles abc, afc are of the same altitude,

$$bc : fc = \triangle abc : \triangle afc. \qquad [\text{VI. } 1]$$

But
$$bc : fc = ac \cdot bc : ac \cdot fc, \qquad [\text{V. } 15]$$
$$\therefore \triangle abc : \triangle afc = ac \cdot bc : ac \cdot fc,$$

and since $\triangle dec = \triangle acf$,

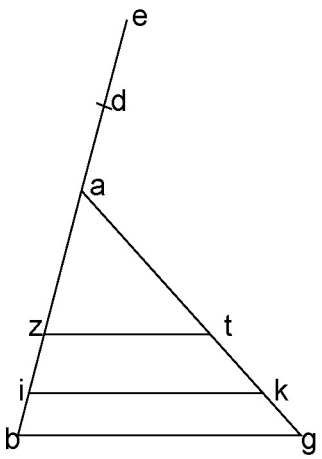
$$\triangle acb : \triangle dce = ac \cdot bc : ac \cdot cf.$$

Again, since the triangles acf, dce are equal and have a common angle, as in the fifteenth theorem of the sixth book of Euclid, the sides are mutually proportional.

$$\therefore ac : dc = ce : cf, \qquad \therefore ac \cdot cf = dc \cdot ce,$$
$$\therefore \triangle acb : \triangle dce = ac \cdot cb : dc \cdot ce.$$

PROPOSITION 2.

23. “To divide a given triangle into three equal parts by two lines parallel to its base” [Leonardo 14, p. 122, l. 8.]



Let *abg* be the given triangle with base *bg*. Produce *ba* to *d* till *ba* = 3 *ad*, and produce *ad* to *e* till *ad* = *de;* then *ae* = ⅔*ba.* Find *az*, a mean proportional between *ba* and *ad*, and *ia* a mean proportional between *ba* and *ae.* Then through z and *i* draw *zt*, *ik* parallel to *bg* and I say that the triangle *abg* is divided into three equal parts of which one is the triangle *azt*, another the quadrilateral *zikt*, the third the quadrilateral *ibgk*.

$$ba : az = az : ad,$$

$$ba : ad = \triangle abg : \triangle azt, \qquad \text{[VI. 19, Porism]}$$

for these triangles are similar.

Now $ba = 3ad; \quad \therefore \triangle abg = 3\triangle azt.$

$$\therefore \triangle azt = \tfrac{1}{3}\triangle abg.$$

Again, $ba : ia = ia : ae;$

Proof: Since $\therefore ba : ae = \triangle$ on ea: similar and similarly situated $\triangle$ on ai.

But triangles *aik, abg* are similar and similarly described on *ai* and *ab*; and

$$ea : ab = 2 : 3.$$

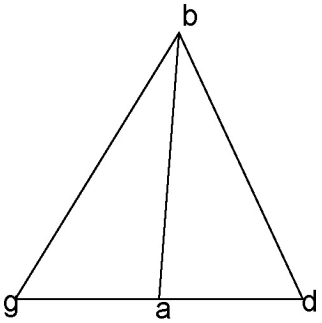
$$\therefore \triangle aik = \tfrac{2}{3}\triangle abg.$$

And since $\triangle azt = \frac{1}{3}\triangle abg$, there remains the quadrilateral $zikt = \frac{1}{3}\triangle abg$. We see that the quadrilateral *ibgk* will be the other third part; hence the triangle *abg* has been divided into three equal parts; "quod oportebat facere."

Leonardo continues: "Et sic per demonstrates modos omnia genera trigonorum possunt diuidi in quatuor partes uel plures" Cf note 45.

PROPOSITION 3.

24. “To divide a given triangle into two equal parts by a line drawn from a given point situated on one of the sides of the triangle” [Leonardo 1, 2, p. no, l. 31; p. 111, ll. 41-43.]

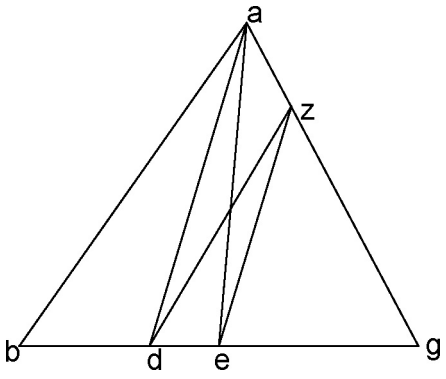


Given the triangle *bgd*; if *a* be the middle point of *gd* the line *ba* will divide the triangle as required; either because the triangles are on equal bases and of the same altitude [I. 38; Leonardo 1], or because

$$\triangle bgd : \triangle bad = bd \cdot dg : bd \cdot da^{88}.$$

Whence $$\triangle bgd = 2\triangle bad.$$

But if the given point be not the middle point of any side, let *abg* be the triangle and *d* the given point nearer to *b* than to *g*. Bisect *bg* at *e* and draw *ad, ae.* Through *e* draw *ez* parallel to *da;* join *dz*. Then the triangle *abg* is bisected by *dz*.



Proof: Since

$$ad \parallel ez, \quad \triangle adz = \triangle ade.$$

To each add $\triangle abd$. Then

$$\begin{aligned} \text{quadl.}abdz &= \triangle abd + \triangle ade, \\ &= \triangle abe. \end{aligned}$$

But $\triangle abe = \frac{1}{2}\triangle abg;$

$\therefore$ quadl.$abdz = \frac{1}{2}\triangle abg;$ and the triangle *zdg* is the other half of the triangle *abg*. Therefore the triangle *abg* is divided into two equal parts by the line *dz* drawn from the point d;

"ut oportebat facere."

Then follows a numerical example.

PROPOSITION 4.

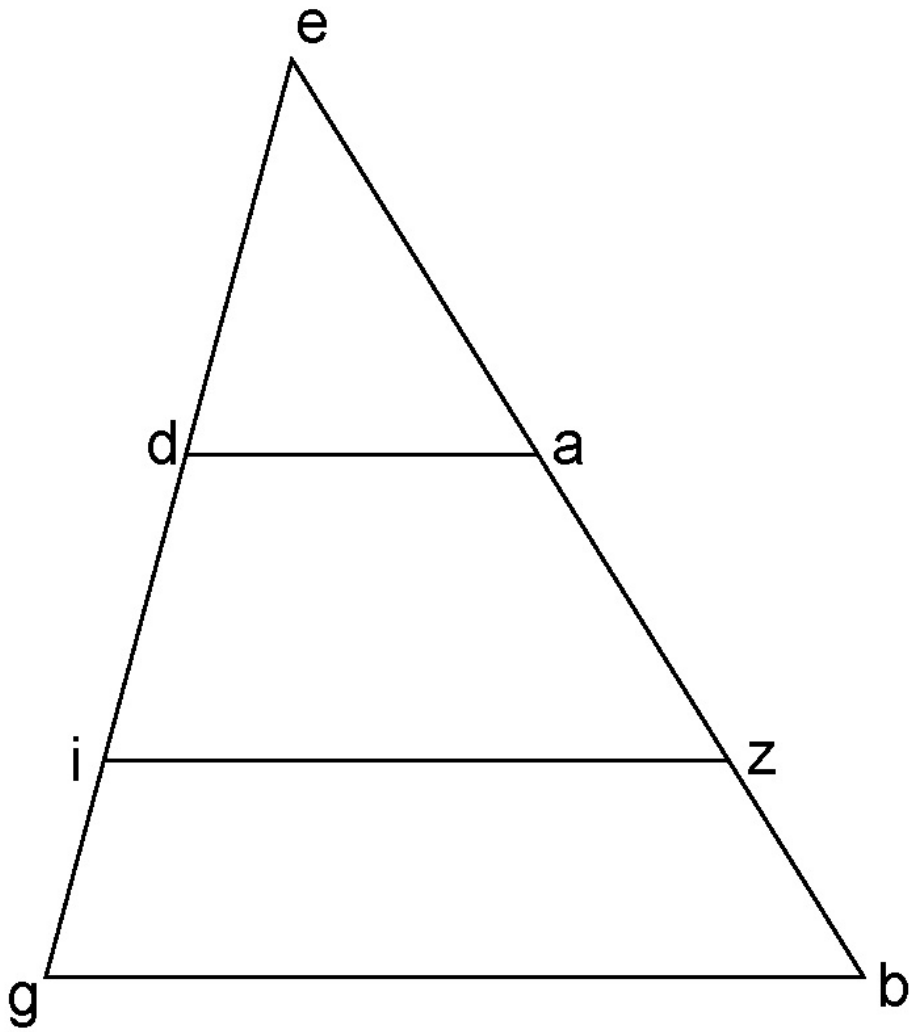
25. “To divide a given trapezium into two equal parts by a line parallel to its base.” [Leonardo 23, p. 125, ll. 37-38.]

Let *abgd* be the given trapezium with parallel sides *ad, bg, ad* being the lesser. It is required to bisect the trapezium by a line parallel to the base *bg*.

Let *gd, ba,* produced, meet in a point *e*. Determine z such that

$$ze^2 = \tfrac{1}{2}(eb^2 + ea^2).^{90}$$

Through z draw *zi* parallel to *gb*. I say that the trapezium *abgd* is divided into two equal parts by the line *zi* parallel to the base *bg*.



Proof: For since

$$2ze^2 = eb^2 + ea^2,$$

and all the triangles are similar,

$$2\triangle ezi = \triangle ebg + \triangle ead. \qquad \text{[VI. 19]}$$

From the triangle ebg take away the triangle ezi. Then

$$\triangle ezi = \text{quadl.}zbgi + \triangle eda.$$

And taking away from the equals the triangle eda, we get

$$\text{quadl.}ai = \text{quadl.}zg.$$

Therefore the trapezium $abgd$ is divided into two equal parts by the line zi parallel to its base. Q. O. F.

A numerical example then follows.

PROPOSITION 5.

26. "And we divide the given trapezium into three equal parts as we divide the triangle, by a construction analogous to the preceding construction." [Leonardo 33, p. 134, ll. 14-15.]

Let *abgd* be the trapezium with parallel sides *ad, bg* and other sides *ba, gd* produced to meet in *e*. Let *zti* be a line such that

$$zi : it = eb^2 : ea^2.$$[92]

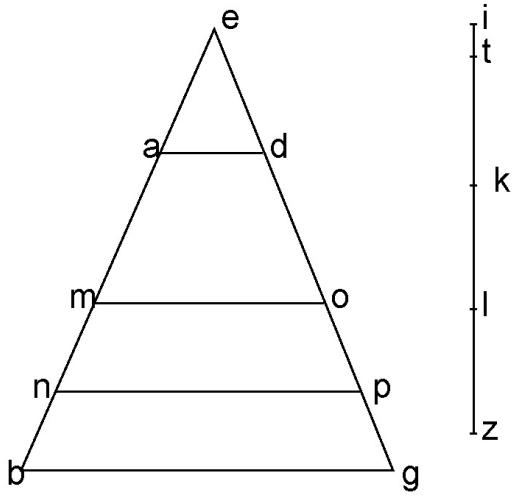
Divide *tz* into three equal parts *tk, kl, lz*. Find in and *n* in *be* such that

$$em^2 : eb^2 = ik : zi,$$
and
$$en^2 : eb^2 = il : zi.$$

Through in and *n* draw *me, up* parallel to the base *bg*. Then I say that the quadrilateral *ag* is divided into three equal parts: *ao, mp, ng*.

Proof: For $eb^2 : ea^2 = \triangle ebg : \triangle ead.$ [VI. 19]

$\therefore zi : it = \triangle ebg : \triangle ead.$ . . . . . [1]



But	$zi : ik = eb^2 : em^2,$	
	$\therefore zi : ik = \triangle ebg : \triangle emo.$	...[2]
So also	$zi : il = \triangle ebg : \triangle enp.$	...[3]
Whence	$it : tk = \triangle ead : \text{quadl.}ao,$ [93]	
and therefore	$tk : kl = \text{quadl.}ao : \text{quadl.}mp.$ [94]	

But	$tk = kl.$ $\therefore \text{quadl.}ao = \text{quadl.}mp.$	
So also	$kl : lz = \text{quadl.}mp : \text{quadl.}ng;$	
and	$kl = lz.$ $\therefore \text{quadl.}mp = \text{quadl.}ng.$	

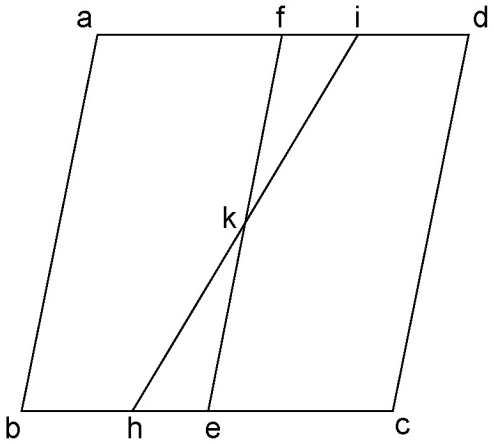
Therefore the quadrilateral is divided into equal quadrilaterals *ao, mp, ng*; "ut prediximus."

Then follows a numerical example.

PROPOSITION 6.

27. "To divide a parallelogram into two equal parts by a straight line drawn from a given point situated on one of the sides of the parallelogram" [Leonardo 16, p. 123, ll. 30-31.]

Let *abcd* be the parallelogram and *i* any point in the side *ad*. Bisect *ad*



in f and bc in e. Join fe. Then the parallelogram ac is divided into equal parallelograms ae, fc on equal bases.

Cut off $eh = fi$. Join hi. Then this is the line required.

Leonardo gives two proofs:

I. Let hi meet fe in k. Then [$\triangle$s fki, hke are equal; add to each the pentagon $kfabh$, etc.]

II. Since ae, fc are $\square$s, $af = be$ and $fd = ec$. But

$$fd = \tfrac{1}{2}ad.$$
$$\therefore fd = af = ec.$$

And since $fi = he, \quad ai = ch.$

So also $di = bh,$ and hi is common.

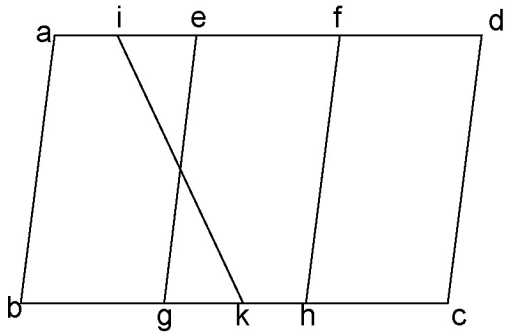
$\therefore$ quadl.$iabh$ = quadl.$ihcd$.[95]

Similarly if the given point were between *a* and *f,* [etc.; or on any other side]. And thus a parallelogram can be divided into two equal parts by a straight line drawn from a given point situated on any one of its sides.

PROPOSITION 7.

28. “To cut off a certain fraction from a given parallelogram by a straight line drawn from a given point situated on one of the sides of the parallelogram” [Leonardo 20 (the case where the fraction is one-third), p. 124, ll. 24-26.]

Let *abcd* be the given parallelogram. Suppose it be required to cut off a third of this parallelogram, by a straight line drawn from *i,* in the side *ad*.



(The figure here is a combination of two in the original.)

Trisect ad in e and f and through e, f draw eg, fh parallel to dc; [then these lines trisect the $\square$. If the point i be in the line ad, at either e or f, then the problem is solved. But if it be between a and e, draw ik to bisect the $\square ah$ (Prop. 6), etc. Similarly if i were between e and f, or between f and d].

After finishing these cases Leonardo concludes:

“eodem modo potest omne paralilogramum diuidi in quatuor uel plures partes equales”

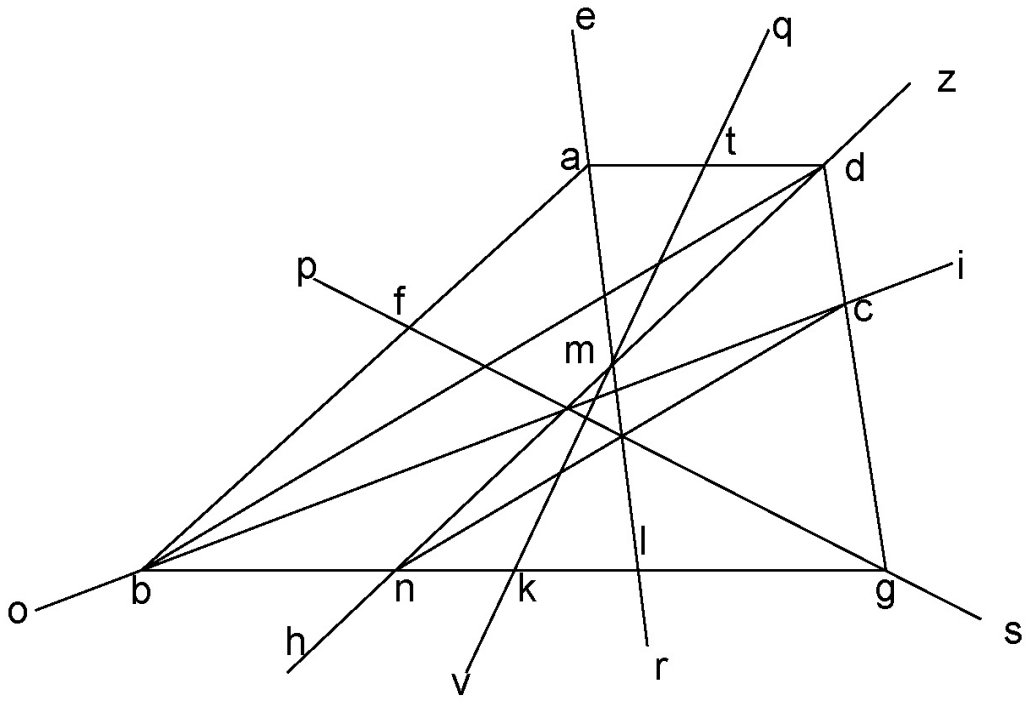
The construction in this proposition is limited to the case where “a certain fraction” is the reciprocal of an integer. But more generally, if the fraction were *m: n* (the ratio of the lengths of two given lines), we could proceed in a very similar way: Divide *ad* in *e,* internally, so that *ae: ed = m: n — in* (*n > m*). In *ad* cut off *ef = ae* and through *f* draw *fh* parallel to *ab.* Then, as before, the problem is reduced to Proposition 6.

If the point *e* should fall at *i* or in the interval *ai* the part cut off from the parallelogram by the required line would be in the form of a triangle which might be determined by I. 44.

PROPOSITION 8.

29. “To divide a given trapezium into two equal parts by a straight line drawn from a given point situated on the longer of the sides of the trapezium” [Part of Leonardo 27, p. 127, ll. 2-3.]

This enunciation means, apparently, “from a given point situated on the longer of the [parallel] sides.” At any rate Leonardo gives constructions for the cases when the given point is on any side. These I shall take up successively. The figure is made from more than one of Leonardo’s, and there is a slight change in the lettering.



Let *ad* be the shorter of the parallel sides *ad, bg*, which are bisected in *t* and *k* respectively. Join *tk*. Then if *bt, gt* be joined, [it is clear, from triangles on equal bases and between the same parallels, that *tk* bisects the trapezium]. [This is Leonardo 24, p. 126, l. 31.]

Next consider the given point as any point on the shorter side [Leonardo 25, p. 127, ll. 2-3].

First let the point be at the angle *a.* Cut off *kl* in kg, equal to *at*. Join *al*, meeting *tk* in *m;* then the quadrilateral is divided as required by *al*. For [the triangles *atm, mkl* are equal in all respects, etc.].

Similarly if *d* were the given point; in *kb* cut off *kn* equal to *td*, and *dn* divides the quadrilateral into two equal parts which is proved as in the preceding case.

[Were the given point anywhere between *a* and *t* the other end of the bisecting line would be between *k* and *l.* Similarly if the given point were between *t* and d, the corresponding point would be between *k* and n.]

Although not observed by Favaro, Leonardo now considers:

If the given point be in the side *bg*; either *l,* or n, or a point between *l* and n, then the above construction is at once applicable.

Suppose, however, that the given point were at *b* or in the segment *bn,* at *g* or in the segment *Ig.* First consider the given point at b. Join *bd* and through *n* draw *nc* parallel to *bd* to meet *gd* in *c.* Join *bc.* Then *bc* bisects the trapezium. For *[abnd* is half of the trapezium *ag*, and the triangle *bnd* equals the triangle *bdc* etc.].

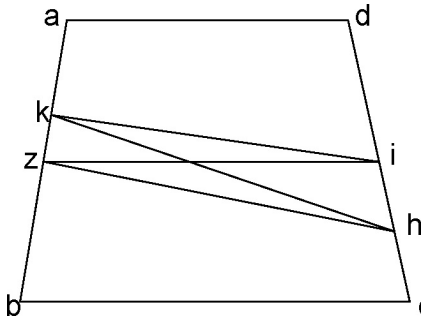
Similarly from a given point between *b* and *n*, a line could be drawn meeting *gd* between *c* and *d,* and dividing the quadrilateral into two equal parts.

So also from *g* a line *gf* could be drawn [etc.]; and similarly for a given point between *g* and *l*.

Leonardo then concludes (p. 127, ll. 37-40):

“Jam ostensum est quomodo in duo equa quadrilatera duorum equidistantium laterum diuidi debeant a linea protracta ab omni dato puncto super lineas equidistantes ipsius; nunc uero ostendamus quomodo diuidantur a linea egrediente a dato puncto super reliqua latera”

This is overlooked by Favaro, though implied in his 27 [Leonardo, p. 129, l. 4]. I may add Leonardo's discussion of the above proposition although it does not seem to be called for by Euclid.



Let the point be in the side *gd*. For *g* or *c* or *d* or any point between *c* and *d* the above constructions clearly suffice. Let us, then, now consider the given point *h* as between *c* and *g*. Draw the line *iz* parallel to *gb* to bisect the trapezium (Prop. 4). Suppose *h* were between *g* and i. Join *zh*. Through *i* draw *ik* parallel to *hz*, and meeting *ab* in *k*.

(The lettering of the original figure is somewhat changed.)

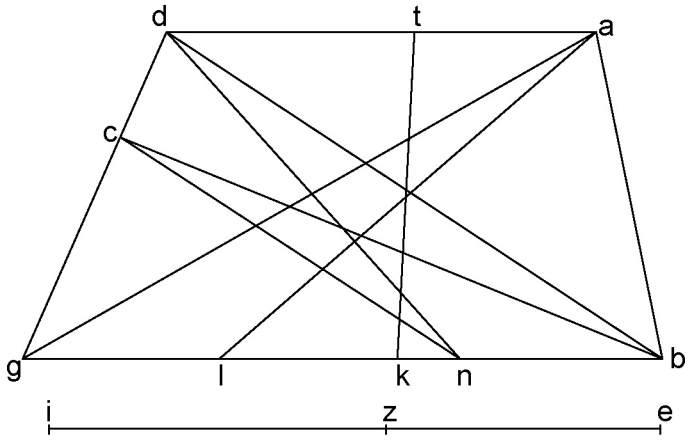
Join *hk*, then [this is the line required; since $\triangle izh = \triangle kzh,$ etc.].

[Similarly if *h* were between *i* and d.] [So also for points on the line ab.]

PROPOSITION 9.

30. “To cut off a certain fraction from a given trapezium by a straight line drawn from a given point situated on the longer side of the trapezium” [Leonardo 30, 3 1, p. 133, ll. 17-19, 31.]

I shall interpret “longer side” as in Proposition 8, and lead up to the consideration of any given point on *bg* after discussing the cases of points on the shorter side *ad*.



(This figure is made from three of Leonardo’s.)

Suppose it be required to divide the trapezium in the ratio *ez: zi*. Divide *ad*, *bg* in the points *t, k,* respectively, such that $at : td = ez : zi = bk : kg.$

Join *tk*. Then by joining *bt* and *gt* [it is easily seen by VI. 1 and V. 12, that the trapezium *ag* is divided by *tk* in the ratio *ez: zi*].

If the given point be at *a* or *d,* make *kl* = *at* and *gn* = *bl*. Join *al, dn.* [Adding the quadrilateral *ak* to the congruent triangles with equal sides *at, kl*, we find *al* divides the trapezium in the required ratio. Then from VI. 1, *dn* does the same.]

> As in Proposition 8, for any point t' between a and t, or t and d, we have a corresponding point k' between l and k or n and k, such that the line $t'k'$ divides the trapezium in the given ratio.

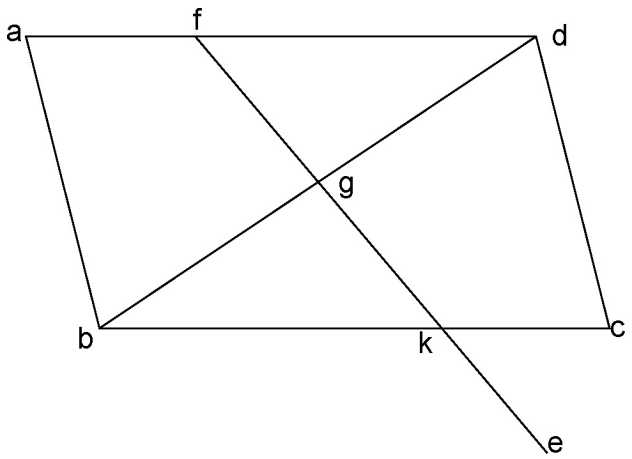
If the given point be in *bg* at *l* or *n* or between *l* and *n,* the above reasoning suffices.

Suppose however that the given point were at *b.* Join *bd*. Through *n* draw *nc* parallel to *bd*. Join *bc.* Then *bc* divides the trapezium in the required ratio. Similarly for the point *g* and for any point between *b* and n, or between *g* and l.

Some of the parts which I have filled in above are covered by the general final statement: “nec non et diuidemus ipsum quadrilaterum ab omni puncto dato super aliquod laterum ipsius—” (Page 134, ll. 10-11. Compare Proposition 13.)

PROPOSITION 10.

31. “To divide a parallelogram into two equal parts by a straight line drawn from a given point outside the parallelogram” [Leonardo 18, p. 124, ll. 5-7.]



Let *abcd* be the given parallelogram and *e* the point outside. Join *bd* and bisect it in *g*. Join *eg* meeting *bc* in *k* and produce it to meet *ad* in *f*. Then the parallelogram has been divided into two equal parts by the line drawn through e, as may be proved by superposition; and one half is the quadrilateral *fabk*, the other, the quadrilateral *fkcd*.

PROPOSITION 11.

32. "To cut off a certain fraction from a parallelogram by a straight line drawn from a given point outside of the parallelogram"

This proposition is not explicitly formulated by Leonardo; but the general method he would have employed seems obvious from what has gone before.

Suppose it were required to cut off one-third of the given parallelogram ac by a line drawn through a point e outside of the parallelogram. Then by the method of Proposition 7, form a parallelogram two-thirds of ac. There are four such parallelograms with centres g_1, g_2, g_3, g_4. Lines l_1, l_2, l_3, l_4 through each one of these points and e will bisect a parallelogram (Proposition 10).

There are several cases to consider with regard to the position of e but it may be readily shown that, in one case at least, there is a line $l_i (i = 1, 2, 3, 4)$, which will cut off a third of the parallelogram ac.

Similarly for one-fourth, one-fifth, or any other fraction such as $m : n$ which represents the ratio of lengths of given lines.

PROPOSITION 12.

33. “To divide a given trapezium into two equal parts by a straight line drawn from a point which is not situated on the longer side of the trapezium. It is necessary that the point be situated beyond the points of concourse of the two sides of the trapezium” [Leonardo 28, p. 129, ll. 2-4, and another, unnumbered.]

PROPOSITION 13.

34. “To cut off a certain fraction from a (parallel-) trapezium by a straight line which passes through a given point lying inside or outside the trapezium but so that a straight line can be drawn through it cutting both *the parallel sides of the trapezium*.” [Part of Leonardo 32, p. 134, ll. 11-12.]

We first take up Leonardo’s discussion of Proposition 12.

In the figure of Proposition 8, suppose *al* to be produced in the directions of the points *e* and r; *tk* in the directions of *q* and v, *dn* of z and *h, cb* of *i* and *o, gf* of *s* and *p.* Then for [any such exterior points e, q, *z, i, s, r, v, h, o, p,* lines are drawn bisecting the trapezium].

If the given point, x, were anywhere in the section of the plane above *ad* and between *ea* and *dz*, the line joining *x* to in would [by the same reasoning as in Proposition 8] bisect the trapezium. Similarly for all points below *nl* and between *hn* and *Ir* [— so also for all points within the triangles amd, nml].

This seems to be all that Euclid’s Proposition 12 calls for. But just as Leonardo considers Proposition 8 for the general case with the given point any- where on the perimeter of the trapezium, so here, he discusses the constructions for drawing a line from any point inside or outside of a trapezium to divide it into two equal parts.

Leonardo does not give any details of the discussion of Euclid’s Proposition 13, but after presentation of the cases given in Proposition 9 concludes: “et diuidemus ipsum quadrilaterum ab omni puncto dato super aliquod lat- erum ipsius, et etiam ab omni puncto dato infra uel extra” [Leonardo 32, p. 134, ll. 10-12].

From Leonardo’s discussion in Propositions 8, 9, 12, not only are the necessary steps for the construction of 13 (indicated in the Woepcke note above) evident, but also those for the more general cases, not considered by Euclid, where restrictions are not imposed on the position of the given point.

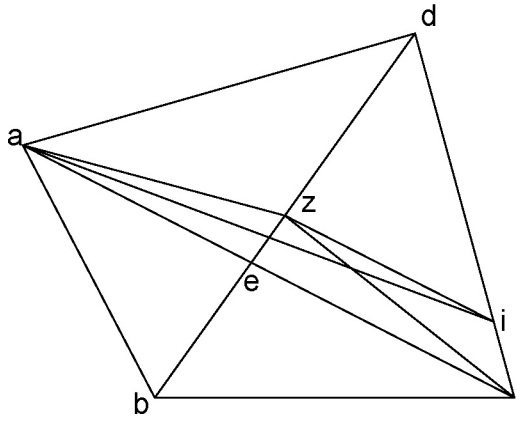
PROPOSITION 14.

35. “To divide a given quadrilateral into two equal parts by a straight line drawn from a given vertex of the quadrilateral” [Leonardo 36, p. 138, ll. 10-11.]

Let *abcd* be the quadrilateral and *a* the given vertex. Draw the diagonal *bd*, meeting the diagonal *ac* in *e.* If *be, ed* are equal, [ac divides the quadrilateral as required].

If *be* be not equal to *ed*, make *bz* = *zd*.

Draw *zi* || *ac* to meet *dc* in *i.* Join *ai*. Then the quadrilateral *abcd* is divided as required by the line *ai*.



Proof: Join *az* and zc. Then the triangles *abz, azd* are respectively equal to the triangles *cbz, cdz*.

Therefore the quadrilateral *abcz* is one-half the quadrilateral *abcd*.

And since the triangles *azc, aic* are on the same base and between the same parallels *ac, zi*, they are equal.

To each add the triangle *abc*.

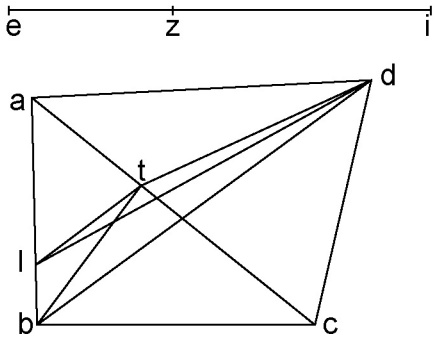
Then the quadrilateral *abcz* is equal to the quadrilateral *abci*. But the quadrilateral *abcz* is one-half of the quadrilateral *abcd*. Therefore *abci* is one-half of the quadrilateral *abcd*;— “ut oportet.”

PROPOSITION 15.

36. “To cut off a certain fraction from a given quadrilateral by a line drawn from a given vertex of the quadrilateral.” [Leonardo 40, p. 140, ll. 36-37.]

Let the given fraction be as *ez: zi*, and let the quadrilateral be *abcd* and the given vertex *d.* Divide *ac* in *t* such that at: tc = ez: zi.

If *bd* pass through *t* [then *bd* is the line required].



But if *bd* do not pass through *t* it will intersect either *ct* or ta; let it intersect *ct*. Join *bt*, *td*.

Then quadl tbcd: quadl tbad = ct: ta = ez: zi.

Draw *tl* parallel to the diagonal *bd*, and join *dl*. Then the quadrilaterals *Ibcd, tbcd* are equal and the construction has been made as required; for ct: ta = ez: zi = quadl. Ibcd: A dal.

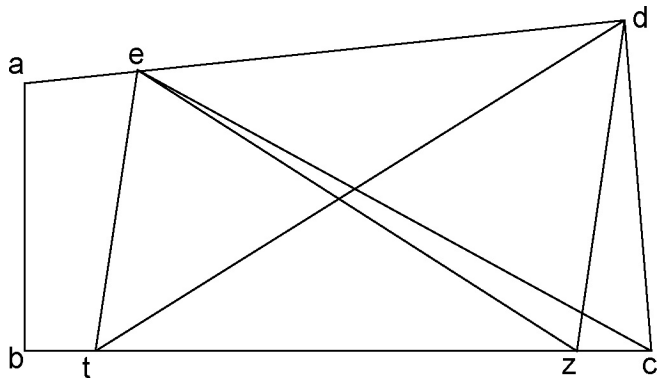
And if *bd* intersect *ta* [a similar construction may be given to divide the given quadrilateral, by a line through d, into a quadrilateral and triangle in the required ratio].

Leonardo then gives the construction for dividing a quadrilateral in a given ratio by a line drawn through a point which divides a side of the quadrilateral in the given ratio.

PROPOSITION 16.

37. “To divide a given quadrilateral into two equal parts by a straight line drawn from a given point situated on one of the sides of the quadrilateral” [Leonardo 37, p. 138, ll. 28-29.]

Let *abcd* be the given quadrilateral, *e* the given point. Divide *ac* into two equal parts by the line *dt* [Prop. 14]. Join *et*. The line *et* either is, or is not, parallel to *dc*.



(Two of Leonardo’s figures are combined in one, here.)

If *et* be parallel to *dc,* join ec. Then the quadrilateral *ac* [is bisected by the line ec, etc.].

If *et* be not parallel to *dc,* draw *dz* || *et*. Join *ez*. Then *ac* [is bisected by the line *ez*, etc.].

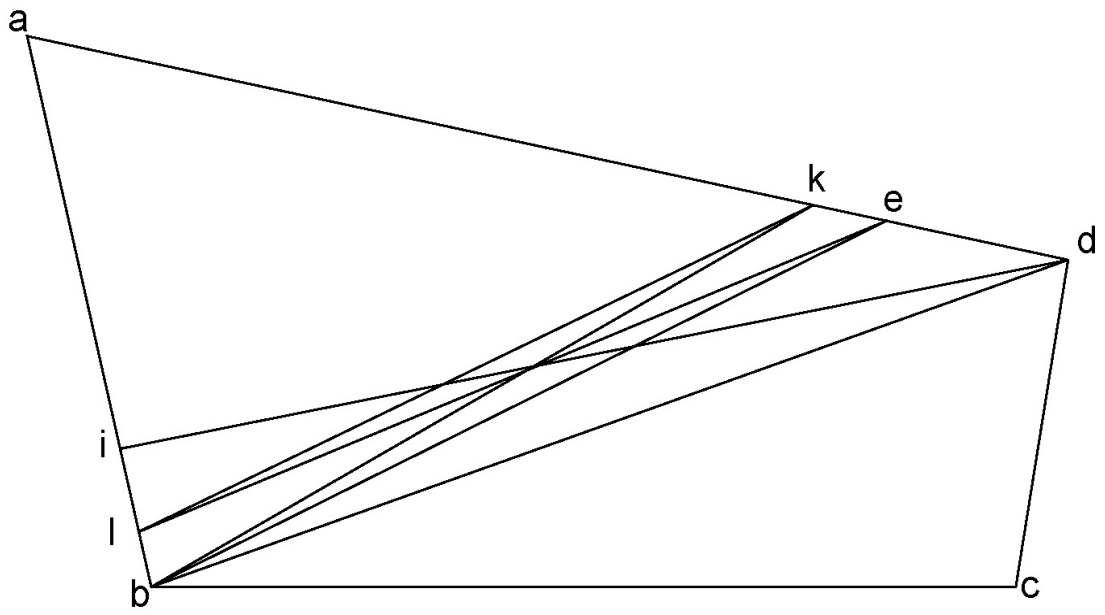
Leonardo does not consider the case of failure of this construction, namely when dz falls outside the quadrilateral. Suppose in such a case that the problem were solved by a line joining e to a point z' (not shown in the figure) on dc. Through t, draw $tt' \parallel cd$. Join ct'. Then $\triangle ct'd = \triangle ctd = \triangle edz'$. Whence $\triangle et'c = \triangle ez'c$, or $t'z' \parallel ce$. Therefore from t', z' may be found and the solution in this case is also possible, indeed in more than one way, but it is not in Euclid's manner to consider this question.

Should the diagonal db bisect the quadrilateral ac, the discussion is similar to the above.

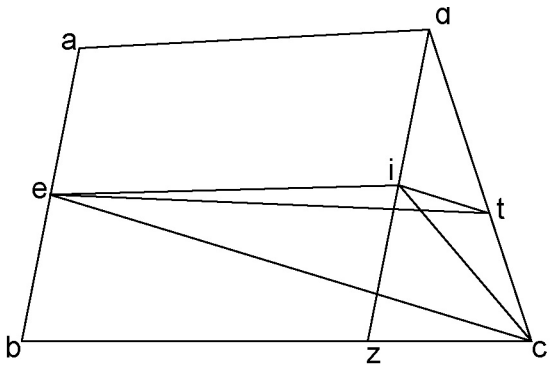
But if the line drawn from d to bisect the quadrilateral meet the side ab in i, draw bk bisecting the quadrilateral ac.

If k be not the given point, it will be between k and d or between k and a.

In the first case join be and through k draw $kl \parallel eb$. Join el [then el is the required bisector].



If the point *e* be between *a* and *k* [a similar construction with the line through *k* parallel to *be,* and meeting *bc* in m, leads to the solution by the line *em*].



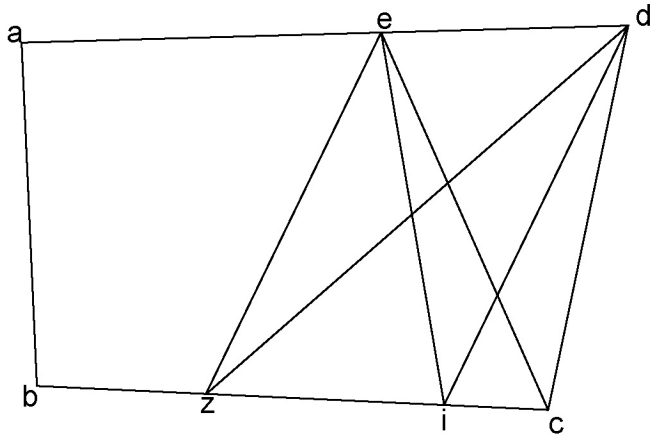
Were e at the middle of a side such as ab, draw $dz \parallel ab$ and bisect dz in i. Join ei, ci and ec. Through i draw $it \parallel ec$. Join et; then et [bisects the quadrilateral ac, since $\triangle itc = \triangle ite$, etc.].

If dz were to fall outside the quadrilateral, draw from c the parallel to ba; and so on.

PROPOSITION 17.

38. "To cut off a certain fraction from a quadrilateral by a straight line drawn from a given point situated on one of the sides of the quadrilateral" [Leonardo 39, p. 140, ll. 11-12.]

Let *abcd* be the given quadrilateral and suppose it be required to cut off one-third by a line drawn from the point *e* in the side *ad*.



Draw dz cutting off one-third of ac [Prop. 15].

Join ez, ec.

If $ez \parallel dc$, then ecd [is the required part cut off, etc.].

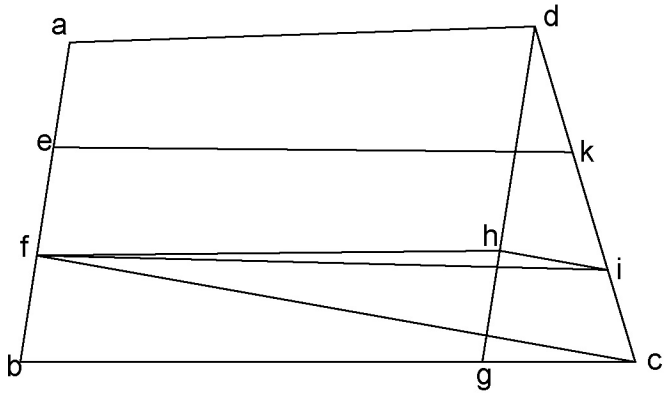
But if ez be not parallel to dc, draw $di \parallel ez$ and join ei. [Then this is the line required, etc.]

The case when ei cuts dc is not taken up but it may be considered as in the last proposition.

So also to divide ac into any ratio: draw dz dividing it in that ratio (Prop. 15), and then proceed as above.

A particular case which Leonardo gives may be added.

Let ab be divided into three equal parts ae, ef, fb; draw $dg \parallel ab$ and cut off $gh = \frac{1}{3}gd$. Join fc and through h draw $hi \parallel fc$, meeting dc in i. Join fi; and the quadrilateral $fbci$ will be one-third of the quadrilateral ac. [As in latter part of Prop. 16.]



Then *ek* may be drawn to bisect the quadrilateral *afid* [Prop. 16], and thus the quadrilateral *abcd* will be divided into three equal portions which are the quadrilaterals *ak*, *ei*, *fc*.

PROPOSITION 18.

39. “To apply to a straight line a rectangle equal to the rectangle contained by AB, AC and deficient by a square.”

“After having done what was required, if some one ask, How is it possible to apply to the line *AB* a rectangle such that the rectangle *AE. EB*

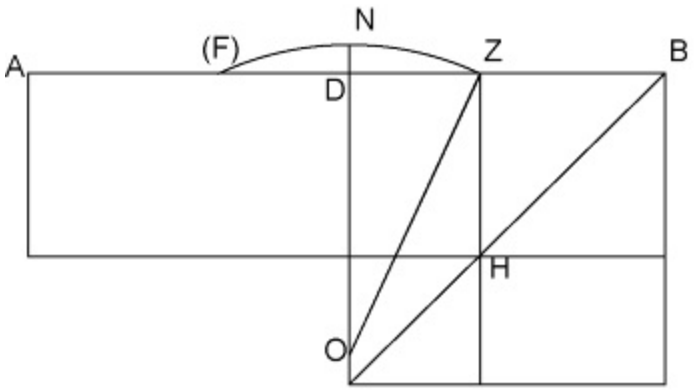
is equal to the rectangle $AB \cdot AC$ and deficient by a square—we say that it is impossible, because AB is greater than BE and AC greater than AE, and consequently the rectangle $BA \cdot AC$ greater than the rectangle $AE \cdot EB$. Then when one applies to the line AB a parallelogram equal to the rectangle $AB \cdot AC$ the rectangle $AZ \cdot ZB$ is.....[104].”

In this problem it is required to find in the given line AB a point Z such that

$$AB \cdot ZB - ZB^2 [= AZ \cdot ZB \text{ by II. 3; cf. x. 16 lemma}] = AB \cdot AC^{105}.$$

Find, by II. 14, the side, b, of a square equal in area to the rectangle *AB. AC*, then the problem is exactly equivalent to that of which a simple solution was given by Simson:

To apply a rectangle which shall be equal to a given square, to a given straight line, deficient by a square: but the given square must not be greater than that upon the half of the given line.



Bisect AB in D, and if the square on AD be equal to the square on b, the thing required is done. But if it be not equal to it, AD must be greater than b according to the determination. Then draw DO perpendicular to AB and equal to b; produce OD to N so that $ON = DB$ (or $\frac{1}{2}a$); and with O as centre and radius ON describe a circle cutting DB in Z.

Then ZB (or x) is found, and therefore the required rectangle AH.

For the rectangle $AZ \cdot ZB$ together with the square on DZ is equal to the square on DB, [II. 5]

i. e. to the square on OZ,

i. e. to the squares on OD, DZ. [I. 47]

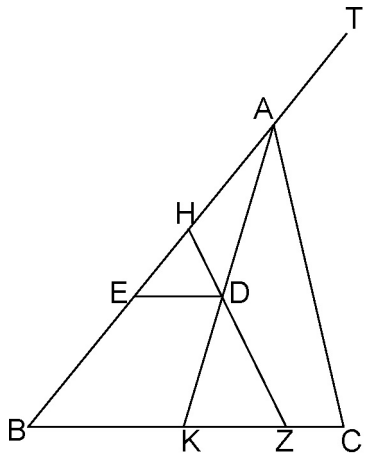
Whence the rectangle $AZ \cdot ZB$ is equal to the square on OD.

Wherefore the rectangle AH equals the given square upon b (i.e.. the rectangle $AB.\ AC$) and has been applied to the given straight line AB, deficient by the square HB.

PROPOSITION 19.

40. “To divide a given triangle into two equal parts by a line which passes through a point situated in the interior of the triangle” [Leonardo 3, p. 115, ll. 7-10.]

“Let the given triangle be *ABC*, and the given point in the interior of this triangle, D.



It is required to draw through *D* a straight line which divides the triangle *ABC* into two equal parts.

Draw from the point *D* a line parallel to the line *BC*, as *DE*, and Apply to *DE* a rectangle equal to half of the rectangle *AB*. *BC*, such as

$$TB \cdot DE \left[TB = \frac{AB \cdot BC}{2DE} \right].$$

Apply to the line TB a parallelogram equal to the rectangle $BT \cdot BE$ and deficient by a square[1071]. [Prop. 18]

Let the rectangle applied be

$$BH \,.\, HT[(TB - HT) \,.\, HT = TB \,.\, BE].$$

Draw the line HD and produce it to Z.

Then this is the line required and the triangle ABC is divided into two equal parts HBZ and $HZCA$.

Demonstration. The rectangle $TB.BE$ is equal to the rectangle $TH.HB$, whence it follows that

$$BT : TH = HB : BE;$$

then *dividendo*[108] $\quad TB : BH = BH : HE.$

But $\quad BH : HE = BZ : ED;$ [VI. 2]

therefore $\quad TB : BH = BZ : ED.$

Consequently the rectangle $TB \,.\, ED$ is equal to the rectangle $BH \,.\, BZ$. But the rectangle $TB \,.\, ED$ is equal to half the rectangle $AB \,.\, BC$; and

$$BH \,.\, BZ : AB \,.\, BC = \triangle HBZ : \triangle ABC^{88},$$

since the angle B is common. The triangle HBZ is, then, half the triangle ABC.

Therefore the triangle ABC is divided into two equal parts BHZ and $AHZC$.

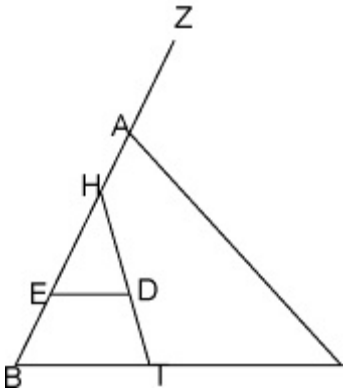
If, in applying to TB a parallelogram equal to the rectangle $TB.BE$ and of which the complement is a square, we obtain the rectangle $AB \,.\, AT$[109],

we may demonstrate in an analogous manner, by drawing the line *AD* and prolonging it to *K*, that the triangle *ABK* is one-half of the triangle *ABC*. And this is what was required to be demonstrated”

PROPOSITION 20.

41. “To cut off a certain fraction from a given triangle by a line drawn from a given point situated in the interior of the triangle” [Leonardo 10, p. 121, ll. 1-2.]

“Let *ABC* be the given triangle and *D* the given point in the interior of the triangle. It is required to pass through the point *D* a straight line which cuts off a certain fraction of the triangle *ABC*.



“Let the certain fraction be one-third. Draw from the point *D* a line parallel to the line *BC*, as *DE*, and apply to *DE* a rectangle equal to one-third of the rectangle *AB*. *BC*. Let this be

$$BZ \,.\, ED \left[BZ = \frac{AB \,.\, BC}{3 \,.\, ED} \right].$$

Then apply to ZB a rectangle equal to the rectangle $ZB \,.\, BE$ and deficient by a square. [Prop. 18.] Let the rectangle applied be the rectangle

$$BH \,.\, HZ \left[(ZB - HZ)\, HZ = ZB \,.\, BE \right].$$

Draw the line *HD* and produce it to *T*.

“On proceeding as above we may demonstrate that the triangle *HTB* is one-third of the triangle *ABC*; and by means of an analogous construction to this we may divide the triangle in any ratio. But this is what it is required to do.”

PROPOSITION 21.

42. “Given the four lines A, B, C, D and that the product of A and D is greater than the product of B and C; I say that the ratio of A to B will be greater than the ratio of C to D”.

Given $A \cdot D > B \cdot C$. To prove $A : B > C : D$.

Let the lines A, D be adjacent sides of a rectangle; and let there be another rectangle with side B lying along A and side C along D. Then either A is greater than B, or D greater than C, for otherwise the rectangle $A \cdot D$ would not be greater than the rectangle $B \cdot C$.

Let then $A > B$. To D apply the rectangle $B \cdot C$ and we get a rectangle

$$A' \cdot D = B \cdot C; \qquad \text{[I. 44–45]}$$

then
$$A' : B = C : D. \qquad \text{[VII. 19]}$$

But since
$$A > A',$$

$$A : B > A' : B; \qquad \text{[V. 8]}$$

$$\therefore A : B > C : D. \qquad \text{[V. 13]}$$

Q. E. D.

Pappus remarks: Conversely if $A : B > C : D$, $A \cdot D > B \cdot C$. The proof follows at once.

For, find A' such that $A' : B = C : D$;

then $A : B > A' : B$,

and $A > A'$. But $A' \cdot D = B \cdot C$. $\therefore A \cdot D > B \cdot C$. Q. E. D.

PROPOSITION 22.

43. “And when the product of A and D is less than the product of B and C, then the ratio of A to B is less than the ratio of C to D.”

From the above proof we evidently have

$$C : D > A : B,$$

that is,

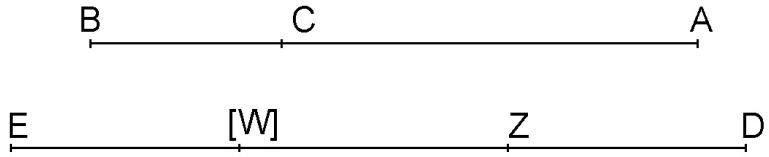
$$A : B < C : D.$$

Conversely, as above, if $A : B < C : D$, $A \cdot B < C \cdot D$.

It is really this converse, and not the proposition, which Euclid uses in Proposition 26. Proclus remarks (page 407) that the converses of Euclid’s Elements, I. 35, 36, about parallelograms, are unnecessary “because it is easy to see that the method would be the same, and therefore the reader may properly be left to prove them for himself.” No doubt similar comment is justifiable here.

PROPOSITION 23.

". "Given any two straight lines and on these lines the points A, B, and D, E; and let the ratio of AB to BC be greater than the ratio of DE: EZ; I say that dividendo the ratio of AC to CB will be greater than the ratio of DZ to ZE"



Given $AB : BC > DE : EZ.$

To prove $AC : CB > DZ : ZE.$

To AB, BC, DE find a fourth proportional $EW.$ [VI. 12]

Then $AB : BC = DE : EW.$ .....(1)

But $AB : BC > DE : EZ;$

$\therefore DE : EW > DE : EZ.$ [V. 13]

$\therefore EW < EZ.$ [V. 8]

From (1) $AC : CB = DW : WE;$ .....(2) [V. 17]

since $DW > DZ, \quad DW : WE > DZ : WE.$ [V. 8]

$\therefore AC : CB > DZ : WE.$ [V. 13]

But $WE < ZE; \quad \therefore DZ : WE > DZ : ZE.$ [V. 8]

$\therefore AC : CB > DZ : ZE.$ From (2) and [V. 17]

Q. E. D.

PROPOSITION 24.

45. "And in an exactly analogous manner I say that when the ratio of AC to CB is greater than the ratio of DZ to ZE, we shall have compo- nendo the ratio of AB to BC is greater than the ratio of DE to EZ"

Given $AC : CB > DZ : ZE.$

To prove $AB : BC > DE : EZ.$

Determine W, as before, such that

$$AB : BC = DE : EW.$$

Then $AC : CB = DW : WE.$ [V. 17]

$\therefore\ DW : WE > DZ : ZE.$ .....(1) [V. 13]

Now either $EW > EZ$ or $EW < EZ.$

If $EW > EZ$, $DW < DZ$, and

$$DW : EW < DZ : EW.$$ [V. 8]

So much the more is

$$DW : EW < DZ : EZ$$ [V. 8]

which contradicts (1).

$$\therefore EW < EZ.$$

But $AB : BC = DE : EW,$

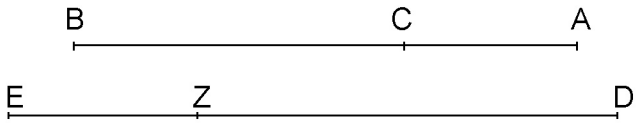
and $DE : EW > DE : EZ;$ [V. 8]

$\therefore AB : BC > DE : EZ.$ [V. 13]

Q. E. D.

PROPOSITION 25.

46. "*Suppose again that the ratio of AB to BC were less than the*



ratio of DE to EZ; dividendo *the ratio of AC to CB will be less than the ratio of DE to ZE*."

Just as the proof of Proposition 22 was contained in that for Proposition 21, so here, the proof required is contained in the proof of Proposition 23. Similarly the converse of Proposition 25 flows out of 24.

PROPOSITION 26.

47. "To divide a given triangle into two equal parts by a line drawn from a given point situated outside the triangle" [Leonardo 4, p. 116, ll. 35-36.]

Let the triangle be *abg* and *d* the point outside.

Join *ad* and let *ad* meet *bg* in *e.* If *be* = *eg,* what was required is done. For the triangles *abe, aeg* being on equal bases and of the same altitude are equal in area.

But if *be* be not equal to *eg*, let it be greater, and draw through *d*, parallel to *bg*, a line meeting *ab* produced in z.

Since $be > \frac{1}{2}bg,$

$$\text{area } ab \cdot be > \tfrac{1}{2} \text{ area } ab \cdot bg; \qquad [Cf. \text{ VII. } 17]$$

much more then is

$$\text{area } ab \cdot zd > \tfrac{1}{2} \text{ area } ab \cdot bg, \qquad \text{since } zd > be.$$

Now take

$$\text{area } ib \cdot zd = \tfrac{1}{2} \text{ area } ab \cdot bg; \qquad [\text{I. } 44]$$

then $\text{area } ab \cdot be > \text{ area } ib \cdot zd,$

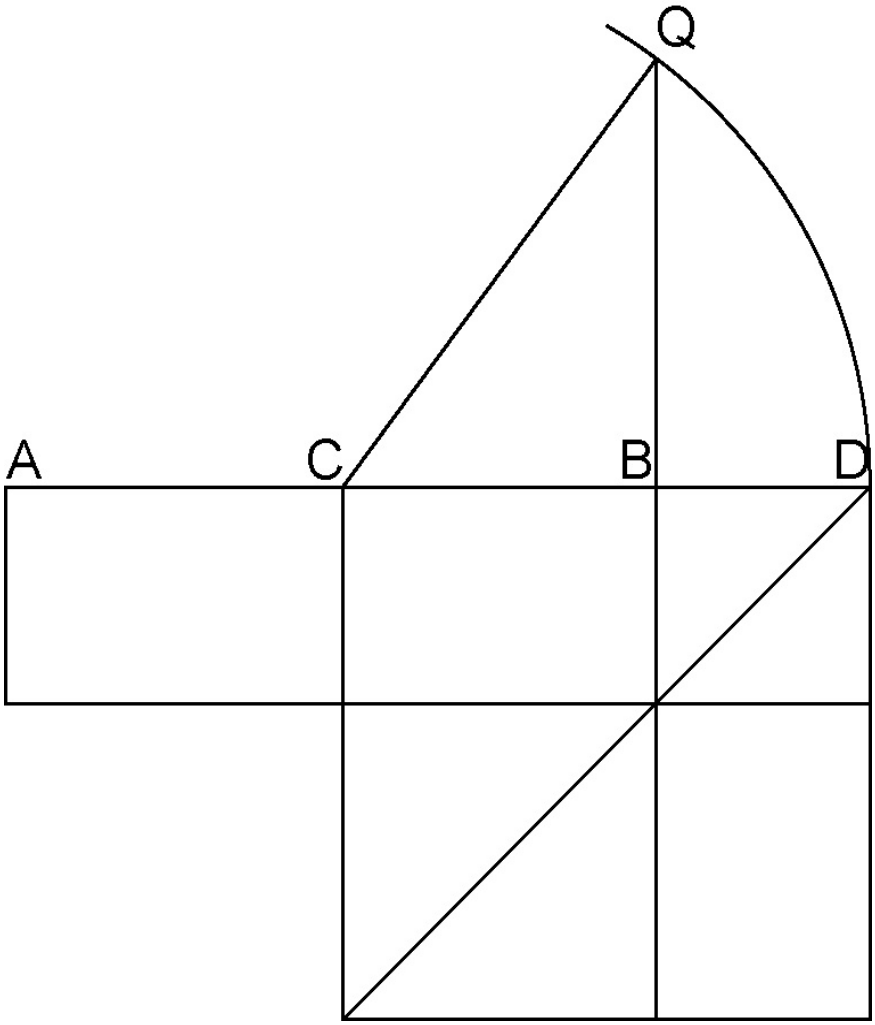
and $zd : be < ba : bi$[114]. [Prop. 21 or 22]

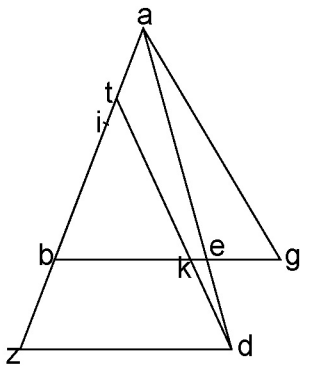
But $zd : be = za : ab,$ [VI. 4]

$\therefore zb : ba < ai : ib;$ [V. 13 and Prop. 25]

or $\text{area } zb \cdot bi < \text{area } ba \cdot ai.$ [Converse of Prop. 22]

Apply a rectangle equal to the rectangle *zb. bi* to the line *bi,* but exceeding by a square; that is to *bi* apply a line such that when multiplied by itself and by *bi* the sum will be equal to the product of *zb* and *bi*; let *ti* be the side of the square.





$$\text{area } zb \,.\, bi = bi \,.\, ti + ti^2 = \text{area } bt \,.\, ti,$$

Draw the straight line *tkd.* Since

$$zb : bt = ti : ib, \qquad [\text{VII. } 19]$$

or $$zt : bt = bt : bi. \qquad [\text{V. } 18]$$

But $$zt : bt = zd : bk, \qquad [\text{VI. } 4]$$

$$\therefore zd : bk = bt : bi,$$

and $$\text{area } kb \cdot bt = \text{area } zd \cdot bi.$$

But $$\text{area } zd \cdot bi = \tfrac{1}{2} \text{ area } ab \cdot bg,$$

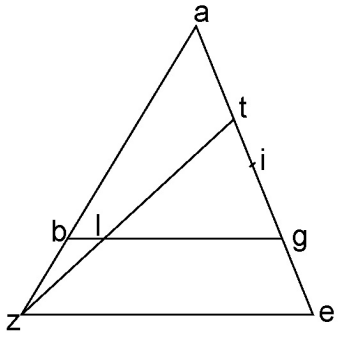
$$\therefore \triangle tbk = \tfrac{1}{2} \triangle abg.$$[88]

Therefore the triangle *abg* is divided by a line drawn from the point *d,* that is, by the line *tkd*, into two equal parts one of which is the triangle *tbk*, and the other the quadrilateral *tkga.*

Q. E. F.

Leonardo now gives a numerical example. He then continues:

[If the point *d* were on one side, *ab,* produced at say, z], through z draw *ze* parallel to *bg* and meeting *ag* produced in e.



Make

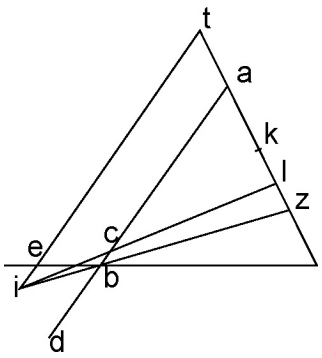
$$\text{area } ze \cdot gi = \tfrac{1}{2} \text{ area } ag \cdot gb,$$

and apply a rectangle, equal to the rectangle $eg \cdot gi$, to the line gi, but exceeded by a square;

then $$eg \cdot gi = gt \cdot ti.$$

Join *tz*, then [this is the required line. The proof is step for step as in the first case].

Leonardo then remarks: “Que etiam demonstrentur in numeris,” and proceeds to a numerical example. Thereafter he continues:



But let the sides *ab, gb* of the triangle be produced to *d* and *e* respectively; and let *i* be the given point in the angle *ebd* from which a line is to be drawn dividing the triangle *abg* into two equal parts. Join *ib* and

produce it to meet ag in z. If $az = zg$, the triangle abg is divided into two equal parts by the line iz. But [if $az > zg$,] let za produced meet, in the point t, the line drawn through i parallel to ab.

Since $za > \frac{1}{2}ag$, area $ab \cdot az > \frac{1}{2}$ area $ba \cdot ag$.
Make area $it \cdot ak = \frac{1}{2}$area $ba \cdot ag$;
then make area $al \cdot kl =$ area $ta \cdot ak$.

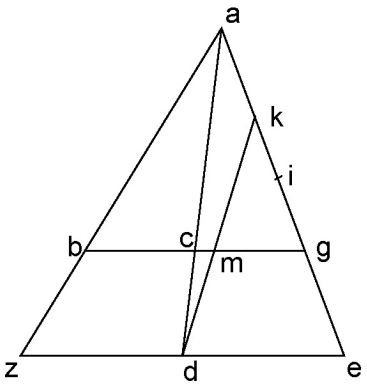
Join *il.* Then as above the triangle *abg* is divided into two equal parts by the line *il*, one part the triangle lac, the other the quadrilateral *lcbg.*

To this statement Leonardo adds nothing further. The proof that *k* lies between *a* and z, and *l* between *k* and *z,* follows as in the first part.

PROPOSITION 27.

48. “To cut off a certain fraction of a triangle by a straight line drawn from a given point situated outside of the triangle “ [Leonardo 11, p. 121, ll. 22-23.]

Let *abg* be the given triangle and *d* the given point outside. It is required to cut off from the triangle a certain fraction, say one-third, by a line drawn through d. Join *ad*, cutting *bg* in *c.* If either *bc* or *cg* be one-third of *bg,* then the line *ad* through the point *d* cuts off one-third of the triangle *abg.* But if this be not the case produce *ab, ag* to meet in z and *e* respectively the line drawn through *d* parallel to *bg*.



Make

$$\text{area } de \cdot gi = \tfrac{1}{3} \text{ area } ag \cdot gb,$$

and apply to the line gi a rectangle equal to the rectangle $eg \cdot gi$, but exceeded by a square; then

$$eg \cdot gi = ik \cdot kg.$$

Draw the line *kmd. I* say that the triangle *kmg* is one-third of the triangle *abg*.

Proof. For since

$$\text{area } eg \cdot gi = \text{ area } gk \cdot ki,$$

$$eg : gk = ki : ig.$$

Hence $ek : gk = gk : gi.$ [V. 17]

But $ek : kg = de : gm;$ [VI. 2]

$$\therefore ed : gm = gk : gi.$$

$$\therefore \text{ area } gk \cdot gm = \text{ area } de \cdot gi.$$

But $\text{area } de \cdot gi = \frac{1}{3} \text{ area } ag \cdot gb;$

$$\therefore \text{ area } gk \cdot gm = \tfrac{1}{3} \text{ area } ag \cdot gb.$$

And since

$$\text{area } gk \cdot gm : \text{ area } ag \cdot gb = \triangle kgm : \triangle agb^{88},$$

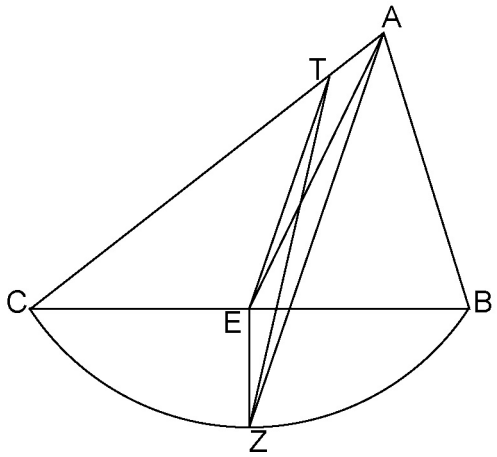
$$\triangle kgm = \tfrac{1}{3} \triangle agb.$$

In a similar manner any part of a triangle may be cut off by a straight line drawn from a given point, on a side of the triangle produced, or within two produced sides.

PROPOSITION 28.

49. “To divide into two equal parts a given figure bounded by an arc of a circle and by two straight lines which form a given angle” [Leonardo 57, p. 148, ll. 13-14.]

“Let *ABC* be the given figure bounded by the arc *BC* and by the two lines *AB, AC* which form the angle *BAC*. It is required to draw a straight line which will divide the figure *ABC* into two equal parts.



“Draw the line *BC* and bisect it at *E*. Through the point *E* draw a line perpendicular to *BC*, as *EZ*, and draw the line *AE*. Then because *BE* is equal to *EC*, the area *BZE* is equal to the area *EZC*, and the triangle *ABE* is equal to the triangle *AEC*. Then the figure *ABZE* will equal the figure *ZCAE*. If the line *AE* lie in *EZ* produced, the figure will be divided into two equal parts *ABZE* and *CAEZ*. But if the line *AE* be not in the line *ZE* produced, join *A* to *Z* by a straight line and through the point *E* draw a line, as *ET*, parallel to the line *AZ*. Finally draw the line *TZ*. I say, that the line *TZ* is that which it is required to find, and that the figure *ABC* is divided into two equal parts *ABZT* and *ZCT*.

“For since the two triangles *TZA* and *EZA* are constructed on the same base *AZ* and contained between the same parallels *AZ*, *TE*: the triangle *ZTA* is equal to the triangle *AEZ*. Then, adding to each the common part *AZB*, we

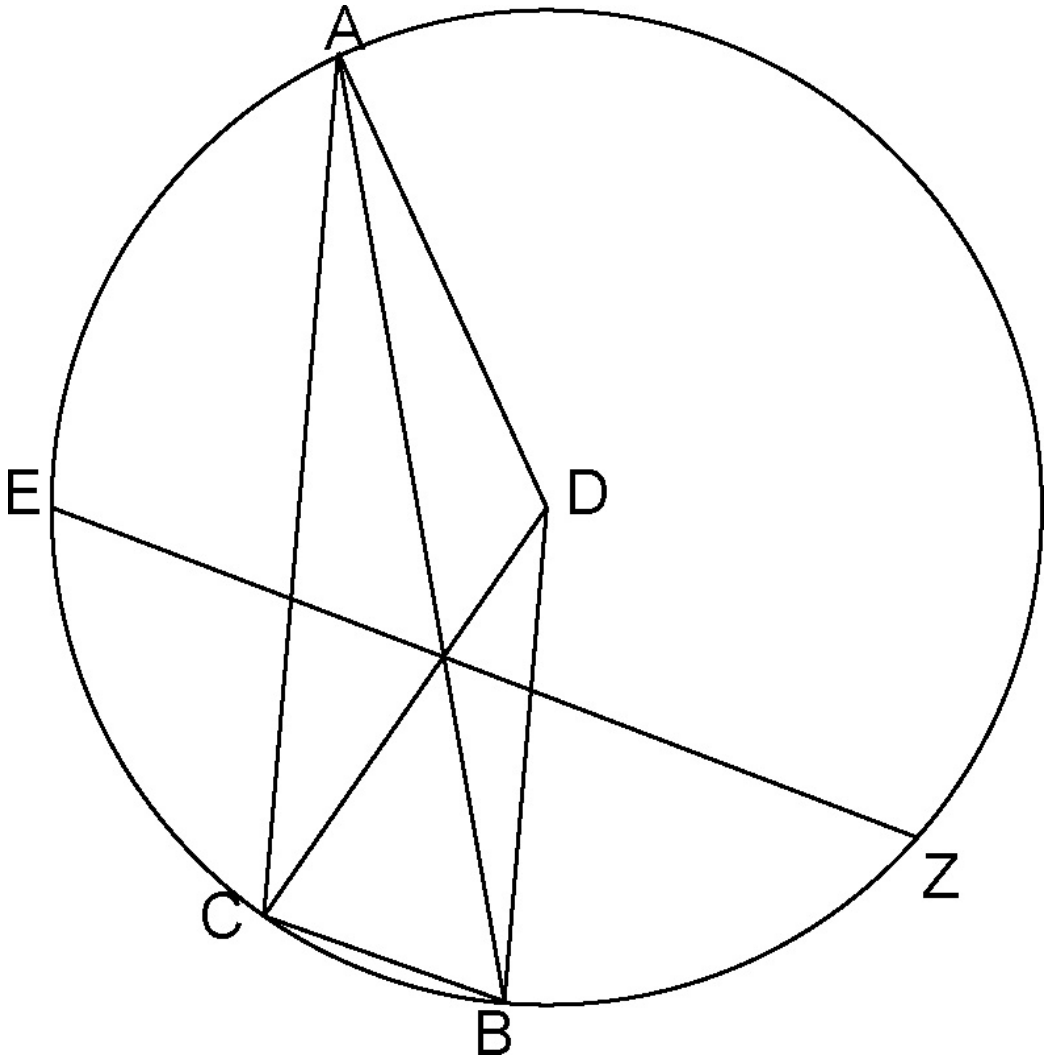
have *TZBA* equal to *ABZE*. But this latter figure was half of the figure *ABC*; consequently the line *ZT* is the line sought and *BZCA* is divided into two equal parts *ABZT, TZC*, which was to be demonstrated" Leonardo's proof is practically word for word as the above. He gives two figures and in each he uses the *Greek* succession of letters.

It is doubtless to this Proposition and the next that reference is made in the account of Proclus [Art. I].

PROPOSITION 29.

50. “To draw in a given circle two parallel lines cutting off a certain fraction from the circle.” [Leonardo 51 (the case where the fraction is one-third), p. 146, ll. 37-38.]

“Let the certain fraction be one-third, and the circle be *ABC*. It is required to do that which is about to be explained.



“Construct the side of the triangle (regular) inscribed in this circle. Let this be *AC*. Draw the two lines *AD*, *DC* and draw through the point *D* a line parallel to the line *AC*, such as *DB*. Draw the line *CB*. Divide the arc *AC* into two equal parts at the point *E*, and draw from the point *E* the chord *EZ*

parallel to the line *BC*. Finally draw the line *AB*. I say that we have two parallel lines *EZ, CB* cutting off a third of the circle *ABC*, viz the figure *ZBCE*.

"*Demonstration.* The line *AC* being parallel to the line *DB*, the triangle *DAC* will be equal to the triangle *BAC*; add to each the segment of the circle *AEC*; the whole figure *DAEC* will be equal to the whole figure *BAEC*. But the figure *DAEC* is one-third of the circle. Consequently the figure *BAEC* is also one-third of the circle. Since *EZ* is parallel to *CB*, the arc *EC* will be equal to the arc *BZ*; but *EC* is equal to EA, hence *EA* equals *ZB*. Add to these equal parts the arc *ECB*; the whole arc *AB* will equal the whole arc *EZ*. Consequently the line *AB* will be equal to the line *EZ*, and the segment of the circle *AECB* will be equal to the segment of the circle *ECBZ*. Taking away the common segment *BC*, there remains the figure *EZBC* equal to the figure *BAEC*. But the figure *BAEC* was one-third of the circle *ABC*. Then the figure *EZBC* is one-third of the circle *ABC*; which was to be demonstrated.

"When it is required to cut off a quarter of a circle, or a fifth or any other definite fraction, by means of two parallel lines, we construct in this circle the side of a square or of the pentagon (regular) inscribed in the circle and we draw from the centre to the extremities of this side the two straight lines as above. (The remainder of) the construction will be analogous to that which has gone before The statement and form of discussion of this proposition are not wholly satisfactory. For "a certain fraction" in the enunciation we should rather expect "one-third," as in Leonardo; while at the conclusion of the proof might possibly occur a remark to the effect that a similar construction would apply when the certain fraction was one-quarter [by means of IV. 6], one-fifth [IV. 11], one-sixth [IV. 15], or one-fifteenth [IV. 16], but is it conceivable that Euclid added "or any other definite fraction"? Moreover the lack of definition of *D* and certain matters of form seem to further indicate that

modification of the original has taken place in its passage through Arabian channels.

On the other hand Leonardo presents the proposition as if drawn from the pure well of Euclid undefiled. Here is his discussion. (I have substituted *C* for his b, and *B* for his g.)

"And if, by means of two parallel lines, we wish to cut off from a circle *ACB,* whose centre is D, a given part which is one-third, draw the line *AC*, the side of an equilateral triangle inscribed in the circle *abg.* Through the centre *D* draw *DB* parallel to this line and join *CB*. Bisect the arc *AC* at *E* and draw *EZ* parallel to *bg*. I say that the figure contained between the lines *CB* and *EZ* and the arcs *EC* and *BZ* is one-third part of the circle *ACB*.

"*Proof:* Draw the lines *DA* and *DB* and *AB*.

"The triangles *BAC* and *DAC* are equal. To each add the portion *ABE*. Then the figure bounded by the lines *BA* and *BC* and the arc *AEC* is equal to the sector *DAEC* which is a third part of the circle *ABC*.

"Therefore the figure bounded by the lines *BA* and *BC* and the arc *AEC* is a third part of the circle.

"And since the lines *CB* and *EZ* are parallel, the arcs *EC* and *BZ* are equal. But arc *EC* is equal to arc *AE*. Therefore arc *AE* is equal to the arc *BZ*. To each add the arc *EB,* and then the arc *AECB* will be equal to the arc *ECBZ*.

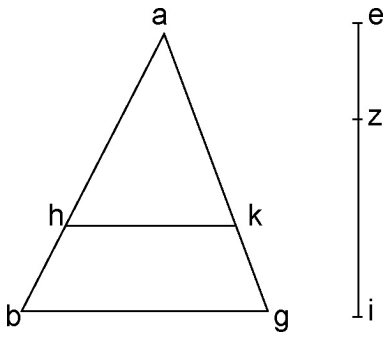
"Hence the portion *EZBC* of the circle is equal to the portion *ABCE*. Take away the common part between the line *CB* and the arc *BC* and there remains the figure, bounded by the lines *BC* and *EZ* and the arcs *CE* and *BZ*, which is the third part of the circle since it is equal to the figure bounded by the lines *BA* and *BC* and the arc *AEC*; quod oportebat ostendere."

In his *μετρικά* (III. 18) Heron of Alexandria considers the problem: *To divide the area of a circle into three equal parts by two straight lines.* He remarks that "it is clear that the problem is not rational"; nevertheless "on

account of its practical use" he proceeds to give an approximate solution. By discussion similar to that above he finds the figure *BCEA,* formed by the triangle *BCA* and the segment *CEA,* to be one-third of the circle. Neglecting the smaller segment with chord *BC*, we have, that *BA* cuts off "approximately" one-third of the circle. Similarly a second chord from *B* might be drawn to cut off another third of the circle, and the approximate solution be completed.

PROPOSITION 30.

51."To divide a given triangle into two parts by a line parallel to its base, such that the ratio of one of the two parts to the other is equal to a given ratio."



Although Leonardo does not explicitly formulate this problem or the next, the method to be employed is clearly indicated in the discussion of Proposition 5 (Art. 26).

Let *abg* be the triangle which is to be divided in the given ratio *ez: zi*, by a line parallel to *bg*. Divide *ab* in *h* such that

$$ah^2 : ab^2 = ez : ei^{92}.$$

Draw $hk \parallel bg$ and meeting ag in k. Then the triangles ahk and abg are similar and

$$\triangle ahk : \triangle abg = ah^2 : ab^2. \qquad [\text{VI. } 19]$$

But $$ah^2 : ab^2 = ez : ei,$$

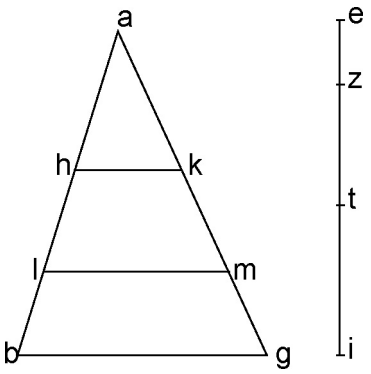
$$\therefore \triangle ahk : \triangle abg = ez : ei;$$

whence $$\triangle ahk : \text{quadl.}hbgk = ez : zi; \qquad [\text{V. } 16, 17]$$

and the triangle *abg* has been divided as required.

PROPOSITION 31.

52. "To divide a given triangle by lines parallel to its base into parts which have given ratios to one another."



Again in the manner of Proposition 5, suppose it be required to divide the triangle *abg* into three parts in the ratio *ez: zt: ti*. Then determine the points *h, l* in *ab* such that

$$ah^2 : ab^2 = ez : ei^{92};$$

and $$al^2 : ab^2 = et : ei.$$

Then $$ah^2 : al^2 = ez : et. \qquad [\text{V. } 16, 20]$$

$$\therefore \triangle ahk : \triangle alm = ez : et,$$

and $$\therefore \triangle ahk : \text{quadl.}hlmk = ez : zt.$$

Similarly

$$\triangle alm : \text{quadl.}lbgm = et : ti.$$

But $$\triangle ahk : \triangle alm = ez : et.$$

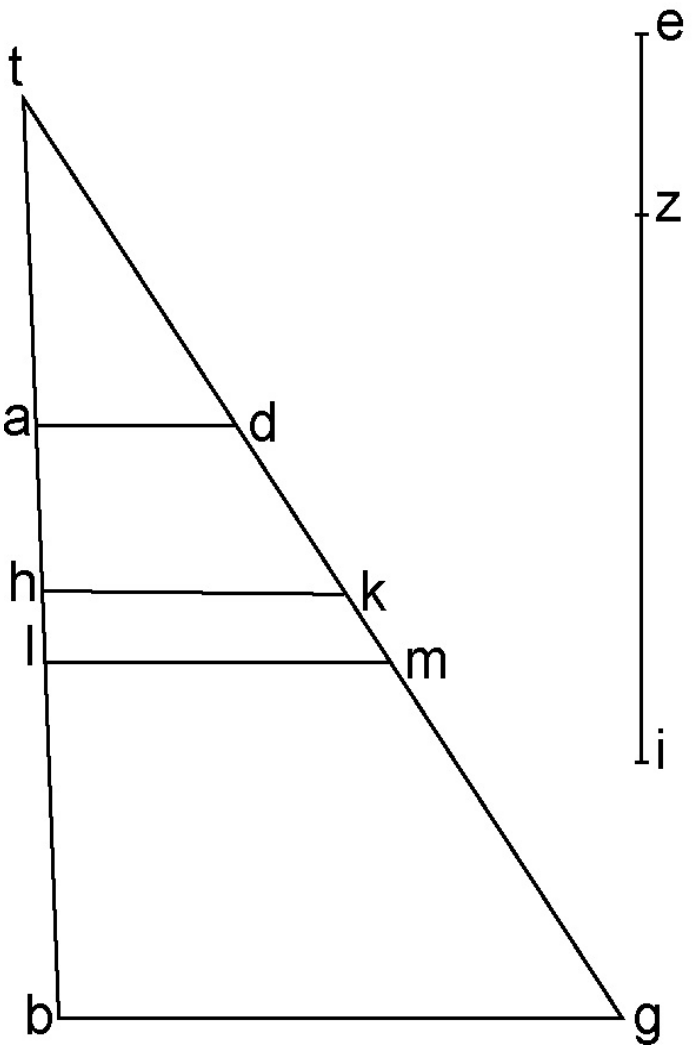
$$\therefore \triangle ahk : \text{quadl.}lbgm = ez : ti. \qquad [\text{V. } 20]$$

Hence, $$\triangle ahk : \text{quadl.}hlmk : \text{quadl.}lbgm = ez : zt : ti,$$ and the triangle *abg* has been divided into three parts in a given ratio to one another. So also for any number of parts which have given ratios to one another.

PROPOSITION 32.

53. “To divide a given trapezium by a line parallel to its base, into two parts such that the ratio of one of these parts to the other is equal to a given ratio” [Leonardo 29, p. 131, ll. 41-42.]

Let *abgd* be the trapezium which is to be divided in the ratio *ez: zi* by a line parallel to the base. Produce the sides *ba, gd* through *a* and *d* to meet in *t.*



Make $$tl^2 : at^2 = zi : ez^{92},$$

and $$ht^2 : (bt^2 + tl^2) = ez : ei.$$

Through l, h, draw lm, hk parallel to bg and ad. Then I say that the quadrilateral ag is divided in the given ratio, $ez : zi$, by the line hk.

Proof: For since the triangles tlm, tad are similar

$$tl^2 : at^2 = \triangle tlm : \triangle tad;$$

but $$tl^2 : at^2 = zi : ez;$$

$$\therefore zi : ez = \triangle tlm : \triangle tad.$$

Whence $$ei : ez = (\triangle tlm + \triangle tad) : \triangle tad, \qquad [\text{V. } 18]$$

or $$ez : ei = \triangle tad : (\triangle tlm + \triangle tad). \qquad [\text{V. } 16]$$

But by construction

$$ez : ei = ht^2 : (bt^2 + tl^2),$$

and $$ht^2 : (bt^2 + tl^2) = \triangle thk : (\triangle tbg + \triangle tlm). \qquad [\text{VI. } 19]$$

$$\therefore ez : ei = \triangle thk : (\triangle tbg + \triangle tlm).$$

But $$\triangle thk = \triangle tad + \text{quadl.}ak.$$

Similarly

$$\triangle tbg + \triangle tlm = \text{quadl.}ag + \triangle tad + \triangle tlm.$$

$$\therefore ez : ei = (\text{quadl.}ak + \triangle tad) : (\text{quadl.}ag + \triangle tad + \triangle tlm).$$

But $$ez : ei = \triangle tad : (\triangle tad + \triangle tlm);$$

$$\therefore ez : ei = \text{quadl.}ak : \text{quadl.}ag; \qquad [\text{V. } 11, 19]$$

whence $$ez : zi = \text{quadl.}ak : \text{quadl.}hg.$$

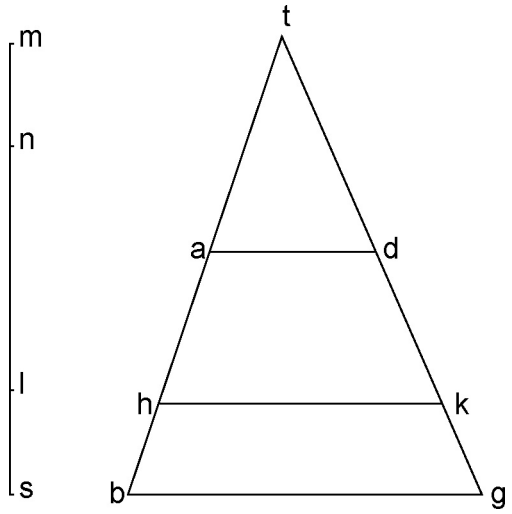
And the trapezium has been divided in the given ratio.

Then follows a numerical example and this alternative construction and proof:

Draw mls such that,

$$ms : ls = tb^2 : ta^2,^{92}$$

and divide ml in n, such that ln is to nm in the given proportion.



In tb determine h such that

$$th^2 : tb^2 = ns : sm.$$

Draw $hk \parallel bg$. Then,

$$\text{quadl.}ak : \text{quadl.}hg = ln : nm.$$

Proof: For, $tb^2 : ta^2 = \triangle tbg : \triangle tad$; [VI. 19]

and $ms : ls = tb^2 : ta^2$.

$\therefore ms : ls = \triangle tbg : \triangle tad$. . . . . [1]

Again, since $tb^2 : th^2 = ms : sn$,

and $\triangle tbg : \triangle thk = tb^2 : th^2$,

$\therefore ms : ns = \triangle tbg : \triangle thk$; . . . . . [2]

$\therefore sm : nm = \triangle tbg : \text{quadl.}hg$. [V. 16, 21] . . . . . [3]

But $sm : ls = \triangle tbg : \triangle tad$[1181],

or $ms : \triangle tbg = ls : \triangle tad$,

while	$ms : \triangle tbg = ns : \triangle thk;$	[from [2]]
	$\therefore ls : ns = \triangle tad : \triangle thk.$	. . . . . [4]
From [3]	$ms : \triangle tbg = nm : \text{quadl.}hg.$	
But from [4]	$sl : ln = \triangle tad : \text{quadl.}ak,$	[v. 16, 21]
	$\therefore sl : \triangle tad = ln : \text{quadl.}ak.$	
But from [1]	$ms : \triangle tbg = sl : \triangle tad,$	
	$\therefore ms : \triangle tbg = ln : \text{quadl.}ak;$	
	$\therefore mn : \text{quadl.}hg = ln : \text{quadl.}ak;$	
	$\therefore ln : nm = \text{quadl.}ak : \text{quadl.}hg.$	

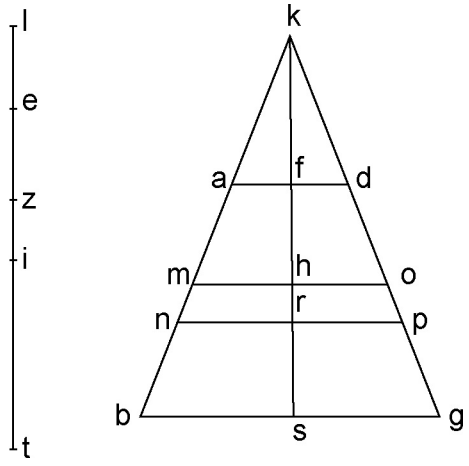
Hence the quadrilateral *ag* is divided by the line *hk,* parallel to the base *bg*, in the given proportion as the number *In* is to the number *nm*. Which was to be done.

Then follows a numerical example.

PROPOSITION 33.

54. "To divide a given trapezium, by lines parallel to its base, into parts which have given ratios to one another" [Leonardo 35, p. 137, ll. 67.]

Let *abgd* be the given trapezium and [let *ez: zi: it* denote



the ratios of the three parts into which the trapezium is to be divided by lines parallel to the base *bg*]. Let *ba, gd* produced meet in *k* and find *l* such that

$$bk^2 : ak^2 = tl : el.$$

Then determine m and n such that

$$bk^2 : km^2 = tl : lz,$$

and

$$bk^2 : kn^2 = tl : il.$$

Through m, n draw lines mo, np parallel to bg. In the same manner as above

$$\text{quadl.}ao : \text{quadl.}mp = ez : zi;$$

and

$$\text{quadl.}mp : \text{quadl.}ng = zi : it.$$

Then follows a numerical example in which the line *kfhrs,* perpendicular to *bg,* is introduced into the figure.

Here is a proof of the Proposition:

By construction, V. 16 and VI. 19,

$\triangle kbg : \triangle kad = tl : el.$ ..... [1]

So also $\triangle kbg : \triangle kmo = tl : lz,$ ..... [2]

and $\triangle kbg : \triangle knp = tl : il$ ..... [3]

From [1] $\triangle kad : \triangle kbg = el : tl.$

But from [2] $\triangle kbg : \triangle kmo = tl : lz;$

hence, by [V. 20], $\triangle kad : \triangle kmo = el : lz,$ ..... [4]

or alternately $\triangle kmo : \triangle kad = lz : el.$

Hence, *separando*, $\text{quadl.}ao : \triangle kad = ez : el.$ ..... [5]

So also from [2] and [3]

$\triangle kmo : \triangle knp = lz : il;$

and $\triangle kmo : \text{quadl.}mp = lz : iz.$

But from [4] $\triangle kad : \triangle kmo = el : lz;$

therefore, by [V. 20], $\triangle kad : \text{quadl.}mp = el : iz$ ..... [6]

Hence from [5], by [V. 20],

$\text{quadl.}ao : \text{quadl.}mp = ez : zi.$

Again, from [3], $\text{quadl.}ng : \triangle kbg = ti : tl;$

and since from [1], $\triangle kbg : \triangle kad = tl : el,$

we have $\text{quadl.}ng : \triangle kad = it : el.$

Hence from [6], by [V. 20], we get

$\text{quadl.}ng : \text{quadl.}mp = it : zi,$

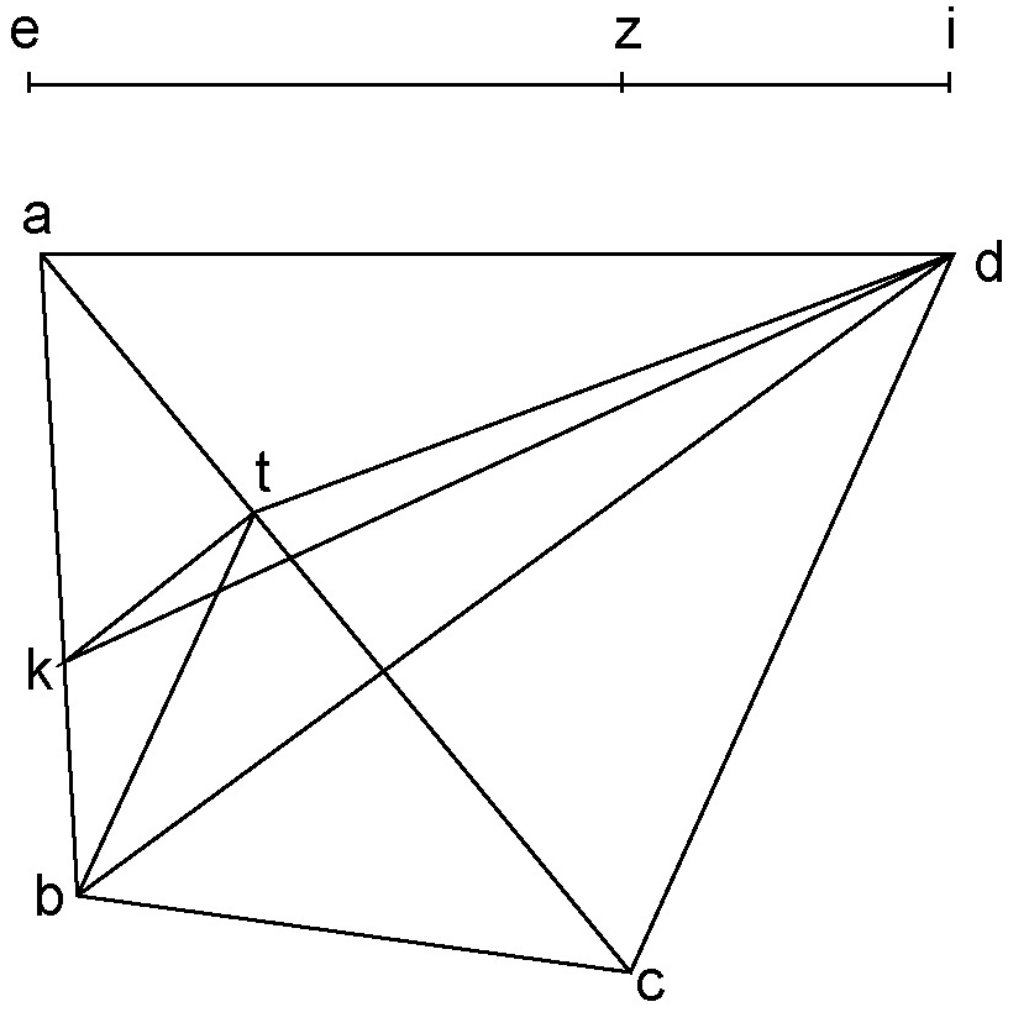
or alternately $\text{quadl.}mp : \text{quadl.}ng = zi : it.$

And since $\text{quadl.}ao : \text{quadl.}mp = ez : zi$, the trapezium ag has been divided by lines parallel to the base ag, into three parts which are in the required ratios to one another. Q. E. F.

PROPOSITION 34.

55. “To divide a given quadrilateral, by a line drawn from a given vertex of the quadrilateral, into two parts such that the ratio of one of these parts to the other is equal to a given ratio” [Leonardo 40, p. 140, ll. 36-37.]

Let *abcd* be the given quadrilateral, and *ez: zi* the given ratio. It is required to draw from the angle *d* a line to divide the quadrilateral in the ratio *ez: zi*.



Draw the diagonal ac and on it find t such that

$$ct : at = ez : zi.$$

Draw the diagonal bd. Then if bd pass through t the quadrilateral is divided as required, in the ratio $ez : zi$.

For,

$$\begin{aligned} \triangle dct : \triangle dta &= ct : ta, \\ &= \triangle cbt : \triangle tba. \\ \therefore ct : ta &= \triangle dcb : \triangle abd. \qquad [\text{V. } 18] \end{aligned}$$

But $$ct : ta = ez : zi;$$

$$\therefore ez : zi = \triangle bdc : \triangle bda;$$

and the quadrilateral *ac* is divided, by a line drawn from a given angle, in a given ratio.

But if *bd* do not pass through *t,* it will cut *ca* either between *c* and *t* or between *t* and *a.* Consider first when *bd* cuts *ct*. Join *bt* and *td*. Then,

$$\text{quadl.}tbcd : \text{quadl.}tbad = ct : ta = ez : zi.$$

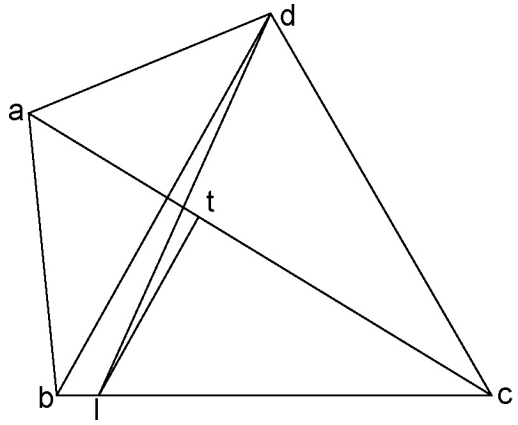
Draw $tk \parallel bd$ and join dk. Then

$$\begin{aligned} \text{quadl.}kbcd &= \text{quadl.}tbcd; \\ \therefore \text{quadl.}kbcd &: \triangle dak = ez : zi, \end{aligned}$$

and the line *dk* has been drawn as required.

If the diagonal *bd* cut *at*, through *t* draw *tl* parallel to the diagonal *bd*. Join *dl*. Then as before,

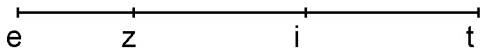
$$ct : ta = ez : zi = \triangle dcl : \text{quadl.}abld.$$



Hence in every case the quadrilateral has been divided as required by a line drawn from d. Similarly for any other vertex of the quadrilateral.

PROPOSITION 35.

56. "*To divide a given quadrilateral by lines drawn from a given vertex of the quadrilateral into parts which are in given ratios to one another*" Although Leonardo does not explicitly formulate this problem, the method he would have followed is clear from his discussion of the last Proposition. Let *abcd* be the quadrilateral to be divided, by lines drawn from d, into three parts in the ratios to one another of *ez: zi: it*.



Divide ca at points r, t so that

$$cr : rt : ta = ez : zi : it.$$

Through *r, t* draw lines parallel to *bd*, and meeting *bc* (or *ab)* in *l* and *ab* (or *bc)* in *m.*

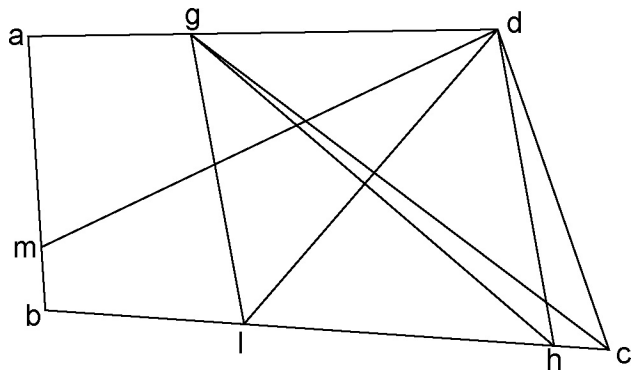
Then as above *dl*, *dm* divide the quadrilateral as required.

We may proceed in a similar manner to divide the quadrilateral *abcd*, by lines drawn from the angular point *d*, into any number of parts in given ratios to one another.

PROPOSITION 36.

57. "Having resolved those problems which have gone before, we are in a position to divide a given quadrilateral in a given ratio or in given ratios by a line or by lines drawn from a given point situated on one of the sides of the quadrilateral, due regard being paid to the conditions mentioned above."

This problem, also, is not formulated by Leonardo; but from his discussion of Euclid's Propositions 16, 17 and of his own 41, the methods of construction which Euclid might have employed are clearly somewhat as follows.



Let *abcd* be the given quadrilateral and *g* the given point.

(1) Let it be required to divide $abcd$ into two parts in the ratio $ez : zi$ by a line drawn through a point g in the side ad.

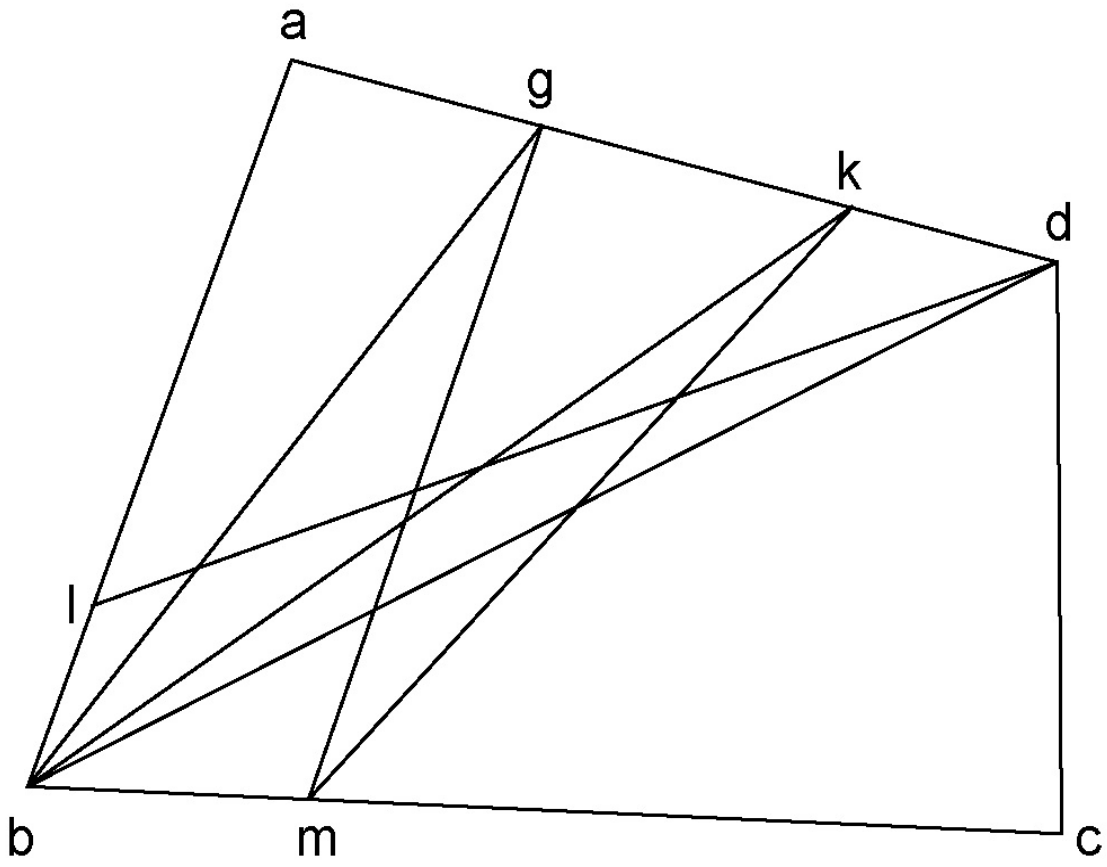
Draw dl such that $\triangle cld : \text{quadl.}lbad = ez : zi$. [Prop. 34]

Join gl. If $gl \parallel dc$, join gc, then this line divides the quadrilateral as required.

If gl be not parallel to dc draw $dh \parallel gl$, and meeting bc in h. Join gh. Then gh divides the quadrilateral as required.

If dh fell outside the quadrilateral draw $ll' \parallel cd$ (not indicated in the figure) to meet ad in l'. Draw $l'z' \parallel gc$ to meet dc in z'. Then gz' is the line required.

The above reasoning is on the assumption that *dl* meets *bc* in l. Suppose now it meet *ab* in l. Join *bd* and draw *bk* such that $\text{quadl.}bcdk : \triangle kab = ez : zi$.



If *k* do not coincide with *g* there are two cases to consider according as *k* is between *g* and *d* or between *g* and *a.* Consider the former case.

Through *k* draw *km* parallel to *gb* and meeting *bc* in *m.* Join *gm*. Then this is the required line dividing the quadrilateral *ac* in such a way that

$$\text{quadl.}abmg : \text{quadl.}gmcd = ez : zi.$$

Similarly if *k* were between *g* and *a*.

(2) Let it be required to divide *abcd* into, say, three parts in the ratios *ez: zi: it*, by lines through any point *g* in the side *ad* (first figure).

Draw *dl, dm* dividing the quadrilateral *ac* into three parts such that

$$\triangle amd : \text{quadl.}dmbl : \triangle dlc = ez : zi : it.$$

There are various cases to consider according as *l* and in are both on *bc,* both on *ab,* or one on *ab* and one on *bc.* The method will be obvious from working out one case, say the last.

Join *gc, gl*. If *gl* be parallel to *cd*, *gc* cuts off the triangle *gdc* such that

$$\triangle gdc : \text{quadl.}abcg = it : ei \qquad (= ez + zi). \qquad \text{[V. 24]}$$

If *gl* be not parallel to *dc,* draw *dh* parallel to *gl* and meeting *bc* in h; then *gh* divides the quadrilateral in such a way that $\text{quadl.}gdch : \text{quadl.}ghba = it : ei.$

Then apply Proposition 34 to draw from *g* a line to divide the quadrilateral *abhg* in the ratio of *ez: zi*.

Hence from *g* are drawn two lines which divide the quadrilateral *abcd* into three parts whose areas are in the ratios of *ez: zi: it*.

The case when *dh* meets *bc* produced may be considered as above.

We could proceed in a similar manner if the quadrilateral *abcd* were to be divided by lines drawn from *g*, into a greater number of parts in given ratios.

The enunciation of this proposition is a manifest corruption of what Euclid may have given. Such clauses as those at the beginning and end he would only have included in the discussion of the construction and proof.

After the enunciation of Proposition 36, Woepcke's translation of the Arabian MS. concludes as follows:

"End of the treatise. We have confined ourselves to giving the enunciations without the demonstrations, because the demonstrations are easy"

Optics

Translated by Harry Edwin Burton, 1945

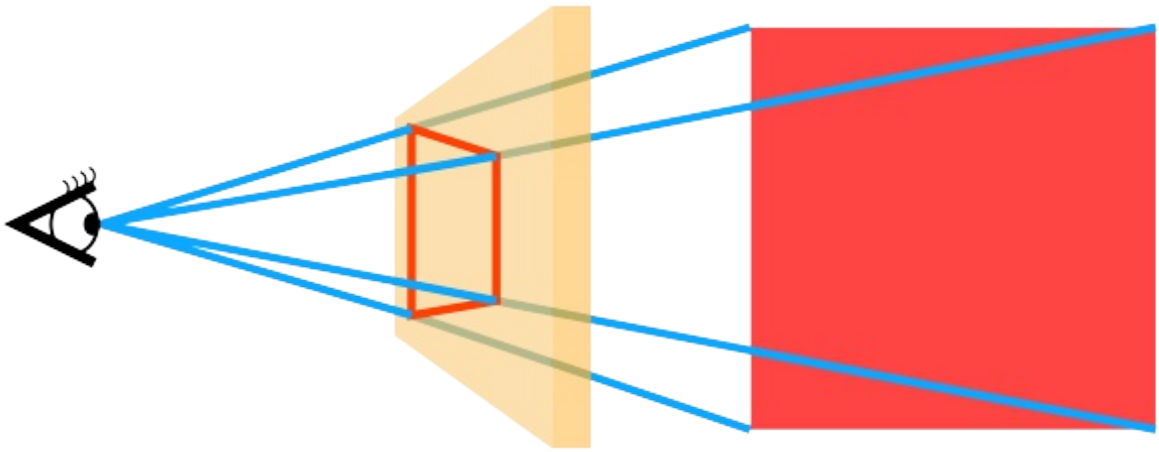
Euclid's book on Ὀπτικά deals chiefly with the geometry of vision, with little reference to either the physical or psychological aspects of sight. No Western scientist had previously given mathematical attention to vision and Euclid's *Optics* would go on to influence the work of later Greek, Islamic and Western European Renaissance scientists and artists. Ptolemy used Euclid's mathematical treatment of vision and his idea of a visual cone in combination with physical theories in his own *Optics*. Renaissance artists such as Brunelleschi, Alberti and Dürer were inspired by Euclid's *Optics* when producing their own work on linear perspective.

Although several writers before Euclid had developed theories of vision, their works were mostly philosophical in nature, lacking the mathematics that Euclid introduced in his work for the first time. Prior to Euclid, such works were concerned primarily with the physical dimension of vision. Whereas Plato and Empedocles thought of the visual ray as "luminous and ethereal emanation", Euclid's treatment of vision in a mathematical manner formed part of the larger Hellenistic trend to quantify a whole range of scientific fields.

Similar to Euclid's more famous work on geometry, *Elements*, the treatise opens with a small number of definitions and postulates, which are then used to prove, by deductive reasoning, a body of geometric propositions about vision. According to Euclid, the eye sees objects that are within its visual cone. The visual cone is made up of straight lines, or visual rays, extending outward from the eye. These visual rays are discrete, but we perceive a

continuous image since our eyes move very quickly. As visual rays are discrete, it is possible for small objects to lie unseen between them. For example, this accounts for the difficulty in searching for a dropped needle. Although the needle may be within one's field of view, until the eye's visual rays fall upon the needle, it will not be seen. Discrete visual rays also explain the sharp or blurred appearance of objects.

The text considers perspective and how an object appears in space relative to the eye. For example, in proposition 8, Euclid argues that the perceived size of an object is not related to its distance from the eye by a simple proportion.



Euclid suggests that visual rays proceed from the eyes onto objects, and that the different visual properties of the objects were determined by how the visual rays struck them. Here the red square is an actual object, while the yellow plane shows how the object is perceived.

DEFINITIONS

1. Let it be assumed that lines drawn directly from the eye pass through a space of great extent;

2. and that the form of the space included within our vision is a cone, with its apex in the eye and its base at the limits of our vision;

3. and that those things upon which the vision falls are seen, and that those things upon which the vision does not fall are not seen;

4. and that those things seen within a larger angle appear larger, and those seen within a smaller angle appear smaller, and those seen within equal angles appear to be of the same size;

5. and that things seen within the higher visual range appear higher, while those within the lower range appear lower;

6. and, similarly, that those seen within the visual range on the right appear on the right, while those within that on the left appear on the left;

7. but that things seen within several angles appear to be more clear.

Nothing that is seen is seen at once in its entirety. (Fig. 1).

For let the thing seen be *AD,* and let the eye be *B,* from which let the rays of vision fall, *BA, BG, BK,* and *BD.* So, since the rays of vision, as they fall, diverge from one another, they could not fall in continuous line upon *AD;* so that there would be spaces also in *AD* upon which the rays of vision would not fall. So *AD* will not be seen in its entirety at the same time. But it seems to be seen all at once because the rays of vision shift rapidly.

Objects located nearby are seen more clearly than objects of equal size located at a distance. (Fig. 2.)

Let *B* represent the eye and let *GD* and *KL* represent the objects seen; and we must understand that they are equal and parallel, and let *GD* be nearer to the eye; and let the rays of vision fall, *BG, BD, BK,* and *BL.* For we could not say that the rays falling from the eye upon *KL* will pass through the points *G* and *D.* For in the triangle *BDLKGB* the line *KL* would be longer than the line *GD;* but they are supposed to be of equal length. So *GD* is seen by more rays of the eye than *KL.* So *GD* will appear more clear than *KL;* for objects seen within more angles appear more clear.

Every object seen has a certain limit of distance, and when this is reached it is seen no longer. (Fig. 3.)

For let the eye be *B,* and let the object seen be *GD.* I say that *GD,* placed at a certain distance, will be seen no longer. For let *GD* lie midway in the divergence of the rays, at the limit of which is *K.* So, none of the rays from *B* will fall upon *K.* And the thing upon which rays do not fall is not seen. Therefore, every object seen has a certain limit of distance, and, when this is reached, the object is seen no longer.

Of equal spaces located upon the same straight line, those seen from a greater distance appear shorter. (Fig. 4.)

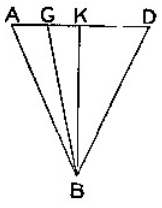


FIG. 1.

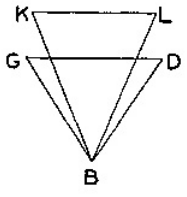


FIG. 2.

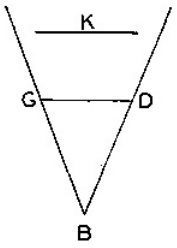


FIG. 3.

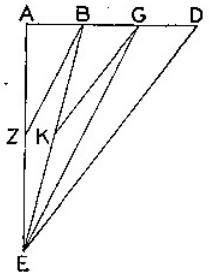


FIG. 4.

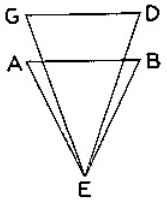


FIG. 5.

Let *AB, BG,* and *GD* represent equal spaces upon one straight line, and let the perpendicular *AE* be drawn, upon which let *E* represent the eye. I say that *AB* will appear longer than *BG* and *BG* longer than *GD.* For let the rays fall,

EB, EG, and *ED,* and through the point *B* let *BZ* be drawn parallel to the straight line *GE.* Now *AZ* is equal to *EZ.* For, since parallel to *GE,* one side of the triangle *AEG,* the straight line *BZ* has been drawn, it follows also that *EZ* is related to *ZA* as *GB* to *BA.* So, as has been said, *AZ* is equal to *ZE.* But the side *BZ* is longer than *ZA*; and so, it is longer than *ZE.* Therefore the angle *ZEB* is greater than the angle *ZBE;* and the angle *ZBE* is equal to the angle *BEG*", and the angle *ZEB* is greater than the angle *GEB.* So, *AB* will seem longer than *BG.* Again in the same way, if through the point *G* a line parallel to *DE* is drawn, *BG* will seem longer than *GD.*

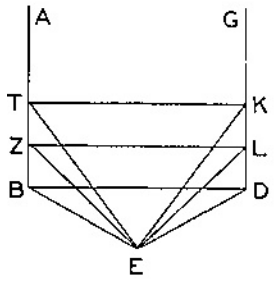


FIG. 6.

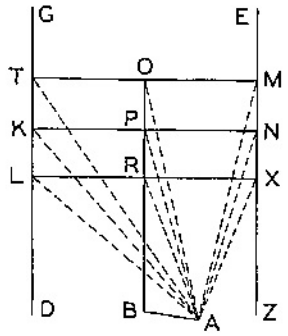


FIG. 6a.

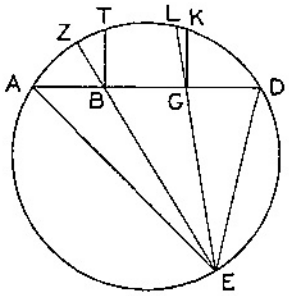


FIG. 7.

Objects of equal size unequally distant appear unequal and the one lying nearer to the eye always appears larger. (Fig. 5.)

Let there be two objects of equal size, *AB* and *GD,* and let the eye be indicated by *E,* from which let the objects be unequally distant, and let *AB* be nearer. I say that *AB* will appear larger. Let the rays fall, *EA, EB, EG,* and *ED.* Now, since things seen within greater angles appear larger, and the angle *AEB* is greater than the angle *GED, AB* will appear to be larger than *GD.*

Parallel lines, when seen from a distance, appear not to be equally distant from each other. (Fig. 6.)

Let there be two parallel lines, *AB* and *GD,* and let the eye be indicated by *E.* I say that *AB* and *GD* appear not to be parallel, and that the nearer space always appears greater than that farther away. Let the rays fall, *EB, EZ, ET, ED, EL,* and *EK,* and let these points be joined by the straight lines, *BD, ZL,*

and *TK.* Now, since the angle *BED* is greater than the angle *ZEL, BD* appears longer than *ZL.* Again, since the angle *ZEL* is greater than the angle *TEK, ZL* appears longer than *TK.* So the space *BD* appears greater than the space *ZL* and the space *ZL* greater than the space *TK.* Therefore, lines equally distant from each other will no longer seem to be parallel, but will seem to be of varying distance apart.

In spaces lying at a higher level let the perpendicular AB be let down from the point A to the plane lying below, and let there be the parallels LX, KN, and TM. I say that so also the lines GD and EZ appear to be not parallel. (Fig. 6a.)

Let a perpendicular *BR* be drawn from *B* to *LX,* and let *BR* be continued to O, and let the rays fall, *AL, AK, AT, AX, AN,* and *AM,* and let *AR, AP,* and *AO* be connected. Now, since a straight line, *AR,* from a higher point has been connected with *RX,* then *AR* is a perpendicular upon *RX,* and *AO* upon *OM,* and *AP* upon *PN.* So, *ARX, APN,* and *AOM* are right-angled triangles. Since, then, they are right-angled triangles, and since *PN* is equal to *RX,* and since *PA* is longer than *AR,* the angle *XAR* is greater than the angle *PAN.* Therefore, *RX* will appear longer than *PN.* Similarly, *RL* appears longer than *PK.* So, the whole line *LX* will appear longer than the whole line *KN.* And thus the lines *DG* and *ZE* will seem to be a varying distance apart.

Objects of equal size upon the same straight line, if not placed next to each other and if unequally distant from the eye, appear unequal. (Fig. 7.)

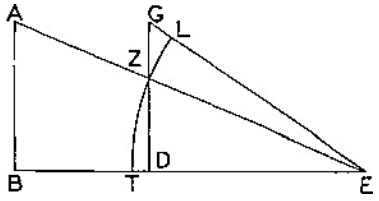
Let there be two objects of equal size, *AB* and *GD,* upon the same straight line, but not next to each other, and unequally distant from the eye, *E,* and let the rays *EA* and *ED* fall upon them and let *EA* be longer than *ED.* I say that *GD* will appear larger than *AB.* Let the rays *EB* and *EG* fall upon them, and let the circle, *AED,* be circumscribed about the triangle, *AED.* And let the straight lines *BZ* and *GL* be added as a continuation of the straight lines *EB*

and *EG,* and from the points *B* and *G* let the equal straight lines *BT* and *GK* be drawn at right angles. And *AB* is equal to *GD,* but also the angle *ABT* is equal to the angle *DGK.* And so the arc *AT* is equal to the arc *DK.* Thus, the arc *KD* is greater than the arc *ZA.* So the arc *LD* is much greater than the arc *ZA.* But upon the arc *ZA* rests the angle *AEZ,* and upon the arc *LD* rests the angle *LED.* So the angle *LED* is greater than the angle *AEZ.* But within the angle *AEZ, AB* is seen, and within the angle *LED, GD* is seen. Therefore, *GD* appears larger than *AB.*

Lines of equal length and parallel, if placed at unequal distances from the eye, are not seen in proportion to the dista?ices. (Fig. 8.)

Let there be two lines, *AB* and *GD,* unequally distant from the eye *E.* I say that, as it appears, *BE* is not in the same relation to *ED* as *GD* is to *AB.* For let the rays fall, *AE* and *EG,* and with *E* as the central point and at the distance *EZ* let an arc be drawn, *LZT.* Now, since the triangle *EZG* is greater than the section *EZL,* and since the triangle *EZD* is less than the section *EZT,* the triangle *EZG,* compared with the section *EZL,* has a greater ratio than the triangle *EZD,* when compared with the section *EZT.*

FIG. 8.



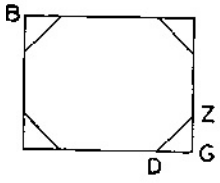


FIG. 9.

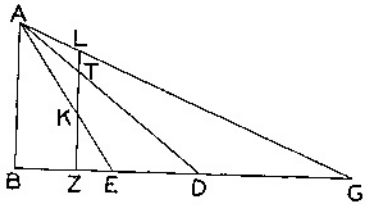


FIG. 10.

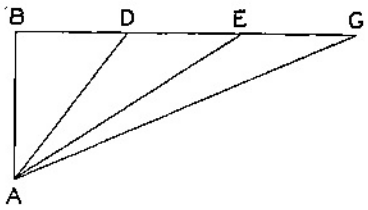


FIG. 11.

And, alternately, the triangle *EZG,* compared with the triangle *EZD,* has a greater ratio than the section *EZL* has, when compared with the section *EZT,* and, when put together, the triangle *EGD,* compared with the triangle *EZD,* has a greater ratio than the section *ELT,* compared with the section *EZT.* But

as the triangle *EDG* is to the triangle *EZD,* so is *GD* to *DZ.* But *GD* is equal to *AB,* and, as *AB* is to *DZ,* so is *BE* to *ED.* Now, *BE* in comparison with *ED* has a greater ratio than the section *ELT* in comparison with the section *EZT.* And as one section is to the other section, so the angle *LET* is to the angle *ZET.* So the straight line *BE,* compared with the straight line *ED,* has a greater ratio than the angle *LET* compared with the angle *ZET.* And from the angle *LET, GD* is seen, and from the angle *ZET, AB* is seen. So lines of equal length are not seen in proportion to the distances.

Rectangular objects, when seen from a distance, appear round. (Fig. 9.)

For let the rectangle *BG,* standing upright, be seen from a distance. Now, since each thing that is seen has a certain limit of distance beyond which it is no longer seen, the angle *G* is not seen, but only the points *D* and *Z* are visible. In the same way also in the case of the other angles this will happen. So that the whole thing will appear round.

In the case of flat surfaces lying below the level of the eye, the more remote parts appear higher. (Fig. 10.)

Let the eye be *A,* at a level higher than *BEDG,* and let the rays fall, *AB, AE, AD,* and *AG,* of which rays let *AB* be perpendicular upon the plane below. I say that *GD* appears higher than *DE,* and *DE* higher than *BE.* For somewhere upon the line *BE* let the point *Z* be taken, and let the perpendicular *ZL* be drawn. Since the lines of vision fall upon *ZL* before they reach *ZG,* let the line *AG* meet *ZL* at the point *L,* the line *AD* at the point *T,* and the line *AE* at the point *K.* Now, since *L* is higher than *T* and *T* higher than *K,* but *G* on the same line as L, and *D* on the same line as *T,* and *E* on the same line as *K,* and, since between the lines *AG* and *AD, DG* is seen, and between the lines *AD* and *AE, DE* is seen, *GD* appears higher than *DE.* And, similarly, *DE* will appear higher than *BE;* for things seen by higher rays appear to be higher.

And it is clear that things seen on a higher plane will appear to be concave.

In the case of flat surfaces located above the level of the eye, the parts farther away appear lower. (Fig. 11.)

Let the eye be indicated by *A* in a position lower than the plane *BG,* and let the rays fall, *AB, AD, AE,* and *AG,* and of these let *AB* be a perpendicular to the assumed plane. I say that *GE* appears lower than *ED.* Now, according to the theorem before stated, the ray *AG* is lower than *AE, AE* lower than *AD,* and *AD* lower than *AB.* But between the lines *G A* and *AE, GE* is seen, and between *EA* and *AD, ED* is seen, and between *DA* and *BA, DB* is seen. So *GE* appears lower than *ED,* and *ED* lower than *DB.*

In the case of lines extending forward, those on the right seem to be inclined toward the left, and those on the left seem to be inclined toward the right. (Fig. 12.)

Let two lines be seen, *AB* and *GD,* and let the eye be indicated by *E,* from which let the rays fall, *ET, EK, EA, EZ, EL,* and *EG.* I say that *EZ, EL,* and *EG* seem to be inclined toward the left, and *ET, EK,* and *EA* toward the right. For since *EZ* is more to the right than *EL,* and *EL* more than *EG,* hence *EG* seems to be inclined to the left of *EL,* and *EL* to the left of *EZ.* Similarly, also *EA, EK,* and *ET* seem to be inclined to the right.

In the case of lines of equal length, lying below the level of the eye, those lying farther away appear to be higher. (Fig. 13.)

Let there be lines of equal length, *AB, GD,* and *EZ,* and let the eye be indicated by *L,* located above these lines, and let the rays fall, *LA, LG,* and *LE.* I say that *AB* appears higher than *GD,* and *GD* higher than *EZ.* For, since *LA* is higher than *LG,* and *LG* higher than *LE,* and since the points *A, G,* and *E* are on the lines *LA, LG,* and *LE,* and since the lines *AB, GD,* and *EZ* extend from the points *A, G,* and *E,* then *AB* seems higher than *GD,* and *GD* higher than *EZ.*

In the case of lines of equal length lying above the level of the eye, those farther away appear lower. (Fig. 14.)

Let there be lines of equal length, *AB, GD,* and *EZ,* lying above the level of the eye, *L.* I say that *AB* appears lower than *GD,* and *GD* lower than *EZ.* Let the rays fall, *LB, LD,* and *LZ.* So, since the ray *LB* is lower than *LD,* and *LD* lower than *LZ,* but the points *B, D,* and *Z* are on the lines *LB, LD,* and *LZ,* and since the lines *AB, GD,* and *EZ* extend from *B, D,* and *Z,* then *AB* appears lower than *GD,* and *GD* lower than *EZ.*

In the case of objects below the level of the eye which rise one above another, as the eye approaches the objects, the taller one appears to gain height, but as the eye recedes, the shorter one appears to gain. (Fig. 15.)

Let there be two unequal lines, *AB* and *GD,* and let *AB* be the taller, and let *E* be the eye, and from this let a ray, *EZ,* fall through *G.* Now, since below the level of the eye and below the ray *EZ, ZB,* and *GD* are seen, *AB* appears above *GD* by the length of *AZ.* Let the eye be moved nearer and let it be *L,* and from this let the ray *LT* fall through *G.* Then, since below the level of the eye and below the ray *LT, GD,* and *TB* are seen, *AB* will appear taller than *GD* by the length of *A T.* And from the point *E, AB* was seen to be taller by the length of *AZ,* and *A T* is longer than *AZ.* So, as the eye approaches the objects, the taller one appears to gain height, but, as the eye recedes, the shorter one appears to gain.

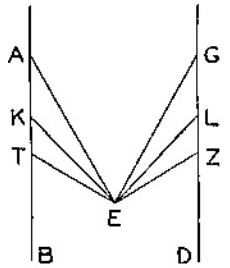


FIG. 12.

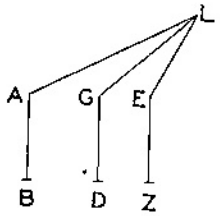


FIG. 13.

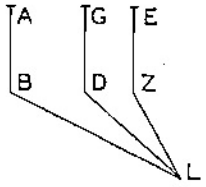


FIG. 14.

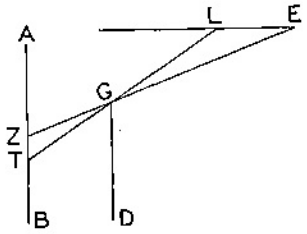


FIG. 15.

In the case of objects of unequal size above the level of the eye which rise one above another, as the eye approaches the objects, the shorter one appears to gain height, but, as the eye recedes, the taller one appears to gain. (Fig. 16.)

Let there be the lines *AB* and *GD,* of unequal length, and of these let *AB* be the taller. Let *E* represent the eye, and from this let the ray *EZ* fall through *G.* Now, since the lines *ZB* and *GD* are included under the ray *EZ, BZ* and *GD* appear equal to each other. So, *AB* appears taller than *GD* by the length of *AZ.* Now let the eye be moved nearer, and let it be *L,* from which let the ray *LT* fall through *G.* Now, since *BT* and *GD* are included under the ray *LT,* and since *ZB* and *GD* are included under the ray *EZ,* and since *ZA* is longer than *A T,* as the eye approaches the objects the shorter one appears to gain height, but, as the eye recedes, the taller one appears to gain.

In the case of objects that rise one above another, if the eye approaches the lesser one or recedes from it at a right angle, the taller one will always appear to rise above the shorter one by an equal amount. (Fig. 17.)

Let there be two lines of unequal length, *AB* and *GD,* and let *AB* be the longer, and let the eye be *Z,* located on a line at right angles to the end of *GD,* that is, at *G.* I say that whether the eye *Z* approaches or recedes, if it is on the same horizontal line, *AB* will seem to be taller than *GD* by an equal amount. For, let the ray *ZE* fall through *G.* Thus, *AB* appears taller than *GD* by the length of *AE.* Now let the eye be moved, and let it be farther away, and let it be *L* on the horizontal line. So, the ray, falling from the eye *L,* will pass through the point *G* and will be continued as far as the point *E,* and *AB* will rise above *GD* by the same amount.

To know how great is a given elevation when the sun is shining. (Fig. 18.)

Let the given elevation be AB, and we have to know how great it is. Let the eye be *D,* and let *GA* be a ray of the sun falling upon the end of the line

AB, and let it be prolonged as far as the eye *D.* And let *DB* be the shadow of *AB.* And let there be a second line, *EZ,* meeting the ray, but not at all illuminated by it below the end of the line at *Z.* So, into the triangle *ABD* has been fitted a second triangle, *EZD.* Thus, as *DE* is to *ZE,* so is *DB* to *BA.* But the ratio of *DE* to *EZ* is known; and, therefore, the ratio of *DB* to *BA* is known. Moreover, *DB* is known; so, *AB* also is known.

To know how great is a given elevation when the sun is not shining. (Fig. 19.)

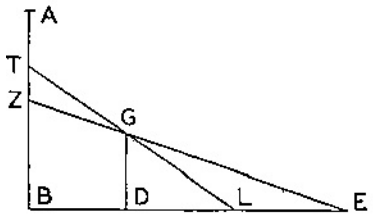


FIG. 16.

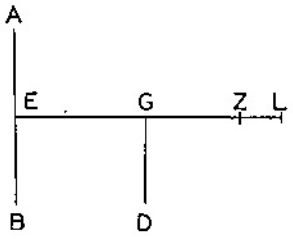


FIG. 17.

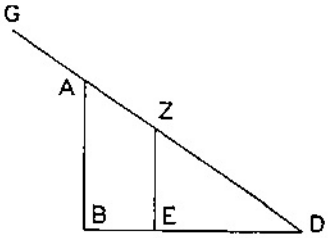


FIG. 18.

Let there be a certain elevation, *AB,* and let the eye be *G,* and let it be necessary to know how high is *AB* when the sun is not shining. Let a mirror be placed, *DZ,* and let *DB* be prolonged in a straight line continuous with *ED,* to the point where it touches *B,* the end of *AB,* and let a ray fall, *GL,* from the eye *G,* and let it be reflected to the point where it touches *A,* the end of *AB,* and let *ET* be a prolongation of *DE,* and from *G* let the perpendicular *GT* be drawn upon *ET.* Now, since the ray *GL* has fallen and the ray *LA* has been reflected, they have been reflected at equal angles, as is said in the *Catoptrica;* thus, the angle *GLT* is equal to the angle *ALB.* But also the angle *ABL* is equal to the angle *GTL;* and the remaining angle *LGT* is equal to the remaining angle *LAB.* So, the triangle *ALB* is equiangular with the triangle *GLT.* But the sides of equiangular triangles are proportionate. Thus, as *GT* is to *TL,* so is *AB* to *BL.* But the ratio of *GT* to *TL* is known; and the ratio of *BA* to *BL* is known. But *LB* is known. And so, *AB* is known.

To know how great is a given depth. (Fig. 20.)

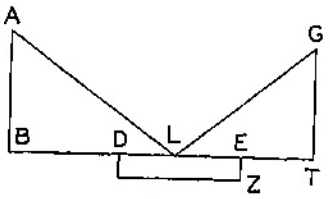


Fig. 19.

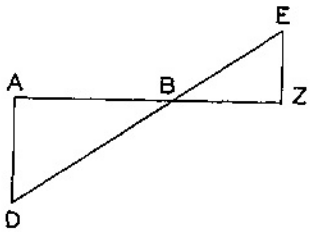


Fig. 20.

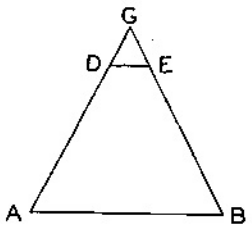


Fig. 21.

Let the given depth be *AD,* and let the eye be *E,* and let it be necessary to know how great is the depth. Let a ray of the sun fall before the eye, meeting the plane at the point *B,* and the depth at the point *D.* And let *BZ* be continued from *B* in a straight line, and let the perpendicular *EZ* be drawn from *E* to the horizontal line *BZ.* Now, since the angle *EZB* is equal to the angle *BAD* and the angle *ABD* is equal to the angle *EBZ,* the third angle also, *BEZ,* is equal to the angle *ADB.* So, the triangle *ADB* is equiangular with the triangle *BEZ.* And thus the sides will be proportionate. Then, as *EZ* is to *ZB,* so is *DA* to *AB.* But the ratio of *EZ* to *ZB* is known; and the ratio of *DA* to *AB* is known. And also *AB* is known. And thus *AD* is known.

To know how great is a given length. (Fig. 21.)

Let the given length be *AB,* and let the eye be *G,* and let it be necessary to know how great is the length of *AB.* Let the rays fall, *G A* and *GB,* and near the eye, *G,* let the point *D* be taken somewhere upon the ray *(GA),* and through the point *D* let the horizontal line *DE* be drawn, parallel to *AB.* Now, since *DE* has been drawn parallel to *BA,* one of the sides of the triangle *ABG,* as *GD* is to *DE,* so is *GA* to *AB.* But the ratio of *GD* to *DE* is known; and the ratio of *AG* to *AB* is known. And AG is known. So, *AB* also is known.

If an arc of a circle is placed on the same plane as the eye, the arc appears to be a straight line. (Fig, 22.)

Let *BG* be an arc lying on the same plane as the eye *A,* from which let the rays fall, *AB, AD, AE, AZ, AL, AT,* and *AG.* I say that the arc *BG* appears to be a straight line. Let *K* be the center of the arc, and let the straight lines *KB,*

KD, KE, KZ, KL, KT, and *KG* be drawn. Now, since *KB* is seen within the angle *KAB,* and *KD* within the angle *KAD, KB* will appear longer than *KD,* and *KD* longer than *KE,* and *KE* longer than *KZ,* and, on the other side, *KG* will appear longer than *KT,* and *KT* longer than *KL,* and *KL* longer than *KZ.* On this account, *KA* remaining a straight line, *BG* is always a perpendicular. And the same things will happen also in the case of a concave arc.

An addition. It is possible to say these things also with reference to the visual rays themselves, that the ray between the eye *A* and the diameter is shortest, and that the ray nearer to it is always shorter than the one farther away. And the same things happen when *AZ* is a perpendicular upon the diameter. On this account the arc gives an impression of a straight line, and especially if it should be seen from a greater distance, so that we do not perceive the curve. On this account ropes not stretched tightly, when seen from the side, appear to sag, but seen from below, seem to be straight, and also the shadows of rings lying on the same plane with the source of light become straight.

An addition. If an arc of a circle is placed on the same plane as the eye, the arc appears to be a straight line. (Fig. *22a.)*

Let *BG* be an arc, and let *D* be the eye, on the same plane as the arc *BG,* from which let the rays fall, *DB, DZ,* and *DG.* So, since nothing that is seen is seen at once in its entirety, *BZ* is a straight line. Similarly, also *ZG.* Thus, the whole arc, *BG,* will seem straight.

Of a sphere seen in whatever way by one eye, less than a hemisphere is always seen, and the part of the sphere that is seen itself appears as an arc. (Fig. 23.)

Let there be a sphere, of which *A* is the center, and let *B* be the eye. And let *A* and *B* be joined, and let the plane be continued along the line *BA.* So it will make a circular section. Let it make the circle *GDTL,* and around the

diameter *AB* let the circle *GBD* be inscribed, and let the straight lines be drawn, *GB, BD, AD,* and *AG.* Now, since *AGB* is a semicircle, the angle *AGB* is a right angle; similarly, also the angle *BDA.* So, the lines *GB* and *BD* touch the sphere. Nowj let *G* and *D* be joined, and let *LT* be drawn through the point *A* parallel to *GD.* So, the angles at *K* are right angles.

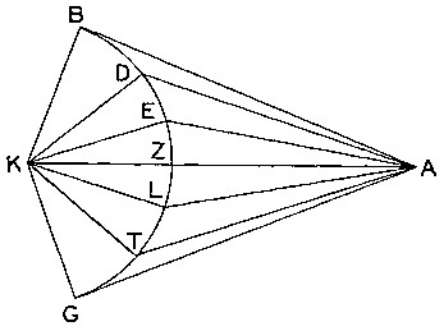


FIG. 22.

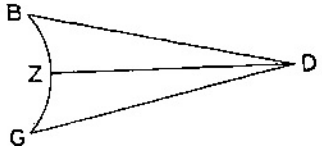
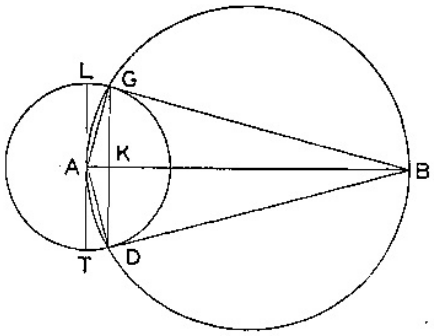


FIG. 22a

FIG. 23.



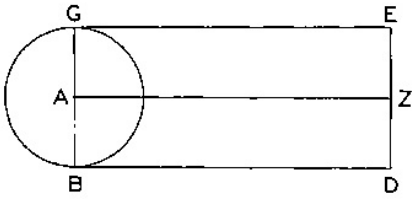


FIG. 25.

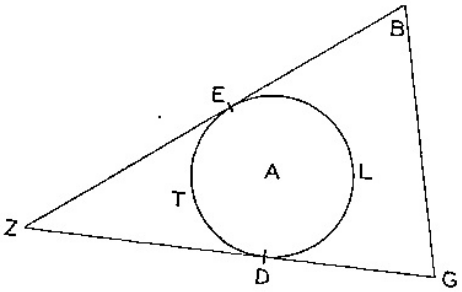


FIG. 26.

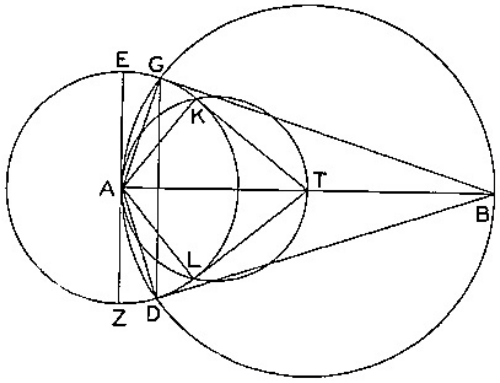


FIG. 24.

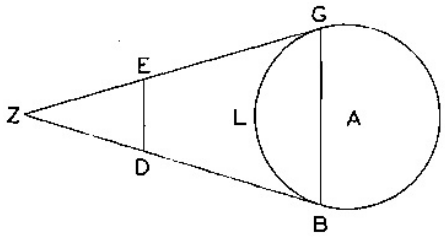


FIG. 27.

Now, with *AB* remaining in its place, if the triangle *BGK* is revolved about the right angle *K*, and is restored to the same position from which it started,

the line *BG* will touch the sphere at one point and the line *KG* will make a circular section. So, an arc will be seen in the sphere. And I say that it is less than a hemisphere. For, since *LT* is a semicircle, *GD* is less than a semicircle. And the same part of the sphere is seen by the rays *BG* and *BD.* So, *GD* is less than a hemisphere; and it is seen by the rays *BG* and *BD.*

When the eye approaches the sphere, the part seen will be less, but will seem to be more. (Fig. 24.)

Let there be a sphere, of which the center is *A,* and let the eye be *B,* from which let the straight line *AB* be drawn. And around *AB* let the circle *GBD* be inscribed, and from the point *A* let the straight line *EZ* be drawn, perpendicular to the straight line *AB* in either direction, and let the plane be produced along *EZ* and *AB.* So it will make a circular section. Let it be *GEZD,* and let *GA, AD, DB, BG,* and *GD* be drawn. So, according to the theorem given before, the angles at the points *G* and *D* are right angles. Thus, *BG* and *BD,* whatever rays there are, touch the sphere. And the part of the sphere, *GD,* is seen by the eye, *B.* Now let the eye he moved nearer to the sphere, and let it be at *T,* from which let the straight line *TA* be drawn, and let the circle *ALK* be inscribed, and let the straight lines *TK, KA, AL,* and *LT* be drawn. Now, similarly, by the eye at *T* is seen the part of the sphere, *KL,* and by the eye at *B* the part *GD* was seen. And *KL* is less than *GD.* So, as the eye approaches, the part seen is less. But it seems to be more; for the angle *KTL* is greater than the angle *GBD.*

When a sphere is seen by both eyes, if the diameter of the sphere is equal to the straight line marking the distance of the eyes from each other, the whole hemisphere will be seen. (Fig. 25.)

Let there be a sphere, of which *A* is the center, and on the sphere let the circle *BG* be inscribed about the center *A,* and let *BG* be drawn as its diameter, and at right angles from *B* and *G* let lines be drawn, *BD* and *GE,*

and let *DE* be parallel to *BG,* and upon this (*DE),* let *D* and *E* represent the eyes. I say that the complete hemisphere will be seen. Through *A* let *AZ* be drawn parallel to each of the lines, *BD* and *GE*; then *ABZD* is a parallelogram. Now, if the inscribed figure is revolved and then restored to the same position whence it started, *AZ* remaining in its place, it will start from *B* and *B* will come over *G,* and the figure formed under *AB* will be a circle through the center of the sphere. So a hemisphere will be seen by the eyes *D* and *E.*

If the distance between the eyes is greater than the diameter of the sphere, more than the hemisphere will be seen. (Fig. 26.)

Let there be a sphere, of which *A* is the center, and about the center *A* let the circle *ETDL* be inscribed, and let the eyes be *B* and *G,* and let the distance between the eyes *B* and *G* be greater than the diameter of the sphere, and let *B* and *G* be joined. I say that more than the hemisphere will be seen. Let the rays fall, *BE* and *GD,* and let them be produced beyond the points *E* and *D;* they approach each other because the diameter is less than *BG.* So let them meet at the point *Z.* Now, since from some point outside the circle the straight lines *ZE* and *ZD* have touched the circumference, *DTE* is less than a semicircle. So, *ELD* is more than a semicircle. But *ELD* is seen by the eyes at *B* and *G.* So, more than half of the circle will be seen by the eyes at *B* and *G.* The same part of the sphere too will be seen.

If the distance between the eyes is less than the diameter of the sphere, less than a hemisphere will be seen. (Fig. 27.)

Let there be a sphere, the center of which is the mark *A,* and let the circle *BG* be inscribed around the mark *A,* and let the distance between the eyes, *DE,* be less than the diameter of the sphere, and from the eyes let the same rays be drawn, *DB* and *EG,* touching the sphere. I say that less than a hemisphere will be seen. For let *BD* and *GE* be produced; they will fall upon

the section *GLB,* since *DE* is less than the diameter of the sphere. Let them meet at the point *Z.* Now, since from a certain mark, *Z,* the straight lines have fallen, *ZG* and *ZB, BLG* is less than a semicircle. But the section of the sphere corresponds to the section *BLG.* So, the rays include less than a hemisphere.

If a cylinder in whatever way is seen by one eye, less than half a cylinder will be seen. (Fig. 28.)

Let there be a cylinder, and let the mark *A* be the center of its base, and let the circle *BG* be inscribed around *A,* and let the eye *D* be on the same plane as *BG,* the base of the cylinder, and from *D* to *A* let the line *DA* be drawn, and from *D* let the rays *DB* and *DG* be drawn, and let them touch the cylinder, and from the marks *B* and *G* let the sides of the cylinder rise at right angles, and let the plane through *DB* and *BE* and the one through *DG* and *GZ* be produced. So, neither of them cuts the cylinder; for *DB, DG* and *BE, GZ* touch the cylinder. Therefore, *BG* is seen by the rays *BD* and *DG,* and *BG* is less than a semicircle. In the same way less than half a cylinder will be seen.

If a cylinder should be seen by two eyes, it is clear that what has been said regarding the sphere will be true also in this case.

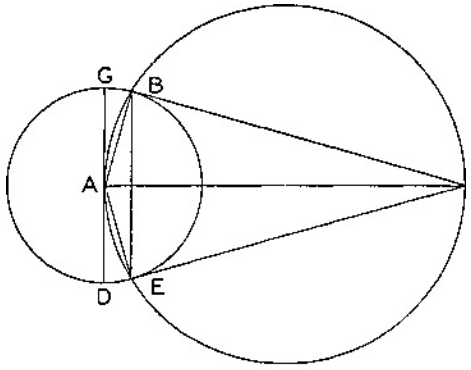


FIG. 28a.

An addition. (Fig. 28a.) Let there be a circle, and let *A* be its center, and outside let there be the point *Z,* and from *A* to *Z* let *AZ* be drawn. And from the point *A* let *GD* be drawn at right angles to *AZ* in both directions; so *GD* is the diameter of the circle. And around *AZ* let the circle *ABZE* be inscribed, and let *AB, BZ, ZE,* and *EA* be drawn. So, *ZB* and *ZE* touch the circle, since

the angles at the points *B* and *E* are right angles. Now, since from a certain point *Z* the rays *ZB* and *ZE* have fallen upon the circumference of the circle, the part of the circle, *BE,* will be seen. But *GBED* is a semicircle. Hence, *BE* is less than a semicircle.

But this theorem was concerned with cones and cylinders. For, if from the points *B* and *E* the sides of the cylinders rise at right angles, the rays will touch them at the points where they fall, and the part of the vision indicated by *BGDE* will be cut off, while the part of the semicircle indicated by *BE* will be seen. So, the same lesser part of cones also will be seen.

When the eye is moved nearer the cylinder, the part of the cylinder reached by the rays is less, but a greater part will appear to be visible. (Fig. 29.)

Let there by a cylinder, the base of which is the circle *BG,* and let *A* be the center, and let *E* be the eye, from which let *EA* be drawn to the center, and let the rays fall, *EB* and *EG,* and from the points *B* and *G* let the sides of the cylinder, *GZ* and *BD,* rise at right angles. Now, according to what has been said before, *DBGZ* is less than half a cylinder; and it is seen by the eye, *E.* Now let the eye, *T,* be moved nearer. I say that the part reached by the eye, *T,* seems to be greater than *ZBGD,* but is actually less. Let the rays fall, *TK* and *TL,* and from the points *K* and *L* let the sides of the cylinder rise at right angles, *KM* and *LN.* Now the part of the cylinder, *MKLN,* will be seen by the rays *TK* and *TL.* But *ZGBD* was seen by the rays *EB* and *EG.* And *ZGBD* is greater than *MKLN,* but it seems to be less, since the angle at *T* is greater than the angle at *E.*

If a cone having a circle as its base and the axis at right angles to the base is seen by one eye, less than half the cone will be seen. (Fig. 30.)

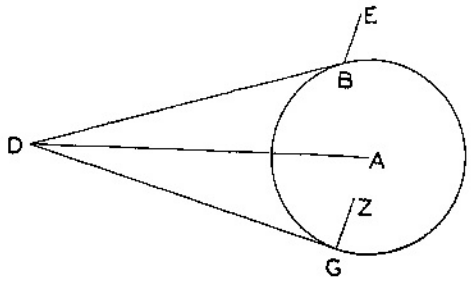


FIG. 28.

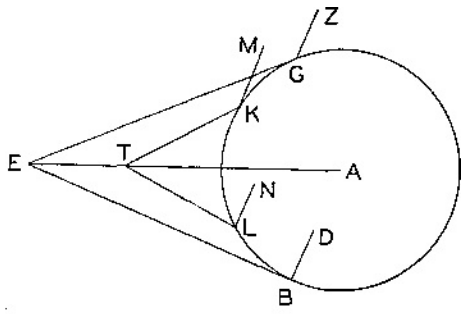


FIG. 29.

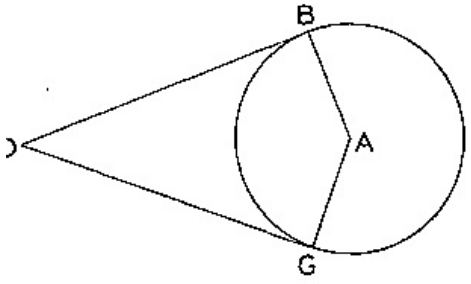


FIG. 30.

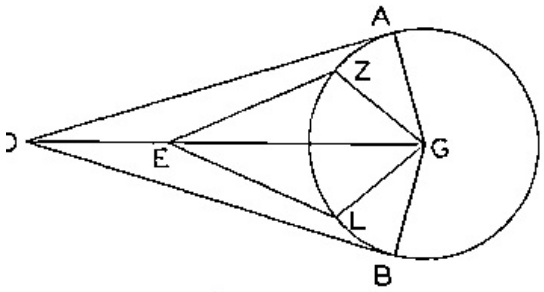


FIG. 31.

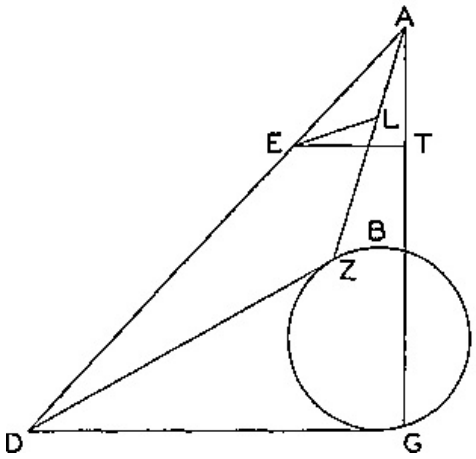


FIG. 32.

Let there be a cone, the base of which is the circle *BG,* and the apex the point *A,* and let *D* be the eye, from which let the rays fall, *DB* and *DG.* And

since the rays *DG* and *DB* have fallen, touching *BG, BG* is less than a semicircle, according to what has been proved before. From the apex of the cone, *A,* let the sides of the cone, *AB* and AG, be drawn to the points *B* and *G.* So the part included by the straight lines, *AB* and *AG,* and the arc *BG* is less than half a cone, since *BG* is less than a semicircle. Less than half a cone will be seen.

If the eye is brought nearer on the same plane on which is the base of the cone, the part included within the vision will be less, but a larger part will seem to be visible. (Fig. 31.)

Let there be a cone, the base of which is the circle *AB,* and the apex the point *G,* and let the eye be *D,* and let *G* be taken also as the center of the circle, and let the straight line, *DG,* be drawn, and let the rays fall, *DA* and *DB,* and let the sides of the cone be drawn, *AG* and *GB.* So, under the eye, *D,* and the lines of vision, *DA* and *DB,* the part of the cone indicated by *ABG* is included, and it is less than half a cone. Now let the eye come nearer, and let it be *E,* and let the rays fall, *EZ* and *EL,* and let the sides be drawn, *ZG* and *GL.* So again, under the eye *E* and the lines of vision, *EZ* and *EL,* the part of the cone *ZGL* is included. And *ZGL* is less than *ABG;* but it seems to be more, since the angle *ZEL* is greater than the angle *ADB.*

And it is clear that in the case of a cone, when seen by both eyes, the same things will occur as in the case of the sphere and the cylinder, when seen in the same way.

If rays fall from the eye upon the base of the cone, and if from the rays that fall and touch the cone straight lines are drawn from the points of contact over the surface of the cone to the apex of it, and if planes are produced through the lines that have been drawn and that fall from the eye upon the base of the cone, and if the eye is placed upon their point of contact, that is, upon the common section of the planes, the part of the cone that is seen will

be quite the same, if the eye remains upon a plane parallel to the original plane. (Fig. 32.)

Let there be a cone, the base of which is the circle *BG* and the apex the point *A,* and let the eye be *D,* from which let the rays fall, *DZ* and *DG,* and from the points of contact, *Z* and *G,* let the sides of the cone, *ZA* and *GA,* be drawn to the apex of the cone, and let the plane through *DZ* and *ZA,* and the one through *GD* and *G A* be produced. Thus it will make a straight line the common section. Let it be *AED.* I say that, if the eye is moved upon the line *AED,* the same part of the cone will be seen, as much as was seen by the rays *DG* and *DZ.* For upon the line *AED* let the eye rest at *E,* from which let rays fall upon the cone. They will reach *AZ* and *AG,* since the eye rests upon a parallel plane, and the rays travel along straight lines. For, if they fall outside of *AG* and *AZ,* the rays will be broken; which is impossible. So, let *ET* and *EL* stand. Now, since the rays move upon a parallel plane in straight lines, and since things seen at equal angles appear equal, and since all the parallel rays upon the straight line *AED* contain equal angles, an equal part of the cone will be seen.

Again, if the eye is shifted from a low position, when it is raised, the part of the cone that is seen will be greater, but will seem to be less, but, on the other hand, if the eye is lowered, the part seen will be less, but will seem to be greater. (Fig. 33.)

Let there be a cone, the base of which is the circle *BG,* and the apex the point *A,* and let *BA* and *AG* be the sides of the cone. Let *B* and *G* be joined, and let *BL* continue the line of *BG,* and through the point *T* taken at random let *TK* be drawn parallel to *AB.* I say that the part of the cone that is seen will be greater, but will seem to be less, if the eye is placed at the point *T* than if it is placed at *K.*

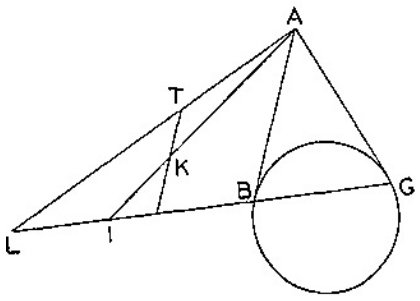


FIG. 33.

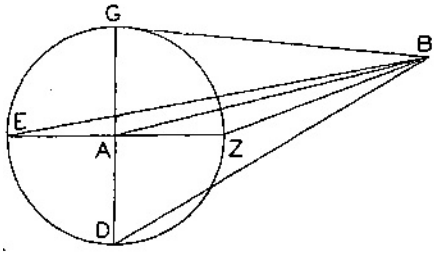


FIG. 34.

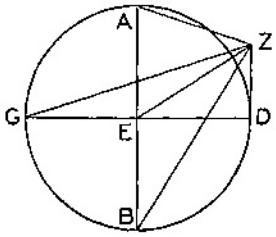


FIG. 34a.

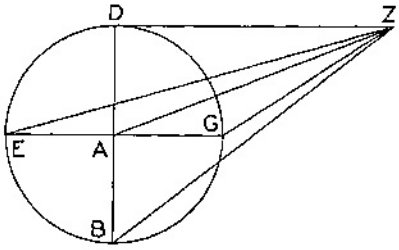


FIG. 34b.

Let *A* and *K* be joined and *A* and *T,* and let *A T* be produced to *L* and *AK* to *I.* So, if the eye is placed at *L* and then at *I,* the parts of the cone that are seen will seem unequal, and the part seen from *L* will be greater, and the part seen from *I,* though less, will seem greater. But the part seen from *L* is equal to the part seen from *T,* and the part seen from *I* is equal to the part seen from *K,* as has been shown before. So, if the eye is placed at *T,* the part of the cone seen will be greater than if the eye is placed at *K,* but it will seem to be less.

If a straight line is erected from the center of a circle, at right angles to the plane of the circle, and the eye is placed upon this, all the diameters crossing upon the plane of the circle will appear equal. (Fig. 34.)

Let there be a circle, the center of which is the point *A,* and from it let the line *AB* be erected at right angles to the plane of the circle, and upon this let the eye *B* rest. I say that the diameters will appear equal. Let there be two diameters, *GD* and *EZ,* and let the lines be drawn, *BG, BE, BD,* and *BZ.* Now, since *ZA* is equal to *AG,* and *AB* is common to them, and the angles are right angles, then the base *ZB* is equal to the base *BG,* and the angles at the bases are equal. So the angle at *ZB, BA* is equal to the angle at *AB, BG.* Similarly,

also the angle at *EBA* is equal to the angle at *ABD.* Then the angle at *GB, BD* is equal to the angle at *EB, BZ.* And the parts seen at equal angles appear equal. Then *GD* is equal to *EZ.*

And if the line drawn from the center is not at right angles to the plane, but is equal to the line from the center (that is, the radius of the circle), all the diameters will appear equal. (Fig. 34a.)

Let there be a circle, *ABGD,* and in it let two diameters be drawn, *AB* and *GD,* and, leading from the point *E,* let there be a line, *ZE,* upon which is the eye *Z,* not at right angles, but equal to each one of the lines from the center, and let the rays be drawn, *ZA, ZG, ZB,* and *ZD.* Now, since *BE* is equal to *EZ,* but also *EA* is equal to *EZ,* the three lines, *EZ, EA,* and *EB,* are equal. So the semicircle drawn in the plane through *AB* and *EZ* about the diameter *AB* will go through *Z.* So the angle at *AZ, ZB* is a right angle. Similarly, also the angle at *GZ, ZD* is a right angle. The right angles are equal, and things seen at the same angles appear equal. So *AB* will appear equal to *GD.*

But let AZ be not equal to the line from the center (that is, the radius), and not at right angles to the plane of the circle, but let it make equal the angles DAZ and ZAG, and the angles EAZ and ZAB. I say that even so the diameters making the equal angles will appear equal. (Fig. 34&.)

For, since *GA, AZ* are equal to *ZA, AD,* and since *BA, AZ* are equal to *ZA, AE,* and since the angles are equal, then the base *DZ* is equal to the base *ZG;* so that also the angle *DZA* is equal to the angle *AZG.* Now similarly we shall show that also the angle *EZA* is equal to the angle *AZG.* So, the whole angle *DZB* is equal to *EZG.* So that also the diameters *DB* and *EG* will appear equal.

But if the line falling from the eye to the center of the circle is not at right angles to the plane of the circle nor equal to the radius and does not enclose

equal angles, the diameters with which it makes unequal angles will appear unequal. (Fig. 35.)

Let there be a circle *ABGD,* and let two diameters be drawn, *AG* and *BD,* cutting each other at right angles at the point *E,* and let the line *ZE,* starting from the point *E,* upon which rests the eye, be neither at right angles to the plane nor equal to the radius nor, with *AG* and *DB,* enclosing right angles. I say that the diameters *AG* and *DB* will appear unequal. For let *ZG, ZA, ZD,* and *ZB* be drawn. So the line *EZ* is either greater than the radius or less. On this account either the angle at *DZ, ZB* is greater than the angle at *GZ, ZA,* or the angle at *GZ, ZA* is greater than the angle at *DZ, ZB,* as we shall next prove. Thus the diameters will appear unequal.

Theorem. Let there be a circle, and let the center of it be the point A, and let B be the eye, and from this let a perpendicular drawn to the circle not fall upon the center A, but outside, and let this be BG, and from A let AG be drawn to G, and from A let AB be drawn to B. I say that of all the angles formed by the straight lines drawn through A and making an angle with the line AB, the least is the angle at GA, AB. (Fig. 35a.)

FIG. 35.

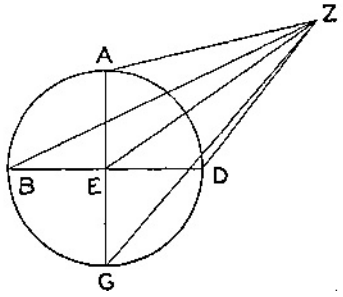
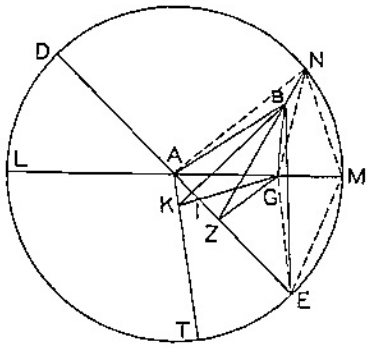


FIG. 35a.



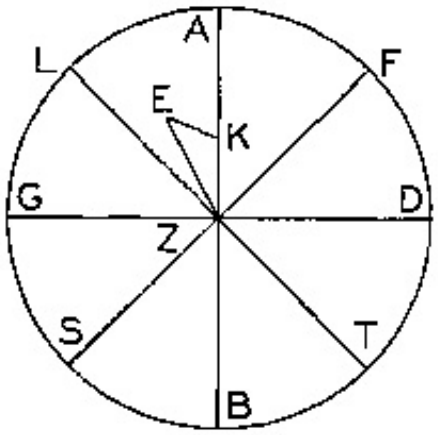


FIG. 35b.

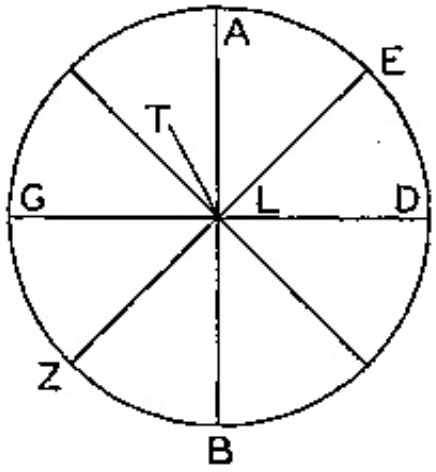


FIG. 35d.

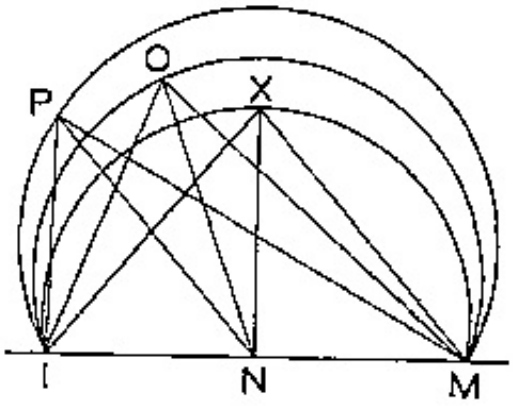


FIG. 35c.

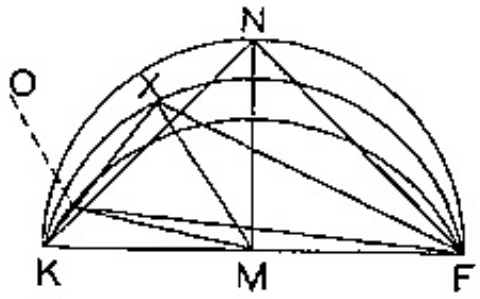


FIG. 35e.

Let the straight line *DAE* be drawn through *A*.

I say that the angle *GAB* is less than the angle *EAB*.

For from *G* to *DE* let the perpendicular *GZ* be drawn in the plane, and let *B* and *Z* be joined. Then *BZ* also is a perpendicular upon *DE*. Now, since the angle *GZA* is a right angle, the angle *AGZ* is less than a right angle. And the greater side subtends the greater angle. So, *AG* is greater than *AZ*. But the angle *AG, GB,* and the angle *BZ, ZA,* are right angles; so that *GB* and *BZ* are unequal. And, therefore, the angle *ZA, AB* is greater than the angle *G A, AB*. Now, similarly, it will be proved that of all the angles formed by the straight lines drawn through *A* and making an angle with the straight line *AB,* the least is the angle *G A, AB*.

And it is clear that, if also some other straight line is drawn through *A,* as *AT,* which is farther than *AZ* from *AG,* the angle *BAT* will be greater than the angle *BAZ.* For again, if the perpendicular *GK* is drawn to *AT, B* and *K* being joined will be similarly perpendicular to *AT.* And, since *AI* is greater than *AK* (for it subtends the right angle *AKI), AZ* is much greater than *AK.* And the angles *BZA* and *BKA* are right angles. So *BZ* is less than *BK,* on account of the fact that the distances from *BZ, ZA* and the distances from *BK, KA,* are equal to the distance from *BA,* and equal to one another, and again the angle *BAK* is greater than the angle *BAZ.* And of all the angles made with *BA* by the lines drawn through *A,* the greatest is the angle *BAL, G A* being produced to *L,* since also the angle *BAG* is less than all. And the lines are equal that are equally distant on either side from *MA,* the line which forms the least angle with *BA.* For, let *MN* be equal to *EM,* and let *EM, MN, EG, GN, BA, BN,* and A A be drawn. Now, since *MN* is equal to *ME,* and *MG* is common, and they form equal angles, then *EG* also is equal to *GN.* And *GB* is a common perpendicular. So, *EB* is equal to *BN.* But also *EA* is equal to *AN;* and *AB* is common. And the angle *EAB* is equal to the angle *NAB.*

Let there be a circle, ABGD, the center of which is Z, and in this let straight lines be drawn from A, B, G, and D, cutting each other at right angles, and let E be the eye, from which let there be a line to the center, joined at right angles to GD, but with AB let it form any angle; and let this line, EZ, be longer than the radius of the circle. I say that the diameters AB and GD will appear unequal, that GD will appear longer and AB shorter, and always the one nearer to the shorter one will appear shorter than the one farther away, and only two diameters will appear equal, those equally distant on either side of the shorter one. (Figs. 35b and 35c.)

For, since *GD* is at right angles to both *AB* and *EZ,* also all the planes produced through *GD* are at right angles to the planes through *EZ* and *AB;* so

that it is true also of the plane of the circle lying below, upon which is *GD.* So, let a perpendicular be drawn from the point *E* to the plane below. It falls upon the common section of the planes, *AB.* Now let *EK* also fall, and let *IM,* equal to the diameter of the circle, be drawn and let it be cut in two at the point *N,* and from *N,* at right angles to *IM,* let the straight line *NX* be erected, and let *NX* be equal to *EZ.* Now the arc inscribed around *IM* and passing through *X* is greater than a semicircle, since the line *NX* is longer than either *IN* or *NM.* Let the arc be *IXM,* and let *XI* and *XM* be joined. The angle at *X* formed by the straight lines *IX* and *XM* is equal to the angle at the point *E* formed by *E* and by *GD.* On the straight line *IN* and at the point *N* let the angle *IN, NO* be formed, equal to the angle *LZ, ZE,* and let *NO* be equal to *EZ,* and let IO and *OM* be joined, and let the arc *IOM* be inscribed around the triangle *IOM.* Now the angle at the point O will be equal to the angle at *E,* formed by *LET.* Again on the straight line *IN* and at the point *N* upon it let the angle *IN, NP* be formed equal to the angle *AZE,* and let *NP* be equal to *EZ,* and let *IP* and *PM* be joined, and around the triangle *IPM* let the arc of a circle *IPM* be inscribed. Now the angle at the point *P* will be equal to the angle *AEB.* Now, since the angle at *X* is greater than the angle at O, but the angle at the point *X* is equal to the angle *GED,* and the angle at O is equal to the angle *LET, GD* will appear longer than *LT.* Again, since the angle at the point O is equal to the angle *LET,* anti the angle at *P* is equal to the angle *AEB,* and since the angle at O is greater than the angle at *P,* also the angle *LET* is greater than the angle *AEB.* So *LT* will appear longer than *AB.* Now of all the straight lines drawn through *Z* and making angles with *EZ, GD* will seem to be the longest and *AB* the shortest, because, of the angles formed at *E, GET)* is the greatest and *AEB* is the least. And only one other, *FES,* will be proved equal to the angle *LET, AF,* which is equal to *LA,* being excluded, and *FZ* being joined and produced to *S.*

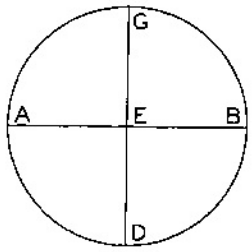


FIG. 36.

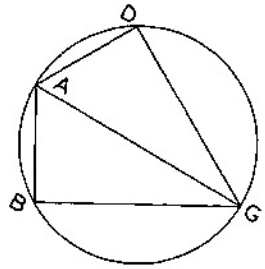


FIG. 37.

And this is clear from the angles at *X, O,* and *P.* For the least of these is *P,* since the angle *PNI* is equal to the least angle, *EZA,* and the greatest is *X,* on account of the fact that *NX* is a perpendicular, the longest of the straight lines drawn through *N* in the arc *IXM,* and because the arc *IXM* falls above the equal line, and because *X* is the innermost point and *P* the outermost, since no angle is less than *PNI.* And the angle *EZF,* being equal to the angle *EZL,* as has been proved, also the adjacent angle, *EZS,* is equal to the angle *EZT,* that is, to the angle *ONM.* So that each of the angles, *FES* and *LET,* is equal to the angle at *O.* So, *LT* will appear equal to *FS.*

Let the line drawn from the eye to the center be less than the radius. Now about the diameters the reverse is true; for the one appearing longer before now will appear shorter, and the shorter one will appear longer. (Figs. 35d and 35e.)

Let there be a circle, *ABGD,* and let two diameters be drawn, *AB* and *GD,* cutting each other at right angles, and let another diameter, *EZ,* be drawn at random, and let the eye be *T,* from which let the line *LT* be drawn to the center, and let it be shorter than either of the radii. And let *KF* be drawn, equal to the diameter of the circle, and let it be cut in two at *M,* and from the point *M* let *MN* be erected at right angles, and let *MN* be equal to *TL,* and let an arc of a circle, *NKF,* be inscribed about *KF* and the point *N.* Now it is less than a semicircle, since *MN is* less than the radius. Now the angle at *N* formed by *KN* and *FN* will be equal to the angle at *T* formed by GT and *TD.* Again, let the angle *KMX* be equal to the angle *ELT,* and let *MX* be equal to

LT, and let an arc, *KXF,* be inscribed about *KF* and the point *X.* Now the angle at the point *X* formed by *KXF* is equal to the angle at *T* formed by *ZTE.* Again, let the angle *KM, MO* be equal to the angle *AL, LT,* and let *MO* be equal to *LT,* and about *KF* and O let an arc be inscribed. Now the angle at O formed by *KOF* will be equal to the angle at *T* formed by *ATB. So,* since the angle at O is greater than the angle at *X,* and the angle at O is equal to the angle at *T* formed by *ATB,* and the angle at *X* is equal to the angle at *T* formed by *ETZ, AB* will appear longer than *EZ.* Again, since the angle at *T* formed by *ET, TZ* is greater than the angle at *T* formed by *GTD, EZ* will appear longer than *GD.*

The wheels of the chariots appear sometimes circular, sometimes distorted. (Fig. 36.)

Let there be the wheel *ABGD,* and let the diameters be drawn, *BA* and *GD,* cutting each other at right angles at the point *E,* and let the eye not be on the plane of the circle. Now, if the line drawn from the eye to the center is at right angles to the plane or is equal to the radius, all the diameters will appear equal; so that the wheel appears circular. But if the line drawn from the eye to the center is not at right angles to the plane or equal to the radius, the diameters will appear unequal, one longer and one shorter, and to every other one which is drawn between the longer and shorter diameters only one other will appear equal, the one drawn to the opposite side; so that the wheel appears distorted.

There is a place where, if the eye remains in the same position, while the thing seen is moved, the thing seen always appears of the same size. (Fig. 37.)

Let the eye be *A,* and the size of the thing seen be indicated by *BG,* and from the eye let rays fall, *AB* and *AG,* and about *ABG* let a circle, *ABG,* be inscribed. I say that there is a place where, if the eye remains in the same

position, while the thing seen is moved, the thing seen always appears of the same size.

For let the object be moved and let it be indicated by the line *DG,* and let *AD* be equal to *AB.* Now, since *BA* is equal to *AD* and *BG* to *GD,* also *BAG* is equal to *DAG.* For they are upon equal arcs; so that they are equal. So the thing seen will appear of the same size.

And the same thing will happen also if the eye should remain at the center of the circle, and the thing seen should move upon the arc.

There is a place where, if the position of the eye is changed, while the thing seen remains in the same place, the thing seen always appears of the same size. (Fig. 38.)

For let *BG* be the thing seen and let *Z* be the eye, from which let rays fall, *ZB* and *ZG,* and about the triangle *BZG* let the arc of a circle, *BZG,* be inscribed, and let the eye, *Z,* be shifted to *D,* and let the rays fall, *DB* and *DG.* Now the angle *D* is equal to the angle *Z;* for they are on the same arc. And things seen at the same angle appear equal. So *BG* will always appear of the same size, when the eye is shifted upon the arc *BDG.*

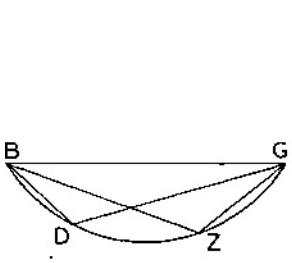


FIG. 38.

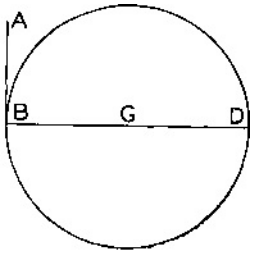


FIG. 39.

If some object is perpendicular to the plane lying below, and if the eye is placed upon some point of the plane, and if the thing seen is shifted upon the circumference of a circle which has the eye as a center, the thing seen will always appear of the same size when it is moved to a position parallel to its original position. (Fig. 39.)

Let *AB* be some object that is seen, located at right angles to the plane, and let *G* be the eye. And let *G* and *B* be joined, and with *G* as the center and *GB* as the radius let the circle *BD* be inscribed. I say that if, upon the circumference of the circle, the object AB is moved, as seen by the eye *G, AB* will appear to be of the same size. For *AB* is a perpendicular and makes a right angle with *BG,* and all the straight lines falling from the center *G* upon the circumference of the circle make equal angles. So the object seen will appear to be of the same size.

And if from the center *G* a straight line rises at a right angle and the eye is placed upon this, and the object seen is moved along the circumference of the circle, parallel to the straight line upon which is the eye, the thing seen will always appear to be of the same size.

But if the thing seen is not at right angles to the plane lying below, and, being equal to the radius, is moved upon the circumference of the circle, it will appear sometimes equal to itself, sometimes unequal, if it is shifted to a position parallel to its original position. (Fig. 40.)

Let *AD* be a circle and let the mark *D* be taken on its circumference, and let the line *DZ,* which is equal to the radius, rise not at right angles to the circle, and lçt *E* be the eye. I say that *DZ,* if it is moved on the circumference of the circle, will appear sometimes equal, sometimes greater, sometimes less. Now through *E,* which is the center, let *GE* be drawn, parallel to *DZ,* and let *EG* be equal to *DZ.* And from the point *G* let the perpendicular *GL* be drawn to the plane lying below and let it reach the plane at the point *L.* And let *EL* be connected and produced, and let it reach the circumference at the point *A,* and through *A* let *AB* be drawn parallel to *GE,* and let *AB* be equal to *DZ.* I say that *AB,* of all the straight lines moved upon the circumference of the circle, will appear the shortest. For let the straight lines, *ED, GZ, GB, EB,* and *ZE,* be drawn. Now, since *GE* is parallel and equal to *AB,* also *EA* is

equal and parallel to *GB.* So, *AEGB* is a parallelogram. For the same reasons also *EDZG* is a parallelogram. And it remains to prove that the same thing appears smaller and greater. Now it is clear that the angle *GEA* is less than the angle *GED,* since it has been proved that of all the straight lines drawn through the center and making an angle, the angle *GEA* is the smallest. So it is smaller than the angle *GED* also. And the angle *BEA* is half the angle *GEA*; for *BE* is an equilateral parallelogram; and the angle *ZED* is half the angle *GED;* for *ZE* also is an equilateral parallelogram. And so the angle *BEA* is less than the angle *ZED.* So that also the object *AB* will appear less than the object *DZ.*

And it is clear from the thesis before proved that it will appear least at the point *A,* and greatest when diametrically opposite to the point *A,* and equal at an equal distance on either side of the point *A.*

But if the thing seen is at right angles to the plane lying below, and the eye moves upon the circumference of a circle which has as its center the point at which the object meets the plane, the thing seen will always appear of the same size. (Fig. 41)

Let the object seen be *AB,* at right angles to the plane lying below, and let the eye be *G.* And with *B* as a center and the radius *BG,* let the circle *GD* be inscribed. I say that if *G* moves upon the circumference of the circle, AB will always appear the same. And this is clear. For all the rays from the point G falling upon AB fall at equal angles, since the angle at B is a right angle. So the thing seen will appear of the same size.

If the thing seen remains in its original position and the eye moves in a straight line that is oblique to the object seen, the object appears sometimes of the same size, sometimes of different size. (Fig. 42.)

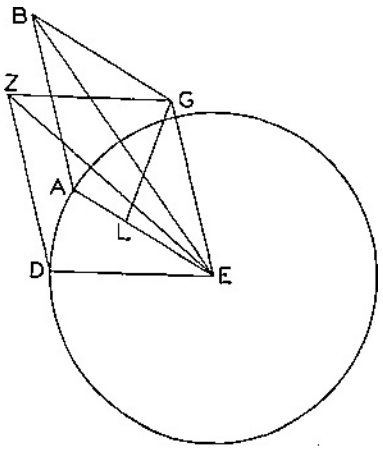


FIG. 40.

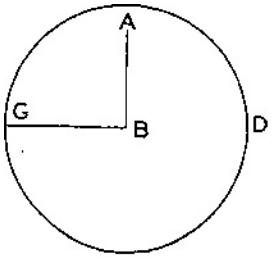


FIG. 41.

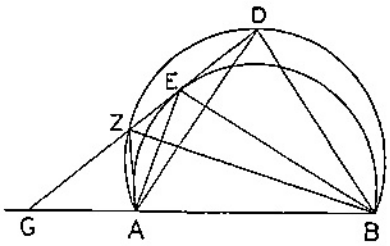


FIG. 42.

Let *AB* be the thing seen and let *E* be the eye, and let *GD* be a straight oblique line, and let *GA* be a continuation of *BA* in a straight line, and let it meet *DG* at *G,* and let the eye be moved upon it. I say that *AB* appears sometimes of the same size, sometimes of different size. For, let *GE* be taken as a mean proportional of the lines *BG* and *G A,* and let *E* be the eye and let it be moved, and let it be on the same straight line at *D.* I say that the thing seen from *E* and *D* appears to be of different size. Let the straight lines *AE, EB, AD,* and *BD* be drawn, and about the triangle *AEB* let the arc *AEB* be inscribed, and let the angle formed by *GA* and *AZ* be equal to the angle formed by *GD* and *DB,* and let *BZ* be joined. Now *B, A, Z,* and *D* are points in a circle. So, since the angle *AEB* is greater than the angle *AZB,* and the angle *AZB* is equal to the angle formed by *AD* and *DB,* since it is on the same arc, also the angle *AEB* is greater than the angle *ADB.* But under the angle *ADB, AB* is seen when the eye is at *D,* and under the angle *AEB* the same *AB*

is seen when the eye is at *E.* So the thing seen appears of different size when the eye moves along the straight line *ED.*

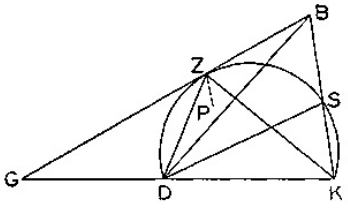


FIG. 42a.

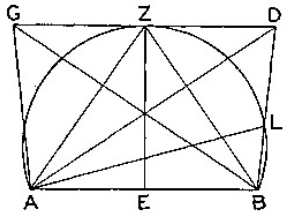


FIG. 43.

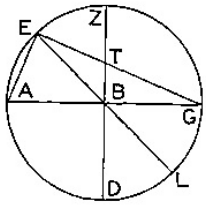


FIG. 44.

And it is clear also that when the eye moves along the line *EG,* the thing seen appears of different size, and that it appears largest from the position at *E,* but always larger in a position nearer to *E* on either of the straight lines, *ED* and *EG,* and of the same size at *Z* and *D* and places taken in like manner to them, on account of the fact that the angles are on the same arc.

In addition. (Fig. 42a.) For let *KD* be the thing seen and let *BG* be a straight line falling upon *KD* produced. Let *GZ* be taken as a mean proportional of *GD* and *GK,* and let *ZK* and *ZD* be drawn, and about *KD* let an arc be inscribed, which contains the angle *KZD.* Now it will touch the straight line *BG,* since as *KG* is to *GZ,* so is *GZ* to *GD.* Now let the eye be at the point *B,* and let *DB* and *BK* be added. And let *S* and *D* be joined. So the angle *P* is equal to the angle *S;* for they are on the same arc. And the angle S is greater than the angle *B;* and so the angle *P* is greater than the angle *B.* Thus the eye being at *Z, KD* appears greater than when the eye is at *B.*

And the same thing will happen, even if the straight line is parallel with the object seen. (Fig. 43.)

Let *AB* be the object seen and let it be cut in two at the point *E,* and from E let *EZ* be erected at right angles to *AB,* upon which line let the eye rest at *Z,* and let the straight lines *ZA* and *ZB* be drawn, and about the triangle *AZB* let the arc *AZB* be inscribed, and let *ZD* be drawn through *Z* parallel to *AB,* and let the eye move to *D,* and let the rays fall, *DA* and *DB.* I say that from the points *D* and *Z* things will appear of different size. Let *A* and *L* be joined.

Now, since the angle *AZB* is equal to the angle *ALB,* but the angle *ALB* is greater than the angle *ADB,* the angle *AZB* also is greater than the angle *ADB.* And *AB* is seen under the angle *AZB* when the eye is at *Z,* and similarly under the angle *ADB* when the eye is at *D.* So the thing seen appears of different size from the points *D* and *Z.*

And if *ZG* is drawn equal to *DZ,* the object appears smaller from *G* than from *Z,* but from *G* and *D* it appears the same.

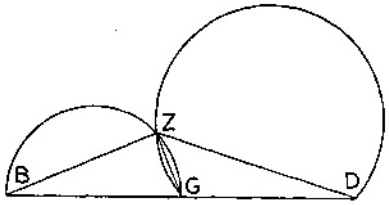
There are places where, if the eye is moved to them, objects that are of equal size and close to one another appear sometimes of the same size, sometimes of different sizes. (Fig. 44.)

Let *T* be the eye, and let *AB* and *BG* be the objects, and from *B* let *BZ* be drawn at right angles, and let it be produced to *D.* Now it is clear that if the eye is placed on any part whatever of the line *ZD, AB* and *BG* will appear equal. Now let the eye be moved and let it be *E.* I say that from E the objects appear of unequal size. Let the rays fall, *EA, EB,* and *EG,* and around the triangle *AGE* let the circle *AEDG* be inscribed, and let *BL* be a continuation of *EB.* Now, since the arc *AD* is equal to the arc *DG,* and since the arc *ADL* is greater than the arc *LG, AB* will appear greater than *BG.* And if the eye moves upon the line *EL,* the objects will appear similarly unequal; and if it is placed upon the parts of the circle outside the perpendicular, the objects appear unequal, and if it is placed outside the circle, not in line with *DZ,* they appear unequal.

In addition. (Fig. 44a.) For let *BG* be equal to *GD,* and about *BG* let the semicircle *BZG* be inscribed, and about *GD* let *GZD,* more than a semicircle, be inscribed; and it is clear that it will cut the semicircle before mentioned. And it is possible to inscribe upon *GD* an arc greater than a semicircle. For if we assume a certain acute angle, it is possible for us to inscribe upon *GD* an arc of a circle containing an angle equal to the acute angle below, as (is clear)

from the 33rd proposition of the third book of the Elements, and the arc imposed upon it will be greater than a semicircle, as (is clear) from the 31st proposition of the third book of the Elements. And let *BZ, ZG,* and *ZD* be drawn. So the angle in the semicircle is greater than the angle in the greater arc. And the things seen at a greater angle appear larger; so *BG* appears larger than *GD.* But it was equal. There is, then, a common place, where, if the eye is placed there, equal things appear unequal. But they will appear equal when.... (The remainder of this sentence is corrupt in the Greek text, and cannot be translated.)

FIG. 44*a*.



There is a certain place from which objects of unequal size appear equal. (Fig. 45.)

For let *BG* be greater than *GD,* and around *BG* let an arc greater than a semicircle be inscribed, and around *GD* an arc like the one around *BG,* that is, one containing an angle equal to the one in *BZG.* Now the arcs will cut each other. Let them cut each other at *Z,* and let *ZB, ZG,* and *ZD* be drawn. So, since the angles in the similar arcs are equal to each other, also the angles in the arcs *BZG* and *GZD* are equal to each other. And the things seen under equal angles appear equal. For if the eye is placed at the point *Z, BG* would appear equal to *GD.* But it really is greater. So there is a common place, from which objects of unequal size appear equal.

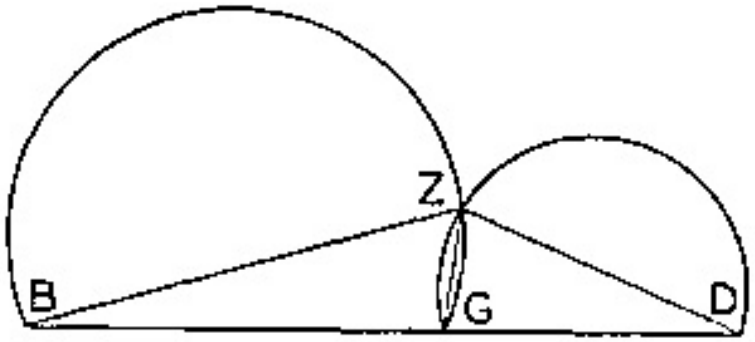
There are places to which the eye may be moved and objects of equal size that are perpendicular to the plane below appear sometimes of equal size, sometimes of unequal size. (Fig. 46.)

Let *AB* and *GD* be objects of equal size perpendicular to the plane below. I say that there is a place where, if the eye is located there, *AB* and *GD* appear equal. Let *BD* be drawn from *B* to *D,* and let it be cut in two at the point *E,* and from *E* let *EZ* be drawn at right angles to *DB.* I say that if the eye is placed upon *EZ, AB* and *GD* will appear equal. For let the eye rest upon *EZ,* and let *Z* mark the position of the eye, and let the rays fall, *ZA, ZB, ZE, ZD,* and *ZG.* Now the straight line *ZB* is equal to *ZD.* But also *AB* is equal to *GD.* So the two lines, *AB* and *BZ,* are equal to the two, *GD* and *DZ.* And they contain equal angles; for the angle *BZA* is equal to the angle *DZG.* So AB and *GD* will appear equal.

I say, however, that they may appear also unequal.

Now let the eye move, and let it be indicated by *L,* and let *L* and *E* be joined, and let the rays fall, *LB, LA, LG,* and *LD.* So, *LB* is longer than *LD.* From *LB* let *BT* be taken, equal to *LD,* and let *A* and *T* be joined. Now the angle *BTA* is equal to the angle *GLD.* But the angle *BTA* is greater than the angle *BLA,* the exterior angle greater than the interior. And so the angle *GLD* is greater than the angle *BLA.* Therefore, *GD* will appear greater than *AB.*

Fig. 45.



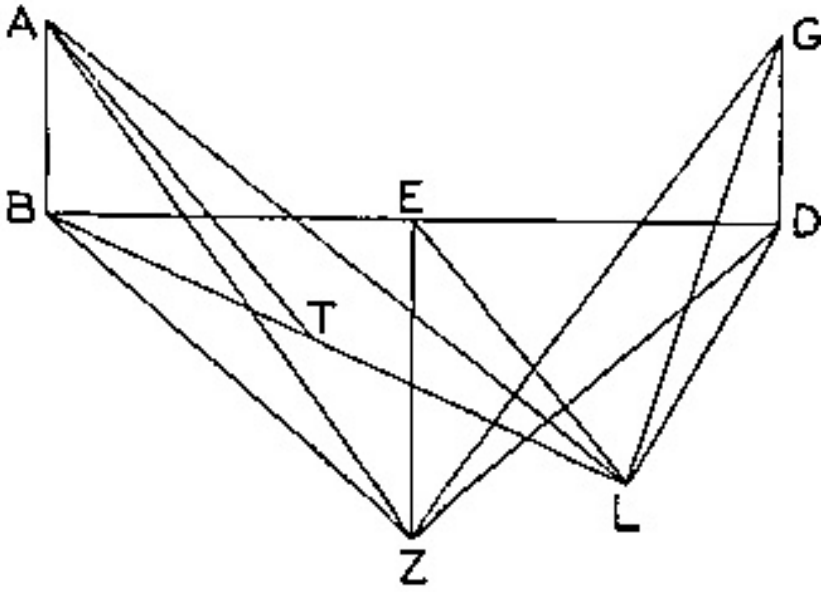
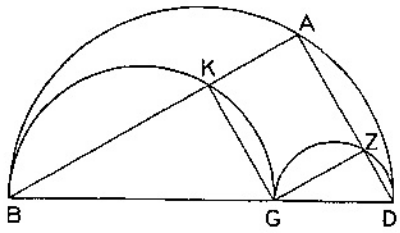


Fig. 46.

Fig. 47.



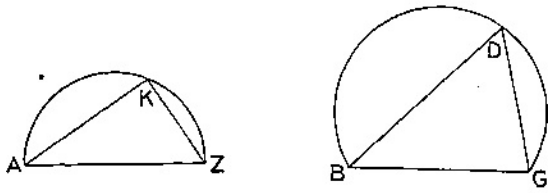


Fig. 48.

There are certain places where the eye may be located, and objects of unequal size placed in contact will together appear equal to each of the

unequal objects. (Fig. 47.)

For let *BG* be larger than *GD,* and about *BG* and *GD* let semicircles be inscribed and another about the whole *BD.* So the angle in the semicircle *BAD* is equal to the angle in *BKG; for each of* them is a right angle. So *BG* appears equal to *BD.* And in like manner also *BD* appears equal to *GD,* when the eyes are upon the semicircles *BAD* and *GZD.* There are, then, certain places where two objects of unequal size, when placed in contact, together appear equal to each of the unequal objects.

To find places from which an object of equal size appears half or a quarter as large, or in general appears of a size proportionate to the cutting of the angle. (Fig. 48.)

Let *AZ* be equal to *BG,* and about *AZ* let a semicircle be inscribed, and in it let the right angle *K* be drawn; and let *BG* be equal to *AZ,* and about *BG* let an arc be inscribed which will contain half of the angle at *K.* So the angle *K* is double the angle *D.* So *AZ* appears twice the size of *BG,* when the eyes are upon the arcs *AKZ* and *BDG.*

Let AB be the object seen. I say that there are places where the eye may be located and the same thing appears sometimes half the size, sometimes the whole size, sometimes quarter size, and in general in the given proportion. (Fig. 49.)

About *AB* let the circle *AEB* be inscribed, so that *AB* is not the diameter, and let the center of the circle be taken and let it be *G,* upon which let the eye rest, and let the straight lines *AG* and *GB* be drawn. So AB is seen under the angle *AGB.* Now let the eye rest upon the circumference of the circle and let it be indicated by *E,* and let the rays fall, *EA* and *EB.* Now, since the angle *AGB* is double the angle *AEB,* from *G, AB* seems twice as large as when seen from *E.* Similarly also, it will appear a fourth as large if one angle is a fourth

as great as the other angle, and (it will always be seen) in the given proportion.

If objects move at equal speed, and have their ends on the same side of a straight line which is at right angles to their course, as they advance toward a line drawn through the point where the eye is located, which is parallel to the straight line before mentioned, the one farther away from the eye will seem to be ahead of the nearer one, but when they have passed (the direct line of vision), the one that was in the lead will seem to follow, and the one that followed will seem to be in the lead. (Fig. 50.)

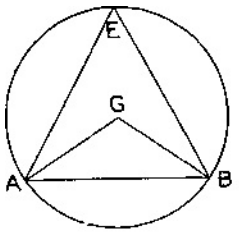


FIG. 49.

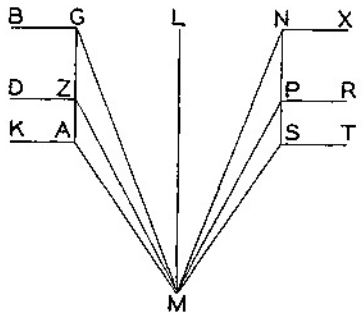


FIG. 50.

For let *BG, DZ,* and *KA* move with equal speed, having their ends, *G, Z,* and *A,* on the same side of a straight line, *G A,* which is at right angles to their course, and from the eye, *M,* let *ML* be drawn parallel to *GA,* and let *MG, MZ,* and *MA* be drawn. So *BG* seems to be in the lead and *KA* seems to follow, on account of the fact that, of the rays falling from the eye, *MG,* leading to *G,* seems to have diverged (to the right) more than the other rays. So *BG* will seem to be in the lead of the moving objects, as has been said. But when *BG, DZ,* and *KA* have passed (the direct line of vision), and have become *NX, PR,* and *ST,* let the rays fall, *MN, MP,* and *MS.* So *NX* seems to have moved in the direction of *N* (that is, toward the left), on account of the fact that the ray *MN* has been diverted to *N* more than the other rays; so *ST* has moved in the direction of *T* (that is, toward the right), on account of the fact that *MS* has been diverted toward *T* more than the other rays. So when *BG,* which seemed to be in the lead, has become *NX,* it will seem to follow,

but when *KA,* which seemed to follow, has become *ST,* it will seem to be in the lead.

If, when several objects move at unequal speed, the eye also moves in the same direction, some objects, moving with the same speed as the eye, will seem to stand still, others, moving more slowly, will seem to move in the opposite direction, and others, moving more quickly, will seem to move ahead. (Fig. 51.)

For, let *B, G,* and *D* move at unequal speed, and let *B* move most slowly, and *G* at the same speed as the eye, *K,* and *D* more quickly than *G.* And from the eye, *K,* let the rays fall, *KB, KG,* and *KD.* So, *G,* moving with the eye, will seem to stand still, and *B,* left behind, will seem to move in the opposite direction, and *D,* which moves more quickly than these, will seem to move forward; for it will be more distant from these.

If, when certain objects are moved, one is obviously not moved, the object that is not moved will seem to move backward. (Fig. 52.)

For, let *B* and *D* move, and let *G* remain unmoved, and from the eye let the rays fall, *ZB, ZG,* and *ZD.* So *B,* as it moves, will be nearer to *G,* and *D,* receding, will be farther away; therefore, *G* will seem to move in the opposite direction.

When the eye moves nearer the object seen, the object will seem to grow larger. (Fig. 53.)

For, the eye being at *Z,* let *BG* be seen by the rays *ZB* and *ZG,* and let the eye move nearer to *BG,* and let it be at *D,* and let the same thing be seen by the rays *DB* and *DG.* So the angle *D* is greater than the angle *Z;* and things seen under a wider angle appear larger. Therefore *BG* will seem to be larger when the eye is at *D,* than when it is at *Z.*

When objects move at equal speed, those more remote seem to move more slowly. (Fig. 54.)

For let *B* and *K* move at equal speed, and from the eye, *A,* let rays be drawn, *AG, AD,* and *AZ.* So, *B* has longer rays than *K* extending from the eye. Therefore it will cover a greater distance, and, later, passing the line of vision, *AZ,* will seem to move more slowly.

An addition. (Fig. 54a.) For let two points, *A* and *B,* move on parallel straight lines, and let *Z* be the eye, from which let the rays fall, *ZA, ZB, ZE,* and *ZD.* I say that the more remote object, *A,* seems to move more slowly than *B.* For, since *AZ* and *ZD* form a smaller angle than *ZB* and *ZE, BE* appears greater than *AD.* (The following sentence is corrupt in the Greek text, and no satisfactory translation is possible.) Therefore, if we extend the ray *ZE* in a straight line, since in the case of objects moving at the same speed *B* reaches the ray *ZE* later than the things moving at the same speed, the more remote objects seem to move more slowly.

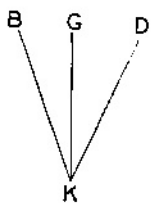


FIG. 51.

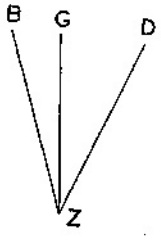


FIG. 52.

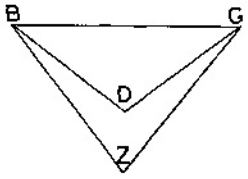


FIG. 53.

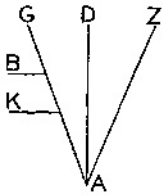


FIG. 54.

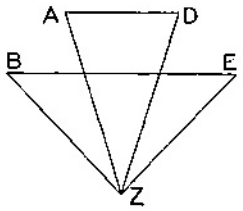


FIG. 54*a*.

An addition. (Fig. 54*b).* Let two points, *A* and *B,* move evenly on the parallel straight lines, *AD* and *BE;* they will cover the same distances in the same time. So, let *AD* and *BE* be equal distances, and from the eye, *Z,* let the rays fall, *ZA, ZD, ZB,* and *ZE.* Now, since the angle *AZD* is less than the angle *BZE,* therefore the distance *AD* will appear less than the distance *BE.* So that *A* will appear to move more slowly.

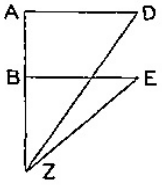


FIG. 54*b*.

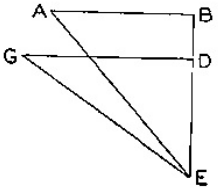


FIG. 55.

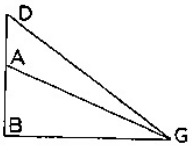
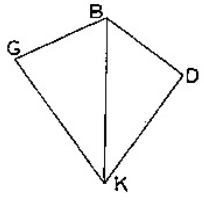


FIG. 56.



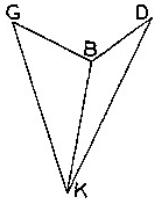


FIG. 57.

If the eye remains at rest, while the things seen are moved, the more remote of the things seen will seem to be left behind. (Fig. 55.) —

Let *A* and *G* be the things seen, located on the straight lines, *AB* and *GD,* and let *E* be the eye, from which let the rays fall, *EG, ED, EA,* and *EB.* I say that the object at *A* will seem to have been left behind. Let *ED* be extended to a point where it meets *AB,* and let it be *EB.* Now, since the angle *GEB* is greater than the angle *AEB,* the distance *GD* appears greater than *AB.* So that if the eye remains at *E,* the rays, moving in the direction of *A* and *G,* will pass *A* more quickly than *G.* So *AB* will seem to be left behind.

Objects increased in size will seem to approach the eye. (Fig. 56.)

Let *AB* represent the size of the object seen, and let *G* be the eye, from which let the rays fall, *GA* and *GB.* And let *BA* be increased, and let it be *BD,* and let the ray fall, *GD.* Now, since the angle *BGD* is greater than the angle *BGA, BD* appears greater than *BA.* But things thought to be greater than themselves seem to be increased, and the things nearer the eye appear greater. So objects increased in size will seem to approach the eye.

When things lie at the same distance and the edges are not in line with the middle, it makes the whole figure sometimes concave, sometimes convex. (Fig. 57.)

For let *GBD* be seen by the eye located at *K,* and let the rays fall, *KG, KB,* and *KD.* So the whole figure will seem to be concave. Now let the thing seen in the middle be moved back, and let it be nearer the eye. So *DBG* will seem to be convex.

If from the meeting-point of the diameters of a square a perpendicular is drawn, and the eye rests upon this, the sides of the square will appear equal and the diameters also will appear equal. (Fig. 58.)

Let *ABGD* be a square and let the diagonals *DB* and *GA* be drawn, and from *E* let the straight line *EZ* be drawn perpendicular to the plane, and upon

this line let *Z* indicate the position of the eye, and let the rays fall, *ZA, ZB, ZG,* and *ZD.* Now, since *DE* is equal to *EG,* and *EZ* is common to them, and the angles are right angles, the base *ZG* is equal to the base *DZ,* and of the angles at the bases those are equal upon which equal sides subtend. So the angle *EZG* is equal to the angle *EZD.* Therefore *EG* will appear equal to *ED.* Similarly also the angle *AZE* is equal to the angle *BZE.* Therefore *AG* will appear equal to *BD.* Again, since *GZ* is equal to *ZB,* but also *AB* to *GD,* the three are equal to the three others, and one angle is equal to another angle. So one side will appear equal to the other, as also the remaining sides will appear equal.

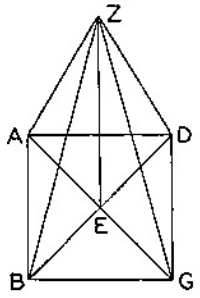


FIG. 58.

But if a line is drawn from the eye to the meeting-point of the diameters, and if it is not perpendicular to the plane or equal to each one of the lines drawn from the meeting- point to the angles of the square, and if it does not make equal angles with them, the diameters will appear unequal. For similarly we shall prove what happens, just as in the case of circles.

The Greek Text

Present-day ruins of the Serapeum of Alexandria, where the Library of Alexandria moved part of its collection after it ran out of storage space in the main building — Euclid was the celebrated founder of the mathematical school at Alexandria.

Elements

In this section of the eBook, readers can view the original Greek text of Euclid's *Στοιχεῖα* (*Elements)*. You may wish to Bookmark this page for future reference.

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Προτάσεις ιη΄.

BOOK I.

Ὅροι κγ΄.

α΄. Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.

β΄. Γραμμὴ δὲ μῆκος ἀπλατές.

γ΄. Γραμμῆς δὲ πέρατα σημεῖα.

δ΄. Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ᾽ ἑαυτῆς σημείοις κεῖται.

ε΄. Ἐπιφάνεια δέ ἐστιν, ὃ μῆκος καὶ πλάτος μόνον ἔχει.

ς΄. Ἐπιφανείας δὲ πέρατα γραμμαί.

ζ΄. Ἐπίπεδος ἐπιφάνειά ἐστιν, ἥτις ἐξ ἴσου ταῖς ἐφ᾽ ἑαυτῆς εὐθείαις κεῖται.

η΄. Ἐπίπεδος δὲ γωνία ἐστὶν ἡ ἐν ἐπιπέδῳ δύο γραμμῶν ἁπτομένων ἀλλήλων καὶ μὴ ἐπ᾽ εὐθείας κειμένων πρὸς ἀλλήλας τῶν γραμμῶν κλίσις.

θ΄. Ὅταν δὲ αἱ περιέχουσαι τὴν γωνίαν γραμμαὶ εὐθεῖαι ὦσιν, εὐθύγραμμος καλεῖται ἡ γωνία.

ι΄. Ὅταν δὲ εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστι, καὶ ἡ ἐφεστηκυῖα εὐθεῖα κάθετος καλεῖται, ἐφ᾽ ἣν ἐφέστηκεν.

ια΄. Ἀμβλεῖα γωνία ἐστὶν ἡ μείζων ὀρθῆς.

ιβ΄. Ὀξεῖα δὲ ἡ ἐλάσσων ὀρθῆς.

ιγ΄. Ὅρος ἐστίν, ὅ τινός ἐστι πέρας.

ιδ΄. Σχῆμά ἐστι τὸ ὑπό τινος ἢ τινων ὅρων περιεχόμενον.

ιε΄. Κύκλος ἐστὶ σχῆμα ἐπίπεδον ὑπὸ μιᾶς γραμμῆς περιεχόμενον [ἣ καλεῖται περιφέρεια], πρὸς ἣν ἀφ᾽ ἑνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἱ προσπίπτουσαι εὐθεῖαι [πρὸς τὴν τοῦ κύκλου περιφέρειαν] ἴσαι ἀλλήλαις εἰσίν.

ις΄. Κέντρον δὲ τοῦ κύκλου τὸ σημεῖον καλεῖται.

ιζ΄. Διάμετρος δὲ τοῦ κύκλου ἐστὶν εὐθεῖά τις διὰ τοῦ κέντρου ἠγμένη καὶ περατουμένη ἐφ᾽ ἑκάτερα τὰ μέρη ὑπὸ τῆς τοῦ κύκλου περιφερείας, ἥτις καὶ δίχα τέμνει τὸν κύκλον.

ιη΄. Ἡμικύκλιον δέ ἐστι τὸ περιεχόμενον σχῆμα ὑπό τε τῆς διαμέτρου καὶ τῆς ἀπολαμβανομένης ὑπ᾽ αὐτῆς περιφερείας. κέντρον δὲ τοῦ ἡμικυκλίου τὸ αὐτό, ὃ καὶ τοῦ κύκλου ἐστίν.

ιθ΄. Σχήματα εὐθύγραμμά ἐστι τὰ ὑπὸ εὐθειῶν περιεχόμενα, τρίπλευρα μὲν τὰ ὑπὸ τριῶν, τετράπλευρα δὲ τὰ ὑπὸ τεσσάρων, πολύπλευρα δὲ τὰ ὑπὸ πλειόνων ἢ τεσσάρων εὐθειῶν περιεχόμενα.

κ΄. Τῶν δὲ τριπλεύρων σχημάτων ἰσόπλευρον μὲν τρίγωνόν ἐστι τὸ τὰς τρεῖς ἴσας ἔχον πλευράς, ἰσοσκελὲς δὲ τὸ τὰς δύο μόνας ἴσας ἔχον πλευράς, σκαληνὸν δὲ τὸ τὰς τρεῖς ἀνίσους ἔχον πλευράς.

κα΄. Ἔτι δὲ τῶν τριπλεύρων σχημάτων ὀρθογώνιον μὲν τρίγωνόν ἐστι τὸ ἔχον ὀρθὴν γωνίαν, ἀμβλυγώνιον δὲ τὸ ἔχον ἀμβλεῖαν γωνίαν, ὀξυγώνιον δὲ τὸ τὰς τρεῖς ὀξείας ἔχον γωνίας.

κβ΄. Τῶν δὲ τετραπλεύρων σχημάτων τετράγωνον μέν ἐστιν, ὃ ἰσόπλευρόν τέ ἐστι καὶ ὀρθογώνιον, ἑτερόμηκες δέ, ὃ ὀρθογώνιον μέν, οὐκ ἰσόπλευρον δέ, ῥόμβος δέ, ὃ ἰσόπλευρον μέν, οὐκ ὀρθογώνιον δέ, ῥομβοειδὲς δὲ τὸ τὰς ἀπεναντίον πλευράς τε καὶ γωνίας ἴσας ἀλλήλαις ἔχον, ὃ οὔτε ἰσόπλευρόν ἐστιν οὔτε ὀρθογώνιον: τὰ δὲ παρὰ ταῦτα τετράπλευρα τραπέζια καλείσθω.

κγ΄. Παράλληλοί εἰσιν εὐθεῖαι, αἵτινες ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι καὶ ἐκβαλλόμεναι εἰς ἄπειρον ἐφ᾽ ἑκάτερα τὰ μέρη ἐπὶ μηδέτερα συμπίπτουσιν ἀλλήλαις.

Αἰτήματα ε΄.

α΄. Ἠιτήσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.

β΄. Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ᾽ εὐθείας ἐκβαλεῖν.

γ΄. Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι.

δ΄. Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.

ε΄. Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ᾽ ἄπειρον συμπίπτειν, ἐφ᾽ ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

Κοιναί Ἔνοιαι θ΄.

α΄. Τὰ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα.

β΄. Καὶ ἐὰν ἴσοις ἴσα προστεθῇ, τὰ ὅλα ἐστὶν ἴσα.

γ΄. Καὶ ἐὰν ἀπὸ ἴσων ἴσα ἀφαιρεθῇ, τὰ καταλειπόμενά ἐστιν ἴσα.

δ΄. [Καὶ ἐὰν ἀνίσοις ἴσα προστεθῇ, τὰ ὅλα ἐστὶν ἄνισα.

ε΄. Καὶ τὰ τοῦ αὐτοῦ διπλάσια ἴσα ἀλλήλοις ἐστίν.

ς΄. Καὶ τὰ τοῦ αὐτοῦ ἡμίση ἴσα ἀλλήλοις ἐστίν.]

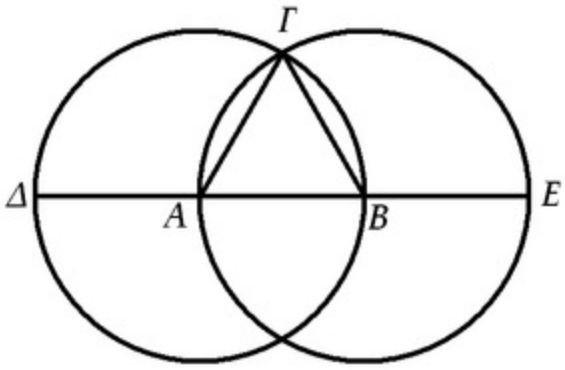
ζ΄. Καὶ τὰ ἐφαρμόζοντα ἐπ᾽ ἄλληλα ἴσα ἀλλήλοις ἐστίν.

η΄. Καὶ τὸ ὅλον τοῦ μέρους μεῖζον [ἐστιν].

θ΄. Καὶ δύο εὐθεῖαι χωρίον οὐ περιέχουσιν.

Προτάσεις μη΄.

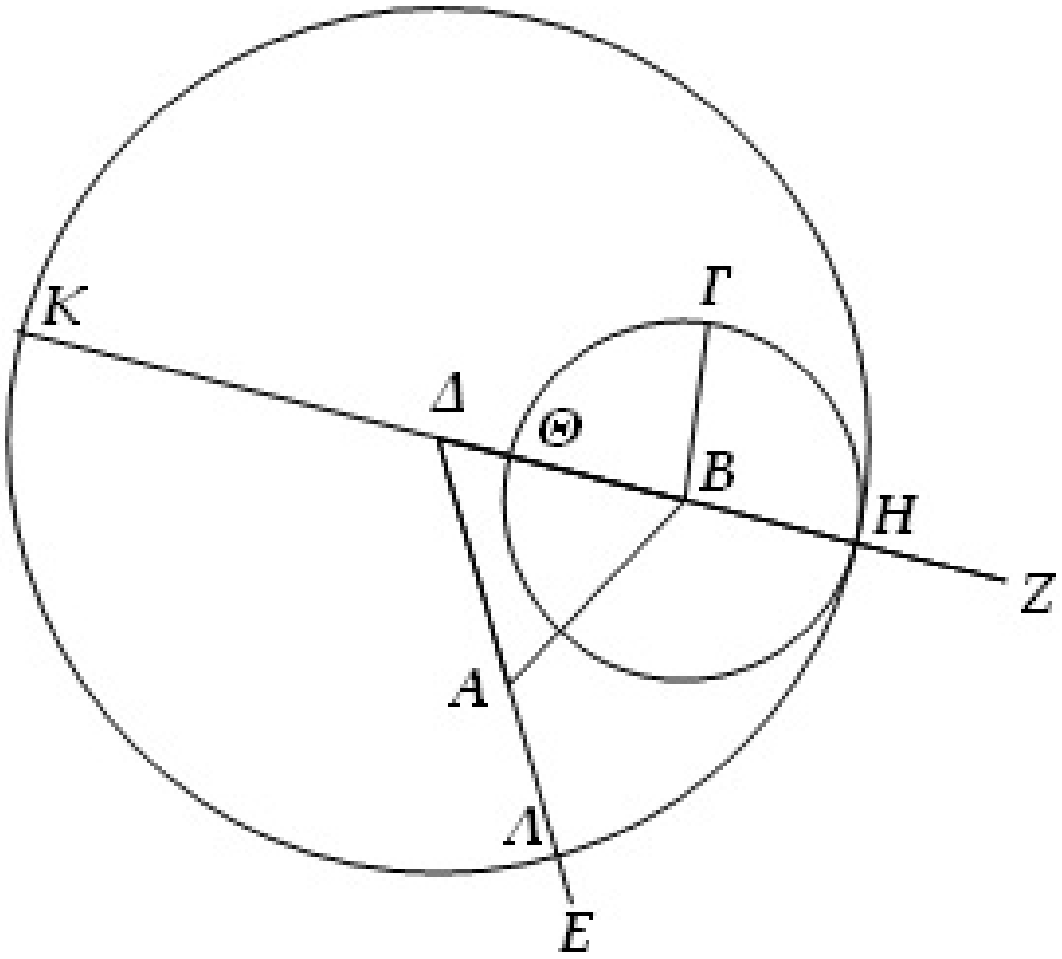
α΄. Ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συστήσασθαι.



Ἔστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ. Δεῖ δὴ ἐπὶ τῆς ΑΒ εὐθείας τρίγωνον ἰσόπλευρον συστήσασθαι. Κέντρῳ μὲν τῷ Α διαστήματι δὲ τῷ ΑΒ κύκλος γεγράφθω ὁ ΒΓΔ, καὶ πάλιν κέντρῳ μὲν τῷ Β διαστήματι δὲ τῷ ΒΑ κύκλος γεγράφθω ὁ ΑΓΕ, καὶ ἀπὸ τοῦ Γ σημείου, καθ’ ὃ τέμνουσιν ἀλλήλους

οἱ κύκλοι, ἐπὶ τὰ Α, Β σημεῖα ἐπεζεύχθωσαν εὐθεῖαι αἱ ΓΑ, ΓΒ. Καὶ ἐπεὶ τὸ Α σημεῖον κέντρον ἐστὶ τοῦ ΓΔΒ κύκλου, ἴση ἐστὶν ἡ ΑΓ τῇ ΑΒ· πάλιν, ἐπεὶ τὸ Β σημεῖον κέντρον ἐστὶ τοῦ ΓΑΕ κύκλου, ἴση ἐστὶν ἡ ΒΓ τῇ ΒΑ. ἐδείχθη δὲ καὶ ἡ ΓΑ τῇ ΑΒ ἴση· ἑκατέρα ἄρα τῶν ΓΑ, ΓΒ τῇ ΑΒ ἐστὶν ἴση. τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ ΓΑ ἄρα τῇ ΓΒ ἐστὶν ἴση· αἱ τρεῖς ἄρα αἱ ΓΑ, ΑΒ, ΒΓ ἴσαι ἀλλήλαις εἰσίν. Ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον, καὶ συνέσταται ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τῆς ΑΒ. [Ἐπὶ τῆς δοθείσης ἄρα εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συνέσταται]· ὅπερ ἔδει ποιῆσαι.

β΄. Πρὸς τῷ δοθέντι σημείῳ τῇ δοθείσῃ εὐθείᾳ ἴσην εὐθεῖαν θέσθαι.



Ἔστω τὸ μὲν δοθὲν σημεῖον τὸ Α, ἡ δὲ δοθεῖσα εὐθεῖα ἡ ΒΓ· δεῖ δὴ πρὸς τῷ Α σημείῳ τῇ δοθείσῃ εὐθείᾳ τῇ ΒΓ ἴσην εὐθεῖαν θέσθαι.

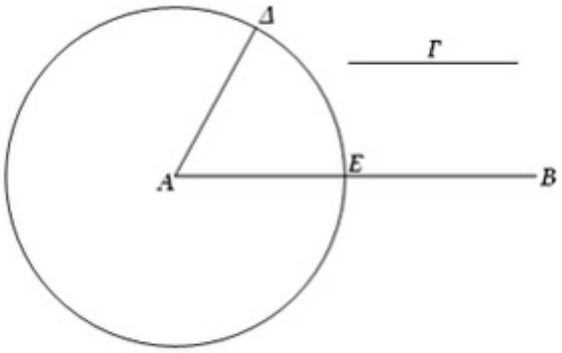
Ἐπεζεύχθω γὰρ ἀπὸ τοῦ Α σημείου ἐπὶ τὸ Β σημεῖον εὐθεῖα ἡ ΑΒ, καὶ συνεστάτω ἐπ᾽ αὐτῆς τρίγωνον ἰσόπλευρον τὸ ΔΑΒ, καὶ ἐκβεβλήσθωσαν ἐπ᾽ εὐθείας ταῖς ΔΑ, ΔΒ εὐθεῖαι αἱ ΑΕ, ΒΖ, καὶ κέντρῳ μὲν τῷ Β διαστήματι δὲ τῷ ΒΓ κύκλος γεγράφθω ὁ ΓΗΘ, καὶ πάλιν κέντρῳ τῷ Δ καὶ διαστήματι τῷ ΔΗ κύκλος γεγράφθω ὁ ΗΚΛ.

Ἐπεὶ οὖν τὸ Β σημεῖον κέντρον ἐστὶ τοῦ ΓΗΘ κύκλου, ἴση ἐστὶν ἡ ΒΓ τῇ ΒΗ. πάλιν, ἐπεὶ τὸ Δ σημεῖον κέντρον ἐστὶ τοῦ ΚΛΗ κύκλου, ἴση ἐστὶν ἡ ΔΛ τῇ ΔΗ, ὧν ἡ ΔΑ τῇ ΔΒ ἴση ἐστίν. λοιπὴ ἄρα ἡ ΑΛ λοιπῇ τῇ ΒΗ ἐστὶν ἴση.

ἐδείχθη δὲ καὶ ἡ ΒΓ τῇ ΒΗ ἴση· ἑκατέρα ἄρα τῶν ΑΛ, ΒΓ τῇ ΒΗ ἐστὶν ἴση. τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ ΑΛ ἄρα τῇ ΒΓ ἐστὶν ἴση.

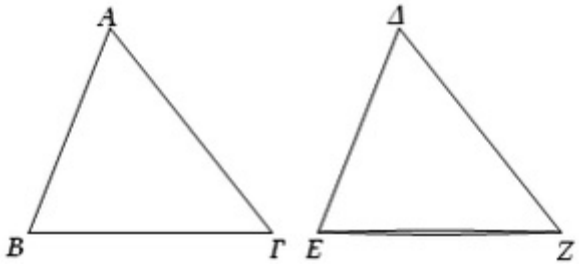
Πρὸς ἄρα τῷ δοθέντι σημείῳ τῷ Α τῇ δοθείσῃ εὐθείᾳ τῇ ΒΓ ἴση εὐθεῖα κεῖται ἡ ΑΛ· ὅπερ ἔδει ποιῆσαι.

γ΄. Δύο δοθεισῶν εὐθειῶν ἀνίσων ἀπὸ τῆς μείζονος τῇ ἐλάσσονι ἴσην εὐθεῖαν ἀφελεῖν.



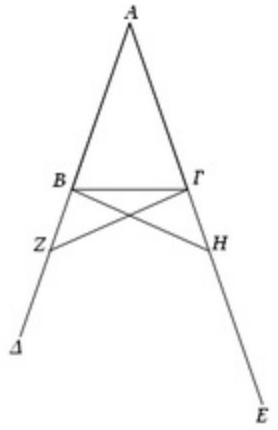
Ἔστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι ἄνισοι αἱ ΑΒ, Γ, ὧν μείζων ἔστω ἡ ΑΒ· δεῖ δὴ ἀπὸ τῆς μείζονος τῆς ΑΒ τῇ ἐλάσσονι τῇ Γ ἴσην εὐθεῖαν ἀφελεῖν. Κείσθω πρὸς τῷ Α σημείῳ τῇ Γ εὐθείᾳ ἴση ἡ ΑΔ· καὶ κέντρῳ μὲν τῷ Α διαστήματι δὲ τῷ ΑΔ κύκλος γεγράφθω ὁ ΔΕΖ. Καὶ ἐπεὶ τὸ Α σημεῖον κέντρον ἐστὶ τοῦ ΔΕΖ κύκλου, ἴση ἐστὶν ἡ ΑΕ τῇ ΑΔ· ἀλλὰ καὶ ἡ Γ τῇ ΑΔ ἐστιν ἴση. ἑκατέρα ἄρα τῶν ΑΕ, Γ τῇ ΑΔ ἐστιν ἴση· ὥστε καὶ ἡ ΑΕ τῇ Γ ἐστιν ἴση. Δύο ἄρα δοθεισῶν εὐθειῶν ἀνίσων τῶν ΑΒ, Γ ἀπὸ τῆς μείζονος τῆς ΑΒ τῇ ἐλάσσονι τῇ Γ ἴση ἀφῄρηται ἡ ΑΕ· ὅπερ ἔδει ποιῆσαι.

δ΄. Ἐὰν δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δυσὶ πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ τὴν γωνίαν τῇ γωνίᾳ ἴσην ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τῇ βάσει ἴσην ἕξει, καὶ τὸ τρίγωνον τῷ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν.



Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ τὰς δύο πλευρὰς τὰς ΑΒ, ΑΓ ταῖς δυσὶ πλευραῖς ταῖς ΔΕ, ΔΖ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ τὴν μὲν ΑΒ τῇ ΔΕ τὴν δὲ ΑΓ τῇ ΔΖ καὶ γωνίαν τὴν ὑπὸ ΒΑΓ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἴσην. λέγω, ὅτι καὶ βάσις ἡ ΒΓ βάσει τῇ ΕΖ ἴση ἐστίν, καὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν, ἡ μὲν ὑπὸ ΑΒΓ τῇ ὑπὸ ΔΕΖ, ἡ δὲ ὑπὸ ΑΓΒ τῇ ὑπὸ ΔΖΕ. Ἐφαρμοζομένου γὰρ τοῦ ΑΒΓ τριγώνου ἐπὶ τὸ ΔΕΖ τρίγωνον καὶ τιθεμένου τοῦ μὲν Α σημείου ἐπὶ τὸ Δ σημεῖον τῆς δὲ ΑΒ εὐθείας ἐπὶ τὴν ΔΕ, ἐφαρμόσει καὶ τὸ Β σημεῖον ἐπὶ τὸ Ε διὰ τὸ ἴσην εἶναι τὴν ΑΒ τῇ ΔΕ· ἐφαρμοσάσης δὴ τῆς ΑΒ ἐπὶ τὴν ΔΕ ἐφαρμόσει καὶ ἡ ΑΓ εὐθεῖα ἐπὶ τὴν ΔΖ διὰ τὸ ἴσην εἶναι τὴν ὑπὸ ΒΑΓ γωνίαν τῇ ὑπὸ ΕΔΖ· ὥστε καὶ τὸ Γ σημεῖον ἐπὶ τὸ Ζ σημεῖον ἐφαρμόσει διὰ τὸ ἴσην πάλιν εἶναι τὴν ΑΓ τῇ ΔΖ. ἀλλὰ μὴν καὶ τὸ Β ἐπὶ τὸ Ε ἐφηρμόκει· ὥστε βάσις ἡ ΒΓ ἐπὶ βάσιν τὴν ΕΖ ἐφαρμόσει. εἰ γὰρ τοῦ μὲν Β ἐπὶ τὸ Ε ἐφαρμόσαντος τοῦ δὲ Γ ἐπὶ τὸ Ζ ἡ ΒΓ βάσις ἐπὶ τὴν ΕΖ οὐκ ἐφαρμόσει, δύο εὐθεῖαι χωρίον περιέξουσιν· ὅπερ ἐστὶν ἀδύνατον. ἐφαρμόσει ἄρα ἡ ΒΓ βάσις ἐπὶ τὴν ΕΖ καὶ ἴση αὐτῇ ἔσται· ὥστε καὶ ὅλον τὸ ΑΒΓ τρίγωνον ἐπὶ ὅλον τὸ ΔΕΖ τρίγωνον ἐφαρμόσει καὶ ἴσον αὐτῷ ἔσται, καὶ αἱ λοιπαὶ γωνίαι ἐπὶ τὰς λοιπὰς γωνίας ἐφαρμόσουσι καὶ ἴσαι αὐταῖς ἔσονται, ἡ μὲν ὑπὸ ΑΒΓ τῇ ὑπὸ ΔΕΖ ἡ δὲ ὑπὸ ΑΓΒ τῇ ὑπὸ ΔΖΕ. Ἐὰν ἄρα δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ τὴν γωνίαν τῇ γωνίᾳ ἴσην ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τῇ βάσει ἴσην ἕξει, καὶ τὸ τρίγωνον τῷ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ὅπερ ἔδει δεῖξαι.

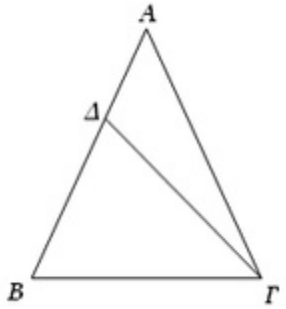
ε΄. Τῶν ἰσοσκελῶν τριγώνων αἱ πρὸς τῇ βάσει γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ προσεκβληθεισῶν τῶν ἴσων εὐθειῶν αἱ ὑπὸ τὴν βάσιν γωνίαι ἴσαι ἀλλήλαις ἔσονται.



Ἔστω τρίγωνον ἰσοσκελὲς τὸ ΑΒΓ ἴσην ἔχον τὴν ΑΒ πλευρὰν τῇ ΑΓ πλευρᾷ, καὶ προσεκβεβλήσθωσαν ἐπ᾽ εὐθείας ταῖς ΑΒ, ΑΓ εὐθεῖαι αἱ ΒΔ, ΓΕ· λέγω, ὅτι ἡ μὲν ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΑΓΒ ἴση ἐστίν, ἡ δὲ ὑπὸ ΓΒΔ τῇ ὑπὸ ΒΓΕ. Εἰλήφθω γὰρ ἐπὶ τῆς ΒΔ τυχὸν σημεῖον τὸ Ζ, καὶ ἀφῃρήσθω ἀπὸ τῆς μείζονος τῆς ΑΕ τῇ ἐλάσσονι τῇ ΑΖ ἴση ἡ ΑΗ, καὶ ἐπεζεύχθωσαν αἱ ΖΓ, ΗΒ εὐθεῖαι. Ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν ΑΖ τῇ ΑΗ ἡ δὲ ΑΒ τῇ ΑΓ, δύο δὴ αἱ ΖΑ, ΑΓ δυσὶ ταῖς ΗΑ, ΑΒ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνίαν κοινὴν περιέχουσι τὴν ὑπὸ ΖΑΗ· βάσις ἄρα ἡ ΖΓ βάσει τῇ ΗΒ ἴση ἐστίν, καὶ τὸ ΑΖΓ τρίγωνον τῷ ΑΗΒ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν, ἡ μὲν ὑπὸ ΑΓΖ τῇ ὑπὸ ΑΒΗ, ἡ δὲ ὑπὸ ΑΖΓ τῇ ὑπὸ ΑΗΒ. καὶ ἐπεὶ ὅλη ἡ ΑΖ ὅλῃ τῇ ΑΗ ἐστιν ἴση, ὧν ἡ ΑΒ τῇ ΑΓ ἐστιν ἴση, λοιπὴ ἄρα ἡ ΒΖ λοιπῇ τῇ ΓΗ ἐστιν ἴση. ἐδείχθη δὲ καὶ ἡ ΖΓ τῇ ΗΒ ἴση· δύο δὴ αἱ ΒΖ, ΖΓ δυσὶ ταῖς ΓΗ, ΗΒ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΒΖΓ γωνίᾳ τῇ ὑπὸ ΓΗΒ ἴση, καὶ βάσις αὐτῶν κοινὴ ἡ ΒΓ· καὶ τὸ ΒΖΓ ἄρα τρίγωνον τῷ ΓΗΒ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἐστὶν ἡ μὲν ὑπὸ ΖΒΓ

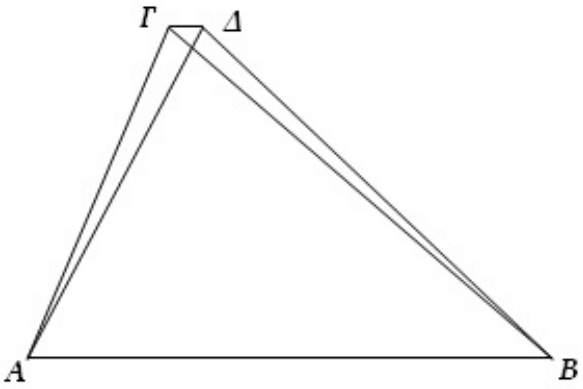
τῇ ὑπὸ ΗΓΒ ἡ δὲ ὑπὸ ΒΓΖ τῇ ὑπὸ ΓΒΗ. ἐπεὶ οὖν ὅλη ἡ ὑπὸ ΑΒΗ γωνία ὅλῃ τῇ ὑπὸ ΑΓΖ γωνίᾳ ἐδείχθη ἴση, ὧν ἡ ὑπὸ ΓΒΗ τῇ ὑπὸ ΒΓΖ ἴση, λοιπὴ ἄρα ἡ ὑπὸ ΑΒΓ λοιπῇ τῇ ὑπὸ ΑΓΒ ἐστιν ἴση· καί εἰσι πρὸς τῇ βάσει τοῦ ΑΒΓ τριγώνου. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΖΒΓ τῇ ὑπὸ ΗΓΒ ἴση· καί εἰσιν ὑπὸ τὴν βάσιν. Τῶν ἄρα ἰσοσκελῶν τριγώνων αἱ πρὸς τῇ βάσει γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ προσεκβληθεισῶν τῶν ἴσων εὐθειῶν αἱ ὑπὸ τὴν βάσιν γωνίαι ἴσαι ἀλλήλαις ἔσονται· ὅπερ ἔδει δεῖξαι.

ς΄. Ἐὰν τριγώνου αἱ δύο γωνίαι ἴσαι ἀλλήλαις ὦσιν, καὶ αἱ ὑπὸ τὰς ἴσας γωνίαις ὑποτείνουσαι πλευραὶ ἴσαι ἀλλήλαις ἔσονται.



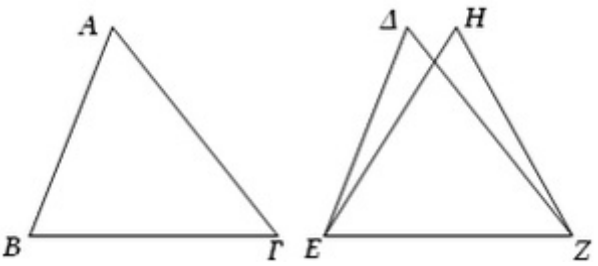
Ἔστω τρίγωνον τὸ ΑΒΓ ἴσην ἔχον τὴν ὑπὸ ΑΒΓ γωνίαν τῇ ὑπὸ ΑΓΒ γωνίᾳ· λέγω, ὅτι καὶ πλευρὰ ἡ ΑΒ πλευρᾷ τῇ ΑΓ ἐστιν ἴση. Εἰ γὰρ ἄνισός ἐστιν ἡ ΑΒ τῇ ΑΓ, ἡ ἑτέρα αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ ΑΒ, καὶ ἀφῃρήσθω ἀπὸ τῆς μείζονος τῆς ΑΒ τῇ ἐλάττονι τῇ ΑΓ ἴση ἡ ΔΒ, καὶ ἐπεζεύχθω ἡ ΔΓ. Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΔΒ τῇ ΑΓ κοινὴ δὲ ἡ ΒΓ, δύο δὴ αἱ ΔΒ, ΒΓ δύο ταῖς ΑΓ, ΓΒ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ γωνία ἡ ὑπὸ ΔΒΓ γωνίᾳ τῇ ὑπὸ ΑΓΒ ἐστιν ἴση· βάσις ἄρα ἡ ΔΓ βάσει τῇ ΑΒ ἴση ἐστίν, καὶ τὸ ΔΒΓ τρίγωνον τῷ ΑΓΒ τριγώνῳ ἴσον ἔσται, τὸ ἔλασσον τῷ μείζονι· ὅπερ ἄτοπον· οὐκ ἄρα ἄνισός ἐστιν ἡ ΑΒ τῇ ΑΓ· ἴση ἄρα. Ἐὰν ἄρα τριγώνου αἱ δύο γωνίαι ἴσαι ἀλλήλαις ὦσιν, καὶ αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι πλευραὶ ἴσαι ἀλλήλαις ἔσονται· ὅπερ ἔδει δεῖξαι.

ζ΄. Ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθεῖαι ἴσαι ἑκατέρα ἑκατέρᾳ οὐ συσταθήσονται πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι ταῖς ἐξ ἀρχῆς εὐθείαις.



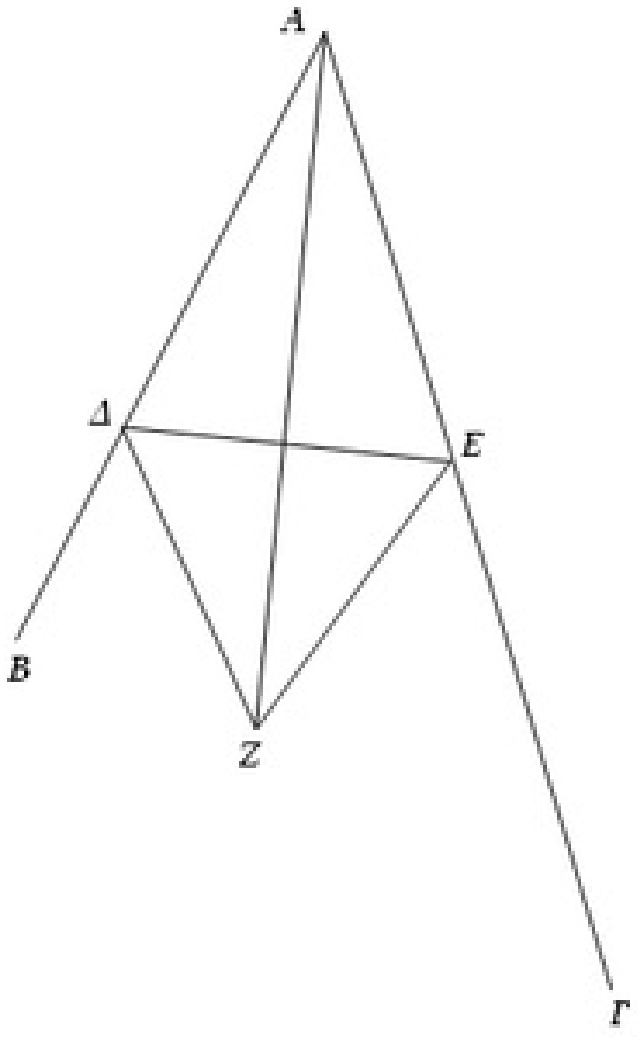
Εἰ γὰρ δυνατόν, ἐπὶ τῆς αὐτῆς εὐθείας τῆς ΑΒ δύο ταῖς αὐταῖς εὐθείαις ταῖς ΑΓ, ΓΒ ἄλλαι δύο εὐθεῖαι αἱ ΑΔ, ΔΒ ἴσαι ἑκατέρα ἑκατέρᾳ συνεστάτωσαν πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ τῷ τε Γ καὶ Δ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι, ὥστε ἴσην εἶναι τὴν μὲν ΓΑ τῇ ΔΑ τὸ αὐτὸ πέρας ἔχουσαν αὐτῇ τὸ Α, τὴν δὲ ΓΒ τῇ ΔΒ τὸ αὐτὸ πέρας ἔχουσαν αὐτῇ τὸ Β, καὶ ἐπεζεύχθω ἡ ΓΔ. Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΑΓ τῇ ΑΔ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΑΓΔ τῇ ὑπὸ ΑΔΓ· μείζων ἄρα ἡ ὑπὸ ΑΔΓ τῆς ὑπὸ ΔΓΒ· πολλῷ ἄρα ἡ ὑπὸ ΓΔΒ μείζων ἐστὶ τῆς ὑπὸ ΔΓΒ. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ΓΒ τῇ ΔΒ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΓΔΒ γωνίᾳ τῇ ὑπὸ ΔΓΒ. ἐδείχθη δὲ αὐτῆς καὶ πολλῷ μείζων· ὅπερ ἐστὶν ἀδύνατον. Οὐκ ἄρα ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθεῖαι ἴσαι ἑκατέρα ἑκατέρᾳ συσταθήσονται πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι ταῖς ἐξ ἀρχῆς εὐθείαις· ὅπερ ἔδει δεῖξαι.

η΄. Ἐὰν δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρα, ἔχῃ δὲ καὶ τὴν βάσιν τῇ βάσει ἴσην, καὶ τὴν γωνίαν τῇ γωνίᾳ ἴσην ἕξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην.



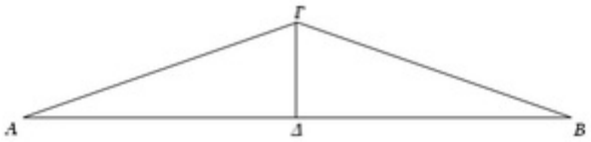
Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ τὰς δύο πλευρὰς τὰς ΑΒ, ΑΓ ταῖς δύο πλευραῖς ταῖς ΔΕ, ΔΖ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ, τὴν μὲν ΑΒ τῇ ΔΕ τὴν δὲ ΑΓ τῇ ΔΖ· ἐχέτω δὲ καὶ βάσιν τὴν ΒΓ βάσει τῇ ΕΖ ἴσην· λέγω, ὅτι καὶ γωνία ἡ ὑπὸ ΒΑΓ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἐστιν ἴση. Ἐφαρμοζομένου γὰρ τοῦ ΑΒΓ τριγώνου ἐπὶ τὸ ΔΕΖ τρίγωνον καὶ τιθεμένου τοῦ μὲν Β σημείου ἐπὶ τὸ Ε σημεῖον τῆς δὲ ΒΓ εὐθείας ἐπὶ τὴν ΕΖ ἐφαρμόσει καὶ τὸ Γ σημεῖον ἐπὶ τὸ Ζ διὰ τὸ ἴσην εἶναι τὴν ΒΓ τῇ ΕΖ· ἐφαρμοσάσης δὴ τῆς ΒΓ ἐπὶ τὴν ΕΖ ἐφαρμόσουσι καὶ αἱ ΒΑ, ΓΑ ἐπὶ τὰς ΕΔ, ΔΖ. εἰ γὰρ βάσις μὲν ἡ ΒΓ ἐπὶ βάσιν τὴν ΕΖ ἐφαρμόσει, αἱ δὲ ΒΑ, ΑΓ πλευραὶ ἐπὶ τὰς ΕΔ, ΔΖ οὐκ ἐφαρμόσουσιν ἀλλὰ παραλλάξουσιν ὡς αἱ ΕΗ, ΗΖ, συσταθήσονται ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθεῖαι ἴσαι ἑκατέρα ἑκατέρᾳ πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι. οὐ συνίστανται δέ· οὐκ ἄρα ἐφαρμοζομένης τῆς ΒΓ βάσεως ἐπὶ τὴν ΕΖ βάσιν οὐκ ἐφαρμόσουσι καὶ αἱ ΒΑ, ΑΓ πλευραὶ ἐπὶ τὰς ΕΔ, ΔΖ. ἐφαρμόσουσιν ἄρα· ὥστε καὶ γωνία ἡ ὑπὸ ΒΑΓ ἐπὶ γωνίαν τὴν ὑπὸ ΕΔΖ ἐφαρμόσει καὶ ἴση αὐτῇ ἔσται. Ἐὰν ἄρα δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ τὴν βάσιν τῇ βάσει ἴσην ἔχῃ, καὶ τὴν γωνίαν τῇ γωνίᾳ ἴσην ἕξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην· ὅπερ ἔδει δεῖξαι.

θ΄. Τὴν δοθεῖσαν γωνίαν εὐθύγραμμον δίχα τεμεῖν.



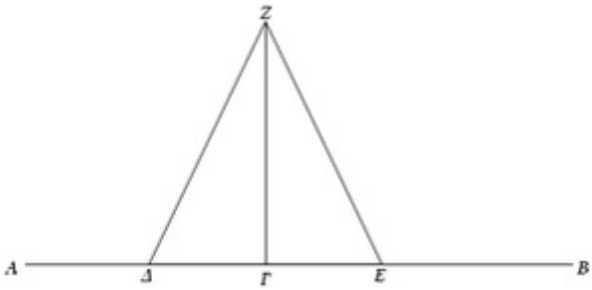
Ἔστω ἡ δοθεῖσα γωνία εὐθύγραμμος ἡ ὑπὸ ΒΑΓ. δεῖ δὴ αὐτὴν δίχα τεμεῖν. Εἰλήφθω ἐπὶ τῆς ΑΒ τυχὸν σημεῖον τὸ Δ, καὶ ἀφῃρήσθω ἀπὸ τῆς ΑΓ τῇ ΑΔ ἴση ἡ ΑΕ, καὶ ἐπεζεύχθω ἡ ΔΕ, καὶ συνεστάτω ἐπὶ τῆς ΔΕ τρίγωνον ἰσόπλευρον τὸ ΔΕΖ, καὶ ἐπεζεύχθω ἡ ΑΖ· λέγω, ὅτι ἡ ὑπὸ ΒΑΓ γωνία δίχα τέτμηται ὑπὸ τῆς ΑΖ εὐθείας. Ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΑΔ τῇ ΑΕ, κοινὴ δὲ ἡ ΑΖ, δύο δὴ αἱ ΔΑ, ΑΖ δυσὶ ταῖς ΕΑ, ΑΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ. καὶ βάσις ἡ ΔΖ βάσει τῇ ΕΖ ἴση ἐστίν· γωνία ἄρα ἡ ὑπὸ ΔΑΖ γωνίᾳ τῇ ὑπὸ ΕΑΖ ἴση ἐστίν. Ἡ ἄρα δοθεῖσα γωνία εὐθύγραμμος ἡ ὑπὸ ΒΑΓ δίχα τέτμηται ὑπὸ τῆς ΑΖ εὐθείας· ὅπερ ἔδει ποιῆσαι.

ι΄. Τὴν δοθεῖσαν εὐθεῖαν πεπερασμένην δίχα τεμεῖν.



Ἔστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ· δεῖ δὴ τὴν ΑΒ εὐθεῖαν πεπερασμένην δίχα τεμεῖν. Συνεστάτω ἐπ᾽ αὐτῆς τρίγωνον ἰσόπλευρον τὸ ΑΒΓ, καὶ τετμήσθω ἡ ὑπὸ ΑΓΒ γωνία δίχα τῇ ΓΔ εὐθείᾳ· λέγω, ὅτι ἡ ΑΒ εὐθεῖα δίχα τέτμηται κατὰ τὸ Δ σημεῖον. Ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΒ, κοινὴ δὲ ἡ ΓΔ, δύο δὴ αἱ ΑΓ, ΓΔ δύο ταῖς ΒΓ, ΓΔ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΑΓΔ γωνίᾳ τῇ ὑπὸ ΒΓΔ ἴση ἐστίν· βάσις ἄρα ἡ ΑΔ βάσει τῇ ΒΔ ἴση ἐστίν. Ἡ ἄρα δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ δίχα τέτμηται κατὰ τὸ Δ· ὅπερ ἔδει ποιῆσαι.

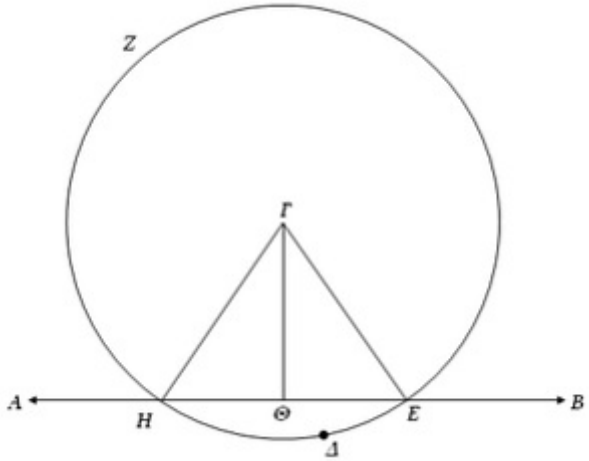
ια΄. Τῇ δοθείσῃ εὐθείᾳ ἀπὸ τοῦ πρὸς αὐτῇ δοθέντος σημείου πρὸς ὀρθὰς γωνίας εὐθεῖαν γραμμὴν ἀγαγεῖν.



Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ τὸ δὲ δοθὲν σημεῖον ἐπ᾽ αὐτῆς τὸ Γ· δεῖ δὴ ἀπὸ τοῦ Γ σημείου τῇ ΑΒ εὐθείᾳ πρὸς ὀρθὰς γωνίας εὐθεῖαν γραμμὴν ἀγαγεῖν. Εἰλήφθω ἐπὶ τῆς ΑΓ τυχὸν σημεῖον τὸ Δ, καὶ κείσθω τῇ ΓΔ ἴση ἡ ΓΕ, καὶ συνεστάτω ἐπὶ τῆς ΔΕ τρίγωνον ἰσόπλευρον τὸ ΖΔΕ, καὶ ἐπεζεύχθω ἡ ΖΓ· λέγω, ὅτι τῇ δοθείσῃ εὐθείᾳ τῇ ΑΒ ἀπὸ τοῦ πρὸς αὐτῇ δοθέντος σημείου τοῦ Γ πρὸς ὀρθὰς γωνίας εὐθεῖα γραμμὴ ἦκται ἡ ΖΓ. Ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΔΓ τῇ ΓΕ, κοινὴ δὲ ἡ ΓΖ, δύο δὴ αἱ ΔΓ, ΓΖ δυσὶ ταῖς ΕΓ, ΓΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ βάσις ἡ ΔΖ βάσει τῇ ΖΕ ἴση ἐστίν· γωνία ἄρα ἡ ὑπὸ ΔΓΖ γωνίᾳ τῇ ὑπὸ ΕΓΖ ἴση ἐστίν· καί εἰσιν ἐφεξῆς. ὅταν δὲ εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν

ἴσων γωνιῶν ἐστιν· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ ΔΓΖ, ΖΓΕ. Τῇ ἄρα δοθείσῃ εὐθείᾳ τῇ ΑΒ ἀπὸ τοῦ πρὸς αὐτῇ δοθέντος σημείου τοῦ Γ πρὸς ὀρθὰς γωνίας εὐθεῖα γραμμὴ ἦκται ἡ ΓΖ· ὅπερ ἔδει ποιῆσαι.

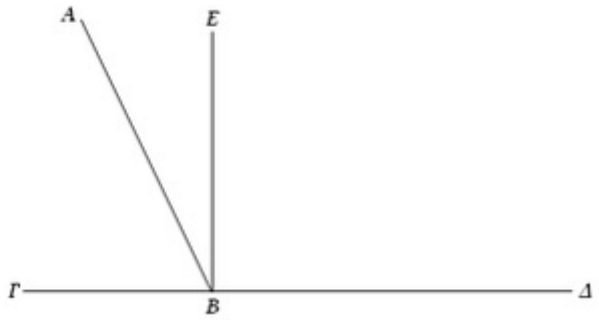
ιβ΄. Ἐπὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον ἀπὸ τοῦ δοθέντος σημείου, ὃ μή ἐστιν ἐπ᾽ αὐτῆς, κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν.



Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἄπειρος ἡ ΑΒ τὸ δὲ δοθὲν σημεῖον, ὃ μή ἐστιν ἐπ᾽ αὐτῆς, τὸ Γ· δεῖ δὴ ἐπὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον τὴν ΑΒ ἀπὸ τοῦ δοθέντος σημείου τοῦ Γ, ὃ μή ἐστιν ἐπ᾽ αὐτῆς, κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν. Εἰλήφθω γὰρ ἐπὶ τὰ ἕτερα μέρη τῆς ΑΒ εὐθείας τυχὸν σημεῖον τὸ Δ, καὶ κέντρῳ μὲν τῷ Γ διαστήματι δὲ τῷ ΓΔ κύκλος γεγράφθω ὁ ΕΖΗ, καὶ τετμήσθω ἡ ΕΗ εὐθεῖα δίχα κατὰ τὸ Θ, καὶ ἐπεζεύχθωσαν αἱ ΓΗ, ΓΘ, ΓΕ εὐθεῖαι· λέγω, ὅτι ἐπὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον τὴν ΑΒ ἀπὸ τοῦ δοθέντος σημείου τοῦ Γ, ὃ μή ἐστιν ἐπ᾽ αὐτῆς, κάθετος ἦκται ἡ ΓΘ. Ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΗΘ τῇ ΘΕ, κοινὴ δὲ ἡ ΘΓ, δύο δὴ αἱ ΗΘ, ΘΓ δύο ταῖς ΕΘ, ΘΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ βάσις ἡ ΓΗ βάσει τῇ ΓΕ ἐστιν ἴση· γωνία ἄρα ἡ ὑπὸ ΓΘΗ γωνίᾳ τῇ ὑπὸ ΕΘΓ ἐστιν ἴση. καί εἰσιν ἐφεξῆς. ὅταν δὲ εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστιν, καὶ ἡ ἐφεστηκυῖα εὐθεῖα κάθετος καλεῖται ἐφ᾽ ἣν ἐφέστηκεν. Ἐπὶ τὴν δοθεῖσαν ἄρα εὐθεῖαν ἄπειρον τὴν ΑΒ ἀπὸ τοῦ

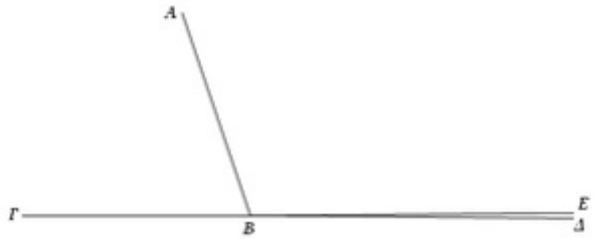
δοθέντος σημείου τοῦ Γ, ὃ μή ἐστιν ἐπ᾽ αὐτῆς, κάθετος ἦκται ἡ ΓΘ· ὅπερ ἔδει ποιῆσαι.

ιγ΄. Ἐὰν εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα γωνίας ποιῇ, ἤτοι δύο ὀρθὰς ἢ δυσὶν ὀρθαῖς ἴσας ποιήσει.



Εὐθεῖα γάρ τις ἡ ΑΒ ἐπ᾽ εὐθεῖαν τὴν ΓΔ σταθεῖσα γωνίας ποιείτω τὰς ὑπὸ ΓΒΑ, ΑΒΔ· λέγω, ὅτι αἱ ὑπὸ ΓΒΑ, ΑΒΔ γωνίαι ἤτοι δύο ὀρθαί εἰσιν ἢ δυσὶν ὀρθαῖς ἴσαι. Εἰ μὲν οὖν ἴση ἐστὶν ἡ ὑπὸ ΓΒΑ τῇ ὑπὸ ΑΒΔ, δύο ὀρθαί εἰσιν. εἰ δὲ οὔ, ἤχθω ἀπὸ τοῦ Β σημείου τῇ ΓΔ [εὐθείᾳ] πρὸς ὀρθὰς ἡ ΒΕ· αἱ ἄρα ὑπὸ ΓΒΕ, ΕΒΔ δύο ὀρθαί εἰσιν· καὶ ἐπεὶ ἡ ὑπὸ ΓΒΕ δυσὶ ταῖς ὑπὸ ΓΒΑ, ΑΒΕ ἴση ἐστίν, κοινὴ προσκείσθω ἡ ὑπὸ ΕΒΔ· αἱ ἄρα ὑπὸ ΓΒΕ, ΕΒΔ τρισὶ ταῖς ὑπὸ ΓΒΑ, ΑΒΕ, ΕΒΔ ἴσαι εἰσίν. πάλιν, ἐπεὶ ἡ ὑπὸ ΔΒΑ δυσὶ ταῖς ὑπὸ ΔΒΕ, ΕΒΑ ἴση ἐστίν, κοινὴ προσκείσθω ἡ ὑπὸ ΑΒΓ· αἱ ἄρα ὑπὸ ΔΒΑ, ΑΒΓ τρισὶ ταῖς ὑπὸ ΔΒΕ, ΕΒΑ, ΑΒΓ ἴσαι εἰσίν. ἐδείχθησαν δὲ καὶ αἱ ὑπὸ ΓΒΕ, ΕΒΔ τρισὶ ταῖς αὐταῖς ἴσαι· τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ αἱ ὑπὸ ΓΒΕ, ΕΒΔ ἄρα ταῖς ὑπὸ ΔΒΑ, ΑΒΓ ἴσαι εἰσίν· ἀλλὰ αἱ ὑπὸ ΓΒΕ, ΕΒΔ δύο ὀρθαί εἰσιν? καὶ αἱ ὑπὸ ΔΒΑ, ΑΒΓ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσίν. Ἐὰν ἄρα εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα γωνίας ποιῇ, ἤτοι δύο ὀρθὰς ἢ δυσὶν ὀρθαῖς ἴσας ποιήσει· ὅπερ ἔδει δεῖξαι.

ιδ΄. Ἐὰν πρός τινι εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ δύο εὐθεῖαι μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δυσὶν ὀρθαῖς ἴσας ποιῶσιν, ἐπ᾽ εὐθείας ἔσονται ἀλλήλαις αἱ εὐθεῖαι.



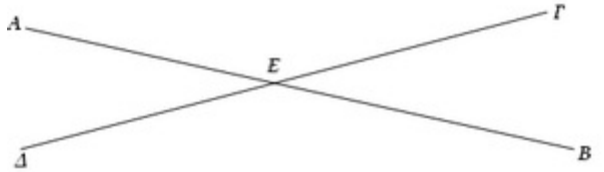
Πρὸς γάρ τινι εὐθείᾳ τῇ ΑΒ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Β δύο εὐθεῖαι αἱ ΒΓ, ΒΔ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας τὰς ὑπὸ ΑΒΓ, ΑΒΔ δύο ὀρθαῖς ἴσας ποιείτωσαν· λέγω, ὅτι ἐπ᾽ εὐθείας ἐστὶ τῇ ΓΒ ἡ ΒΔ.

Εἰ γὰρ μή ἐστι τῇ ΒΓ ἐπ᾽ εὐθείας ἡ ΒΔ, ἔστω τῇ ΓΒ ἐπ᾽ εὐθείας ἡ ΒΕ.

Ἐπεὶ οὖν εὐθεῖα ἡ ΑΒ ἐπ᾽ εὐθεῖαν τὴν ΓΒΕ ἐφέστηκεν, αἱ ἄρα ὑπὸ ΑΒΓ, ΑΒΕ γωνίαι δύο ὀρθαῖς ἴσαι εἰσίν· εἰσὶ δὲ καὶ αἱ ὑπὸ ΑΒΓ, ΑΒΔ δύο ὀρθαῖς ἴσαι· αἱ ἄρα ὑπὸ ΓΒΑ, ΑΒΕ ταῖς ὑπὸ ΓΒΑ, ΑΒΔ ἴσαι εἰσίν. κοινὴ ἀφῃρήσθω ἡ ὑπὸ ΓΒΑ· λοιπὴ ἄρα ἡ ὑπὸ ΑΒΕ λοιπῇ τῇ ὑπὸ ΑΒΔ ἐστιν ἴση, ἡ ἐλάσσων τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἐπ᾽ εὐθείας ἐστὶν ἡ ΒΕ τῇ ΓΒ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἄλλη τις πλὴν τῆς ΒΔ· ἐπ᾽ εὐθείας ἄρα ἐστὶν ἡ ΓΒ τῇ ΒΔ.

Ἐὰν ἄρα πρός τινι εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ δύο εὐθεῖαι μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δυσὶν ὀρθαῖς ἴσας ποιῶσιν, ἐπ᾽ εὐθείας ἔσονται ἀλλήλαις αἱ εὐθεῖαι· ὅπερ ἔδει δεῖξαι.

ιε΄. Ἐὰν δύο εὐθεῖαι τέμνωσιν ἀλλήλας, τὰς κατὰ κορυφὴν γωνίας ἴσας ἀλλήλαις ποιοῦσιν.



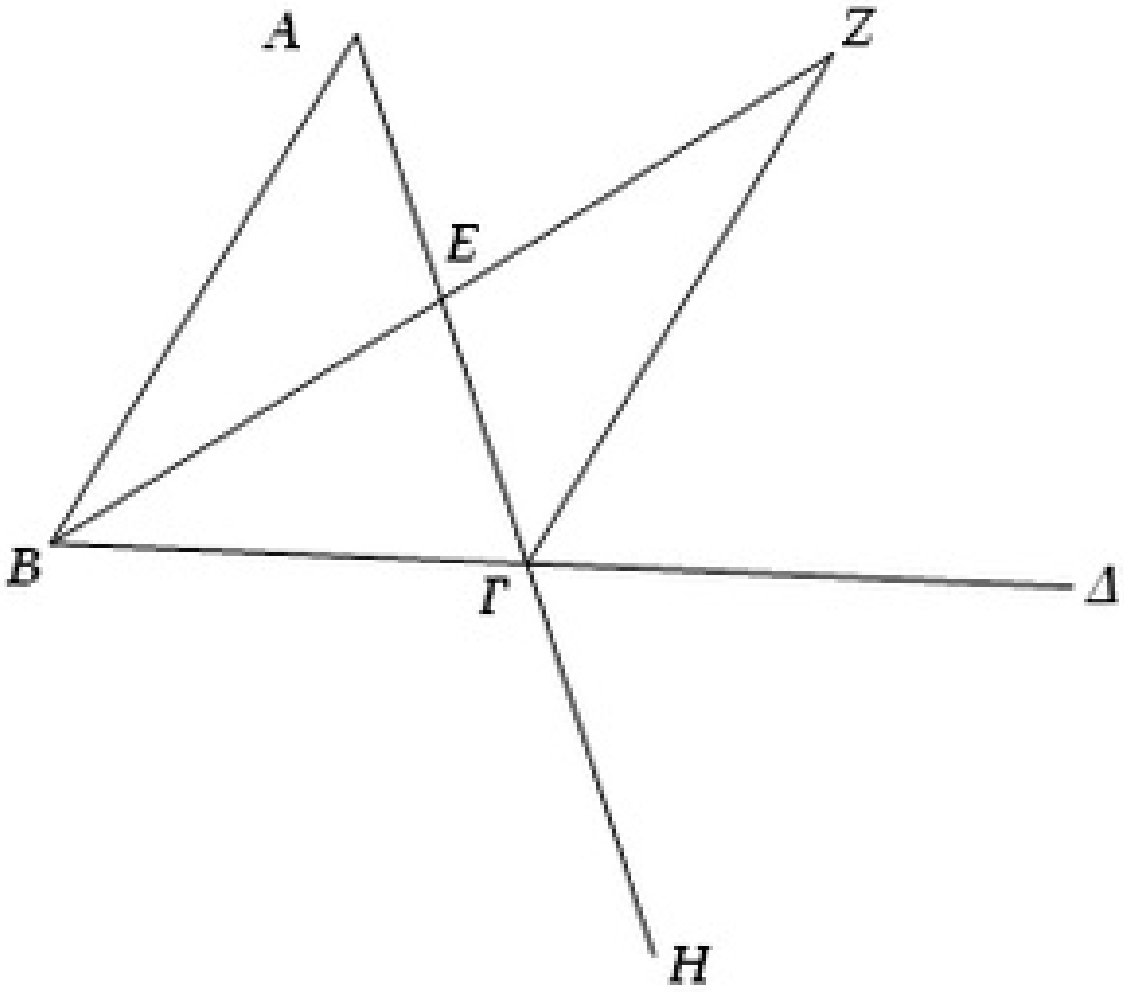
Δύο γὰρ εὐθεῖαι αἱ ΑΒ, ΓΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε σημεῖον· λέγω, ὅτι ἴση ἐστὶν ἡ μὲν ὑπὸ ΑΕΓ γωνία τῇ ὑπὸ ΔΕΒ, ἡ δὲ ὑπὸ ΓΕΒ τῇ ὑπὸ ΑΕΔ.

Ἐπεὶ γὰρ εὐθεῖα ἡ ΑΕ ἐπ᾽ εὐθεῖαν τὴν ΓΔ ἐφέστηκε γωνίας ποιοῦσα τὰς ὑπὸ ΓΕΑ, ΑΕΔ, αἱ ἄρα ὑπὸ ΓΕΑ, ΑΕΔ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. πάλιν, ἐπεὶ εὐθεῖα ἡ ΔΕ ἐπ᾽ εὐθεῖαν τὴν ΑΒ ἐφέστηκε γωνίας ποιοῦσα τὰς ὑπὸ ΑΕΔ, ΔΕΒ, αἱ ἄρα ὑπὸ ΑΕΔ, ΔΕΒ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. ἐδείχθησαν δὲ καὶ αἱ ὑπὸ ΓΕΑ, ΑΕΔ δυσὶν ὀρθαῖς ἴσαι· αἱ ἄρα ὑπὸ ΓΕΑ, ΑΕΔ ταῖς ὑπὸ ΑΕΔ, ΔΕΒ ἴσαι εἰσίν. κοινὴ ἀφῃρήσθω ἡ ὑπὸ ΑΕΔ· λοιπὴ ἄρα ἡ ὑπὸ ΓΕΑ λοιπῇ τῇ ὑπὸ ΒΕΔ ἴση ἐστίν· ὁμοίως δὴ δειχθήσεται, ὅτι καὶ αἱ ὑπὸ ΓΕΒ, ΔΕΑ ἴσαι εἰσίν.

Ἐὰν ἄρα δύο εὐθεῖαι τέμνωσιν ἀλλήλας, τὰς κατὰ κορυφὴν γωνίας ἴσας ἀλλήλαις ποιοῦσιν· ὅπερ ἔδει δεῖξαι.

[Πόρισμα Ἐκ δὴ τούτου φανερὸν ὅτι, ἐὰν δύο εὐθεῖαι τέμνωσιν ἀλλήλας, τὰς πρὸς τῇ τομῇ γωνίας τέτρασιν ὀρθαῖς ἴσας ποιήσουσιν.]

ις΄. Παντὸς τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία ἑκατέρας τῶν ἐντὸς καὶ ἀπεναντίον γωνιῶν μείζων ἐστίν.



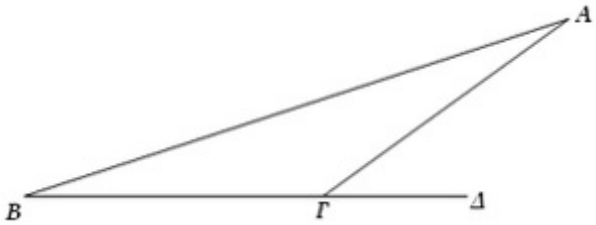
Ἔστω τρίγωνον τὸ ΑΒΓ, καὶ προσεκβεβλήσθω αὐτοῦ μία πλευρὰ ἡ ΒΓ ἐπὶ τὸ Δ· λέγω, ὅτι ἡ ἐκτὸς γωνία ἡ ὑπὸ ΑΓΔ μείζων ἐστὶν ἑκατέρας τῶν ἐντὸς καὶ ἀπεναντίον τῶν ὑπὸ ΓΒΑ, ΒΑΓ γωνιῶν.

Τετμήσθω ἡ ΑΓ δίχα κατὰ τὸ Ε, καὶ ἐπιζευχθεῖσα ἡ ΒΕ ἐκβεβλήσθω ἐπ᾽ εὐθείας ἐπὶ τὸ Ζ, καὶ κείσθω τῇ ΒΕ ἴση ἡ ΕΖ, καὶ ἐπεζεύχθω ἡ ΖΓ, καὶ διήχθω ἡ ΑΓ ἐπὶ τὸ Η.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν ΑΕ τῇ ΕΓ, ἡ δὲ ΒΕ τῇ ΕΖ, δύο δὴ αἱ ΑΕ, ΕΒ δυσὶ ταῖς ΓΕ, ΕΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΑΕΒ γωνίᾳ τῇ ὑπὸ ΖΕΓ ἴση ἐστίν· κατὰ κορυφὴν γάρ· βάσις ἄρα ἡ ΑΒ βάσει τῇ ΖΓ ἴση ἐστίν, καὶ τὸ ΑΒΕ τρίγωνον τῷ ΖΕΓ τριγώνῳ ἐστὶν ἴσον, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ

ὑποτείνουσιν· ἴση ἄρα ἐστὶν ἡ ὑπὸ ΒΑΕ τῇ ὑπὸ ΕΓΖ. μείζων δέ ἐστιν ἡ ὑπὸ ΕΓΔ τῆς ὑπὸ ΕΓΖ· μείζων ἄρα ἡ ὑπὸ ΑΓΔ τῆς ὑπὸ ΒΑΕ. ὁμοίως δὴ τῆς ΒΓ τετμημένης δίχα δειχθήσεται καὶ ἡ ὑπὸ ΒΓΗ, τουτέστιν ἡ ὑπὸ ΑΓΔ, μείζων καὶ τῆς ὑπὸ ΑΒΓ. Παντὸς ἄρα τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία ἑκατέρας τῶν ἐντὸς καὶ ἀπεναντίον γωνιῶν μείζων ἐστίν· ὅπερ ἔδει δεῖξαι.

ιζ΄. Παντὸς τριγώνου αἱ δύο γωνίαι δύο ὀρθῶν ἐλάσσονές εἰσι πάντῃ μεταλαμβανόμεναι.

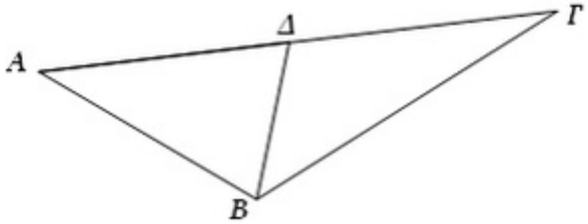


Ἔστω τρίγωνον τὸ ΑΒΓ· λέγω, ὅτι τοῦ ΑΒΓ τριγώνου αἱ δύο γωνίαι δύο ὀρθῶν ἐλάττονές εἰσι πάντῃ μεταλαμβανόμεναι.

Ἐκβεβλήσθω γὰρ ἡ ΒΓ ἐπὶ τὸ Δ.

Καὶ ἐπεὶ τριγώνου τοῦ ΑΒΓ ἐκτός ἐστι γωνία ἡ ὑπὸ ΑΓΔ, μείζων ἐστὶ τῆς ἐντὸς καὶ ἀπεναντίον τῆς ὑπὸ ΑΒΓ. κοινὴ προσκείσθω ἡ ὑπὸ ΑΓΒ· αἱ ἄρα ὑπὸ ΑΓΔ, ΑΓΒ τῶν ὑπὸ ΑΒΓ, ΒΓΑ μείζονές εἰσιν. ἀλλ’ αἱ ὑπὸ ΑΓΔ, ΑΓΒ δύο ὀρθαῖς ἴσαι εἰσίν· αἱ ἄρα ὑπὸ ΑΒΓ, ΒΓΑ δύο ὀρθῶν ἐλάσσονές εἰσιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ ὑπὸ ΒΑΓ, ΑΓΒ δύο ὀρθῶν ἐλάσσονές εἰσι καὶ ἔτι αἱ ὑπὸ ΓΑΒ, ΑΒΓ. Παντὸς ἄρα τριγώνου αἱ δύο γωνίαι δύο ὀρθῶν ἐλάσσονές εἰσι πάντῃ μεταλαμβανόμεναι· ὅπερ ἔδει δεῖξαι.

ιη΄. Παντὸς τριγώνου ἡ μείζων πλευρὰ τὴν μείζονα γωνίαν ὑποτείνει.

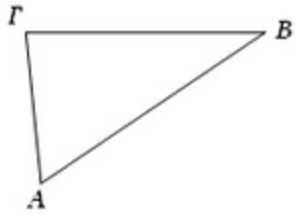


Ἔστω γὰρ τρίγωνον τὸ ΑΒΓ μείζονα ἔχον τὴν ΑΓ πλευρὰν τῆς ΑΒ· λέγω, ὅτι καὶ γωνία ἡ ὑπὸ ΑΒΓ μείζων ἐστὶ τῆς ὑπὸ ΒΓΑ. Ἐπεὶ γὰρ μείζων ἐστὶν ἡ ΑΓ τῆς ΑΒ, κείσθω τῇ ΑΒ ἴση ἡ ΑΔ, καὶ ἐπεζεύχθω ἡ ΒΔ.

Καὶ ἐπεὶ τριγώνου τοῦ ΒΓΔ ἐκτός ἐστι γωνία ἡ ὑπὸ ΑΔΒ, μείζων ἐστὶ τῆς ἐντὸς καὶ ἀπεναντίον τῆς ὑπὸ ΔΓΒ· ἴση δὲ ἡ ὑπὸ ΑΔΒ τῇ ὑπὸ ΑΒΔ, ἐπεὶ καὶ πλευρὰ ἡ ΑΒ τῇ ΑΔ ἐστιν ἴση· μείζων ἄρα καὶ ἡ ὑπὸ ΑΒΔ τῆς ὑπὸ ΑΓΒ· πολλῷ ἄρα ἡ ὑπὸ ΑΒΓ μείζων ἐστὶ τῆς ὑπὸ ΑΓΒ.

Παντὸς ἄρα τριγώνου ἡ μείζων πλευρὰ τὴν μείζονα γωνίαν ὑποτείνει· ὅπερ ἔδει δεῖξαι.

ιθ΄. Παντὸς τριγώνου ὑπὸ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει.

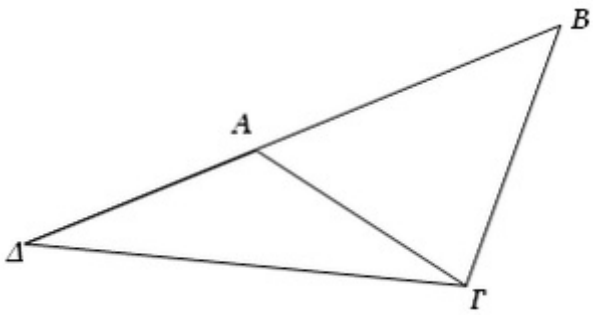


Ἔστω τρίγωνον τὸ ΑΒΓ μείζονα ἔχον τὴν ὑπὸ ΑΒΓ γωνίαν τῆς ὑπὸ ΒΓΑ· λέγω, ὅτι καὶ πλευρὰ ἡ ΑΓ πλευρᾶς τῆς ΑΒ μείζων ἐστίν.

Εἰ γὰρ μή, ἤτοι ἴση ἐστὶν ἡ ΑΓ τῇ ΑΒ ἢ ἐλάσσων· ἴση μὲν οὖν οὐκ ἔστιν ἡ ΑΓ τῇ ΑΒ· ἴση γὰρ ἂν ἦν καὶ γωνία ἡ ὑπὸ ΑΒΓ τῇ ὑπὸ ΑΓΒ· οὐκ ἔστι δέ· οὐκ ἄρα ἴση ἐστὶν ἡ ΑΓ τῇ ΑΒ. οὐδὲ μὴν ἐλάσσων ἐστὶν ἡ ΑΓ τῆς ΑΒ· ἐλάσσων γὰρ ἂν ἦν καὶ γωνία ἡ ὑπὸ ΑΒΓ τῆς ὑπὸ ΑΓΒ· οὐκ ἔστι δέ· οὐκ ἄρα ἐλάσσων ἐστὶν ἡ ΑΓ τῆς ΑΒ. ἐδείχθη δέ, ὅτι οὐδὲ ἴση ἐστίν. μείζων ἄρα ἐστὶν ἡ ΑΓ τῆς ΑΒ.

Παντὸς ἄρα τριγώνου ὑπὸ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει· ὅπερ ἔδει δεῖξαι.

κ΄. Παντὸς τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι.



Ἔστω γὰρ τρίγωνον τὸ ΑΒΓ· λέγω, ὅτι τοῦ ΑΒΓ τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, αἱ μὲν ΒΑ, ΑΓ τῆς ΒΓ, αἱ δὲ ΑΒ, ΒΓ τῆς ΑΓ, αἱ δὲ ΒΓ, ΓΑ τῆς ΑΒ.

Διήχθω γὰρ ἡ ΒΑ ἐπὶ τὸ Δ σημεῖον, καὶ κείσθω τῇ ΓΑ ἴση ἡ ΑΔ, καὶ ἐπεζεύχθω ἡ ΔΓ. Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΔΑ τῇ ΑΓ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΑΔΓ τῇ ὑπὸ ΑΓΔ· μείζων ἄρα ἡ ὑπὸ ΒΓΔ τῆς ὑπὸ ΑΔΓ· καὶ ἐπεὶ τρίγωνόν ἐστι τὸ ΔΓΒ μείζονα ἔχον τὴν ὑπὸ ΒΓΔ γωνίαν τῆς ὑπὸ ΒΔΓ, ὑπὸ δὲ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει, ἡ ΔΒ ἄρα τῆς ΒΓ ἐστι μείζων. ἴση δὲ ἡ ΔΑ τῇ ΑΓ· μείζονες ἄρα αἱ ΒΑ, ΑΓ τῆς ΒΓ· ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ μὲν ΑΒ, ΒΓ τῆς ΓΑ μείζονές εἰσιν, αἱ δὲ ΒΓ, ΓΑ τῆς ΑΒ.

Παντὸς ἄρα τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι· ὅπερ ἔδει δεῖξαι.

κα΄. Ἐὰν τριγώνου ἐπὶ μιᾶς τῶν πλευρῶν ἀπὸ τῶν περάτων δύο εὐθεῖαι ἐντὸς συσταθῶσιν, αἱ συσταθεῖσαι τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ἐλάττονες μὲν ἔσονται, μείζονα δὲ γωνίαν περιέξουσιν.

Τριγώνου γὰρ τοῦ ΑΒΓ ἐπὶ μιᾶς τῶν πλευρῶν τῆς ΒΓ ἀπὸ τῶν περάτων τῶν Β, Γ δύο εὐθεῖαι ἐντὸς συνεστάτωσαν αἱ ΒΔ, ΔΓ· λέγω, ὅτι αἱ ΒΔ, ΔΓ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν τῶν ΒΑ, ΑΓ ἐλάσσονες μέν εἰσιν, μείζονα δὲ γωνίαν περιέχουσι τὴν ὑπὸ ΒΔΓ τῆς ὑπὸ ΒΑΓ.

Διήχθω γὰρ ἡ ΒΔ ἐπὶ τὸ Ε. καὶ ἐπεὶ παντὸς τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσιν, τοῦ ΑΒΕ ἄρα τριγώνου αἱ δύο πλευραὶ αἱ ΑΒ, ΑΕ τῆς ΒΕ μείζονές εἰσιν· κοινὴ προσκείσθω ἡ ΕΓ· αἱ ἄρα ΒΑ, ΑΓ τῶν ΒΕ, ΕΓ μείζονές εἰσιν. πάλιν, ἐπεὶ τοῦ ΓΕΔ τριγώνου αἱ δύο πλευραὶ αἱ ΓΕ, ΕΔ τῆς ΓΔ μείζονές εἰσιν, κοινὴ προσκείσθω ἡ ΔΒ· αἱ ΓΕ, ΕΒ ἄρα τῶν ΓΔ, ΔΒ μείζονές εἰσιν. ἀλλὰ τῶν ΒΕ, ΕΓ μείζονες ἐδείχθησαν αἱ ΒΑ, ΑΓ· πολλῷ ἄρα αἱ ΒΑ, ΑΓ τῶν ΒΔ, ΔΓ μείζονές εἰσιν.

Πάλιν, ἐπεὶ παντὸς τριγώνου ἡ ἐκτὸς γωνία τῆς ἐντὸς καὶ ἀπεναντίον μείζων ἐστίν, τοῦ ΓΔΕ ἄρα τριγώνου ἡ ἐκτὸς γωνία ἡ ὑπὸ ΒΔΓ μείζων ἐστὶ τῆς ὑπὸ ΓΕΔ. διὰ ταὐτὰ τοίνυν καὶ τοῦ ΑΒΕ τριγώνου ἡ ἐκτὸς γωνία ἡ ὑπὸ ΓΕΒ μείζων ἐστὶ τῆς ὑπὸ ΒΑΓ. ἀλλὰ τῆς ὑπὸ ΓΕΒ μείζων ἐδείχθη ἡ ὑπὸ ΒΔΓ· πολλῷ ἄρα ἡ ὑπὸ ΒΔΓ μείζων ἐστὶ τῆς ὑπὸ ΒΑΓ. Ἐὰν ἄρα τριγώνου ἐπὶ μιᾶς τῶν πλευρῶν ἀπὸ τῶν περάτων δύο εὐθεῖαι ἐντὸς συσταθῶσιν, αἱ συσταθεῖσαι τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ἐλάττονες μέν εἰσιν, μείζονα δὲ γωνίαν περιέχουσιν· ὅπερ ἔδει δεῖξαι.

κβ΄. Ἐκ τριῶν εὐθειῶν, αἵ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις [εὐθείαις], τρίγωνον συστήσασθαι· δεῖ δὲ τὰς δύο τῆς λοιπῆς μείζονας εἶναι πάντῃ μεταλαμβανομένας [διὰ τὸ καὶ παντὸς τριγώνου τὰς δύο πλευρὰς τῆς λοιπῆς μείζονας εἶναι πάντῃ μεταλαμβανομένας].

Ἔστωσαν αἱ δοθεῖσαι τρεῖς εὐθεῖαι αἱ Α, Β, Γ, ὧν αἱ δύο τῆς λοιπῆς μείζονες ἔστωσαν πάντῃ μεταλαμβανόμεναι, αἱ μὲν Α, Β τῆς Γ, αἱ δὲ Α, Γ τῆς Β, καὶ ἔτι αἱ Β, Γ τῆς Α· δεῖ δὴ ἐκ τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συστήσασθαι. Ἐκκείσθω τις εὐθεῖα ἡ ΔΕ πεπερασμένη μὲν κατὰ τὸ Δ

ἄπειρος δὲ κατὰ τὸ Ε, καὶ κείσθω τῇ μὲν Α ἴση ἡ ΔΖ, τῇ δὲ Β ἴση ἡ ΖΗ, τῇ δὲ Γ ἴση ἡ ΗΘ· καὶ κέντρῳ μὲν τῷ Ζ, διαστήματι δὲ τῷ ΖΔ κύκλος γεγράφθω ὁ ΔΚΛ· πάλιν κέντρῳ μὲν τῷ Η, διαστήματι δὲ τῷ ΗΘ κύκλος γεγράφθω ὁ ΚΛΘ, καὶ ἐπεζεύχθωσαν αἱ ΚΖ, ΚΗ· λέγω, ὅτι ἐκ τριῶν εὐθειῶν τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συνέσταται τὸ ΚΖΗ.

Ἐπεὶ γὰρ τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ΔΚΛ κύκλου, ἴση ἐστὶν ἡ ΖΔ τῇ ΖΚ· ἀλλὰ ἡ ΖΔ τῇ Α ἐστιν ἴση. καὶ ἡ ΚΖ ἄρα τῇ Α ἐστιν ἴση. πάλιν, ἐπεὶ τὸ Η σημεῖον κέντρον ἐστὶ τοῦ ΛΚΘ κύκλου, ἴση ἐστὶν ἡ ΗΘ τῇ ΗΚ· ἀλλὰ ἡ ΗΘ τῇ Γ ἐστιν ἴση· καὶ ἡ ΚΗ ἄρα τῇ Γ ἐστιν ἴση. ἐστὶ δὲ καὶ ἡ ΖΗ τῇ Β ἴση· αἱ τρεῖς ἄρα εὐθεῖαι αἱ ΚΖ, ΖΗ, ΗΚ τρισὶ ταῖς Α, Β, Γ ἴσαι εἰσίν.

Ἐκ τριῶν ἄρα εὐθειῶν τῶν ΚΖ, ΖΗ, ΗΚ, αἵ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις εὐθείαις ταῖς Α, Β, Γ, τρίγωνον συνέσταται τὸ ΚΖΗ· ὅπερ ἔδει ποιῆσαι.

κγ΄. Πρὸς τῇ δοθείσῃ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ ἴσην γωνίαν εὐθύγραμμον συστήσασθαι.

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ πρὸς αὐτῇ σημεῖον τὸ Α, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ ὑπὸ ΔΓΕ· δεῖ δὴ πρὸς τῇ δοθείσῃ εὐθείᾳ τῇ ΑΒ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ τῇ ὑπὸ ΔΓΕ ἴσην γωνίαν εὐθύγραμμον συστήσασθαι.

Εἰλήφθω ἐφ’ ἑκατέρας τῶν ΓΔ, ΓΕ τυχόντα σημεῖα τὰ Δ, Ε, καὶ ἐπεζεύχθω ἡ ΔΕ· καὶ ἐκ τριῶν εὐθειῶν, αἵ εἰσιν ἴσαι τρισὶ ταῖς ΓΔ, ΔΕ, ΓΕ, τρίγωνον συνεστάτω τὸ ΑΖΗ, ὥστε ἴσην εἶναι τὴν μὲν ΓΔ τῇ ΑΖ, τὴν δὲ ΓΕ τῇ ΑΗ, καὶ ἔτι τὴν ΔΕ τῇ ΖΗ.

Ἐπεὶ οὖν δύο αἱ ΔΓ, ΓΕ δύο ταῖς ΖΑ, ΑΗ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ βάσις ἡ ΔΕ βάσει τῇ ΖΗ ἴση, γωνία ἄρα ἡ ὑπὸ ΔΓΕ γωνίᾳ τῇ ὑπὸ ΖΑΗ ἐστιν ἴση. Πρὸς ἄρα τῇ δοθείσῃ εὐθείᾳ τῇ ΑΒ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ τῇ ὑπὸ ΔΓΕ ἴση γωνία εὐθύγραμμος συνέσταται ἡ ὑπὸ ΖΑΗ· ὅπερ ἔδει ποιῆσαι.

κδ΄. Ἐὰν δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ, τὴν δὲ γωνίαν τῆς γωνίας μείζονα ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τῆς βάσεως μείζονα ἕξει.

Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ τὰς δύο πλευρὰς τὰς ΑΒ, ΑΓ ταῖς δύο πλευραῖς ταῖς ΔΕ, ΔΖ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ, τὴν μὲν ΑΒ τῇ ΔΕ τὴν δὲ ΑΓ τῇ ΔΖ, ἡ δὲ πρὸς τῷ Α γωνία τῆς πρὸς τῷ Δ γωνίας μείζων ἔστω· λέγω, ὅτι καὶ βάσις ἡ ΒΓ βάσεως τῆς ΕΖ μείζων ἐστίν. Ἐπεὶ γὰρ μείζων ἡ ὑπὸ ΒΑΓ γωνία τῆς ὑπὸ ΕΔΖ γωνίας, συνεστάτω πρὸς τῇ ΔΕ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Δ τῇ ὑπὸ ΒΑΓ γωνίᾳ ἴση ἡ ὑπὸ ΕΔΗ, καὶ κείσθω ὁποτέρᾳ τῶν ΑΓ, ΔΖ ἴση ἡ ΔΗ, καὶ ἐπεζεύχθωσαν αἱ ΕΗ, ΖΗ.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν ΑΒ τῇ ΔΕ, ἡ δὲ ΑΓ τῇ ΔΗ, δύο δὴ αἱ ΒΑ, ΑΓ δυσὶ ταῖς ΕΔ, ΔΗ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΒΑΓ γωνίᾳ τῇ ὑπὸ ΕΔΗ ἴση· βάσις ἄρα ἡ ΒΓ βάσει τῇ ΕΗ ἐστιν ἴση. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΔΖ τῇ ΔΗ, ἴση ἐστὶ καὶ ἡ ὑπὸ ΔΗΖ γωνία τῇ ὑπὸ ΔΖΗ· μείζων ἄρα ἡ ὑπὸ ΔΖΗ τῆς ὑπὸ ΕΗΖ· πολλῷ ἄρα μείζων ἐστὶν ἡ ὑπὸ ΕΖΗ τῆς ὑπὸ ΕΗΖ. καὶ ἐπεὶ τρίγωνόν ἐστι τὸ ΕΖΗ μείζονα ἔχον τὴν ὑπὸ ΕΖΗ γωνίαν τῆς ὑπὸ ΕΗΖ, ὑπὸ δὲ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει, μείζων ἄρα καὶ πλευρὰ ἡ ΕΗ τῆς ΕΖ. ἴση δὲ ἡ ΕΗ τῇ ΒΓ· μείζων ἄρα καὶ ἡ ΒΓ τῆς ΕΖ.

Ἐὰν ἄρα δύο τρίγωνα τὰς δύο πλευρὰς δυσὶ πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ, τὴν δὲ γωνίαν τῆς γωνίας μείζονα ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τῆς βάσεως μείζονα ἕξει· ὅπερ ἔδει δεῖξαι.

κε΄. Ἐὰν δύο τρίγωνα τὰς δύο πλευρὰς δυσὶ πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ, τὴν δὲ βάσιν τῆς βάσεως μείζονα ἔχῃ, καὶ τὴν γωνίαν τῆς γωνίας μείζονα ἕξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην.

Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ τὰς δύο πλευρὰς τὰς ΑΒ, ΑΓ ταῖς δύο πλευραῖς ταῖς ΔΕ, ΔΖ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ, τὴν μὲν ΑΒ τῇ ΔΕ, τὴν

δὲ ΑΓ τῇ ΔΖ· βάσις δὲ ἡ ΒΓ βάσεως τῆς ΕΖ μείζων ἔστω· λέγω, ὅτι καὶ γωνία ἡ ὑπὸ ΒΑΓ γωνίας τῆς ὑπὸ ΕΔΖ μείζων ἐστίν·

Εἰ γὰρ μή, ἤτοι ἴση ἐστὶν αὐτῇ ἢ ἐλάσσων· ἴση μὲν οὖν οὐκ ἔστιν ἡ ὑπὸ ΒΑΓ τῇ ὑπὸ ΕΔΖ· ἴση γὰρ ἂν ἦν καὶ βάσις ἡ ΒΓ βάσει τῇ ΕΖ· οὐκ ἔστι δέ. οὐκ ἄρα ἴση ἐστὶ γωνία ἡ ὑπὸ ΒΑΓ τῇ ὑπὸ ΕΔΖ· οὐδὲ μὴν ἐλάσσων ἐστὶν ἡ ὑπὸ ΒΑΓ τῆς ὑπὸ ΕΔΖ· ἐλάσσων γὰρ ἂν ἦν καὶ βάσις ἡ ΒΓ βάσεως τῆς ΕΖ· οὐκ ἔστι δέ· οὐκ ἄρα ἐλάσσων ἐστὶν ἡ ὑπὸ ΒΑΓ γωνία τῆς ὑπὸ ΕΔΖ. ἐδείχθη δὲ ὅτι οὐδὲ ἴση· μείζων ἄρα ἐστὶν ἡ ὑπὸ ΒΑΓ τῆς ὑπὸ ΕΔΖ.

Ἐὰν ἄρα δύο τρίγωνα τὰς δύο πλευρὰς δυσὶ πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκάτερᾳ, τὴν δὲ βάσιν τῆς βάσεως μείζονα ἔχῃ, καὶ τὴν γωνίαν τῆς γωνίας μείζονα ἕξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην· ὅπερ ἔδει δεῖξαι.

κς΄. Ἐὰν δύο τρίγωνα τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην ἤτοι τὴν πρὸς ταῖς ἴσαις γωνίαις ἢ τὴν ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν, καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει [ἑκατέραν ἑκατέρᾳ] καὶ τὴν λοιπὴν γωνίαν τῇ λοιπῇ γωνίᾳ.

Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ τὰς δύο γωνίας τὰς ὑπὸ ΑΒΓ, ΒΓΑ δυσὶ ταῖς ὑπὸ ΔΕΖ, ΕΖΔ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ, τὴν μὲν ὑπὸ ΑΒΓ τῇ ὑπὸ ΔΕΖ, τὴν δὲ ὑπὸ ΒΓΑ τῇ ὑπὸ ΕΖΔ· ἐχέτω δὲ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην, πρότερον τὴν πρὸς ταῖς ἴσαις γωνίαις τὴν ΒΓ τῇ ΕΖ· λέγω, ὅτι καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει ἑκατέραν ἑκατέρᾳ, τὴν μὲν ΑΒ τῇ ΔΕ τὴν δὲ ΑΓ τῇ ΔΖ, καὶ τὴν λοιπὴν γωνίαν τῇ λοιπῇ γωνίᾳ, τὴν ὑπὸ ΒΑΓ τῇ ὑπὸ ΕΔΖ.

Εἰ γὰρ ἄνισός ἐστιν ἡ ΑΒ τῇ ΔΕ, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ ΑΒ, καὶ κείσθω τῇ ΔΕ ἴση ἡ ΒΗ, καὶ ἐπεζεύχθω ἡ ΗΓ.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν ΒΗ τῇ ΔΕ, ἡ δὲ ΒΓ τῇ ΕΖ, δύο δὴ αἱ ΒΗ, ΒΓ δυσὶ ταῖς ΔΕ, ΕΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΗΒΓ γωνίᾳ τῇ ὑπὸ ΔΕΖ ἴση ἐστίν· βάσις ἄρα ἡ ΗΓ βάσει τῇ ΔΖ ἴση ἐστίν, καὶ τὸ ΗΒΓ

τρίγωνον τῷ ΔΕΖ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὑφ' ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἡ ὑπὸ ΗΓΒ γωνία τῇ ὑπὸ ΔΖΕ. ἀλλὰ ἡ ὑπὸ ΔΖΕ τῇ ὑπὸ ΒΓΑ ὑπόκειται ἴση· καὶ ἡ ὑπὸ ΒΓΗ ἄρα τῇ ὑπὸ ΒΓΑ ἴση ἐστίν, ἡ ἐλάσσων τῇ μείζονι· ὅπερ ἀδύνατον. οὐκ ἄρα ἄνισός ἐστιν ἡ ΑΒ τῇ ΔΕ. ἴση ἄρα. ἔστι δὲ καὶ ἡ ΒΓ τῇ ΕΖ ἴση· δύο δὴ αἱ ΑΒ, ΒΓ δυσὶ ταῖς ΔΕ, ΕΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΑΒΓ γωνίᾳ τῇ ὑπὸ ΔΕΖ ἐστιν ἴση· βάσις ἄρα ἡ ΑΓ βάσει τῇ ΔΖ ἴση ἐστίν, καὶ λοιπὴ γωνία ἡ ὑπὸ ΒΑΓ τῇ λοιπῇ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἴση ἐστίν.

Ἀλλὰ δὴ πάλιν ἔστωσαν αἱ ὑπὸ τὰς ἴσας γωνίας πλευραὶ ὑποτείνουσαι ἴσαι, ὡς ἡ ΑΒ τῇ ΔΕ· λέγω πάλιν, ὅτι καὶ αἱ λοιπαὶ πλευραὶ ταῖς λοιπαῖς πλευραῖς ἴσαι ἔσονται, ἡ μὲν ΑΓ τῇ ΔΖ, ἡ δὲ ΒΓ τῇ ΕΖ καὶ ἔτι ἡ λοιπὴ γωνία ἡ ὑπὸ ΒΑΓ τῇ λοιπῇ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἴση ἐστίν.

Εἰ γὰρ ἄνισός ἐστιν ἡ ΒΓ τῇ ΕΖ, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων, εἰ δυνατόν, ἡ ΒΓ, καὶ κείσθω τῇ ΕΖ ἴση ἡ ΒΘ, καὶ ἐπεζεύχθω ἡ ΑΘ. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΒΘ τῇ ΕΖ ἡ δὲ ΑΒ τῇ ΔΕ, δύο δὴ αἱ ΑΒ, ΒΘ δυσὶ ταῖς ΔΕ, ΕΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνίας ἴσας περιέχουσιν· βάσις ἄρα ἡ ΑΘ βάσει τῇ ΔΖ ἴση ἐστίν, καὶ τὸ ΑΒΘ τρίγωνον τῷ ΔΕΖ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὑφ' ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἐστὶν ἡ ὑπὸ ΒΘΑ γωνία τῇ ὑπὸ ΕΖΔ. ἀλλὰ ἡ ὑπὸ ΕΖΔ τῇ ὑπὸ ΒΓΑ ἐστιν ἴση· τριγώνου δὴ τοῦ ΑΘΓ ἡ ἐκτὸς γωνία ἡ ὑπὸ ΒΘΑ ἴση ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΒΓΑ· ὅπερ ἀδύνατον. οὐκ ἄρα ἄνισός ἐστιν ἡ ΒΓ τῇ ΕΖ· ἴση ἄρα. ἐστὶ δὲ καὶ ἡ ΑΒ τῇ ΔΕ ἴση. δύο δὴ αἱ ΑΒ, ΒΓ δύο ταῖς ΔΕ, ΕΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνίας ἴσας περιέχουσι· βάσις ἄρα ἡ ΑΓ βάσει τῇ ΔΖ ἴση ἐστίν, καὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ ἴσον καὶ λοιπὴ γωνία ἡ ὑπὸ ΒΑΓ τῇ λοιπῇ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἴση.

Ἐὰν ἄρα δύο τρίγωνα τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην ἤτοι τὴν πρὸς ταῖς ἴσαις γωνίαις,

ἢ τὴν ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν, καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει καὶ τὴν λοιπὴν γωνίαν τῇ λοιπῇ γωνίᾳ· ὅπερ ἔδει δεῖξαι.

κζ΄. Ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐναλλὰξ γωνίας ἴσας ἀλλήλαις ποιῇ, παράλληλοι ἔσονται ἀλλήλαις αἱ εὐθεῖαι.

Εἰς γὰρ δύο εὐθείας τὰς ΑΒ, ΓΔ εὐθεῖα ἐμπίπτουσα ἡ ΕΖ τὰς ἐναλλὰξ γωνίας τὰς ὑπὸ ΑΕΖ, ΕΖΔ ἴσας ἀλλήλαις ποιείτω· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ.

Εἰ γὰρ μή, ἐκβαλλόμεναι αἱ ΑΒ, ΓΔ συμπεσοῦνται ἤτοι ἐπὶ τὰ Β, Δ μέρη ἢ ἐπὶ τὰ Α, Γ. ἐκβεβλήσθωσαν καὶ συμπιπτέτωσαν ἐπὶ τὰ Β, Δ μέρη κατὰ τὸ Η. τριγώνου δὴ τοῦ ΗΕΖ ἡ ἐκτὸς γωνία ἡ ὑπὸ ΑΕΖ ἴση ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΕΖΗ· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα αἱ ΑΒ, ΓΔ ἐκβαλλόμεναι συμπεσοῦνται ἐπὶ τὰ Β, Δ μέρη. ὁμοίως δὴ δειχθήσεται, ὅτι οὐδὲ ἐπὶ τὰ Α, Γ· αἱ δὲ ἐπὶ μηδέτερα τὰ μέρη συμπίπτουσαι παράλληλοί εἰσιν· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΓΔ.

Ἐὰν ἄρα εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐναλλὰξ γωνίας ἴσας ἀλλήλαις ποιῇ, παράλληλοι ἔσονται αἱ εὐθεῖαι· ὅπερ ἔδει δεῖξαι.

κη΄. Ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὴν ἐκτὸς γωνίαν τῇ ἐντὸς καὶ ἀπεναντίον καὶ ἐπὶ τὰ αὐτὰ μέρη ἴσην ποιῇ ἢ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δυσὶν ὀρθαῖς ἴσας, παράλληλοι ἔσονται ἀλλήλαις αἱ εὐθεῖαι.

Εἰς γὰρ δύο εὐθείας τὰς ΑΒ, ΓΔ εὐθεῖα ἐμπίπτουσα ἡ ΕΖ τὴν ἐκτὸς γωνίαν τὴν ὑπὸ ΕΗΒ τῇ ἐντὸς καὶ ἀπεναντίον γωνίᾳ τῇ ὑπὸ ΗΘΔ ἴσην ποιείτω ἢ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη τὰς ὑπὸ ΒΗΘ, ΗΘΔ δυσὶν ὀρθαῖς ἴσας· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ.

Ἐπεὶ γὰρ ἴση ἐστὶν ἡ ὑπὸ ΕΗΒ τῇ ὑπὸ ΗΘΔ, ἀλλὰ ἡ ὑπὸ ΕΗΒ τῇ ὑπὸ ΑΗΘ ἐστιν ἴση, καὶ ἡ ὑπὸ ΑΗΘ ἄρα τῇ ὑπὸ ΗΘΔ ἐστιν ἴση· καί εἰσιν

ἐναλλάξ· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΓΔ.

Πάλιν, ἐπεὶ αἱ ὑπὸ ΒΗΘ, ΗΘΔ δύο ὀρθαῖς ἴσαι εἰσίν, εἰσὶ δὲ καὶ αἱ ὑπὸ ΑΗΘ, ΒΗΘ δυσὶν ὀρθαῖς ἴσαι, αἱ ἄρα ὑπὸ ΑΗΘ, ΒΗΘ ταῖς ὑπὸ ΒΗΘ, ΗΘΔ ἴσαι εἰσίν· κοινὴ ἀφῃρήσθω ἡ ὑπὸ ΒΗΘ· λοιπὴ ἄρα ἡ ὑπὸ ΑΗΘ λοιπῇ τῇ ὑπὸ ΗΘΔ ἐστιν ἴση· καί εἰσιν ἐναλλάξ· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΓΔ.

Ἐὰν ἄρα εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὴν ἐκτὸς γωνίαν τῇ ἐντὸς καὶ ἀπεναντίον καὶ ἐπὶ τὰ αὐτὰ μέρη ἴσην ποιῇ ἢ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δυσὶν ὀρθαῖς ἴσας, παράλληλοι ἔσονται αἱ εὐθεῖαι· ὅπερ ἔδει δεῖξαι.

κθ΄. Ἡ εἰς τὰς παραλλήλους εὐθείας εὐθεῖα ἐμπίπτουσα τάς τε ἐναλλὰξ γωνίας ἴσας ἀλλήλαις ποιεῖ καὶ τὴν ἐκτὸς τῇ ἐντὸς καὶ ἀπεναντίον ἴσην καὶ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δυσὶν ὀρθαῖς ἴσας.

Εἰς γὰρ παραλλήλους εὐθείας τὰς ΑΒ, ΓΔ εὐθεῖα ἐμπιπτέτω ἡ ΕΖ· λέγω, ὅτι τὰς ἐναλλὰξ γωνίας τὰς ὑπὸ ΑΗΘ, ΗΘΔ ἴσας ποιεῖ καὶ τὴν ἐκτὸς γωνίαν τὴν ὑπὸ ΕΗΒ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΗΘΔ ἴσην καὶ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη τὰς ὑπὸ ΒΗΘ, ΗΘΔ δυσὶν ὀρθαῖς ἴσας.

Εἰ γὰρ ἄνισός ἐστιν ἡ ὑπὸ ΑΗΘ τῇ ὑπὸ ΗΘΔ, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ ὑπὸ ΑΗΘ· κοινὴ προσκείσθω ἡ ὑπὸ ΒΗΘ· αἱ ἄρα ὑπὸ ΑΗΘ, ΒΗΘ τῶν ὑπὸ ΒΗΘ, ΗΘΔ μείζονές εἰσιν. ἀλλὰ αἱ ὑπὸ ΑΗΘ, ΒΗΘ δυσὶν ὀρθαῖς ἴσαι εἰσίν. [καὶ] αἱ ἄρα ὑπὸ ΒΗΘ, ΗΘΔ δύο ὀρθῶν ἐλάσσονές εἰσιν. αἱ δὲ ἀπ’ ἐλασσόνων ἢ δύο ὀρθῶν ἐκβαλλόμεναι εἰς ἄπειρον συμπίπτουσιν· αἱ ἄρα ΑΒ, ΓΔ ἐκβαλλόμεναι εἰς ἄπειρον συμπεσοῦνται· οὐ συμπίπτουσι δὲ διὰ τὸ παραλλήλους αὐτὰς ὑποκεῖσθαι· οὐκ ἄρα ἄνισός ἐστιν ἡ ὑπὸ ΑΗΘ τῇ ὑπὸ ΗΘΔ· ἴση ἄρα. ἀλλὰ ἡ ὑπὸ ΑΗΘ τῇ ὑπὸ ΕΗΒ ἐστιν ἴση· καὶ ἡ ὑπὸ ΕΗΒ ἄρα τῇ ὑπὸ ΗΘΔ ἐστιν ἴση. κοινὴ προσκείσθω ἡ ὑπὸ ΒΗΘ· αἱ ἄρα ὑπὸ ΕΗΒ, ΒΗΘ ταῖς ὑπὸ ΒΗΘ, ΗΘΔ ἴσαι εἰσίν. ἀλλὰ αἱ ὑπὸ ΕΗΒ, ΒΗΘ δύο ὀρθαῖς ἴσαι εἰσίν· καὶ αἱ ὑπὸ ΒΗΘ, ΗΘΔ ἄρα δύο ὀρθαῖς ἴσαι εἰσίν.

Ἡ ἄρα εἰς τὰς παραλλήλους εὐθείας εὐθεῖα ἐμπίπτουσα τάς τε ἐναλλὰξ γωνίας ἴσας ἀλλήλαις ποιεῖ καὶ τὴν ἐκτὸς τῇ ἐντὸς καὶ ἀπεναντίον ἴσην καὶ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δυσὶν ὀρθαῖς ἴσας· ὅπερ ἔδει δεῖξαι.

λ΄. Αἱ τῇ αὐτῇ εὐθείᾳ παράλληλοι καὶ ἀλλήλαις εἰσὶ παράλληλοι.

Ἔστω ἑκατέρα τῶν ΑΒ, ΓΔ τῇ ΕΖ παράλληλος· λέγω, ὅτι καὶ ἡ ΑΒ τῇ ΓΔ ἐστι παράλληλος. Ἐμπιπτέτω γὰρ εἰς αὐτὰς εὐθεῖα ἡ ΗΚ.

Καὶ ἐπεὶ εἰς παραλλήλους εὐθείας τὰς ΑΒ, ΕΖ εὐθεῖα ἐμπέπτωκεν ἡ ΗΚ, ἴση ἄρα ἡ ὑπὸ ΑΗΚ τῇ ὑπὸ ΗΘΖ. πάλιν, ἐπεὶ εἰς παραλλήλους εὐθείας τὰς ΕΖ, ΓΔ εὐθεῖα ἐμπέπτωκεν ἡ ΗΚ, ἴση ἐστὶν ἡ ὑπὸ ΗΘΖ τῇ ὑπὸ ΗΚΔ. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΑΗΚ τῇ ὑπὸ ΗΘΖ ἴση. καὶ ἡ ὑπὸ ΑΗΚ ἄρα τῇ ὑπὸ ΗΚΔ ἐστιν ἴση· καί εἰσιν ἐναλλάξ. παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΓΔ.

[Αἱ ἄρα τῇ αὐτῇ εὐθείᾳ παράλληλοι καὶ ἀλλήλαις εἰσὶ παράλληλοι·] ὅπερ ἔδει δεῖξαι.

λα΄. Διὰ τοῦ δοθέντος σημείου τῇ δοθείσῃ εὐθείᾳ παράλληλον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Ἔστω τὸ μὲν δοθὲν σημεῖον τὸ Α, ἡ δὲ δοθεῖσα εὐθεῖα ἡ ΒΓ· δεῖ δὴ διὰ τοῦ Α σημείου τῇ ΒΓ εὐθείᾳ παράλληλον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Εἰλήφθω ἐπὶ τῆς ΒΓ τυχὸν σημεῖον τὸ Δ, καὶ ἐπεζεύχθω ἡ ΑΔ· καὶ συνεστάτω πρὸς τῇ ΔΑ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ ὑπὸ ΑΔΓ γωνίᾳ ἴση ἡ ὑπὸ ΔΑΕ· καὶ ἐκβεβλήσθω ἐπ’ εὐθείας τῇ ΕΑ εὐθεῖα ἡ ΑΖ.

Καὶ ἐπεὶ εἰς δύο εὐθείας τὰς ΒΓ, ΕΖ εὐθεῖα ἐμπίπτουσα ἡ ΑΔ τὰς ἐναλλὰξ γωνίας τὰς ὑπὸ ΕΑΔ, ΑΔΓ ἴσας ἀλλήλαις πεποίηκεν, παράλληλος ἄρα ἐστὶν ἡ ΕΑΖ τῇ ΒΓ. Διὰ τοῦ δοθέντος ἄρα σημείου τοῦ Α τῇ δοθείσῃ εὐθείᾳ τῇ ΒΓ παράλληλος εὐθεῖα γραμμὴ ἦκται ἡ ΕΑΖ· ὅπερ ἔδει ποιῆσαι.

λβ΄. Παντὸς τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ἴση ἐστίν, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Ἔστω τρίγωνον τὸ ΑΒΓ, καὶ προσεκβεβλήσθω αὐτοῦ μία πλευρὰ ἡ ΒΓ ἐπὶ τὸ Δ· λέγω, ὅτι ἡ ἐκτὸς γωνία ἡ ὑπὸ ΑΓΔ ἴση ἐστὶ δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ταῖς ὑπὸ ΓΑΒ, ΑΒΓ, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι αἱ ὑπὸ ΑΒΓ, ΒΓΑ, ΓΑΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Ἤχθω γὰρ διὰ τοῦ Γ σημείου τῇ ΑΒ εὐθείᾳ παράλληλος ἡ ΓΕ.

Καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΕ, καὶ εἰς αὐτὰς ἐμπέπτωκεν ἡ ΑΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΒΑΓ, ΑΓΕ ἴσαι ἀλλήλαις εἰσίν. πάλιν, ἐπεὶ παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΕ, καὶ εἰς αὐτὰς ἐμπέπτωκεν εὐθεῖα ἡ ΒΔ, ἡ ἐκτὸς γωνία ἡ ὑπὸ ΕΓΔ ἴση ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΑΒΓ. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΑΓΕ τῇ ὑπὸ ΒΑΓ ἴση· ὅλη ἄρα ἡ ὑπὸ ΑΓΔ γωνία ἴση ἐστὶ δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ταῖς ὑπὸ ΒΑΓ, ΑΒΓ.

Κοινὴ προσκείσθω ἡ ὑπὸ ΑΓΒ· αἱ ἄρα ὑπὸ ΑΓΔ, ΑΓΒ τρισὶ ταῖς ὑπὸ ΑΒΓ, ΒΓΑ, ΓΑΒ ἴσαι εἰσίν. ἀλλ’ αἱ ὑπὸ ΑΓΔ, ΑΓΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν· καὶ αἱ ὑπὸ ΑΓΒ, ΓΒΑ, ΓΑΒ ἄρα δυσίν ὀρθαῖς ἴσαι εἰσίν.

Παντὸς ἄρα τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ἴση ἐστίν, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν· ὅπερ ἔδει δεῖξαι.

λγ΄. Αἱ τὰς ἴσας τε καὶ παραλλήλους ἐπὶ τὰ αὐτὰ μέρη ἐπιζευγνύουσαι εὐθεῖαι καὶ αὐταὶ ἴσαι τε καὶ παράλληλοί εἰσιν.

Ἔστωσαν ἴσαι τε καὶ παράλληλοι αἱ ΑΒ, ΓΔ, καὶ ἐπιζευγνύτωσαν αὐτὰς ἐπὶ τὰ αὐτὰ μέρη εὐθεῖαι αἱ ΑΓ, ΒΔ· λέγω, ὅτι καὶ αἱ ΑΓ, ΒΔ ἴσαι τε καὶ παράλληλοί εἰσιν.

Ἐπεζεύχθω ἡ ΒΓ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ, καὶ εἰς αὐτὰς ἐμπέπτωκεν ἡ ΒΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΑΒΓ, ΒΓΔ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΓΔ κοινὴ δὲ ἡ ΒΓ, δύο δὴ αἱ ΑΒ, ΒΓ δύο ταῖς ΒΓ, ΓΔ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΑΒΓ γωνίᾳ τῇ ὑπὸ ΒΓΔ ἴση· βάσις ἄρα ἡ ΑΓ βάσει τῇ ΒΔ ἐστιν ἴση, καὶ τὸ ΑΒΓ τρίγωνον τῷ ΒΓΔ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἡ ὑπὸ ΑΓΒ γωνία τῇ ὑπὸ ΓΒΔ. καὶ ἐπεὶ εἰς δύο εὐθείας τὰς ΑΓ, ΒΔ εὐθεῖα ἐμπίπτουσα ἡ ΒΓ τὰς ἐναλλὰξ γωνίας ἴσας ἀλλήλαις πεποίηκεν, παράλληλος ἄρα ἐστὶν ἡ ΑΓ τῇ ΒΔ. ἐδείχθη δὲ αὐτῇ καὶ ἴση.

Αἱ ἄρα τὰς ἴσας τε καὶ παραλλήλους ἐπὶ τὰ αὐτὰ μέρη ἐπιζευγνύουσαι εὐθεῖαι καὶ αὐταὶ ἴσαι τε καὶ παράλληλοί εἰσιν· ὅπερ ἔδει δεῖξαι.

λδ΄. Τῶν παραλληλογράμμων χωρίων αἱ ἀπεναντίον πλευραί τε καὶ γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ ἡ διάμετρος αὐτὰ δίχα τέμνει.

Ἔστω παραλληλόγραμμον χωρίον τὸ ΑΓΔΒ, διάμετρος δὲ αὐτοῦ ἡ ΒΓ· λέγω, ὅτι τοῦ ΑΓΔΒ παραλληλογράμμου αἱ ἀπεναντίον πλευραί τε καὶ γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ ἡ ΒΓ διάμετρος αὐτὸ δίχα τέμνει.

Ἐπεὶ γὰρ παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ, καὶ εἰς αὐτὰς ἐμπέπτωκεν εὐθεῖα ἡ ΒΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΑΒΓ, ΒΓΔ ἴσαι ἀλλήλαις εἰσίν. πάλιν, ἐπεὶ παράλληλός ἐστιν ἡ ΑΓ τῇ ΒΔ, καὶ εἰς αὐτὰς ἐμπέπτωκεν ἡ ΒΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΑΓΒ, ΓΒΔ ἴσαι ἀλλήλαις εἰσίν. δύο δὴ τρίγωνά ἐστι τὰ ΑΒΓ, ΒΓΔ τὰς δύο γωνίας τὰς ὑπὸ ΑΒΓ, ΒΓΑ δυσὶ ταῖς ὑπὸ ΒΓΔ, ΓΒΔ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν πρὸς ταῖς ἴσαις γωνίαις κοινὴν αὐτῶν τὴν ΒΓ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς ἴσας ἕξει ἑκατέραν ἑκατέρᾳ καὶ τὴν λοιπὴν γωνίαν τῇ λοιπῇ γωνίᾳ· ἴση ἄρα ἡ μὲν ΑΒ πλευρὰ τῇ ΓΔ, ἡ δὲ ΑΓ τῇ ΒΔ, καὶ ἔτι ἴση ἐστὶν ἡ ὑπὸ ΒΑΓ γωνία τῇ ὑπὸ ΓΔΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΒΓΔ, ἡ δὲ ὑπὸ

ΓΒΔ τῇ ὑπὸ ΑΓΒ, ὅλη ἄρα ἡ ὑπὸ ΑΒΔ ὅλῃ τῇ ὑπὸ ΑΓΔ ἐστιν ἴση. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΒΑΓ τῇ ὑπὸ ΓΔΒ ἴση.

Τῶν ἄρα παραλληλογράμμων χωρίων αἱ ἀπεναντίον πλευραί τε καὶ γωνίαι ἴσαι ἀλλήλαις εἰσίν.

Λέγω δή, ὅτι καὶ ἡ διάμετρος αὐτὰ δίχα τέμνει. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΑΒ τῇ ΓΔ, κοινὴ δὲ ἡ ΒΓ, δύο δὴ αἱ ΑΒ, ΒΓ δυσὶ ταῖς ΓΔ, ΒΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΑΒΓ γωνίᾳ τῇ ὑπὸ ΒΓΔ ἴση. καὶ βάσις ἄρα ἡ ΑΓ τῇ ΔΒ ἴση. καὶ τὸ ΑΒΓ [ἄρα] τρίγωνον τῷ ΒΓΔ τριγώνῳ ἴσον ἐστίν.

Ἡ ἄρα ΒΓ διάμετρος δίχα τέμνει τὸ ΑΒΓΔ παραλληλόγραμμον· ὅπερ ἔδει δεῖξαι.

λε΄. Τὰ παραλληλόγραμμα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν.

Ἔστω παραλληλόγραμμα τὰ ΑΒΓΔ, ΕΒΓΖ ἐπὶ τῆς αὐτῆς βάσεως τῆς ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΑΖ, ΒΓ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓΔ τῷ ΕΒΓΖ παραλληλογράμμῳ.

Ἐπεὶ γὰρ παραλληλόγραμμόν ἐστι τὸ ΑΒΓΔ, ἴση ἐστὶν ἡ ΑΔ τῇ ΒΓ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΕΖ τῇ ΒΓ ἐστιν ἴση· ὥστε καὶ ἡ ΑΔ τῇ ΕΖ ἐστιν ἴση· καὶ κοινὴ ἡ ΔΕ· ὅλη ἄρα ἡ ΑΕ ὅλῃ τῇ ΔΖ ἐστιν ἴση. ἔστι δὲ καὶ ἡ ΑΒ τῇ ΔΓ ἴση· δύο δὴ αἱ ΕΑ, ΑΒ δύο ταῖς ΖΔ, ΔΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΖΔΓ γωνίᾳ τῇ ὑπὸ ΕΑΒ ἐστιν ἴση ἡ ἐκτὸς τῇ ἐντός· βάσις ἄρα ἡ ΕΒ βάσει τῇ ΖΓ ἴση ἐστίν, καὶ τὸ ΕΑΒ τρίγωνον τῷ ΔΖΓ τριγώνῳ ἴσον ἔσται· κοινὸν ἀφῃρήσθω τὸ ΔΗΕ· λοιπὸν ἄρα τὸ ΑΒΗΔ τραπέζιον λοιπῷ τῷ ΕΗΓΖ τραπεζίῳ ἐστὶν ἴσον· κοινὸν προσκείσθω τὸ ΗΒΓ τρίγωνον· ὅλον ἄρα τὸ ΑΒΓΔ παραλληλόγραμμον ὅλῳ τῷ ΕΒΓΖ παραλληλογράμμῳ ἴσον ἐστίν.

Τὰ ἄρα παραλληλόγραμμα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

λς΄. Τὰ παραλληλόγραμμα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν.

Ἔστω παραλληλόγραμμα τὰ ΑΒΓΔ, ΕΖΗΘ ἐπὶ ἴσων βάσεων ὄντα τῶν ΒΓ, ΖΗ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΑΘ, ΒΗ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓΔ παραλληλόγραμμον τῷ ΕΖΗΘ.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΕ, ΓΘ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΓ τῇ ΖΗ, ἀλλὰ ἡ ΖΗ τῇ ΕΘ ἐστιν ἴση, καὶ ἡ ΒΓ ἄρα τῇ ΕΘ ἐστιν ἴση. εἰσὶ δὲ καὶ παράλληλοι. καὶ ἐπιζευγνύουσιν αὐτὰς αἱ ΕΒ, ΘΓ· αἱ δὲ τὰς ἴσας τε καὶ παραλλήλους ἐπὶ τὰ αὐτὰ μέρη ἐπιζευγνύουσαι ἴσαι τε καὶ παράλληλοί εἰσι· [καὶ αἱ ΕΒ, ΘΓ ἄρα ἴσαι τέ εἰσι καὶ παράλληλοι]. παραλληλόγραμμον ἄρα ἐστὶ τὸ ΕΒΓΘ. καί ἐστιν ἴσον τῷ ΑΒΓΔ· βάσιν τε γὰρ αὐτῷ τὴν αὐτὴν ἔχει τὴν ΒΓ, καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστὶν αὐτῷ ταῖς ΒΓ, ΑΘ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΕΖΗΘ τῷ αὐτῷ τῷ ΕΒΓΘ ἐστιν ἴσον· ὥστε καὶ τὸ ΑΒΓΔ παραλληλόγραμμον τῷ ΕΖΗΘ ἐστιν ἴσον.

Τὰ ἄρα παραλληλόγραμμα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

λζ΄. Τὰ τρίγωνα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν.

Ἔστω τρίγωνα τὰ ΑΒΓ, ΔΒΓ ἐπὶ τῆς αὐτῆς βάσεως τῆς ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΑΔ, ΒΓ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΒΓ τριγώνῳ.

Ἐκβεβλήσθω ἡ ΑΔ ἐφ’ ἑκάτερα τὰ μέρη ἐπὶ τὰ Ε, Ζ, καὶ διὰ μὲν τοῦ Β τῇ ΓΑ παράλληλος ἤχθω ἡ ΒΕ, διὰ δὲ τοῦ Γ τῇ ΒΔ παράλληλος ἤχθω ἡ ΓΖ. παραλληλόγραμμον ἄρα ἐστὶν ἑκάτερον τῶν ΕΒΓΑ, ΔΒΓΖ· καί εἰσιν ἴσα· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΓ, ΕΖ· καί ἐστι τοῦ μὲν ΕΒΓΑ παραλληλογράμμου ἥμισυ τὸ ΑΒΓ τρίγωνον· ἡ

γὰρ ΑΒ διάμετρος αὐτὸ δίχα τέμνει· τοῦ δὲ ΔΒΓΖ παραλληλογράμμου ἥμισυ τὸ ΔΒΓ τρίγωνον· ἡ γὰρ ΔΓ διάμετρος αὐτὸ δίχα τέμνει. [τὰ δὲ τῶν ἴσων ἡμίση ἴσα ἀλλήλοις ἐστίν]. ἴσον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΒΓ τριγώνῳ.

Τὰ ἄρα τρίγωνα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

λη΄. Τὰ τρίγωνα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν.

Ἔστω τρίγωνα τὰ ΑΒΓ, ΔΕΖ ἐπὶ ἴσων βάσεων τῶν ΒΓ, ΕΖ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΖ, ΑΔ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Ἐκβεβλήσθω γὰρ ἡ ΑΔ ἐφ᾽ ἑκάτερα τὰ μέρη ἐπὶ τὰ Η, Θ, καὶ διὰ μὲν τοῦ Β τῇ ΓΑ παράλληλος ἤχθω ἡ ΒΗ, διὰ δὲ τοῦ Ζ τῇ ΔΕ παράλληλος ἤχθω ἡ ΖΘ. παραλληλόγραμμον ἄρα ἐστὶν ἑκάτερον τῶν ΗΒΓΑ, ΔΕΖΘ· καὶ ἴσον τὸ ΗΒΓΑ τῷ ΔΕΖΘ· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν ΒΓ, ΕΖ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΖ, ΗΘ· καί ἐστι τοῦ μὲν ΗΒΓΑ παραλληλογράμμου ἥμισυ τὸ ΑΒΓ τρίγωνον. ἡ γὰρ ΑΒ διάμετρος αὐτὸ δίχα τέμνει· τοῦ δὲ ΔΕΖΘ παραλληλογράμμου ἥμισυ τὸ ΖΕΔ τρίγωνον· ἡ γὰρ ΔΖ διάμετρος αὐτὸ δίχα τέμνει· [τὰ δὲ τῶν ἴσων ἡμίση ἴσα ἀλλήλοις ἐστίν]. ἴσον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Τὰ ἄρα τρίγωνα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

λθ΄. Τὰ ἴσα τρίγωνα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν.

Ἔστω ἴσα τρίγωνα τὰ ΑΒΓ, ΔΒΓ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη τῆς ΒΓ· λέγω, ὅτι καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν.

Ἐπεζεύχθω γὰρ ἡ ΑΔ· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΔ τῇ ΒΓ.

Εἰ γὰρ μή, ἤχθω διὰ τοῦ Α σημείου τῇ ΒΓ εὐθείᾳ παράλληλος ἡ ΑΕ, καὶ ἐπεζεύχθω ἡ ΕΓ. ἴσον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΕΒΓ τριγώνῳ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς ἐστιν αὐτῷ τῆς ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις. ἀλλὰ τὸ ΑΒΓ τῷ ΔΒΓ ἐστιν ἴσον· καὶ τὸ ΔΒΓ ἄρα τῷ ΕΒΓ ἴσον ἐστὶ τὸ μεῖζον τῷ ἐλάσσονι· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα παράλληλός ἐστιν ἡ ΑΕ τῇ ΒΓ. ὁμοίως δὴ δείξομεν, ὅτι οὐδ' ἄλλη τις πλὴν τῆς ΑΔ· ἡ ΑΔ ἄρα τῇ ΒΓ ἐστι παράλληλος.

Τὰ ἄρα ἴσα τρίγωνα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

μ΄. Τὰ ἴσα τρίγωνα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν.

Ἔστω ἴσα τρίγωνα τὰ ΑΒΓ, ΓΔΕ ἐπὶ ἴσων βάσεων τῶν ΒΓ, ΓΕ καὶ ἐπὶ τὰ αὐτὰ μέρη. λέγω, ὅτι καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν.

Ἐπεζεύχθω γὰρ ἡ ΑΔ· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΔ τῇ ΒΕ.

Εἰ γὰρ μή, ἤχθω διὰ τοῦ Α τῇ ΒΕ παράλληλος ἡ ΑΖ, καὶ ἐπεζεύχθω ἡ ΖΕ. ἴσον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΖΓΕ τριγώνῳ· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν ΒΓ, ΓΕ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΕ, ΑΖ. ἀλλὰ τὸ ΑΒΓ τρίγωνον ἴσον ἐστὶ τῷ ΔΓΕ [τριγώνῳ]· καὶ τὸ ΔΓΕ ἄρα [τρίγωνον] ἴσον ἐστὶ τῷ ΖΓΕ τριγώνῳ τὸ μεῖζον τῷ ἐλάσσονι· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα παράλληλος ἡ ΑΖ τῇ ΒΕ. ὁμοίως δὴ δείξομεν, ὅτι οὐδ' ἄλλη τις πλὴν τῆς ΑΔ· ἡ ΑΔ ἄρα τῇ ΒΕ ἐστι παράλληλος.

Τὰ ἄρα ἴσα τρίγωνα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

μα΄. Ἐὰν παραλληλόγραμμον τριγώνῳ βάσιν τε ἔχῃ τὴν αὐτὴν καὶ ἐν ταῖς αὐταῖς παραλλήλοις ᾖ, διπλάσιόν ἐστι τὸ παραλληλόγραμμον τοῦ τριγώνου.

Παραλληλόγραμμον γὰρ τὸ ΑΒΓΔ τριγώνῳ τῷ ΕΒΓ βάσιν τε ἐχέτω τὴν αὐτὴν τὴν ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἔστω ταῖς ΒΓ, ΑΕ· λέγω, ὅτι διπλάσιόν ἐστι τὸ ΑΒΓΔ παραλληλόγραμμον τοῦ ΒΕΓ τριγώνου.

Ἐπεζεύχθω γὰρ ἡ ΑΓ. ἴσον δή ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΕΒΓ τριγώνῳ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς ἐστιν αὐτῷ τῆς ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΓ, ΑΕ. ἀλλὰ τὸ ΑΒΓΔ παραλληλόγραμμον διπλάσιόν ἐστι τοῦ ΑΒΓ τριγώνου· ἡ γὰρ ΑΓ διάμετρος αὐτὸ δίχα τέμνει· ὥστε τὸ ΑΒΓΔ παραλληλόγραμμον καὶ τοῦ ΕΒΓ τριγώνου ἐστὶ διπλάσιον.

Ἐὰν ἄρα παραλληλόγραμμον τριγώνῳ βάσιν τε ἔχῃ τὴν αὐτὴν καὶ ἐν ταῖς αὐταῖς παραλλήλοις ᾖ, διπλάσιόν ἐστι τὸ παραλληλόγραμμον τοῦ τριγώνου· ὅπερ ἔδει δεῖξαι.

μβ΄. Τῷ δοθέντι τριγώνῳ ἴσον παραλληλόγραμμον συστήσασθαι ἐν τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ.

Ἔστω τὸ μὲν δοθὲν τρίγωνον τὸ ΑΒΓ, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ Δ· δεῖ δὴ τῷ ΑΒΓ τριγώνῳ ἴσον παραλληλόγραμμον συστήσασθαι ἐν τῇ Δ γωνίᾳ εὐθυγράμμῳ.

Τετμήσθω ἡ ΒΓ δίχα κατὰ τὸ Ε, καὶ ἐπεζεύχθω ἡ ΑΕ, καὶ συνεστάτω πρὸς τῇ ΕΓ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Ε τῇ Δ γωνίᾳ ἴση ἡ ὑπὸ ΓΕΖ, καὶ διὰ μὲν τοῦ Α τῇ ΕΓ παράλληλος ἤχθω ἡ ΑΗ, διὰ δὲ τοῦ Γ τῇ ΕΖ παράλληλος ἤχθω ἡ ΓΗ· παραλληλόγραμμον ἄρα ἐστὶ τὸ ΖΕΓΗ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΕ τῇ ΕΓ, ἴσον ἐστὶ καὶ τὸ ΑΒΕ τρίγωνον τῷ ΑΕΓ τριγώνῳ· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν ΒΕ, ΕΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΓ, ΑΗ· διπλάσιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τοῦ ΑΕΓ τριγώνου. ἔστι δὲ καὶ τὸ ΖΕΓΗ παραλληλόγραμμον διπλάσιον τοῦ ΑΕΓ τριγώνου· βάσιν τε γὰρ αὐτῷ τὴν αὐτὴν ἔχει καὶ ἐν ταῖς αὐταῖς ἐστιν αὐτῷ παραλλήλοις· ἴσον ἄρα ἐστὶ τὸ ΖΕΓΗ παραλληλόγραμμον τῷ ΑΒΓ τριγώνῳ. καὶ ἔχει τὴν ὑπὸ ΓΕΖ γωνίαν ἴσην τῇ δοθείσῃ τῇ Δ.

Τῷ ἄρα δοθέντι τριγώνῳ τῷ ΑΒΓ ἴσον παραλληλόγραμμον συνέσταται τὸ ΖΕΓΗ ἐν γωνίᾳ τῇ ὑπὸ ΓΕΖ, ἥτις ἐστὶν ἴση τῇ Δ· ὅπερ ἔδει ποιῆσαι.

μγ΄. Παντὸς παραλληλογράμμου τῶν περὶ τὴν διάμετρον παραλληλογράμμων τὰ παραπληρώματα ἴσα ἀλλήλοις ἐστίν.

Ἔστω παραλληλόγραμμον τὸ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἡ ΑΓ, περὶ δὲ τὴν ΑΓ παραλληλόγραμμα μὲν ἔστω τὰ ΖΘ, ΖΗ, τὰ δὲ λεγόμενα παραπληρώματα τὰ ΒΚ, ΚΔ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΒΚ παραπλήρωμα τῷ ΚΔ παραπληρώματι.

Ἐπεὶ γὰρ παραλληλόγραμμόν ἐστι τὸ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἡ ΑΓ, ἴσον ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΓΔ τριγώνῳ. πάλιν, ἐπεὶ παραλληλόγραμμόν ἐστι τὸ ΕΘ, διάμετρος δὲ αὐτοῦ ἐστιν ἡ ΑΚ, ἴσον ἐστὶ τὸ ΑΕΚ τρίγωνον τῷ ΑΘΚ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΚΖΓ τρίγωνον τῷ ΚΗΓ ἐστιν ἴσον. ἐπεὶ οὖν τὸ μὲν ΑΕΚ τρίγωνον τῷ ΑΘΚ τριγώνῳ ἐστὶν ἴσον, τὸ δὲ ΚΖΓ τῷ ΚΗΓ, τὸ ΑΕΚ τρίγωνον μετὰ τοῦ ΚΗΓ ἴσον ἐστὶ τῷ ΑΘΚ τριγώνῳ μετὰ τοῦ ΚΖΓ· ἔστι δὲ καὶ ὅλον τὸ ΑΒΓ τρίγωνον ὅλῳ τῷ ΑΔΓ ἴσον· λοιπὸν ἄρα τὸ ΒΚ παραπλήρωμα λοιπῷ τῷ ΚΔ παραπληρώματί ἐστιν ἴσον.

Παντὸς ἄρα παραλληλογράμμου χωρίου τῶν περὶ τὴν διάμετρον παραλληλογράμμων τὰ παραπληρώματα ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

μδ΄. Παρὰ τὴν δοθεῖσαν εὐθεῖαν τῷ δοθέντι τριγώνῳ ἴσον παραλληλόγραμμον παραβαλεῖν ἐν τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ.

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ δοθὲν τρίγωνον τὸ Γ, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ Δ· δεῖ δὴ παρὰ τὴν δοθεῖσαν εὐθεῖαν τὴν ΑΒ τῷ δοθέντι τριγώνῳ τῷ Γ ἴσον παραλληλόγραμμον παραβαλεῖν ἐν ἴσῃ τῇ Δ γωνίᾳ.

Συνεστάτω τῷ Γ τριγώνῳ ἴσον παραλληλόγραμμον τὸ ΒΕΖΗ ἐν γωνίᾳ τῇ ὑπὸ ΕΒΗ, ἥ ἐστιν ἴση τῇ Δ· καὶ κείσθω ὥστε ἐπ᾽ εὐθείας εἶναι τὴν ΒΕ τῇ ΑΒ, καὶ διήχθω ἡ ΖΗ ἐπὶ τὸ Θ, καὶ διὰ τοῦ Α ὁποτέρᾳ τῶν ΒΗ, ΕΖ παράλληλος ἤχθω ἡ ΑΘ, καὶ ἐπεζεύχθω ἡ ΘΒ. καὶ ἐπεὶ εἰς παραλλήλους τὰς ΑΘ, ΕΖ εὐθεῖα ἐνέπεσεν ἡ ΘΖ, αἱ ἄρα ὑπὸ ΑΘΖ, ΘΖΕ γωνίαι δυσὶν ὀρθαῖς εἰσιν ἴσαι. αἱ ἄρα ὑπὸ ΒΘΗ, ΗΖΕ δύο ὀρθῶν ἐλάσσονές εἰσιν· αἱ δὲ ἀπὸ ἐλασσόνων ἢ δύο ὀρθῶν εἰς ἄπειρον ἐκβαλλόμεναι συμπίπτουσιν· αἱ ΘΒ, ΖΕ ἄρα ἐκβαλλόμεναι συμπεσοῦνται. ἐκβεβλήσθωσαν καὶ συμπιπτέτωσαν κατὰ τὸ Κ, καὶ διὰ τοῦ Κ σημείου ὁποτέρᾳ τῶν ΕΑ, ΖΘ παράλληλος ἤχθω ἡ ΚΛ, καὶ ἐκβεβλήσθωσαν αἱ ΘΑ, ΗΒ ἐπὶ τὰ Λ, Μ σημεῖα. παραλληλόγραμμον ἄρα ἐστὶ τὸ ΘΛΚΖ, διάμετρος δὲ αὐτοῦ ἡ ΘΚ, περὶ δὲ τὴν ΘΚ παραλληλόγραμμα μὲν τὰ ΑΗ, ΜΕ, τὰ δὲ λεγόμενα παραπληρώματα τὰ ΛΒ, ΒΖ· ἴσον ἄρα ἐστὶ τὸ ΛΒ τῷ ΒΖ. ἀλλὰ τὸ ΒΖ τῷ Γ τριγώνῳ ἐστὶν ἴσον· καὶ τὸ ΛΒ ἄρα τῷ Γ ἐστιν ἴσον. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΗΒΕ γωνία τῇ ὑπὸ ΑΒΜ, ἀλλὰ ἡ ὑπὸ ΗΒΕ τῇ Δ ἐστιν ἴση, καὶ ἡ ὑπὸ ΑΒΜ ἄρα τῇ Δ γωνίᾳ ἐστὶν ἴση.

Παρὰ τὴν δοθεῖσαν ἄρα εὐθεῖαν τὴν ΑΒ τῷ δοθέντι τριγώνῳ τῷ Γ ἴσον παραλληλόγραμμον παραβέβληται τὸ ΛΒ ἐν γωνίᾳ τῇ ὑπὸ ΑΒΜ, ἥ ἐστιν ἴση τῇ Δ· ὅπερ ἔδει ποιῆσαι.

με΄. Τῷ δοθέντι εὐθυγράμμῳ ἴσον παραλληλόγραμμον συστήσασθαι ἐν τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ.

Ἔστω τὸ μὲν δοθὲν εὐθύγραμμον τὸ ΑΒΓΔ, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ Ε· δεῖ δὴ τῷ ΑΒΓΔ εὐθυγράμμῳ ἴσον παραλληλόγραμμον συστήσασθαι ἐν τῇ δοθείσῃ γωνίᾳ τῇ Ε.

Ἐπεζεύχθω ἡ ΔΒ, καὶ συνεστάτω τῷ ΑΒΔ τριγώνῳ ἴσον παραλληλόγραμμον τὸ ΖΘ ἐν τῇ ὑπὸ ΘΚΖ γωνίᾳ, ἥ ἐστιν ἴση τῇ Ε· καὶ παραβεβλήσθω παρὰ τὴν ΗΘ εὐθεῖαν τῷ ΔΒΓ τριγώνῳ ἴσον παραλληλόγραμμον τὸ ΗΜ ἐν τῇ ὑπὸ ΗΘΜ γωνίᾳ, ἥ ἐστιν ἴση τῇ Ε. καὶ ἐπεὶ

ἡ Ε γωνία ἑκατέρᾳ τῶν ὑπὸ ΘΚΖ, ΗΘΜ ἐστιν ἴση, καὶ ἡ ὑπὸ ΘΚΖ ἄρα τῇ ὑπὸ ΗΘΜ ἐστιν ἴση. κοινὴ προσκείσθω ἡ ὑπὸ ΚΘΗ· αἱ ἄρα ὑπὸ ΖΚΘ, ΚΘΗ ταῖς ὑπὸ ΚΘΗ, ΗΘΜ ἴσαι εἰσίν. ἀλλ’ αἱ ὑπὸ ΖΚΘ, ΚΘΗ δυσὶν ὀρθαῖς ἴσαι εἰσίν· καὶ αἱ ὑπὸ ΚΘΗ, ΗΘΜ ἄρα δύο ὀρθαῖς ἴσαι εἰσίν. πρὸς δή τινι εὐθείᾳ τῇ ΗΘ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Θ δύο εὐθεῖαι αἱ ΚΘ, ΘΜ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δύο ὀρθαῖς ἴσας ποιοῦσιν· ἐπ’ εὐθείας ἄρα ἐστὶν ἡ ΚΘ τῇ ΘΜ· καὶ ἐπεὶ εἰς παραλλήλους τὰς ΚΜ, ΖΗ εὐθεῖα ἐνέπεσεν ἡ ΘΗ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΜΘΗ, ΘΗΖ ἴσαι ἀλλήλαις εἰσίν. κοινὴ προσκείσθω ἡ ὑπὸ ΘΗΛ· αἱ ἄρα ὑπὸ ΜΘΗ, ΘΗΛ ταῖς ὑπὸ ΘΗΖ, ΘΗΛ ἴσαι εἰσίν. ἀλλ’ αἱ ὑπὸ ΜΘΗ, ΘΗΛ δύο ὀρθαῖς ἴσαι εἰσίν· καὶ αἱ ὑπὸ ΘΗΖ, ΘΗΛ ἄρα δύο ὀρθαῖς ἴσαι εἰσίν· ἐπ’ εὐθείας ἄρα ἐστὶν ἡ ΖΗ τῇ ΗΛ. καὶ ἐπεὶ ἡ ΖΚ τῇ ΘΗ ἴση τε καὶ παράλληλός ἐστιν, ἀλλὰ καὶ ἡ ΘΗ τῇ ΜΛ, καὶ ἡ ΚΖ ἄρα τῇ ΜΛ ἴση τε καὶ παράλληλός ἐστιν· καὶ ἐπιζευγνύουσιν αὐτὰς εὐθεῖαι αἱ ΚΜ, ΖΛ· καὶ αἱ ΚΜ, ΖΛ ἄρα ἴσαι τε καὶ παράλληλοί εἰσιν· παραλληλόγραμμον ἄρα ἐστὶ τὸ ΚΖΛΜ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ μὲν ΑΒΔ τρίγωνον τῷ ΖΘ παραλληλογράμμῳ, τὸ δὲ ΔΒΓ τῷ ΗΜ, ὅλον ἄρα τὸ ΑΒΓΔ εὐθύγραμμον ὅλῳ τῷ ΚΖΛΜ παραλληλογράμμῳ ἐστὶν ἴσον.

Τῷ ἄρα δοθέντι εὐθυγράμμῳ τῷ ΑΒΓΔ ἴσον παραλληλόγραμμον συνέσταται τὸ ΚΖΛΜ ἐν γωνίᾳ τῇ ὑπὸ ΖΚΜ, ἥ ἐστιν ἴση τῇ δοθείσῃ τῇ Ε· ὅπερ ἔδει ποιῆσαι.

μς΄. Ἀπὸ τῆς δοθείσης εὐθείας τετράγωνον ἀναγράψαι.

Ἔστω ἡ δοθεῖσα εὐθεῖα ἡ ΑΒ· δεῖ δὴ ἀπὸ τῆς ΑΒ εὐθείας τετράγωνον ἀναγράψαι.

Ἤχθω τῇ ΑΒ εὐθείᾳ ἀπὸ τοῦ πρὸς αὐτῇ σημείου τοῦ Α πρὸς ὀρθὰς ἡ ΑΓ, καὶ κείσθω τῇ ΑΒ ἴση ἡ ΑΔ· καὶ διὰ μὲν τοῦ Δ σημείου τῇ ΑΒ παράλληλος ἤχθω ἡ ΔΕ, διὰ δὲ τοῦ Β σημείου τῇ ΑΔ παράλληλος ἤχθω ἡ ΒΕ. Παραλληλόγραμμον ἄρα ἐστὶ τὸ ΑΔΕΒ· ἴση ἄρα ἐστὶν ἡ μὲν ΑΒ τῇ ΔΕ, ἡ δὲ

ΑΔ τῇ ΒΕ. ἀλλὰ ἡ ΑΒ τῇ ΑΔ ἐστιν ἴση· αἱ τέσσαρες ἄρα αἱ ΒΑ, ΑΔ, ΔΕ, ΕΒ ἴσαι ἀλλήλαις εἰσίν· ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΔΕΒ παραλληλόγραμμον. λέγω δή, ὅτι καὶ ὀρθογώνιον. ἐπεὶ γὰρ εἰς παραλλήλους τὰς ΑΒ, ΔΕ εὐθεῖα ἐνέπεσεν ἡ ΑΔ, αἱ ἄρα ὑπὸ ΒΑΔ, ΑΔΕ γωνίαι δύο ὀρθαῖς ἴσαι εἰσίν. ὀρθὴ δὲ ἡ ὑπὸ ΒΑΔ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΑΔΕ. τῶν δὲ παραλληλογράμμων χωρίων αἱ ἀπεναντίον πλευραί τε καὶ γωνίαι ἴσαι ἀλλήλαις εἰσίν· ὀρθὴ ἄρα καὶ ἑκατέρα τῶν ἀπεναντίον τῶν ὑπὸ ΑΒΕ, ΒΕΔ γωνιῶν· ὀρθογώνιον ἄρα ἐστὶ τὸ ΑΔΕΒ. ἐδείχθη δὲ καὶ ἰσόπλευρον.

Τετράγωνον ἄρα ἐστίν· καί ἐστιν ἀπὸ τῆς ΑΒ εὐθείας ἀναγεγραμμένον· ὅπερ ἔδει ποιῆσαι.

μζ΄. Ἐν τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν τετραγώνοις.

Ἔστω τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν ἔχον τὴν ὑπὸ ΒΑΓ γωνίαν· λέγω, ὅτι τὸ ἀπὸ τῆς ΒΓ τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ τετραγώνοις.

Ἀναγεγράφθω γὰρ ἀπὸ μὲν τῆς ΒΓ τετράγωνον τὸ ΒΔΕΓ, ἀπὸ δὲ τῶν ΒΑ, ΑΓ τὰ ΗΒ, ΘΓ, καὶ διὰ τοῦ Α ὁποτέρᾳ τῶν ΒΔ, ΓΕ παράλληλος ἤχθω ἡ ΑΛ· καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΖΓ. καὶ ἐπεὶ ὀρθή ἐστιν ἑκατέρα τῶν ὑπὸ ΒΑΓ, ΒΑΗ γωνιῶν, πρὸς δή τινι εὐθείᾳ τῇ ΒΑ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α δύο εὐθεῖαι αἱ ΑΓ, ΑΗ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δυσὶν ὀρθαῖς ἴσας ποιοῦσιν· ἐπ᾽ εὐθείας ἄρα ἐστὶν ἡ ΓΑ τῇ ΑΗ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΒΑ τῇ ΑΘ ἐστιν ἐπ᾽ εὐθείας. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΔΒΓ γωνία τῇ ὑπὸ ΖΒΑ· ὀρθὴ γὰρ ἑκατέρα· κοινὴ προσκείσθω ἡ ὑπὸ ΑΒΓ· ὅλη ἄρα ἡ ὑπὸ ΔΒΑ ὅλῃ τῇ ὑπὸ ΖΒΓ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΔΒ τῇ ΒΓ, ἡ δὲ ΖΒ τῇ ΒΑ, δύο δὴ αἱ ΔΒ, ΒΑ δύο ταῖς ΖΒ, ΒΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΔΒΑ γωνίᾳ τῇ ὑπὸ ΖΒΓ ἴση· βάσις ἄρα ἡ ΑΔ βάσει τῇ ΖΓ

[ἐστιν] ἴση, καὶ τὸ ΑΒΔ τρίγωνον τῷ ΖΒΓ τριγώνῳ ἐστὶν ἴσον· καὶ [ἐστὶ] τοῦ μὲν ΑΒΔ τριγώνου διπλάσιον τὸ ΒΛ παραλληλόγραμμον· βάσιν τε γὰρ τὴν αὐτὴν ἔχουσι τὴν ΒΔ καὶ ἐν ταῖς αὐταῖς εἰσι παραλλήλοις ταῖς ΒΔ, ΑΛ· τοῦ δὲ ΖΒΓ τριγώνου διπλάσιον τὸ ΗΒ τετράγωνον· βάσιν τε γὰρ πάλιν τὴν αὐτὴν ἔχουσι τὴν ΖΒ καὶ ἐν ταῖς αὐταῖς εἰσι παραλλήλοις ταῖς ΖΒ, ΗΓ. [τὰ δὲ τῶν ἴσων διπλάσια ἴσα ἀλλήλοις ἐστίν·] ἴσον ἄρα ἐστὶ καὶ τὸ ΒΛ παραλληλόγραμμον τῷ ΗΒ τετραγώνῳ. ὁμοίως δὴ ἐπιζευγνυμένων τῶν ΑΕ, ΒΚ δειχθήσεται καὶ τὸ ΓΛ παραλληλόγραμμον ἴσον τῷ ΘΓ τετραγώνῳ· ὅλον ἄρα τὸ ΒΔΕΓ τετράγωνον δυσὶ τοῖς ΗΒ, ΘΓ τετραγώνοις ἴσον ἐστίν. καί ἐστι τὸ μὲν ΒΔΕΓ τετράγωνον ἀπὸ τῆς ΒΓ ἀναγραφέν, τὰ δὲ ΗΒ, ΘΓ ἀπὸ τῶν ΒΑ, ΑΓ. τὸ ἄρα ἀπὸ τῆς ΒΓ πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ πλευρῶν τετραγώνοις.

Ἐν ἄρα τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν [γωνίαν] περιεχουσῶν πλευρῶν τετραγώνοις· ὅπερ ἔδει δεῖξαι.

μη΄. Ἐὰν τριγώνου τὸ ἀπὸ μιᾶς τῶν πλευρῶν τετράγωνον ἴσον ᾖ τοῖς ἀπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν τετραγώνοις, ἡ περιεχομένη γωνία ὑπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ὀρθή ἐστιν.

Τριγώνου γὰρ τοῦ ΑΒΓ τὸ ἀπὸ μιᾶς τῆς ΒΓ πλευρᾶς τετράγωνον ἴσον ἔστω τοῖς ἀπὸ τῶν ΒΑ, ΑΓ πλευρῶν τετραγώνοις· λέγω, ὅτι ὀρθή ἐστιν ἡ ὑπὸ ΒΑΓ γωνία.

Ἤχθω γὰρ ἀπὸ τοῦ Α σημείου τῇ ΑΓ εὐθείᾳ πρὸς ὀρθὰς ἡ ΑΔ καὶ κείσθω τῇ ΒΑ ἴση ἡ ΑΔ, καὶ ἐπεζεύχθω ἡ ΔΓ. ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΑΒ, ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς ΔΑ τετράγωνον τῷ ἀπὸ τῆς ΑΒ τετραγώνῳ. κοινὸν προσκείσθω τὸ ἀπὸ τῆς ΑΓ τετράγωνον· τὰ ἄρα ἀπὸ τῶν ΔΑ, ΑΓ τετράγωνα ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ τετραγώνοις. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΔΑ, ΑΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΔΓ· ὀρθὴ γάρ ἐστιν ἡ ὑπὸ ΔΑΓ γωνία· τοῖς δὲ ἀπὸ τῶν

ΒΑ, ΑΓ ἴσον ἐστὶ τὸ ἀπὸ ΒΓ· ὑπόκειται γάρ· τὸ ἄρα ἀπὸ τῆς ΔΓ τετράγωνον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΒΓ τετραγώνῳ· ὥστε καὶ πλευρὰ ἡ ΔΓ τῇ ΒΓ ἐστιν ἴση· καὶ ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΑΒ, κοινὴ δὲ ἡ ΑΓ, δύο δὴ αἱ ΔΑ, ΑΓ δύο ταῖς ΒΑ, ΑΓ ἴσαι εἰσίν· καὶ βάσις ἡ ΔΓ βάσει τῇ ΒΓ ἴση· γωνία ἄρα ἡ ὑπὸ ΔΑΓ γωνίᾳ τῇ ὑπὸ ΒΑΓ [ἐστιν] ἴση. ὀρθὴ δὲ ἡ ὑπὸ ΔΑΓ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΒΑΓ.

Ἐὰν ἄρα τριγώνου τὸ ἀπὸ μιᾶς τῶν πλευρῶν τετράγωνον ἴσον ᾖ τοῖς ἀπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν τετραγώνοις, ἡ περιεχομένη γωνία ὑπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ὀρθή ἐστιν· ὅπερ ἔδει δεῖξαι.

BOOK II.

Ὅροι β΄.

α΄. Πᾶν παραλληλόγραμμον ὀρθογώνιον περιέχεσθαι λέγεται ὑπὸ δύο τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν εὐθειῶν.

β΄. Παντὸς δὲ παραλληλογράμμου χωρίου τῶν περὶ τὴν διάμετρον αὐτοῦ παραλληλογράμμων ἓν ὁποιονοῦν σὺν τοῖς δυσὶ παραπληρώμασι γνώμων καλείσθω.

Προτάσεις ιδ΄.

α΄. Ἐὰν ᾖσι δύο εὐθεῖαι, τμηθῇ δὲ ἡ ἑτέρα αὐτῶν εἰς ὁσαδηποτοῦν τμήματα, τὸ περιεχόμενον ὀρθογώνιον ὑπὸ τῶν δύο εὐθειῶν ἴσον ἐστὶ τοῖς ὑπό τε τῆς ἀτμήτου καὶ ἑκάστου τῶν τμημάτων περιεχομένοις ὀρθογωνίοις.

Ἔστωσαν δύο εὐθεῖαι αἱ Α, ΒΓ, καὶ τετμήσθω ἡ ΒΓ, ὡς ἔτυχεν, κατὰ τὰ Δ, Ε σημεῖα· λέγω, ὅτι τὸ ὑπὸ τῶν Α, ΒΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ τε ὑπὸ τῶν Α, ΒΔ περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ὑπὸ τῶν Α, ΔΕ καὶ ἔτι τῷ ὑπὸ τῶν Α, ΕΓ.

Ἤχθω γὰρ ἀπὸ τοῦ Β τῇ ΒΓ πρὸς ὀρθὰς ἡ ΒΖ, καὶ κείσθω τῇ Α ἴση ἡ ΒΗ, καὶ διὰ μὲν τοῦ Η τῇ ΒΓ παράλληλος ἤχθω ἡ ΗΘ, διὰ δὲ τῶν Δ, Ε, Γ τῇ ΒΗ παράλληλοι ἤχθωσαν αἱ ΔΚ, ΕΛ, ΓΘ.

Ἴσον δή ἐστι τὸ ΒΘ τοῖς ΒΚ, ΔΛ, ΕΘ. καί ἐστι τὸ μὲν ΒΘ τὸ ὑπὸ τῶν Α, ΒΓ· περιέχεται μὲν γὰρ ὑπὸ τῶν ΗΒ, ΒΓ, ἴση δὲ ἡ ΒΗ τῇ Α· τὸ δὲ ΒΚ τὸ ὑπὸ τῶν Α, ΒΔ· περιέχεται μὲν γὰρ ὑπὸ τῶν ΗΒ, ΒΔ, ἴση δὲ ἡ ΒΗ τῇ Α. τὸ δὲ ΔΛ τὸ ὑπὸ τῶν Α, ΔΕ· ἴση γὰρ ἡ ΔΚ, τουτέστιν ἡ ΒΗ, τῇ Α. καὶ ἔτι ὁμοίως τὸ ΕΘ τὸ ὑπὸ τῶν Α, ΕΓ· τὸ ἄρα ὑπὸ τῶν Α, ΒΓ ἴσον ἐστὶ τῷ τε ὑπὸ Α, ΒΔ καὶ τῷ ὑπὸ Α, ΔΕ καὶ ἔτι τῷ ὑπὸ Α, ΕΓ.

Ἐὰν ἄρα ᾖσι δύο εὐθεῖαι, τμηθῇ δὲ ἡ ἑτέρα αὐτῶν εἰς ὁσαδηποτοῦν τμήματα, τὸ περιεχόμενον ὀρθογώνιον ὑπὸ τῶν δύο εὐθειῶν ἴσον ἐστὶ τοῖς ὑπό τε τῆς ἀτμήτου καὶ ἑκάστου τῶν τμημάτων περιεχομένοις ὀρθογωνίοις· ὅπερ ἔδει δεῖξαι.

β΄. Ἐὰν εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ὑπὸ τῆς ὅλης καὶ ἑκατέρου τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ὅλης τετραγώνῳ.

Εὐθεῖα γὰρ ἡ ΑΒ τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ σημεῖον· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ὑπὸ ΒΑ, ΑΓ

περιεχομένου ὀρθογωνίου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ τετραγώνῳ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔΕΒ, καὶ ἤχθω διὰ τοῦ Γ ὁποτέρᾳ τῶν ΑΔ, ΒΕ παράλληλος ἡ ΓΖ.

Ἴσον δή ἐστι τὸ ΑΕ τοῖς ΑΖ, ΓΕ. καί ἐστι τὸ μὲν ΑΕ τὸ ἀπὸ τῆς ΑΒ τετράγωνον, τὸ δὲ ΑΖ τὸ ὑπὸ τῶν ΒΑ, ΑΓ περιεχόμενον ὀρθογώνιον· περιέχεται μὲν γὰρ ὑπὸ τῶν ΔΑ, ΑΓ, ἴση δὲ ἡ ΑΔ τῇ ΑΒ· τὸ δὲ ΓΕ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· ἴση γὰρ ἡ ΒΕ τῇ ΑΒ. τὸ ἄρα ὑπὸ τῶν ΒΑ, ΑΓ μετὰ τοῦ ὑπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ τετραγώνῳ.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ὑπὸ τῆς ὅλης καὶ ἑκατέρου τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ὅλης τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

γ΄. Ἐὰν εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ὑπὸ τῆς ὅλης καὶ ἑνὸς τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ τε ὑπὸ τῶν τμημάτων περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τοῦ προειρημένου τμήματος τετραγώνῳ.

Εὐθεῖα γὰρ ἡ ΑΒ τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ τε ὑπὸ τῶν ΑΓ, ΓΒ περιεχομένῳ ὀρθογωνίῳ μετὰ τοῦ ἀπὸ τῆς ΒΓ τετραγώνου.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΓΒ τετράγωνον τὸ ΓΔΕΒ, καὶ διήχθω ἡ ΕΔ ἐπὶ τὸ Ζ, καὶ διὰ τοῦ Α ὁποτέρᾳ τῶν ΓΔ, ΒΕ παράλληλος ἤχθω ἡ ΑΖ. ἴσον δή ἐστι τὸ ΑΕ τοῖς ΑΔ, ΓΕ· καί ἐστι τὸ μὲν ΑΕ τὸ ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον· περιέχεται μὲν γὰρ ὑπὸ τῶν ΑΒ, ΒΕ, ἴση δὲ ἡ ΒΕ τῇ ΒΓ· τὸ δὲ ΑΔ τὸ ὑπὸ τῶν ΑΓ, ΓΒ· ἴση γὰρ ἡ ΔΓ τῇ ΓΒ· τὸ δὲ ΔΒ τὸ ἀπὸ τῆς ΓΒ τετράγωνον· τὸ ἄρα ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΓ, ΓΒ περιεχομένῳ ὀρθογωνίῳ μετὰ τοῦ ἀπὸ τῆς ΒΓ τετραγώνου.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ὑπὸ τῆς ὅλης καὶ ἑνὸς τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ τε ὑπὸ τῶν τμημάτων

περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τοῦ προειρημένου τμήματος τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

δ΄. Ἐὰν εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν τμημάτων τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν τμημάτων περιεχομένῳ ὀρθογωνίῳ.

Εὐθεῖα γὰρ γραμμὴ ἡ ΑΒ τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ. λέγω, ὅτι τὸ ἀπὸ τῆς ΑΒ τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν ΑΓ, ΓΒ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ περιεχομένῳ ὀρθογωνίῳ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔΕΒ, καὶ ἐπεζεύχθω ἡ ΒΔ, καὶ διὰ μὲν τοῦ Γ ὁποτέρᾳ τῶν ΑΔ, ΕΒ παράλληλος ἤχθω ἡ ΓΖ, διὰ δὲ τοῦ Η ὁποτέρᾳ τῶν ΑΒ, ΔΕ παράλληλος ἤχθω ἡ ΘΚ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΓΖ τῇ ΑΔ, καὶ εἰς αὐτὰς ἐμπέπτωκεν ἡ ΒΔ, ἡ ἐκτὸς γωνία ἡ ὑπὸ ΓΗΒ ἴση ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΑΔΒ. ἀλλ’ ἡ ὑπὸ ΑΔΒ τῇ ὑπὸ ΑΒΔ ἐστιν ἴση, ἐπεὶ καὶ πλευρὰ ἡ ΒΑ τῇ ΑΔ ἐστιν ἴση· καὶ ἡ ὑπὸ ΓΗΒ ἄρα γωνία τῇ ὑπὸ ΗΒΓ ἐστιν ἴση· ὥστε καὶ πλευρὰ ἡ ΒΓ πλευρᾷ τῇ ΓΗ ἐστιν ἴση· ἀλλ’ ἡ μὲν ΓΒ τῇ ΗΚ ἐστιν ἴση, ἡ δὲ ΓΗ τῇ ΚΒ· καὶ ἡ ΗΚ ἄρα τῇ ΚΒ ἐστιν ἴση· ἰσόπλευρον ἄρα ἐστὶ τὸ ΓΗΚΒ. λέγω δή, ὅτι καὶ ὀρθογώνιον. ἐπεὶ γὰρ παράλληλός ἐστιν ἡ ΓΗ τῇ ΒΚ [καὶ εἰς αὐτὰς ἐμπέπτωκεν εὐθεῖα ἡ ΓΒ], αἱ ἄρα ὑπὸ ΚΒΓ, ΗΓΒ γωνίαι δύο ὀρθαῖς εἰσιν ἴσαι. ὀρθὴ δὲ ἡ ὑπὸ ΚΒΓ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΒΓΗ· ὥστε καὶ αἱ ἀπεναντίον αἱ ὑπὸ ΓΗΚ, ΗΚΒ ὀρθαί εἰσιν. ὀρθογώνιον ἄρα ἐστὶ τὸ ΓΗΚΒ· ἐδείχθη δὲ καὶ ἰσόπλευρον· τετράγωνον ἄρα ἐστίν· καί ἐστιν ἀπὸ τῆς ΓΒ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΘΖ τετράγωνόν ἐστιν· καί ἐστιν ἀπὸ τῆς ΘΗ, τουτέστιν [ἀπὸ] τῆς ΑΓ· τὰ ἄρα ΘΖ, ΚΓ τετράγωνα ἀπὸ τῶν ΑΓ, ΓΒ εἰσιν. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΑΗ τῷ ΗΕ, καί ἐστι τὸ ΑΗ τὸ ὑπὸ τῶν ΑΓ, ΓΒ· ἴση γὰρ ἡ ΗΓ τῇ ΓΒ· καὶ τὸ ΗΕ ἄρα ἴσον ἐστὶ τῷ ὑπὸ ΑΓ, ΓΒ· τὰ ἄρα ΑΗ, ΗΕ ἴσα ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. ἔστι δὲ καὶ τὰ ΘΖ, ΓΚ τετράγωνα ἀπὸ τῶν ΑΓ, ΓΒ· τὰ ἄρα τέσσαρα τὰ ΘΖ, ΓΚ, ΑΗ, ΗΕ ἴσα ἐστὶ

τοῖς τε ἀπὸ τῶν ΑΓ, ΓΒ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ περιεχομένῳ ὀρθογωνίῳ. ἀλλὰ τὰ ΘΖ, ΓΚ, ΑΗ, ΗΕ ὅλον ἐστὶ τὸ ΑΔΕΒ, ὅ ἐστιν ἀπὸ τῆς ΑΒ τετράγωνον· τὸ ἄρα ἀπὸ τῆς ΑΒ τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν ΑΓ, ΓΒ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ περιεχομένῳ ὀρθογωνίῳ.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν τμημάτων τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν τμημάτων περιεχομένῳ ὀρθογωνίῳ· ὅπερ ἔδει δεῖξαι.

[Πόρισμα: Ἐκ δὴ τούτου φανερόν, ὅτι ἐν τοῖς τετραγώνοις χωρίοις τὰ περὶ τὴν διάμετρον παραλληλόγραμμα τετράγωνά ἐστιν].

ε΄. Ἐὰν εὐθεῖα γραμμὴ τμηθῇ εἰς ἴσα καὶ ἄνισα, τὸ ὑπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ἡμισείας τετραγώνῳ.

Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω εἰς μὲν ἴσα κατὰ τὸ Γ, εἰς δὲ ἄνισα κατὰ τὸ Δ· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΔ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΒ τετραγώνῳ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΓΒ τετράγωνον τὸ ΓΕΖΒ, καὶ ἐπεζεύχθω ἡ ΒΕ, καὶ διὰ μὲν τοῦ Δ ὁποτέρᾳ τῶν ΓΕ, ΒΖ παράλληλος ἤχθω ἡ ΔΗ, διὰ δὲ τοῦ Θ ὁποτέρᾳ τῶν ΑΒ, ΕΖ παράλληλος πάλιν ἤχθω ἡ ΚΜ, καὶ πάλιν διὰ τοῦ Α ὁποτέρᾳ τῶν ΓΛ, ΒΜ παράλληλος ἤχθω ἡ ΑΚ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΓΘ παραπλήρωμα τῷ ΘΖ παραπληρώματι, κοινὸν προσκείσθω τὸ ΔΜ· ὅλον ἄρα τὸ ΓΜ ὅλῳ τῷ ΔΖ ἴσον ἐστίν. ἀλλὰ τὸ ΓΜ τῷ ΑΛ ἴσον ἐστίν, ἐπεὶ καὶ ἡ ΑΓ τῇ ΓΒ ἐστιν ἴση· καὶ τὸ ΑΛ ἄρα τῷ ΔΖ ἴσον ἐστίν. κοινὸν προσκείσθω τὸ ΓΘ· ὅλον ἄρα τὸ ΑΘ τῷ ΜΝΞ γνώμονι ἴσον ἐστίν. ἀλλὰ τὸ ΑΘ τὸ ὑπὸ τῶν ΑΔ, ΔΒ ἐστιν· ἴση γὰρ ἡ ΔΘ τῇ ΔΒ· καὶ ὁ ΜΝΞ ἄρα γνώμων ἴσος ἐστὶ τῷ ὑπὸ ΑΔ, ΔΒ. κοινὸν προσκείσθω τὸ ΛΗ, ὅ ἐστιν ἴσον τῷ ἀπὸ τῆς ΓΔ· ὁ ἄρα ΜΝΞ γνώμων καὶ τὸ ΛΗ ἴσα ἐστὶ τῷ ὑπὸ τῶν ΑΔ, ΔΒ περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τῆς ΓΔ τετραγώνῳ. ἀλλὰ ὁ ΜΝΞ γνώμων καὶ τὸ ΛΗ

ὅλον ἐστὶ τὸ ΓΕΖΒ τετράγωνον, ὅ ἐστιν ἀπὸ τῆς ΓΒ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΔ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΒ τετραγώνῳ.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ εἰς ἴσα καὶ ἄνισα, τὸ ὑπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ἡμισείας τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

ς΄. Ἐὰν εὐθεῖα γραμμὴ τμηθῇ δίχα, προστεθῇ δέ τις αὐτῇ εὐθεῖα ἐπ᾽ εὐθείας, τὸ ὑπὸ τῆς ὅλης σὺν τῇ προσκειμένῃ καὶ τῆς προσκειμένης περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς συγκειμένης ἔκ τε τῆς ἡμισείας καὶ τῆς προσκειμένης τετραγώνῳ.

Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω δίχα κατὰ τὸ Γ σημεῖον, προσκείσθω δέ τις αὐτῇ εὐθεῖα ἐπ᾽ εὐθείας ἡ ΒΔ· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΒ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΔ τετραγώνῳ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΓΔ τετράγωνον τὸ ΓΕΖΔ, καὶ ἐπεζεύχθω ἡ ΔΕ, καὶ διὰ μὲν τοῦ Β σημείου ὁποτέρᾳ τῶν ΕΓ, ΔΖ παράλληλος ἤχθω ἡ ΒΗ, διὰ δὲ τοῦ Θ σημείου ὁποτέρᾳ τῶν ΑΒ, ΕΖ παράλληλος ἤχθω ἡ ΚΜ, καὶ ἔτι διὰ τοῦ Α ὁποτέρᾳ τῶν ΓΛ, ΔΜ παράλληλος ἤχθω ἡ ΑΚ.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΑΓ τῇ ΓΒ, ἴσον ἐστὶ καὶ τὸ ΑΛ τῷ ΓΘ. ἀλλὰ τὸ ΓΘ τῷ ΘΖ ἴσον ἐστίν. καὶ τὸ ΑΛ ἄρα τῷ ΘΖ ἐστιν ἴσον. κοινὸν προσκείσθω τὸ ΓΜ· ὅλον ἄρα τὸ ΑΜ τῷ ΝΞΟ γνώμονί ἐστιν ἴσον. ἀλλὰ τὸ ΑΜ ἐστι τὸ ὑπὸ τῶν ΑΔ, ΔΒ· ἴση γάρ ἐστιν ἡ ΔΜ τῇ ΔΒ· καὶ ὁ ΝΞΟ ἄρα γνώμων ἴσος ἐστὶ τῷ ὑπὸ τῶν ΑΔ, ΔΒ [περιεχομένῳ ὀρθογωνίῳ]. κοινὸν προσκείσθω τὸ ΛΗ, ὅ ἐστιν ἴσον τῷ ἀπὸ τῆς ΒΓ τετραγώνῳ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΒ τετραγώνου ἴσον ἐστὶ τῷ ΝΞΟ γνώμονι καὶ τῷ ΛΗ. ἀλλὰ ὁ ΝΞΟ γνώμων καὶ τὸ ΛΗ ὅλον ἐστὶ τὸ ΓΕΖΔ τετράγωνον, ὅ

ἐστιν ἀπὸ τῆς ΓΔ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΒ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΔ τετραγώνῳ.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ δίχα, προστεθῇ δέ τις αὐτῇ εὐθεῖα ἐπ' εὐθείας, τὸ ὑπὸ τῆς ὅλης σὺν τῇ προσκειμένῃ καὶ τῆς προσκειμένης περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς συγκειμένης ἔκ τε τῆς ἡμισείας καὶ τῆς προσκειμένης τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης καὶ τὸ ἀφ' ἑνὸς τῶν τμημάτων τὰ συναμφότερα τετράγωνα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ.

Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ σημεῖον· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῶν ΑΒ, ΒΓ περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τῆς ΓΑ τετραγώνῳ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔΕΒ· καὶ καταγεγράφθω τὸ σχῆμα.

Ἐπεὶ οὖν ἴσον ἐστὶ τὸ ΑΗ τῷ ΗΕ, κοινὸν προσκείσθω τὸ ΓΖ· ὅλον ἄρα τὸ ΑΖ ὅλῳ τῷ ΓΕ ἴσον ἐστίν· τὰ ἄρα ΑΖ, ΓΕ διπλάσιά ἐστι τοῦ ΑΖ. ἀλλὰ τὰ ΑΖ, ΓΕ ὁ ΚΛΜ ἐστι γνώμων καὶ τὸ ΓΖ τετράγωνον· ὁ ΚΛΜ ἄρα γνώμων καὶ τὸ ΓΖ διπλάσιά ἐστι τοῦ ΑΖ. ἔστι δὲ τοῦ ΑΖ διπλάσιον καὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· ἴση γὰρ ἡ ΒΖ τῇ ΒΓ· ὁ ἄρα ΚΛΜ γνώμων καὶ τὸ ΓΖ τετράγωνον ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. κοινὸν προσκείσθω τὸ ΔΗ, ὅ ἐστιν ἀπὸ τῆς ΑΓ τετράγωνον· ὁ ἄρα ΚΛΜ γνώμων καὶ τὰ ΒΗ, ΗΔ τετράγωνα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῶν ΑΒ, ΒΓ περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τῆς ΑΓ τετραγώνῳ. ἀλλὰ ὁ ΚΛΜ γνώμων καὶ τὰ ΒΗ, ΗΔ τετράγωνα ὅλον ἐστὶ τὸ ΑΔΕΒ καὶ τὸ ΓΖ, ἅ ἐστιν ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα· τὰ ἄρα ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα ἴσα ἐστὶ τῷ [τε] δὶς ὑπὸ τῶν ΑΒ, ΒΓ περιεχομένῳ ὀρθογωνίῳ μετὰ τοῦ ἀπὸ τῆς ΑΓ τετραγώνου.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης καὶ τὸ ἀφ' ἑνὸς τῶν τμημάτων τὰ συναμφότερα τετράγωνα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

ζ΄. Ἐὰν εὐθεῖα γραμμή τμηθῇ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης καὶ τὸ ἀφ' ἑνὸς τῶν τμημάτων τὰ συναμφότερα τετράγωνα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ.

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Ἐπεὶ οὖν ἴσον ἐστὶ τὸ ΑΗ τῷ ΗΕ, κοινὸν προσκείσθω τὸ ΓΖ· ὅλον ἄρα τὸ ΑΖ ὅλῳ τῷ ΓΕ ἴσον ἐστίν· τὰ ἄρα ΑΖ, ΓΕ διπλάσιά ἐστι τοῦ ΑΖ. ἀλλὰ τὰ ΑΖ, ΓΕ ὁ ΚΛΜ ἐστι γνώμων καὶ τὸ ΓΖ τετράγωνον· ὁ ΚΛΜ ἄρα γνώμων καὶ τὸ ΓΖ διπλάσιά ἐστι τοῦ ΑΖ. ἔστι δὲ τοῦ ΑΖ διπλάσιον καὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· ἴση γὰρ ἡ ΒΖ τῇ ΒΓ· ὁ ἄρα ΚΛΜ γνώμων καὶ τὸ ΓΖ τετράγωνον ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. κοινὸν προσκείσθω τὸ ΔΗ, ὅ ἐστιν ἀπὸ τῆς ΑΓ τετράγωνον· ὁ ἄρα ΚΛΜ γνώμων καὶ τὰ ΒΗ, ΗΔ τετράγωνα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῶν ΑΒ, ΒΓ περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τῆς ΑΓ τετραγώνῳ. ἀλλὰ ὁ ΚΛΜ γνώμων καὶ τὰ ΒΗ, ΗΔ τετράγωνα ὅλον ἐστὶ τὸ ΑΔΕΒ καὶ τὸ ΓΖ, ἅ ἐστιν ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα· τὰ ἄρα ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα ἴσα ἐστὶ τῷ [τε] δὶς ὑπὸ τῶν ΑΒ, ΒΓ περιεχομένῳ ὀρθογωνίῳ μετὰ τοῦ ἀπὸ τῆς ΑΓ τετραγώνου.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης καὶ τὸ ἀφ' ἑνὸς τῶν τμημάτων τὰ συναμφότερα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῆς ὅλης καὶ τοῦ

εἰρημένου τμήματος περιεχομένω ὀρθογωνίῳ καὶ τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

η΄. Ἐὰν εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ τετράκις ὑπὸ τῆς ὅλης καὶ ἑνὸς τῶν τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνου ἴσον ἐστὶ τῷ ἀπό τε τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος ὡς ἀπὸ μιᾶς ἀναγραφέντι τετραγώνῳ.

Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ σημεῖον· λέγω, ὅτι τὸ τετράκις ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΑΓ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ, ΒΓ ὡς ἀπὸ μιᾶς ἀναγραφέντι τετραγώνῳ.

Ἐκβεβλήσθω γὰρ ἐπ᾽ εὐθείας [τῇ ΑΒ εὐθεῖα] ἡ ΒΔ, καὶ κείσθω τῇ ΓΒ ἴση ἡ ΒΔ, καὶ ἀναγεγράφθω ἀπὸ τῆς ΑΔ τετράγωνον τὸ ΑΕΖΔ, καὶ καταγεγράφθω διπλοῦν τὸ σχῆμα.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΓΒ τῇ ΒΔ, ἀλλὰ ἡ μὲν ΓΒ τῇ ΗΚ ἐστιν ἴση, ἡ δὲ ΒΔ τῇ ΚΝ, καὶ ἡ ΗΚ ἄρα τῇ ΚΝ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΡΠ τῇ ΡΟ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΓ τῇ ΒΔ, ἡ δὲ ΗΚ τῇ ΚΝ, ἴσον ἄρα ἐστὶ καὶ τὸ μὲν ΓΚ τῷ ΚΔ, τὸ δὲ ΗΡ τῷ ΡΝ. ἀλλὰ τὸ ΓΚ τῷ ΡΝ ἐστιν ἴσον· παραπληρώματα γὰρ τοῦ ΓΟ παραλληλογράμμου· καὶ τὸ ΚΔ ἄρα τῷ ΗΡ ἴσον ἐστίν· τὰ τέσσαρα ἄρα τὰ ΔΚ, ΓΚ, ΗΡ, ΡΝ ἴσα ἀλλήλοις ἐστίν. τὰ τέσσαρα ἄρα τετραπλάσιά ἐστι τοῦ ΓΚ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΓΒ τῇ ΒΔ, ἀλλὰ ἡ μὲν ΒΔ τῇ ΒΚ, τουτέστι τῇ ΓΗ ἴση, ἡ δὲ ΓΒ τῇ ΗΚ, τουτέστι τῇ ΗΠ, ἐστιν ἴση, καὶ ἡ ΓΗ ἄρα τῇ ΗΠ ἴση ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΓΗ τῇ ΗΠ, ἡ δὲ ΡΠ τῇ ΡΟ, ἴσον ἐστὶ καὶ τὸ μὲν ΑΗ τῷ ΜΠ, τὸ δὲ ΠΛ τῷ ΡΖ. ἀλλὰ τὸ ΜΠ τῷ ΠΛ ἐστιν ἴσον· παραπληρώματα γὰρ τοῦ ΜΛ παραλληλογράμμου· καὶ τὸ ΑΗ ἄρα τῷ ΡΖ ἴσον ἐστίν· τὰ τέσσαρα ἄρα τὰ ΑΗ, ΜΠ, ΠΛ, ΡΖ ἴσα ἀλλήλοις ἐστίν· τὰ τέσσαρα ἄρα τοῦ ΑΗ ἐστι τετραπλάσια. ἐδείχθη δὲ καὶ τὰ τέσσαρα τὰ ΓΚ, ΚΔ, ΗΡ, ΡΝ τοῦ ΓΚ τετραπλάσια· τὰ ἄρα ὀκτώ, ἃ περιέχει τὸν ΣΤΥ

γνώμονα, τετραπλάσιά ἐστι τοῦ ΑΚ. καὶ ἐπεὶ τὸ ΑΚ τὸ ὑπὸ τῶν ΑΒ, ΒΔ ἐστιν· ἴση γὰρ ἡ ΒΚ τῇ ΒΔ· τὸ ἄρα τετράκις ὑπὸ τῶν ΑΒ, ΒΔ τετραπλάσιόν ἐστι τοῦ ΑΚ. ἐδείχθη δὲ τοῦ ΑΚ τετραπλάσιος καὶ ὁ ΣΤΥ γνώμων· τὸ ἄρα τετράκις ὑπὸ τῶν ΑΒ, ΒΔ ἴσον ἐστὶ τῷ ΣΤΥ γνώμονι. κοινὸν προσκείσθω τὸ ΞΘ, ὅ ἐστιν ἴσον τῷ ἀπὸ τῆς ΑΓ τετραγώνῳ· τὸ ἄρα τετράκις ὑπὸ τῶν ΑΒ, ΒΔ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ ΑΓ τετραγώνου ἴσον ἐστὶ τῷ ΣΤΥ γνώμονι καὶ τῷ ΞΘ. ἀλλὰ ὁ ΣΤΥ γνώμων καὶ τὸ ΞΘ ὅλον ἐστὶ τὸ ΑΕΖΔ τετράγωνον, ὅ ἐστιν ἀπὸ τῆς ΑΔ· τὸ ἄρα τετράκις ὑπὸ τῶν ΑΒ, ΒΔ μετὰ τοῦ ἀπὸ ΑΓ ἴσον ἐστὶ τῷ ἀπὸ ΑΔ τετραγώνῳ· ἴση δὲ ἡ ΒΔ τῇ ΒΓ. τὸ ἄρα τετράκις ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ ΑΓ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΔ, τουτέστι τῷ ἀπὸ τῆς ΑΒ καὶ ΒΓ ὡς ἀπὸ μιᾶς ἀναγραφέντι τετραγώνῳ.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ τετράκις ὑπὸ τῆς ὅλης καὶ ἑνὸς τῶν τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνου ἴσον ἐστὶ τῷ ἀπό τε τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος ὡς ἀπὸ μιᾶς ἀναγραφέντι τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

θ΄. Ἐὰν εὐθεῖα γραμμὴ τμηθῇ εἰς ἴσα καὶ ἄνισα, τὰ ἀπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου.

Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω εἰς μὲν ἴσα κατὰ τὸ Γ, εἰς δὲ ἄνισα κατὰ τὸ Δ· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΔ, ΔΒ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων.

Ἤχθω γὰρ ἀπὸ τοῦ Γ τῇ ΑΒ πρὸς ὀρθὰς ἡ ΓΕ, καὶ κείσθω ἴση ἑκατέρα τῶν ΑΓ, ΓΒ, καὶ ἐπεζεύχθωσαν αἱ ΕΑ, ΕΒ, καὶ διὰ μὲν τοῦ Δ τῇ ΕΓ παράλληλος ἤχθω ἡ ΔΖ, διὰ δὲ τοῦ Ζ τῇ ΑΒ ἡ ΖΗ, καὶ ἐπεζεύχθω ἡ ΑΖ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΕ, ἴση ἐστὶ καὶ ἡ ὑπὸ ΕΑΓ γωνία τῇ ὑπὸ ΑΕΓ. καὶ ἐπεὶ ὀρθή ἐστιν ἡ πρὸς τῷ Γ, λοιπαὶ ἄρα αἱ ὑπὸ ΕΑΓ, ΑΕΓ μιᾷ ὀρθῇ ἴσαι

εἰσίν· καί εἰσιν ἴσαι· ἡμίσεια ἄρα ὀρθῆς ἐστιν ἑκατέρα τῶν ὑπὸ ΓΕΑ, ΓΑΕ. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ὑπὸ ΓΕΒ, ΕΒΓ ἡμίσειά ἐστιν ὀρθῆς· ὅλη ἄρα ἡ ὑπὸ ΑΕΒ ὀρθή ἐστιν. καὶ ἐπεὶ ἡ ὑπὸ ΗΕΖ ἡμίσειά ἐστιν ὀρθῆς, ὀρθὴ δὲ ἡ ὑπὸ ΕΗΖ· ἴση γάρ ἐστι τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΕΓΒ· λοιπὴ ἄρα ἡ ὑπὸ ΕΖΗ ἡμίσειά ἐστιν ὀρθῆς· ἴση ἄρα [ἐστὶν] ἡ ὑπὸ ΗΕΖ γωνία τῇ ὑπὸ ΕΖΗ· ὥστε καὶ πλευρὰ ἡ ΕΗ τῇ ΗΖ ἐστιν ἴση. πάλιν ἐπεὶ ἡ πρὸς τῷ Β γωνία ἡμίσειά ἐστιν ὀρθῆς, ὀρθὴ δὲ ἡ ὑπὸ ΖΔΒ· ἴση γὰρ πάλιν ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΕΓΒ· λοιπὴ ἄρα ἡ ὑπὸ ΒΖΔ ἡμίσειά ἐστιν ὀρθῆς· ἴση ἄρα ἡ πρὸς τῷ Β γωνία τῇ ὑπὸ ΔΖΒ· ὥστε καὶ πλευρὰ ἡ ΖΔ πλευρᾷ τῇ ΔΒ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΕ, ἴσον ἐστὶ καὶ τὸ ἀπὸ ΑΓ τῷ ἀπὸ ΓΕ· τὰ ἄρα ἀπὸ τῶν ΑΓ, ΓΕ τετράγωνα διπλάσιά ἐστι τοῦ ἀπὸ ΑΓ. τοῖς δὲ ἀπὸ τῶν ΑΓ, ΓΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΑ τετράγωνον· ὀρθὴ γὰρ ἡ ὑπὸ ΑΓΕ γωνία· τὸ ἄρα ἀπὸ τῆς ΕΑ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΑΓ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΕΗ τῇ ΗΖ, ἴσον καὶ τὸ ἀπὸ τῆς ΕΗ τῷ ἀπὸ τῆς ΗΖ· τὰ ἄρα ἀπὸ τῶν ΕΗ, ΗΖ τετράγωνα διπλάσιά ἐστι τοῦ ἀπὸ τῆς ΗΖ τετραγώνου. τοῖς δὲ ἀπὸ τῶν ΕΗ, ΗΖ τετραγώνοις ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΖ τετράγωνον· τὸ ἄρα ἀπὸ τῆς ΕΖ τετράγωνον διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΗΖ. ἴση δὲ ἡ ΗΖ τῇ ΓΔ· τὸ ἄρα ἀπὸ τῆς ΕΖ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΓΔ. ἔστι δὲ καὶ τὸ ἀπὸ τῆς ΕΑ διπλάσιον τοῦ ἀπὸ τῆς ΑΓ· τὰ ἄρα ἀπὸ τῶν ΑΕ, ΕΖ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων. τοῖς δὲ ἀπὸ τῶν ΑΕ, ΕΖ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΑΖ τετράγωνον· ὀρθὴ γάρ ἐστιν ἡ ὑπὸ ΑΕΖ γωνία· τὸ ἄρα ἀπὸ τῆς ΑΖ τετράγωνον διπλάσιόν ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ. τῷ δὲ ἀπὸ τῆς ΑΖ ἴσα τὰ ἀπὸ τῶν ΑΔ, ΔΖ· ὀρθὴ γὰρ ἡ πρὸς τῷ Δ γωνία· τὰ ἄρα ἀπὸ τῶν ΑΔ, ΔΖ διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων. ἴση δὲ ἡ ΔΖ τῇ ΔΒ· τὰ ἄρα ἀπὸ τῶν ΑΔ, ΔΒ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ εἰς ἴσα καὶ ἄνισα, τὰ ἀπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου· ὅπερ ἔδει δεῖξαι.

ι΄. Ἐὰν εὐθεῖα γραμμὴ τμηθῇ δίχα, προστεθῇ δέ τις αὐτῇ εὐθεῖα ἐπ᾽ εὐθείας, τὸ ἀπὸ τῆς ὅλης σὺν τῇ προσκειμένῃ καὶ τὸ ἀπὸ τῆς προσκειμένης τὰ συναμφότερα τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς συγκειμένης ἔκ τε τῆς ἡμισείας καὶ τῆς προσκειμένης ὡς ἀπὸ μιᾶς ἀναγραφέντος τετραγώνου.

Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω δίχα κατὰ τὸ Γ, προσκείσθω δέ τις αὐτῇ εὐθεῖα ἐπ᾽ εὐθείας ἡ ΒΔ· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΔ, ΔΒ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων.

Ἤχθω γὰρ ἀπὸ τοῦ Γ σημείου τῇ ΑΒ πρὸς ὀρθὰς ἡ ΓΕ, καὶ κείσθω ἴση ἑκατέρᾳ, τῶν ΑΓ, ΓΒ, καὶ ἐπεζεύχθωσαν αἱ ΕΑ, ΕΒ· καὶ διὰ μὲν τοῦ Ε τῇ ΑΔ παράλληλος ἤχθω ἡ ΕΖ, διὰ δὲ τοῦ Δ τῇ ΓΕ παράλληλος ἤχθω ἡ ΖΔ. καὶ ἐπεὶ εἰς παραλλήλους εὐθείας τὰς ΕΓ, ΖΔ εὐθεῖά τις ἐνέπεσεν ἡ ΕΖ, αἱ ὑπὸ ΓΕΖ, ΕΖΔ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσίν· αἱ ἄρα ὑπὸ ΖΕΒ, ΕΖΔ δύο ὀρθῶν ἐλάσσονές εἰσιν· αἱ δὲ ἀπ᾽ ἐλασσόνων ἢ δύο ὀρθῶν ἐκβαλλόμεναι συμπίπτουσιν· αἱ ἄρα ΕΒ, ΖΔ ἐκβαλλόμεναι ἐπὶ τὰ Β, Δ μέρη συμπεσοῦνται. ἐκβεβλήσθωσαν καὶ συμπιπτέτωσαν κατὰ τὸ Η, καὶ ἐπεζεύχθω ἡ ΑΗ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΕ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΕΑΓ τῇ ὑπὸ ΑΕΓ· καὶ ὀρθὴ ἡ πρὸς τῷ Γ· ἡμίσεια ἄρα ὀρθῆς [ἐστιν] ἑκατέρα τῶν ὑπὸ ΕΑΓ, ΑΕΓ. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ὑπὸ ΓΕΒ, ΕΒΓ ἡμίσειά ἐστιν ὀρθῆς· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΑΕΒ. καὶ ἐπεὶ ἡμίσεια ὀρθῆς ἐστιν ἡ ὑπὸ ΕΒΓ, ἡμίσεια ἄρα ὀρθῆς καὶ ἡ ὑπὸ ΔΒΗ. ἔστι δὲ καὶ ἡ ὑπὸ ΒΔΗ ὀρθή· ἴση γάρ ἐστι τῇ ὑπὸ ΔΓΕ· ἐναλλὰξ γάρ· λοιπὴ ἄρα ἡ ὑπὸ ΔΗΒ ἡμίσειά ἐστιν ὀρθῆς· ἡ ἄρα ὑπὸ ΔΗΒ τῇ ὑπὸ ΔΒΗ ἐστιν ἴση· ὥστε καὶ πλευρὰ ἡ ΒΔ πλευρᾷ τῇ ΗΔ ἐστιν ἴση. πάλιν, ἐπεὶ ἡ ὑπὸ ΕΗΖ ἡμίσειά ἐστιν ὀρθῆς, ὀρθὴ δὲ ἡ πρὸς τῷ Ζ· ἴση γάρ ἐστι τῇ ἀπεναντίον τῇ πρὸς τῷ Γ· λοιπὴ ἄρα ἡ ὑπὸ ΖΕΗ ἡμίσειά ἐστιν ὀρθῆς· ἴση ἄρα ἡ ὑπὸ ΕΗΖ γωνία τῇ ὑπὸ ΖΕΗ· ὥστε καὶ πλευρὰ ἡ ΗΖ πλευρᾷ τῇ ΕΖ ἐστιν ἴση. καὶ ἐπεὶ [ἴση ἐστὶν ἡ ΕΓ τῇ ΓΑ,] ἴσον ἐστὶ [καὶ] τὸ ἀπὸ τῆς ΕΓ τετράγωνον τῷ ἀπὸ τῆς ΓΑ τετραγώνῳ· τὰ ἄρα ἀπὸ τῶν ΕΓ, ΓΑ τετράγωνα

διπλάσιά ἐστι τοῦ ἀπὸ τῆς ΓΑ τετραγώνου. τοῖς δὲ ἀπὸ τῶν ΕΓ, ΓΑ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΑ· τὸ ἄρα ἀπὸ τῆς ΕΑ τετράγωνον διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΑΓ τετραγώνου. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΖΗ τῇ ΕΖ, ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΖΕ· τὰ ἄρα ἀπὸ τῶν ΗΖ, ΖΕ διπλάσιά ἐστι τοῦ ἀπὸ τῆς ΕΖ. τοῖς δὲ ἀπὸ τῶν ΗΖ, ΖΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΗ· τὸ ἄρα ἀπὸ τῆς ΕΗ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΖ. ἴση δὲ ἡ ΕΖ τῇ ΓΔ· τὸ ἄρα ἀπὸ τῆς ΕΗ τετράγωνον διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΓΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΕΑ διπλάσιον τοῦ ἀπὸ τῆς ΑΓ· τὰ ἄρα ἀπὸ τῶν ΑΕ, ΕΗ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων. τοῖς δὲ ἀπὸ τῶν ΑΕ, ΕΗ τετραγώνοις ἴσον ἐστὶ τὸ ἀπὸ τῆς ΑΗ τετράγωνον· τὸ ἄρα ἀπὸ τῆς ΑΗ διπλάσιόν ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ. τῷ δὲ ἀπὸ τῆς ΑΗ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΑΔ, ΔΗ· τὰ ἄρα ἀπὸ τῶν ΑΔ, ΔΗ [τετράγωνα] διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ [τετραγώνων]. ἴση δὲ ἡ ΔΗ τῇ ΔΒ· τὰ ἄρα ἀπὸ τῶν ΑΔ, ΔΒ [τετράγωνα] διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ δίχα, προστεθῇ δέ τις αὐτῇ εὐθεῖα ἐπ' εὐθείας, τὸ ἀπὸ τῆς ὅλης σὺν τῇ προσκειμένῃ καὶ τὸ ἀπὸ τῆς προσκειμένης τὰ συναμφότερα τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς συγκειμένης ἔκ τε τῆς ἡμισείας καὶ τῆς προσκειμένης ὡς ἀπὸ μιᾶς ἀναγραφέντος τετραγώνου· ὅπερ ἔδει δεῖξαι.

ια΄. Τὴν δοθεῖσαν εὐθεῖαν τεμεῖν ὥστε τὸ ὑπὸ τῆς ὅλης καὶ τοῦ ἑτέρου τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον εἶναι τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ.

Ἔστω ἡ δοθεῖσα εὐθεῖα ἡ ΑΒ· δεῖ δὴ τὴν ΑΒ τεμεῖν ὥστε τὸ ὑπὸ τῆς ὅλης καὶ τοῦ ἑτέρου τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον εἶναι τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΒΔΓ, καὶ τετμήσθω ἡ ΑΓ δίχα κατὰ τὸ Ε σημεῖον, καὶ ἐπεζεύχθω ἡ ΒΕ, καὶ διήχθω ἡ ΓΑ ἐπὶ τὸ Ζ, καὶ

κείσθω τῇ ΒΕ ἴση ἡ ΕΖ, καὶ ἀναγεγράφθω ἀπὸ τῆς ΑΖ τετράγωνον τὸ ΖΘ, καὶ διήχθω ἡ ΗΘ ἐπὶ τὸ Κ· λέγω, ὅτι ἡ ΑΒ τέτμηται κατὰ τὸ Θ, ὥστε τὸ ὑπὸ τῶν ΑΒ, ΒΘ περιεχόμενον ὀρθογώνιον ἴσον ποιεῖν τῷ ἀπὸ τῆς ΑΘ τετραγώνῳ.

Ἐπεὶ γὰρ εὐθεῖα ἡ ΑΓ τέτμηται δίχα κατὰ τὸ Ε, πρόσκειται δὲ αὐτῇ ἡ ΖΑ, τὸ ἄρα ὑπὸ τῶν ΓΖ, ΖΑ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΑΕ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΖ τετραγώνῳ. ἴση δὲ ἡ ΕΖ τῇ ΕΒ· τὸ ἄρα ὑπὸ τῶν ΓΖ, ΖΑ μετὰ τοῦ ἀπὸ τῆς ΑΕ ἴσον ἐστὶ τῷ ἀπὸ ΕΒ. ἀλλὰ τῷ ἀπὸ ΕΒ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΒΑ, ΑΕ· ὀρθὴ γὰρ ἡ πρὸς τῷ Α γωνία· τὸ ἄρα ὑπὸ τῶν ΓΖ, ΖΑ μετὰ τοῦ ἀπὸ τῆς ΑΕ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΕ. κοινὸν ἀφῃρήσθω τὸ ἀπὸ τῆς ΑΕ· λοιπὸν ἄρα τὸ ὑπὸ τῶν ΓΖ, ΖΑ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ τετραγώνῳ. καί ἐστι τὸ μὲν ὑπὸ τῶν ΓΖ, ΖΑ τὸ ΖΚ· ἴση γὰρ ἡ ΑΖ τῇ ΖΗ· τὸ δὲ ἀπὸ τῆς ΑΒ τὸ ΑΔ· τὸ ἄρα ΖΚ ἴσον ἐστὶ τῷ ΑΔ. κοινὸν ἀφῃρήσθω τὸ ΑΚ· λοιπὸν ἄρα τὸ ΖΘ τῷ ΘΔ ἴσον ἐστίν. καί ἐστι τὸ μὲν ΘΔ τὸ ὑπὸ τῶν ΑΒ, ΒΘ· ἴση γὰρ ἡ ΑΒ τῇ ΒΔ· τὸ δὲ ΖΘ τὸ ἀπὸ τῆς ΑΘ· τὸ ἄρα ὑπὸ τῶν ΑΒ, ΒΘ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ ΘΑ τετραγώνῳ.

Ἡ ἄρα δοθεῖσα εὐθεῖα ἡ ΑΒ τέτμηται κατὰ τὸ Θ ὥστε τὸ ὑπὸ τῶν ΑΒ, ΒΘ περιεχόμενον ὀρθογώνιον ἴσον ποιεῖν τῷ ἀπὸ τῆς ΘΑ τετραγώνῳ· ὅπερ ἔδει ποιῆσαι.

ιβ΄. Ἐν τοῖς ἀμβλυγωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ἀμβλεῖαν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον μεῖζόν ἐστι τῶν ἀπὸ τῶν τὴν ἀμβλεῖαν γωνίαν περιεχουσῶν πλευρῶν τετραγώνων τῷ περιεχομένῳ δὶς ὑπό τε μιᾶς τῶν περὶ τὴν ἀμβλεῖαν γωνίαν, ἐφ᾽ ἣν ἡ κάθετος πίπτει, καὶ τῆς ἀπολαμβανομένης ἐκτὸς ὑπὸ τῆς καθέτου πρὸς τῇ ἀμβλείᾳ γωνίᾳ.

Ἔστω ἀμβλυγώνιον τρίγωνον τὸ ΑΒΓ ἀμβλεῖαν ἔχον τὴν ὑπὸ ΒΑΓ, καὶ ἤχθω ἀπὸ τοῦ Β σημείου ἐπὶ τὴν ΓΑ ἐκβληθεῖσαν κάθετος ἡ ΒΔ. λέγω, ὅτι τὸ

ἀπὸ τῆς ΒΓ τετράγωνον μεῖζόν ἐστι τῶν ἀπὸ τῶν ΒΑ, ΑΓ τετραγώνων τῷ δὶς ὑπὸ τῶν ΓΑ, ΑΔ περιεχομένῳ ὀρθογωνίῳ.

Ἐπεὶ γὰρ εὐθεῖα ἡ ΓΑ τέτμηται, ὡς ἔτυχεν, κατὰ τὸ Α σημεῖον, τὸ ἄρα ἀπὸ τῆς ΔΓ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΓΑ, ΑΔ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΓΑ, ΑΔ περιεχομένῳ ὀρθογωνίῳ. κοινὸν προσκείσθω τὸ ἀπὸ τῆς ΔΒ· τὰ ἄρα ἀπὸ τῶν ΓΔ, ΔΒ ἴσα ἐστὶ τοῖς τε ἀπὸ τῶν ΓΑ, ΑΔ, ΔΒ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΓΑ, ΑΔ [περιεχομένῳ ὀρθογωνίῳ]. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΓΔ, ΔΒ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΓΒ· ὀρθὴ γὰρ ἡ πρὸς τῷ Δ γωνία· τοῖς δὲ ἀπὸ τῶν ΑΔ, ΔΒ ἴσον τὸ ἀπὸ τῆς ΑΒ· τὸ ἄρα ἀπὸ τῆς ΓΒ τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν ΓΑ, ΑΒ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΓΑ, ΑΔ περιεχομένῳ ὀρθογωνίῳ· ὥστε τὸ ἀπὸ τῆς ΓΒ τετράγωνον τῶν ἀπὸ τῶν ΓΑ, ΑΒ τετραγώνων μεῖζόν ἐστι τῷ δὶς ὑπὸ τῶν ΓΑ, ΑΔ περιεχομένῳ ὀρθογωνίῳ.

Ἐν ἄρα τοῖς ἀμβλυγωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ἀμβλεῖαν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον μεῖζόν ἐστι τῶν ἀπὸ τῶν τὴν ἀμβλεῖαν γωνίαν περιεχουσῶν πλευρῶν τετραγώνων τῷ περιεχομένῳ δὶς ὑπό τε μιᾶς τῶν περὶ τὴν ἀμβλεῖαν γωνίαν, ἐφ’ ἣν ἡ κάθετος πίπτει, καὶ τῆς ἀπολαμβανομένης ἐκτὸς ὑπὸ τῆς καθέτου πρὸς τῇ ἀμβλείᾳ γωνίᾳ· ὅπερ ἔδει δεῖξαι.

ιγ΄. Ἐν τοῖς ὀξυγωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀξεῖαν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἔλαττόν ἐστι τῶν ἀπὸ τῶν τὴν ὀξεῖαν γωνίαν περιεχουσῶν πλευρῶν τετραγώνων τῷ περιεχομένῳ δὶς ὑπό τε μιᾶς τῶν περὶ τὴν ὀξεῖαν γωνίαν, ἐφ’ ἣν ἡ κάθετος πίπτει, καὶ τῆς ἀπολαμβανομένης ἐντὸς ὑπὸ τῆς καθέτου πρὸς τῇ ὀξείᾳ γωνίᾳ.

Ἔστω ὀξυγώνιον τρίγωνον τὸ ΑΒΓ ὀξεῖαν ἔχον τὴν πρὸς τῷ Β γωνίαν, καὶ ἤχθω ἀπὸ τοῦ Α σημείου ἐπὶ τὴν ΒΓ κάθετος ἡ ΑΔ· λέγω, ὅτι τὸ ἀπὸ τῆς ΑΓ τετράγωνον ἔλαττόν ἐστι τῶν ἀπὸ τῶν ΓΒ, ΒΑ τετραγώνων τῷ δὶς ὑπὸ τῶν ΓΒ, ΒΔ περιεχομένῳ ὀρθογωνίῳ.

Ἐπεὶ γὰρ εὐθεῖα ἡ ΓΒ τέτμηται, ὡς ἔτυχεν, κατὰ τὸ Δ, τὰ ἄρα ἀπὸ τῶν ΓΒ, ΒΔ τετράγωνα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῶν ΓΒ, ΒΔ περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τῆς ΔΓ τετραγώνῳ. κοινὸν προσκείσθω τὸ ἀπὸ τῆς ΔΑ τετράγωνον· τὰ ἄρα ἀπὸ τῶν ΓΒ, ΒΔ, ΔΑ τετράγωνα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῶν ΓΒ, ΒΔ περιεχομένῳ ὀρθογωνίῳ καὶ τοῖς ἀπὸ τῶν ΑΔ, ΔΓ τετραγώνοις. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΒΔ, ΔΑ ἴσον τὸ ἀπὸ τῆς ΑΒ· ὀρθὴ γὰρ ἡ πρὸς τῷ Δ γωνίᾳ· τοῖς δὲ ἀπὸ τῶν ΑΔ, ΔΓ ἴσον τὸ ἀπὸ τῆς ΑΓ· τὰ ἄρα ἀπὸ τῶν ΓΒ, ΒΑ ἴσα ἐστὶ τῷ τε ἀπὸ τῆς ΑΓ καὶ τῷ δὶς ὑπὸ τῶν ΓΒ, ΒΔ· ὥστε μόνον τὸ ἀπὸ τῆς ΑΓ ἔλαττόν ἐστι τῶν ἀπὸ τῶν ΓΒ, ΒΑ τετραγώνων τῷ δὶς ὑπὸ τῶν ΓΒ, ΒΔ περιεχομένῳ ὀρθογωνίῳ.

Ἐν ἄρα τοῖς ὀξυγωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀξεῖαν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἔλαττόν ἐστι τῶν ἀπὸ τῶν τὴν ὀξεῖαν γωνίαν περιεχουσῶν πλευρῶν τετραγώνων τῷ περιεχομένῳ δὶς ὑπό τε μιᾶς τῶν περὶ τὴν ὀξεῖαν γωνίαν, ἐφ’ ἣν ἡ κάθετος πίπτει, καὶ τῆς ἀπολαμβανομένης ἐντὸς ὑπὸ τῆς καθέτου πρὸς τῇ ὀξείᾳ γωνίᾳ· ὅπερ ἔδει δεῖξαι.

ιδ΄. Τῷ δοθέντι εὐθυγράμμῳ ἴσον τετράγωνον συστήσασθαι.

Ἔστω τὸ δοθὲν εὐθύγραμμον τὸ Α· δεῖ δὴ τῷ Α εὐθυγράμμῳ ἴσον τετράγωνον συστήσασθαι.

Συνεστάτω γὰρ τῷ Α εὐθυγράμμῳ ἴσον παραλληλόγραμμον ὀρθογώνιον τὸ ΒΔ· εἰ μὲν οὖν ἴση ἐστὶν ἡ ΒΕ τῇ ΕΔ, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν. συνέσταται γὰρ τῷ Α εὐθυγράμμῳ ἴσον τετράγωνον τὸ ΒΔ· εἰ δὲ οὔ, μία τῶν ΒΕ, ΕΔ μείζων ἐστίν. ἔστω μείζων ἡ ΒΕ, καὶ ἐκβεβλήσθω ἐπὶ τὸ Ζ, καὶ κείσθω τῇ ΕΔ ἴση ἡ ΕΖ, καὶ τετμήσθω ἡ ΒΖ δίχα κατὰ τὸ Η, καὶ κέντρῳ τῷ Η, διαστήματι δὲ ἑνὶ τῶν ΗΒ, ΗΖ ἡμικύκλιον γεγράφθω τὸ ΒΘΖ, καὶ ἐκβεβλήσθω ἡ ΔΕ ἐπὶ τὸ Θ, καὶ ἐπεζεύχθω ἡ ΗΘ.

Ἐπεὶ οὖν εὐθεῖα ἡ ΒΖ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Η, εἰς δὲ ἄνισα κατὰ τὸ Ε, τὸ ἄρα ὑπὸ τῶν ΒΕ, ΕΖ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΕΗ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΗΖ τετραγώνῳ. ἴση δὲ ἡ ΗΖ τῇ ΗΘ· τὸ ἄρα ὑπὸ τῶν ΒΕ, ΕΖ μετὰ τοῦ ἀπὸ τῆς ΗΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΗΘ. τῷ δὲ ἀπὸ τῆς ΗΘ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΘΕ, ΕΗ τετράγωνα· τὸ ἄρα ὑπὸ τῶν ΒΕ, ΕΖ μετὰ τοῦ ἀπὸ ΗΕ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΘΕ, ΕΗ. κοινὸν ἀφῃρήσθω τὸ ἀπὸ τῆς ΗΕ τετράγωνον· λοιπὸν ἄρα τὸ ὑπὸ τῶν ΒΕ, ΕΖ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ τετραγώνῳ. ἀλλὰ τὸ ὑπὸ τῶν ΒΕ, ΕΖ τὸ ΒΔ ἐστιν· ἴση γὰρ ἡ ΕΖ τῇ ΕΔ· τὸ ἄρα ΒΔ παραλληλόγραμμον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΘΕ τετραγώνῳ. ἴσον δὲ τὸ ΒΔ τῷ Α εὐθυγράμμῳ. καὶ τὸ Α ἄρα εὐθύγραμμον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ ἀναγραφησομένῳ τετραγώνῳ.

Τῷ ἄρα δοθέντι εὐθυγράμμῳ τῷ Α ἴσον τετράγωνον συνέσταται τὸ ἀπὸ τῆς ΕΘ ἀναγραφησόμενον· ὅπερ ἔδει ποιῆσαι.

BOOK III.

Ὅροι ια΄.

α΄. Ἴσοι κύκλοι εἰσίν, ὧν αἱ διάμετροι ἴσαι εἰσίν, ἢ ὧν αἱ ἐκ τῶν κέντρων ἴσαι εἰσίν.

β΄. Εὐθεῖα κύκλου ἐφάπτεσθαι λέγεται, ἥτις ἁπτομένη τοῦ κύκλου καὶ ἐκβαλλομένη οὐ τέμνει τὸν κύκλον.

γ΄.Κύκλοι ἐφάπτεσθαι ἀλλήλων λέγονται οἵτινες ἁπτόμενοι ἀλλήλων οὐ τέμνουσιν ἀλλήλους.

δ΄. Ἐν κύκλῳ ἴσον ἀπέχειν ἀπὸ τοῦ κέντρου εὐθεῖαι λέγονται, ὅταν αἱ ἀπὸ τοῦ κέντρου ἐπ᾽ αὐτὰς κάθετοι ἀγόμεναι ἴσαι ὦσιν.

ε΄. Μεῖζον δὲ ἀπέχειν λέγεται, ἐφ᾽ ἣν ἡ μείζων κάθετος πίπτει.

ς΄. Τμῆμα κύκλου ἐστὶ τὸ περιεχόμενον σχῆμα ὑπό τε εὐθείας καὶ κύκλου περιφερείας.

ζ΄. Τμήματος δὲ γωνία ἐστὶν ἡ περιεχομένη ὑπό τε εὐθείας καὶ κύκλου περιφερείας.

η΄. Ἐν τμήματι δὲ γωνία ἐστίν, ὅταν ἐπὶ τῆς περιφερείας τοῦ τμήματος ληφθῇ τι σημεῖον καὶ ἀπ᾽ αὐτοῦ ἐπὶ τὰ πέρατα τῆς εὐθείας, ἥ ἐστι βάσις τοῦ τμήματος, ἐπιζευχθῶσιν εὐθεῖαι, ἡ περιεχομένη γωνία ὑπὸ τῶν ἐπιζευχθεισῶν εὐθειῶν.

θ΄. Ὅταν δὲ αἱ περιέχουσαι τὴν γωνίαν εὐθεῖαι ἀπολαμβάνωσί τινα περιφέρειαν, ἐπ᾽ ἐκείνης λέγεται βεβηκέναι ἡ γωνία.

ι΄. Τομεὺς δὲ κύκλου ἐστίν, ὅταν πρὸς τῷ κέντρῳ τοῦ κύκλου συσταθῇ γωνία, τὸ περιεχόμενον σχῆμα ὑπό τε τῶν τὴν γωνίαν περιεχουσῶν εὐθειῶν καὶ τῆς ἀπολαμβανομένης ὑπ᾽ αὐτῶν περιφερείας.

ια΄. Ὅμοια τμήματα κύκλων ἐστὶ τὰ δεχόμενα γωνίας ἴσας, ἢ ἐν οἷς αἱ γωνίαι ἴσαι ἀλλήλαις εἰσίν.

Προτάσεις λζ΄.

α΄. Τοῦ δοθέντος κύκλου τὸ κέντρον εὑρεῖν.

Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓ· δεῖ δὴ τοῦ ΑΒΓ κύκλου τὸ κέντρον εὑρεῖν.

Διήχθω τις εἰς αὐτόν, ὡς ἔτυχεν, εὐθεῖα ἡ ΑΒ, καὶ τετμήσθω δίχα κατὰ τὸ Δ σημεῖον, καὶ ἀπὸ τοῦ Δ τῇ ΑΒ πρὸς ὀρθὰς ἤχθω ἡ ΔΓ καὶ διήχθω ἐπὶ τὸ Ε, καὶ τετμήσθω ἡ ΓΕ δίχα κατὰ τὸ Ζ· λέγω, ὅτι τὸ Ζ κέντρον ἐστὶ τοῦ ΑΒΓ [κύκλου].

Μὴ γάρ, ἀλλ’ εἰ δυνατόν, ἔστω τὸ Η, καὶ ἐπεζεύχθωσαν αἱ ΗΑ, ΗΔ, ΗΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΔΒ, κοινὴ δὲ ἡ ΔΗ, δύο δὴ αἱ ΑΔ, ΔΗ δύο ταῖς ΗΔ, ΔΒ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ βάσις ἡ ΗΑ βάσει τῇ ΗΒ ἐστιν ἴση· ἐκ κέντρου γάρ· γωνία ἄρα ἡ ὑπὸ ΑΔΗ γωνίᾳ τῇ ὑπὸ ΗΔΒ ἴση ἐστίν. ὅταν δὲ εὐθεῖα ἐπ’ εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστιν· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΗΔΒ. ἐστὶ δὲ καὶ ἡ ὑπὸ ΖΔΒ ὀρθή· ἴση ἄρα ἡ ὑπὸ ΖΔΒ τῇ ὑπὸ ΗΔΒ, ἡ μείζων τῇ ἐλάττονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὸ Η κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου. ὁμοίως δὴ δείξομεν, ὅτι οὐδ’ ἄλλο τι πλὴν τοῦ Ζ.

Τὸ Ζ ἄρα σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ [κύκλου].

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν ἐν κύκλῳ εὐθεῖά τις εὐθεῖάν τινα δίχα καὶ πρὸς ὀρθὰς τέμνῃ, ἐπὶ τῆς τεμνούσης ἐστὶ τὸ κέντρον τοῦ κύκλου· ὅπερ ἔδει ποιῆσαι.

β΄. Ἐὰν κύκλου ἐπὶ τῆς περιφερείας ληφθῇ δύο τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐντὸς πεσεῖται τοῦ κύκλου.

Ἔστω κύκλος ὁ ΑΒΓ, καὶ ἐπὶ τῆς περιφερείας αὐτοῦ εἰλήφθω δύο τυχόντα σημεῖα τὰ Α, Β· λέγω, ὅτι ἡ ἀπὸ τοῦ Α ἐπὶ τὸ Β ἐπιζευγνυμένη εὐθεῖα ἐντὸς πεσεῖται τοῦ κύκλου.

Μὴ γάρ, ἀλλ' εἰ δυνατόν, πιπτέτω ἐκτὸς ὡς ἡ ΑΕΒ, καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓ κύκλου, καὶ ἔστω τὸ Δ, καὶ ἐπεζεύχθωσαν αἱ ΔΑ, ΔΒ, καὶ διήχθω ἡ ΔΖΕ.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΔΑ τῇ ΔΒ, ἴση ἄρα καὶ γωνία ἡ ὑπὸ ΔΑΕ τῇ ὑπὸ ΔΒΕ· καὶ ἐπεὶ τριγώνου τοῦ ΔΑΕ μία πλευρὰ προσεκβέβληται ἡ ΑΕΒ, μείζων ἄρα ἡ ὑπὸ ΔΕΒ γωνία τῆς ὑπὸ ΔΑΕ. ἴση δὲ ἡ ὑπὸ ΔΑΕ τῇ ὑπὸ ΔΒΕ· μείζων ἄρα ἡ ὑπὸ ΔΕΒ τῆς ὑπὸ ΔΒΕ. ὑπὸ δὲ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει· μείζων ἄρα ἡ ΔΒ τῆς ΔΕ. ἴση δὲ ἡ ΔΒ τῇ ΔΖ. μείζων ἄρα ἡ ΔΖ τῆς ΔΕ ἡ ἐλάττων τῆς μείζονος· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ἀπὸ τοῦ Α ἐπὶ τὸ Β ἐπιζευγνυμένη εὐθεῖα ἐκτὸς πεσεῖται τοῦ κύκλου. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἐπ' αὐτῆς τῆς περιφερείας· ἐντὸς ἄρα.

Ἐὰν ἄρα κύκλου ἐπὶ τῆς περιφερείας ληφθῇ δύο τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐντὸς πεσεῖται τοῦ κύκλου· ὅπερ ἔδει δεῖξαι.

γ΄. Ἐὰν ἐν κύκλῳ εὐθεῖά τις διὰ τοῦ κέντρου εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου δίχα τέμνῃ, καὶ πρὸς ὀρθὰς αὐτὴν τέμνει· καὶ ἐὰν πρὸς ὀρθὰς αὐτὴν τέμνῃ, καὶ δίχα αὐτὴν τέμνει.

Ἔστω κύκλος ὁ ΑΒΓ, καὶ ἐν αὐτῷ εὐθεῖά τις διὰ τοῦ κέντρου ἡ ΓΔ εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου τὴν ΑΒ δίχα τεμνέτω κατὰ τὸ Ζ σημεῖον· λέγω, ὅτι καὶ πρὸς ὀρθὰς αὐτὴν τέμνει.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ ΑΒΓ κύκλου, καὶ ἔστω τὸ Ε, καὶ ἐπεζεύχθωσαν αἱ ΕΑ, ΕΒ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΖ τῇ ΖΒ, κοινὴ δὲ ἡ ΖΕ, δύο δυσὶν ἴσαι [εἰσίν]. καὶ βάσις ἡ ΕΑ βάσει τῇ ΕΒ ἴση· γωνία ἄρα ἡ ὑπὸ ΑΖΕ γωνίᾳ τῇ ὑπὸ ΒΖΕ ἴση ἐστίν. ὅταν δὲ εὐθεῖα ἐπ' εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις

ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστιν· ἑκατέρα ἄρα τῶν ὑπὸ ΑΖΕ, ΒΖΕ ὀρθή ἐστιν. ἡ ΓΔ ἄρα διὰ τοῦ κέντρου οὖσα τὴν ΑΒ μὴ διὰ τοῦ κέντρου οὖσαν δίχα τέμνουσα καὶ πρὸς ὀρθὰς τέμνει.

Ἀλλὰ δὴ ἡ ΓΔ τὴν ΑΒ πρὸς ὀρθὰς τεμνέτω· λέγω, ὅτι καὶ δίχα αὐτὴν τέμνει, τουτέστιν, ὅτι ἴση ἐστὶν ἡ ΑΖ τῇ ΖΒ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ ἴση ἐστὶν ἡ ΕΑ τῇ ΕΒ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΕΑΖ τῇ ὑπὸ ΕΒΖ. ἐστὶ δὲ καὶ ὀρθὴ ἡ ὑπὸ ΑΖΕ ὀρθῇ τῇ ὑπὸ ΒΖΕ ἴση· δύο ἄρα τρίγωνά ἐστι τὰ ΕΑΖ, ΕΖΒ τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην κοινὴν αὐτῶν τὴν ΕΖ ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει· ἴση ἄρα ἡ ΑΖ τῇ ΖΒ.

Ἐὰν ἄρα ἐν κύκλῳ εὐθεῖά τις διὰ τοῦ κέντρου εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου δίχα τέμνῃ, καὶ πρὸς ὀρθὰς αὐτὴν τέμνει· καὶ ἐὰν πρὸς ὀρθὰς αὐτὴν τέμνῃ, καὶ δίχα αὐτὴν τέμνει· ὅπερ ἔδει δεῖξαι.

δ΄. Ἐὰν ἐν κύκλῳ δύο εὐθεῖαι τέμνωσιν ἀλλήλας μὴ διὰ τοῦ κέντρου οὖσαι, οὐ τέμνουσιν ἀλλήλας δίχα.

Ἔστω κύκλος ὁ ΑΒΓ, καὶ ἐν αὐτῷ εὐθεῖά τις διὰ τοῦ κέντρου ἡ ΓΔ εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου τὴν ΑΒ δίχα τεμνέτω κατὰ τὸ Ζ σημεῖον· λέγω, ὅτι καὶ πρὸς ὀρθὰς αὐτὴν τέμνει.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ ΑΒΓ κύκλου, καὶ ἔστω τὸ Ε, καὶ ἐπεζεύχθωσαν αἱ ΕΑ, ΕΒ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΖ τῇ ΖΒ, κοινὴ δὲ ἡ ΖΕ, δύο δυσὶν ἴσαι [εἰσίν]. καὶ βάσις ἡ ΕΑ βάσει τῇ ΕΒ ἴση· γωνία ἄρα ἡ ὑπὸ ΑΖΕ γωνίᾳ τῇ ὑπὸ ΒΖΕ ἴση ἐστίν. ὅταν δὲ εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστιν· ἑκατέρα ἄρα τῶν ὑπὸ ΑΖΕ, ΒΖΕ ὀρθή ἐστιν. ἡ ΓΔ ἄρα διὰ τοῦ κέντρου οὖσα τὴν ΑΒ μὴ διὰ τοῦ κέντρου οὖσαν δίχα τέμνουσα καὶ πρὸς ὀρθὰς τέμνει.

Ἀλλὰ δὴ ἡ ΓΔ τὴν ΑΒ πρὸς ὀρθὰς τεμνέτω· λέγω, ὅτι καὶ δίχα αὐτὴν τέμνει, τουτέστιν, ὅτι ἴση ἐστὶν ἡ ΑΖ τῇ ΖΒ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ ἴση ἐστὶν ἡ ΕΑ τῇ ΕΒ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΕΑΖ τῇ ὑπὸ ΕΒΖ. ἐστὶ δὲ καὶ ὀρθὴ ἡ ὑπὸ ΑΖΕ ὀρθῇ τῇ ὑπὸ ΒΖΕ ἴση· δύο ἄρα τρίγωνά ἐστι τὰ ΕΑΖ, ΕΖΒ τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην κοινὴν αὐτῶν τὴν ΕΖ ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει· ἴση ἄρα ἡ ΑΖ τῇ ΖΒ.

Ἐὰν ἄρα ἐν κύκλῳ εὐθεῖά τις διὰ τοῦ κέντρου εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου δίχα τέμνῃ, καὶ πρὸς ὀρθὰς αὐτὴν τέμνει· καὶ ἐὰν πρὸς ὀρθὰς αὐτὴν τέμνῃ, καὶ δίχα αὐτὴν τέμνει· ὅπερ ἔδει δεῖξαι.

ε΄. Ἐὰν δύο κύκλοι τέμνωσιν ἀλλήλους, οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον.

Δύο γὰρ κύκλοι οἱ ΑΒΓ, ΓΔΗ τεμνέτωσαν ἀλλήλους κατὰ τὰ Β, Γ σημεῖα. λέγω, ὅτι οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον.

Εἰ γὰρ δυνατόν, ἔστω τὸ Ε, καὶ ἐπεζεύχθω ἡ ΕΓ, καὶ διήχθω ἡ ΕΖΗ, ὡς ἔτυχεν. καὶ ἐπεὶ τὸ Ε σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου, ἴση ἐστὶν ἡ ΕΓ τῇ ΕΖ. πάλιν, ἐπεὶ τὸ Ε σημεῖον κέντρον ἐστὶ τοῦ ΓΔΗ κύκλου, ἴση ἐστὶν ἡ ΕΓ τῇ ΕΗ· ἐδείχθη δὲ ἡ ΕΓ καὶ τῇ ΕΖ ἴση· καὶ ἡ ΕΖ ἄρα τῇ ΕΗ ἐστιν ἴση ἡ ἐλάσσων τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὸ Ε σημεῖον κέντρον ἐστὶ τῶν ΑΒΓ, ΓΔΗ κύκλων.

Ἐὰν ἄρα δύο κύκλοι τέμνωσιν ἀλλήλους, οὐκ ἔστιν αὐτῶν τὸ αὐτὸ κέντρον· ὅπερ ἔδει δεῖξαι.

ς΄. Ἐὰν δύο κύκλοι ἐφάπτωνται ἀλλήλων, οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον.

Δύο γὰρ κύκλοι οἱ ΑΒΓ, ΓΔΕ ἐφαπτέσθωσαν ἀλλήλων κατὰ τὸ Γ σημεῖον· λέγω, ὅτι οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον.

Εἰ γὰρ δυνατόν, ἔστω τὸ Ζ, καὶ ἐπεζεύχθω ἡ ΖΓ, καὶ διήχθω, ὡς ἔτυχεν, ἡ ΖΕΒ.

Ἐπεὶ οὖν τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου, ἴση ἐστὶν ἡ ΖΓ τῇ ΖΒ. πάλιν, ἐπεὶ τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ΓΔΕ κύκλου, ἴση ἐστὶν ἡ ΖΓ τῇ ΖΕ. ἐδείχθη δὲ ἡ ΖΓ τῇ ΖΒ ἴση· καὶ ἡ ΖΕ ἄρα τῇ ΖΒ ἐστιν ἴση, ἡ ἐλάττων τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὸ Ζ σημεῖον κέντρον ἐστὶ τῶν ΑΒΓ, ΓΔΕ κύκλων.

Ἐὰν ἄρα δύο κύκλοι ἐφάπτωνται ἀλλήλων, οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον· ὅπερ ἔδει δεῖξαι.

ζ΄. Ἐὰν κύκλου ἐπὶ τῆς διαμέτρου ληφθῇ τι σημεῖον, ὃ μή ἐστι κέντρον τοῦ κύκλου, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσιν εὐθεῖαί τινες, μεγίστη μὲν ἔσται, ἐφ᾽ ἧς τὸ κέντρον, ἐλαχίστη δὲ ἡ λοιπή, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τῆς διὰ τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν, δύο δὲ μόνον ἴσαι ἀπὸ τοῦ σημείου προσπεσοῦνται πρὸς τὸν κύκλον ἐφ᾽ ἑκάτερα τῆς ἐλαχίστης.

Ἔστω κύκλος ὁ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἔστω ἡ ΑΔ, καὶ ἐπὶ τῆς ΑΔ εἰλήφθω τι σημεῖον τὸ Ζ, ὃ μή ἐστι κέντρον τοῦ κύκλου, κέντρον δὲ τοῦ κύκλου ἔστω τὸ Ε, καὶ ἀπὸ τοῦ Ζ πρὸς τὸν ΑΒΓΔ κύκλον προσπιπτέτωσαν εὐθεῖαί τινες αἱ ΖΒ, ΖΓ, ΖΗ· λέγω, ὅτι μεγίστη μέν ἐστιν ἡ ΖΑ, ἐλαχίστη δὲ ἡ ΖΔ, τῶν δὲ ἄλλων ἡ μὲν ΖΒ τῆς ΖΓ μείζων, ἡ δὲ ΖΓ τῆς ΖΗ.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΕ, ΓΕ, ΗΕ. καὶ ἐπεὶ παντὸς τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσιν, αἱ ἄρα ΕΒ, ΕΖ τῆς ΒΖ μείζονές εἰσιν. ἴση δὲ ἡ ΑΕ τῇ ΒΕ [αἱ ἄρα ΒΕ, ΕΖ ἴσαι εἰσὶ τῇ ΑΖ]· μείζων ἄρα ἡ ΑΖ τῆς ΒΖ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΒΕ τῇ ΓΕ, κοινὴ δὲ ἡ ΖΕ, δύο δὴ αἱ ΒΕ, ΕΖ δυσὶ ταῖς ΓΕ, ΕΖ ἴσαι εἰσίν. ἀλλὰ καὶ γωνία ἡ ὑπὸ ΒΕΖ γωνίας τῆς ὑπὸ ΓΕΖ μείζων. βάσις ἄρα ἡ ΒΖ βάσεως τῆς ΓΖ μείζων ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΓΖ τῆς ΖΗ μείζων ἐστίν.

Πάλιν, ἐπεὶ αἱ ΗΖ, ΖΕ τῆς ΕΗ μείζονές εἰσιν, ἴση δὲ ἡ ΕΗ τῇ ΕΔ, αἱ ἄρα ΗΖ, ΖΕ τῆς ΕΔ μείζονές εἰσιν. κοινὴ ἀφῃρήσθω ἡ ΕΖ· λοιπὴ ἄρα ἡ ΗΖ λοιπῆς τῆς ΖΔ μείζων ἐστίν. μεγίστη μὲν ἄρα ἡ ΖΑ, ἐλαχίστη δὲ ἡ ΖΔ, μείζων δὲ ἡ μὲν ΖΒ τῆς ΖΓ, ἡ δὲ ΖΓ τῆς ΖΗ.

Λέγω, ὅτι καὶ ἀπὸ τοῦ Ζ σημείου δύο μόνον ἴσαι προσπεσοῦνται πρὸς τὸν ΑΒΓΔ κύκλον ἐφ' ἑκάτερα τῆς ΖΔ ἐλαχίστης. συνεστάτω γὰρ πρὸς τῇ ΕΖ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Ε τῇ ὑπὸ ΗΕΖ γωνίᾳ ἴση ἡ ὑπὸ ΖΕΘ, καὶ ἐπεζεύχθω ἡ ΖΘ. ἐπεὶ οὖν ἴση ἐστὶν ἡ ΗΕ τῇ ΕΘ, κοινὴ δὲ ἡ ΕΖ, δύο δὴ αἱ ΗΕ, ΕΖ δυσὶ ταῖς ΘΕ, ΕΖ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΗΕΖ γωνίᾳ τῇ ὑπὸ ΘΕΖ ἴση· βάσις ἄρα ἡ ΖΗ βάσει τῇ ΖΘ ἴση ἐστίν. λέγω δή, ὅτι τῇ ΖΗ ἄλλη ἴση οὐ προσπεσεῖται πρὸς τὸν κύκλον ἀπὸ τοῦ Ζ σημείου. εἰ γὰρ δυνατόν, προσπιπτέτω ἡ ΖΚ. καὶ ἐπεὶ ἡ ΖΚ τῇ ΖΗ ἴση ἐστίν, ἀλλὰ ἡ ΖΘ τῇ ΖΗ [ἴση ἐστίν], καὶ ἡ ΖΚ ἄρα τῇ ΖΘ ἐστιν ἴση, ἡ ἔγγιον τῆς διὰ τοῦ κέντρου τῇ ἀπώτερον ἴση· ὅπερ ἀδύνατον. οὐκ ἄρα ἀπὸ τοῦ Ζ σημείου ἑτέρα τις προσπεσεῖται πρὸς τὸν κύκλον ἴση τῇ ΗΖ· μία ἄρα μόνη.

Ἐὰν ἄρα κύκλου ἐπὶ τῆς διαμέτρου ληφθῇ τι σημεῖον, ὃ μή ἐστι κέντρον τοῦ κύκλου, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσιν εὐθεῖαί τινες, μεγίστη μὲν ἔσται, ἐφ' ἧς τὸ κέντρον, ἐλαχίστη δὲ ἡ λοιπή, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τῆς διὰ τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν, δύο δὲ μόνον ἴσαι ἀπὸ τοῦ αὐτοῦ σημείου προσπεσοῦνται πρὸς τὸν κύκλον ἐφ' ἑκάτερα τῆς ἐλαχίστης· ὅπερ ἔδει δεῖξαι.

η΄. Ἐὰν κύκλου ληφθῇ τι σημεῖον ἐκτός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον διαχθῶσιν εὐθεῖαί τινες, ὧν μία μὲν διὰ τοῦ κέντρου, αἱ δὲ λοιπαί, ὡς ἔτυχεν, τῶν μὲν πρὸς τὴν κοίλην περιφέρειαν προσπιπτουσῶν εὐθειῶν μεγίστη μέν ἐστιν ἡ διὰ τοῦ κέντρου, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τῆς διὰ τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν, τῶν δὲ πρὸς τὴν κυρτὴν περιφέρειαν προσπιπτουσῶν εὐθειῶν ἐλαχίστη μέν ἐστιν ἡ μεταξὺ τοῦ τε σημείου καὶ τῆς διαμέτρου, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τῆς ἐλαχίστης τῆς ἀπώτερόν ἐστιν ἐλάττων, δύο δὲ μόνον

ἴσαι ἀπὸ τοῦ σημείου προσπεσοῦνται πρὸς τὸν κύκλον ἐφ᾽ ἑκάτερα τῆς ἐλαχίστης.

Ἔστω κύκλος ὁ ΑΒΓ, καὶ τοῦ ΑΒΓ εἰλήφθω τι σημεῖον ἐκτὸς τὸ Δ, καὶ ἀπ᾽ αὐτοῦ διήχθωσαν εὐθεῖαί τινες αἱ ΔΑ, ΔΕ, ΔΖ, ΔΓ, ἔστω δὲ ἡ ΔΑ διὰ τοῦ κέντρου. λέγω, ὅτι τῶν μὲν πρὸς τὴν ΑΕΖΓ κοίλην περιφέρειαν προσπιπτουσῶν εὐθειῶν μεγίστη μέν ἐστιν ἡ διὰ τοῦ κέντρου ἡ ΔΑ, μείζων δὲ ἡ μὲν ΔΕ τῆς ΔΖ ἡ δὲ ΔΖ τῆς ΔΓ, τῶν δὲ πρὸς τὴν ΘΛΚΗ κυρτὴν περιφέρειαν προσπιπτουσῶν εὐθειῶν ἐλαχίστη μέν ἐστιν ἡ ΔΗ ἡ μεταξὺ τοῦ σημείου καὶ τῆς διαμέτρου τῆς ΑΗ, ἀεὶ δὲ ἡ ἔγγιον τῆς ΔΗ ἐλαχίστης ἐλάττων ἐστὶ τῆς ἀπώτερον, ἡ μὲν ΔΚ τῆς ΔΛ, ἡ δὲ ΔΛ τῆς ΔΘ.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ ΑΒΓ κύκλου καὶ ἔστω τὸ Μ· καὶ ἐπεζεύχθωσαν αἱ ΜΕ, ΜΖ, ΜΓ, ΜΚ, ΜΛ, ΜΘ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΜ τῇ ΕΜ, κοινὴ προσκείσθω ἡ ΜΔ· ἡ ἄρα ΑΔ ἴση ἐστὶ ταῖς ΕΜ, ΜΔ. ἀλλ᾽ αἱ ΕΜ, ΜΔ τῆς ΕΔ μείζονές εἰσιν· καὶ ἡ ΑΔ ἄρα τῆς ΕΔ μείζων ἐστίν. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΜΕ τῇ ΜΖ, κοινὴ δὲ ἡ ΜΔ, αἱ ΕΜ, ΜΔ ἄρα ταῖς ΖΜ, ΜΔ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΕΜΔ γωνίας τῆς ὑπὸ ΖΜΔ μείζων ἐστίν. βάσις ἄρα ἡ ΕΔ βάσεως τῆς ΖΔ μείζων ἐστίν. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΖΔ τῆς ΓΔ μείζων ἐστίν· μεγίστη μὲν ἄρα ἡ ΔΑ, μείζων δὲ ἡ μὲν ΔΕ τῆς ΔΖ, ἡ δὲ ΔΖ τῆς ΔΓ.

Καὶ ἐπεὶ αἱ ΜΚ, ΚΔ τῆς ΜΔ μείζονές εἰσιν, ἴση δὲ ἡ ΜΗ τῇ ΜΚ, λοιπὴ ἄρα ἡ ΚΔ λοιπῆς τῆς ΗΔ μείζων ἐστίν· ὥστε ἡ ΗΔ τῆς ΚΔ ἐλάττων ἐστίν· καὶ ἐπεὶ τριγώνου τοῦ ΜΛΔ ἐπὶ μιᾶς τῶν πλευρῶν τῆς ΜΔ δύο εὐθεῖαι ἐντὸς συνεστάθησαν αἱ ΜΚ, ΚΔ, αἱ ἄρα ΜΚ, ΚΔ τῶν ΜΛ, ΛΔ ἐλάττονές εἰσιν· ἴση δὲ ἡ ΜΚ τῇ ΜΛ· λοιπὴ ἄρα ἡ ΔΚ λοιπῆς τῆς ΔΛ ἐλάττων ἐστίν. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΔΛ τῆς ΔΘ ἐλάττων ἐστίν· ἐλαχίστη μὲν ἄρα ἡ ΔΗ, ἐλάττων δὲ ἡ μὲν ΔΚ τῆς ΔΛ ἡ δὲ ΔΛ τῆς ΔΘ.

Λέγω, ὅτι καὶ δύο μόνον ἴσαι ἀπὸ τοῦ Δ σημείου προσπεσοῦνται πρὸς τὸν κύκλον ἐφ᾽ ἑκάτερα τῆς ΔΗ ἐλαχίστης· συνεστάτω πρὸς τῇ ΜΔ εὐθείᾳ καὶ

τῷ πρὸς αὐτῇ σημείῳ τῷ Μ τῇ ὑπὸ ΚΜΔ γωνίᾳ ἴση γωνία ἡ ὑπὸ ΔΜΒ καὶ ἐπεζεύχθω ἡ ΔΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΜΚ τῇ ΜΒ, κοινὴ δὲ ἡ ΜΔ, δύο δὴ αἱ ΚΜ, ΜΔ δύο ταῖς ΒΜ, ΜΔ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΚΜΔ γωνίᾳ τῇ ὑπὸ ΒΜΔ ἴση· βάσις ἄρα ἡ ΔΚ βάσει τῇ ΔΒ ἴση ἐστίν. λέγω [δή], ὅτι τῇ ΔΚ εὐθείᾳ ἄλλη ἴση οὐ προσπεσεῖται πρὸς τὸν κύκλον ἀπὸ τοῦ Δ σημείου. εἰ γὰρ δυνατόν, προσπιπτέτω καὶ ἔστω ἡ ΔΝ. ἐπεὶ οὖν ἡ ΔΚ τῇ ΔΝ ἐστιν ἴση, ἀλλ᾽ ἡ ΔΚ τῇ ΔΒ ἐστιν ἴση, καὶ ἡ ΔΒ ἄρα τῇ ΔΝ ἐστιν ἴση, ἡ ἔγγιον τῆς ΔΗ ἐλαχίστης τῇ ἀπώτερον [ἐστιν] ἴση· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα πλείους ἢ δύο ἴσαι πρὸς τὸν ΑΒΓ κύκλον ἀπὸ τοῦ Δ σημείου ἐφ᾽ ἑκάτερα τῆς ΔΗ ἐλαχίστης προσπεσοῦνται.

Ἐὰν ἄρα κύκλου ληφθῇ τι σημεῖον ἐκτός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον διαχθῶσιν εὐθεῖαί τινες, ὧν μία μὲν διὰ τοῦ κέντρου αἱ δὲ λοιπαί, ὡς ἔτυχεν, τῶν μὲν πρὸς τὴν κοίλην περιφέρειαν προσπιπτουσῶν εὐθειῶν μεγίστη μέν ἐστιν ἡ διὰ τοῦ κέντρου, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τῆς διὰ τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν, τῶν δὲ πρὸς τὴν κυρτὴν περιφέρειαν προσπιπτουσῶν εὐθειῶν ἐλαχίστη μέν ἐστιν ἡ μεταξὺ τοῦ τε σημείου καὶ τῆς διαμέτρου, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τῆς ἐλαχίστης τῆς ἀπώτερόν ἐστιν ἐλάττων, δύο δὲ μόνον ἴσαι ἀπὸ τοῦ σημείου προσπεσοῦνται πρὸς τὸν κύκλον ἐφ᾽ ἑκάτερα τῆς ἐλαχίστης· ὅπερ ἔδει δεῖξαι.

θ΄. Ἐὰν κύκλου ληφθῇ τι σημεῖον ἐντός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσι πλείους ἢ δύο ἴσαι εὐθεῖαι, τὸ ληφθὲν σημεῖον κέντρον ἐστὶ τοῦ κύκλου.

Ἔστω κύκλος ὁ ΑΒΓ, ἐντὸς δὲ αὐτοῦ σημεῖον τὸ Δ, καὶ ἀπὸ τοῦ Δ πρὸς τὸν ΑΒΓ κύκλον προσπιπτέτωσαν πλείους ἢ δύο ἴσαι εὐθεῖαι αἱ ΔΑ, ΔΒ, ΔΓ· λέγω, ὅτι τὸ Δ σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου.

Ἐπεζεύχθωσαν γὰρ αἱ ΑΒ, ΒΓ καὶ τετμήσθωσαν δίχα κατὰ τὰ Ε, Ζ σημεῖα, καὶ ἐπιζευχθεῖσαι αἱ ΕΔ, ΖΔ διήχθωσαν ἐπὶ τὰ Η, Κ, Θ, Λ σημεῖα.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΑΕ τῇ ΕΒ, κοινὴ δὲ ἡ ΕΔ, δύο δὴ αἱ ΑΕ, ΕΔ δύο ταῖς ΒΕ, ΕΔ ἴσαι εἰσίν· καὶ βάσις ἡ ΔΑ βάσει τῇ ΔΒ ἴση· γωνία ἄρα ἡ ὑπὸ ΑΕΔ γωνίᾳ τῇ ὑπὸ ΒΕΔ ἴση ἐστίν· ὀρθὴ ἄρα ἑκατέρα τῶν ὑπὸ ΑΕΔ, ΒΕΔ γωνιῶν· ἡ ΗΚ ἄρα τὴν ΑΒ τέμνει δίχα καὶ πρὸς ὀρθάς. καὶ ἐπεί, ἐὰν ἐν κύκλῳ εὐθεῖά τις εὐθεῖάν τινα δίχα τε καὶ πρὸς ὀρθὰς τέμνῃ, ἐπὶ τῆς τεμνούσης ἐστὶ τὸ κέντρον τοῦ κύκλου, ἐπὶ τῆς ΗΚ ἄρα ἐστὶ τὸ κέντρον τοῦ κύκλου. διὰ τὰ αὐτὰ δὴ καὶ ἐπὶ τῆς ΘΛ ἐστι τὸ κέντρον τοῦ ΑΒΓ κύκλου. καὶ οὐδὲν ἕτερον κοινὸν ἔχουσιν αἱ ΗΚ, ΘΛ εὐθεῖαι ἢ τὸ Δ σημεῖον· τὸ Δ ἄρα σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου.

Ἐὰν ἄρα κύκλου ληφθῇ τι σημεῖον ἐντός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσι πλείους ἢ δύο ἴσαι εὐθεῖαι, τὸ ληφθὲν σημεῖον κέντρον ἐστὶ τοῦ κύκλου· ὅπερ ἔδει δεῖξαι.

ι΄. Κύκλος κύκλον οὐ τέμνει κατὰ πλείονα σημεῖα ἢ δύο.

Εἰ γὰρ δυνατόν, κύκλος ὁ ΑΒΓ κύκλον τὸν ΔΕΖ τεμνέτω κατὰ πλείονα σημεῖα ἢ δύο τὰ Β, Η, Ζ, Θ, καὶ ἐπιζευχθεῖσαι αἱ ΒΘ, ΒΗ δίχα τεμνέσθωσαν κατὰ τὰ Κ, Λ σημεῖα· καὶ ἀπὸ τῶν Κ, Λ ταῖς ΒΘ, ΒΗ πρὸς ὀρθὰς ἀχθεῖσαι αἱ ΚΓ, ΛΜ διήχθωσαν ἐπὶ τὰ Α, Ε σημεῖα

Ἐπεὶ οὖν ἐν κύκλῳ τῷ ΑΒΓ εὐθεῖά τις ἡ ΑΓ εὐθεῖάν τινα τὴν ΒΘ δίχα καὶ πρὸς ὀρθὰς τέμνει, ἐπὶ τῆς ΑΓ ἄρα ἐστὶ τὸ κέντρον τοῦ ΑΒΓ κύκλου. πάλιν, ἐπεὶ ἐν κύκλῳ τῷ αὐτῷ τῷ ΑΒΓ εὐθεῖά τις ἡ ΝΞ εὐθεῖάν τινα τὴν ΒΗ δίχα καὶ πρὸς ὀρθὰς τέμνει, ἐπὶ τῆς ΝΞ ἄρα ἐστὶ τὸ κέντρον τοῦ ΑΒΓ κύκλου. ἐδείχθη δὲ καὶ ἐπὶ τῆς ΑΓ, καὶ κατ’ οὐδὲν συμβάλλουσιν αἱ ΑΓ, ΝΞ εὐθεῖαι ἢ κατὰ τὸ Ο· τὸ Ο ἄρα σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου. ὁμοίως δὴ δείξομεν, ὅτι καὶ τοῦ ΔΕΖ κύκλου κέντρον ἐστὶ τὸ Ο· δύο ἄρα κύκλων τεμνόντων ἀλλήλους τῶν ΑΒΓ, ΔΕΖ τὸ αὐτό ἐστι κέντρον τὸ Ο· ὅπερ ἐστὶν ἀδύνατον.

Οὐκ ἄρα κύκλος κύκλον τέμνει κατὰ πλείονα σημεῖα ἢ δύο· ὅπερ ἔδει δεῖξαι.

ια΄. Ἐὰν δύο κύκλοι ἐφάπτωνται ἀλλήλων ἐντός, καὶ ληφθῇ αὐτῶν τὰ κέντρα, ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη εὐθεῖα καὶ ἐκβαλλομένη ἐπὶ τὴν συναφὴν πεσεῖται τῶν κύκλων.

Δύο γὰρ κύκλοι οἱ ΑΒΓ, ΑΔΕ ἐφαπτέσθωσαν ἀλλήλων ἐντὸς κατὰ τὸ Α σημεῖον, καὶ εἰλήφθω τοῦ μὲν ΑΒΓ κύκλου κέντρον τὸ Ζ, τοῦ δὲ ΑΔΕ τὸ Η· λέγω, ὅτι ἡ ἀπὸ τοῦ Η ἐπὶ τὸ Ζ ἐπιζευγνυμένη εὐθεῖα ἐκβαλλομένη ἐπὶ τὸ Α πεσεῖται.

Μὴ γάρ, ἀλλ’ εἰ δυνατόν, πιπτέτω ὡς ἡ ΖΗΘ, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΑΗ.

Ἐπεὶ οὖν αἱ ΑΗ, ΗΖ τῆς ΖΑ, τουτέστι τῆς ΖΘ, μείζονές εἰσιν, κοινὴ ἀφῃρήσθω ἡ ΖΗ· λοιπὴ ἄρα ἡ ΑΗ λοιπῆς τῆς ΗΘ μείζων ἐστίν. ἴση δὲ ἡ ΑΗ τῇ ΗΔ· καὶ ἡ ΗΔ ἄρα τῆς ΗΘ μείζων ἐστὶν ἡ ἐλάττων τῆς μείζονος· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα ἡ ἀπὸ τοῦ Ζ ἐπὶ τὸ Η ἐπιζευγνυμένη εὐθεῖα ἐκτὸς πεσεῖται· κατὰ τὸ Α ἄρα ἐπὶ τῆς συναφῆς πεσεῖται.

Ἐὰν ἄρα δύο κύκλοι ἐφάπτωνται ἀλλήλων ἐντός, [καὶ ληφθῇ αὐτῶν τὰ κέντρα], ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη εὐθεῖα [καὶ ἐκβαλλομένη] ἐπὶ τὴν συναφὴν πεσεῖται τῶν κύκλων· ὅπερ ἔδει δεῖξαι.

ιβ΄. Ἐὰν δύο κύκλοι ἐφάπτωνται ἀλλήλων ἐκτός, ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη διὰ τῆς ἐπαφῆς ἐλεύσεται.

Δύο γὰρ κύκλοι οἱ ΑΒΓ, ΑΔΕ ἐφαπτέσθωσαν ἀλλήλων ἐκτὸς κατὰ τὸ Α σημεῖον, καὶ εἰλήφθω τοῦ μὲν ΑΒΓ κέντρον τὸ Ζ, τοῦ δὲ ΑΔΕ τὸ Η· λέγω, ὅτι ἡ ἀπὸ τοῦ Ζ ἐπὶ τὸ Η ἐπιζευγνυμένη εὐθεῖα διὰ τῆς κατὰ τὸ Α ἐπαφῆς ἐλεύσεται.

Μὴ γάρ, ἀλλ’ εἰ δυνατόν, ἐρχέσθω ὡς ἡ ΖΓΔΗ, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΑΗ.

Ἐπεὶ οὖν τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου, ἴση ἐστὶν ἡ ΖΑ τῇ ΖΓ. πάλιν, ἐπεὶ τὸ Η σημεῖον κέντρον ἐστὶ τοῦ ΑΔΕ κύκλου, ἴση ἐστὶν ἡ ΗΑ τῇ ΗΔ. ἐδείχθη δὲ καὶ ἡ ΖΑ τῇ ΖΓ ἴση· αἱ ἄρα ΖΑ, ΑΗ ταῖς ΖΓ, ΗΔ ἴσαι εἰσίν· ὥστε ὅλη ἡ ΖΗ τῶν ΖΑ, ΑΗ μείζων ἐστίν· ἀλλὰ καὶ ἐλάττων· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ἀπὸ τοῦ Ζ ἐπὶ τὸ Η ἐπιζευγνυμένη εὐθεῖα διὰ τῆς κατὰ τὸ Α ἐπαφῆς οὐκ ἐλεύσεται· δι’ αὐτῆς ἄρα.

Ἐὰν ἄρα δύο κύκλοι ἐφάπτωνται ἀλλήλων ἐκτός, ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη [εὐθεῖα] διὰ τῆς ἐπαφῆς ἐλεύσεται· ὅπερ ἔδει δεῖξαι.

ιγ΄. Κύκλος κύκλου οὐκ ἐφάπτεται κατὰ πλείονα σημεῖα ἢ καθ’ ἕν, ἐάν τε ἐντὸς ἐάν τε ἐκτὸς ἐφάπτηται.

Εἰ γὰρ δυνατόν, κύκλος ὁ ΑΒΓΔ κύκλου τοῦ ΕΒΖΔ ἐφαπτέσθω πρότερον ἐντὸς κατὰ πλείονα σημεῖα ἢ ἓν τὰ Δ, Β.

Καὶ εἰλήφθω τοῦ μὲν ΑΒΓΔ κύκλου κέντρον τὸ Η, τοῦ δὲ ΕΒΖΔ τὸ Θ.

Ἡ ἄρα ἀπὸ τοῦ Η ἐπὶ τὸ Θ ἐπιζευγνυμένη ἐπὶ τὰ Β, Δ πεσεῖται. πιπτέτω ὡς ἡ ΒΗΘΔ. καὶ ἐπεὶ τὸ Η σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓΔ κύκλου, ἴση ἐστὶν ἡ ΒΗ τῇ ΗΔ· μείζων ἄρα ἡ ΒΗ τῆς ΘΔ· πολλῷ ἄρα μείζων ἡ ΒΘ τῆς ΘΔ. πάλιν, ἐπεὶ τὸ Θ σημεῖον κέντρον ἐστὶ τοῦ ΕΒΖΔ κύκλου, ἴση ἐστὶν ἡ ΒΘ τῇ ΘΔ· ἐδείχθη δὲ αὐτῆς καὶ πολλῷ μείζων· ὅπερ ἀδύνατον· οὐκ ἄρα κύκλος κύκλου ἐφάπτεται ἐντὸς κατὰ πλείονα σημεῖα ἢ ἕν.

Λέγω δή, ὅτι οὐδὲ ἐκτός.

Εἰ γὰρ δυνατόν, κύκλος ὁ ΑΓΚ κύκλου τοῦ ΑΒΓΔ ἐφαπτέσθω ἐκτὸς κατὰ πλείονα σημεῖα ἢ ἓν τὰ Α, Γ, καὶ ἐπεζεύχθω ἡ ΑΓ.

Ἐπεὶ οὖν κύκλων τῶν ΑΒΓΔ, ΑΓΚ εἴληπται ἐπὶ τῆς περιφερείας ἑκατέρου δύο τυχόντα σημεῖα τὰ Α, Γ, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐντὸς ἑκατέρου πεσεῖται· ἀλλὰ τοῦ μὲν ΑΒΓΔ ἐντὸς ἔπεσεν, τοῦ δὲ ΑΓΚ ἐκτός·

ὅπερ ἄτοπον· οὐκ ἄρα κύκλος κύκλου ἐφάπτεται ἐκτὸς κατὰ πλείονα σημεῖα ἢ ἕν. ἐδείχθη δέ, ὅτι οὐδὲ ἐντός.

Κύκλος ἄρα κύκλου οὐκ ἐφάπτεται κατὰ πλείονα σημεῖα ἢ [καθ'] ἕν, ἐάν τε ἐντὸς ἐάν τε ἐκτὸς ἐφάπτηται· ὅπερ ἔδει δεῖξαι.

ιδ΄. Ἐν κύκλῳ αἱ ἴσαι εὐθεῖαι ἴσον ἀπέχουσιν ἀπὸ τοῦ κέντρου, καὶ αἱ ἴσον ἀπέχουσαι ἀπὸ τοῦ κέντρου ἴσαι ἀλλήλαις εἰσίν.

Ἔστω κύκλος ὁ ΑΒΓΔ, καὶ ἐν αὐτῷ ἴσαι εὐθεῖαι ἔστωσαν αἱ ΑΒ, ΓΔ· λέγω, ὅτι αἱ ΑΒ, ΓΔ ἴσον ἀπέχουσιν ἀπὸ τοῦ κέντρου.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ ΑΒΓΔ κύκλου καὶ ἔστω τὸ Ε, καὶ ἀπὸ τοῦ Ε ἐπὶ τὰς ΑΒ, ΓΔ κάθετοι ἤχθωσαν αἱ ΕΖ, ΕΗ, καὶ ἐπεζεύχθωσαν αἱ ΑΕ, ΕΓ.

Ἐπεὶ οὖν εὐθεῖά τις διὰ τοῦ κέντρου ἡ ΕΖ εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου τὴν ΑΒ πρὸς ὀρθὰς τέμνει, καὶ δίχα αὐτὴν τέμνει. ἴση ἄρα ἡ ΑΖ τῇ ΖΒ· διπλῆ ἄρα ἡ ΑΒ τῆς ΑΖ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΓΔ τῆς ΓΗ ἐστι διπλῆ· καί ἐστιν ἴση ἡ ΑΒ τῇ ΓΔ· ἴση ἄρα καὶ ἡ ΑΖ τῇ ΓΗ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΕ τῇ ΕΓ, ἴσον καὶ τὸ ἀπὸ τῆς ΑΕ τῷ ἀπὸ τῆς ΕΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΕ ἴσα τὰ ἀπὸ τῶν ΑΖ, ΕΖ· ὀρθὴ γὰρ ἡ πρὸς τῷ Ζ γωνία· τῷ δὲ ἀπὸ τῆς ΕΓ ἴσα τὰ ἀπὸ τῶν ΕΗ, ΗΓ· ὀρθὴ γὰρ ἡ πρὸς τῷ Η γωνία· τὰ ἄρα ἀπὸ τῶν ΑΖ, ΖΕ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΓΗ, ΗΕ, ὧν τὸ ἀπὸ τῆς ΑΖ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΗ· ἴση γάρ ἐστιν ἡ ΑΖ τῇ ΓΗ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΖΕ τῷ ἀπὸ τῆς ΕΗ ἴσον ἐστίν· ἴση ἄρα ἡ ΕΖ τῇ ΕΗ. ἐν δὲ κύκλῳ ἴσον ἀπέχειν ἀπὸ τοῦ κέντρου εὐθεῖαι λέγονται, ὅταν αἱ ἀπὸ τοῦ κέντρου ἐπ' αὐτὰς κάθετοι ἀγόμεναι ἴσαι ὦσιν· αἱ ἄρα ΑΒ, ΓΔ ἴσον ἀπέχουσιν ἀπὸ τοῦ κέντρου.

Ἀλλὰ δὴ αἱ ΑΒ, ΓΔ εὐθεῖαι ἴσον ἀπεχέτωσαν ἀπὸ τοῦ κέντρου, τουτέστιν ἴση ἔστω ἡ ΕΖ τῇ ΕΗ. λέγω, ὅτι ἴση ἐστὶ καὶ ἡ ΑΒ τῇ ΓΔ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι διπλῆ ἐστιν ἡ μὲν ΑΒ τῆς ΑΖ, ἡ δὲ ΓΔ τῆς ΓΗ· καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΕ τῇ ΓΕ, ἴσον ἐστὶ τὸ ἀπὸ τῆς ΑΕ τῷ ἀπὸ τῆς ΓΕ· ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΕ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΕΖ,

ΖΑ, τῷ δὲ ἀπὸ τῆς ΓΕ ἴσα τὰ ἀπὸ τῶν ΕΗ, ΗΓ. τὰ ἄρα ἀπὸ τῶν ΕΖ, ΖΑ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΕΗ, ΗΓ· ὧν τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς ΕΗ ἐστιν ἴσον· ἴση γὰρ ἡ ΕΖ τῇ ΕΗ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΑΖ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΗ· ἴση ἄρα ἡ ΑΖ τῇ ΓΗ· καί ἐστι τῆς μὲν ΑΖ διπλῆ ἡ ΑΒ, τῆς δὲ ΓΗ διπλῆ ἡ ΓΔ· ἴση ἄρα ἡ ΑΒ τῇ ΓΔ.

Ἐν κύκλῳ ἄρα αἱ ἴσαι εὐθεῖαι ἴσον ἀπέχουσιν ἀπὸ τοῦ κέντρου, καὶ αἱ ἴσον ἀπέχουσαι ἀπὸ τοῦ κέντρου ἴσαι ἀλλήλαις εἰσίν· ὅπερ ἔδει δεῖξαι.

ιε΄. Ἐν κύκλῳ μεγίστη μὲν ἡ διάμετρος τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν.

Ἔστω κύκλος ὁ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἔστω ἡ ΑΔ, κέντρον δὲ τὸ Ε, καὶ ἔγγιον μὲν τῆς ΑΔ διαμέτρου ἔστω ἡ ΒΓ, ἀπώτερον δὲ ἡ ΖΗ· λέγω, ὅτι μεγίστη μέν ἐστιν ἡ ΑΔ, μείζων δὲ ἡ ΒΓ τῆς ΖΗ.

Ἤχθωσαν γὰρ ἀπὸ τοῦ Ε κέντρου ἐπὶ τὰς ΒΓ, ΖΗ κάθετοι αἱ ΕΘ, ΕΚ. καὶ ἐπεὶ ἔγγιον μὲν τοῦ κέντρου ἐστὶν ἡ ΒΓ, ἀπώτερον δὲ ἡ ΖΗ, μείζων ἄρα ἡ ΕΚ τῆς ΕΘ. κείσθω τῇ ΕΘ ἴση ἡ ΕΛ, καὶ διὰ τοῦ Λ τῇ ΕΚ πρὸς ὀρθὰς ἀχθεῖσα ἡ ΛΜ διήχθω ἐπὶ τὸ Ν, καὶ ἐπεζεύχθωσαν αἱ ΜΕ, ΕΝ, ΖΕ, ΕΗ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΕΘ τῇ ΕΛ, ἴση ἐστὶ καὶ ἡ ΒΓ τῇ ΜΝ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ μὲν ΑΕ τῇ ΕΜ, ἡ δὲ ΕΔ τῇ ΕΝ, ἡ ἄρα ΑΔ ταῖς ΜΕ, ΕΝ ἴση ἐστίν. ἀλλ’ αἱ μὲν ΜΕ, ΕΝ τῆς ΜΝ μείζονές εἰσιν [καὶ ἡ ΑΔ τῆς ΜΝ μείζων ἐστίν, ἴση δὲ ἡ ΜΝ τῇ ΒΓ· ἡ ΑΔ ἄρα τῆς ΒΓ μείζων ἐστίν. καὶ ἐπεὶ δύο αἱ ΜΕ, ΕΝ δύο ταῖς ΖΕ, ΕΗ ἴσαι εἰσίν, καὶ γωνία ἡ ὑπὸ ΜΕΝ γωνίας τῆς ὑπὸ ΖΕΗ μείζων [ἐστίν], βάσις ἄρα ἡ ΜΝ βάσεως τῆς ΖΗ μείζων ἐστίν. ἀλλὰ ἡ ΜΝ τῇ ΒΓ ἐδείχθη ἴση [καὶ ἡ ΒΓ τῆς ΖΗ μείζων ἐστίν]. μεγίστη μὲν ἄρα ἡ ΑΔ διάμετρος, μείζων δὲ ἡ ΒΓ τῆς ΖΗ.

Ἐν κύκλῳ ἄρα μεγίστη μέν ἐστιν ἡ διάμετρος, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν· ὅπερ ἔδει δεῖξαι.

ις΄. Ἡ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ᾽ ἄκρας ἀγομένη ἐκτὸς πεσεῖται τοῦ κύκλου, καὶ εἰς τὸν μεταξὺ τόπον τῆς τε εὐθείας καὶ τῆς περιφερείας ἑτέρα εὐθεῖα οὐ παρεμπεσεῖται, καὶ ἡ μὲν τοῦ ἡμικυκλίου γωνία ἁπάσης γωνίας ὀξείας εὐθυγράμμου μείζων ἐστίν, ἡ δὲ λοιπὴ ἐλάττων.

Ἔστω κύκλος ὁ ΑΒΓ περὶ κέντρον τὸ Δ καὶ διάμετρον τὴν ΑΒ· λέγω, ὅτι ἡ ἀπὸ τοῦ Α τῇ ΑΒ πρὸς ὀρθὰς ἀπ᾽ ἄκρας ἀγομένη ἐκτὸς πεσεῖται τοῦ κύκλου.

Μὴ γάρ, ἀλλ᾽ εἰ δυνατόν, πιπτέτω ἐντὸς ὡς ἡ ΓΑ, καὶ ἐπεζεύχθω ἡ ΔΓ.

Ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΔΓ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΔΑΓ γωνίᾳ τῇ ὑπὸ ΑΓΔ. ὀρθὴ δὲ ἡ ὑπὸ ΔΑΓ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΑΓΔ· τριγώνου δὴ τοῦ ΑΓΔ αἱ δύο γωνίαι αἱ ὑπὸ ΔΑΓ, ΑΓΔ δύο ὀρθαῖς ἴσαι εἰσίν· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ἀπὸ τοῦ Α σημείου τῇ ΒΑ πρὸς ὀρθὰς ἀγομένη ἐντὸς πεσεῖται τοῦ κύκλου. ὁμοίως δὴ δείξομεν, ὅτι οὐδ᾽ ἐπὶ τῆς περιφερείας· ἐκτὸς ἄρα.

Πιπτέτω ὡς ἡ ΑΕ· λέγω δή, ὅτι εἰς τὸν μεταξὺ τόπον τῆς τε ΑΕ εὐθείας καὶ τῆς ΓΘΑ περιφερείας ἑτέρα εὐθεῖα οὐ παρεμπεσεῖται.

Εἰ γὰρ δυνατόν, παρεμπιπτέτω ὡς ἡ ΖΑ, καὶ ἤχθω ἀπὸ τοῦ Δ σημείου ἐπὶ τὴν ΖΑ κάθετος ἡ ΔΗ. καὶ ἐπεὶ ὀρθή ἐστιν ἡ ὑπὸ ΑΗΔ, ἐλάττων δὲ ὀρθῆς ἡ ὑπὸ ΔΑΗ, μείζων ἄρα ἡ ΑΔ τῆς ΔΗ. ἴση δὲ ἡ ΔΑ τῇ ΔΘ· μείζων ἄρα ἡ ΔΘ τῆς ΔΗ, ἡ ἐλάττων τῆς μείζονος· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα εἰς τὸν μεταξὺ τόπον τῆς τε εὐθείας καὶ τῆς περιφερείας ἑτέρα εὐθεῖα παρεμπεσεῖται.

Λέγω, ὅτι καὶ ἡ μὲν τοῦ ἡμικυκλίου γωνία ἡ περιεχομένη ὑπό τε τῆς ΒΑ εὐθείας καὶ τῆς ΓΘΑ περιφερείας ἁπάσης γωνίας ὀξείας εὐθυγράμμου μείζων ἐστίν, ἡ δὲ λοιπὴ ἡ περιεχομένη ὑπό τε τῆς ΓΘΑ περιφερείας καὶ τῆς ΑΕ εὐθείας ἁπάσης γωνίας ὀξείας εὐθυγράμμου ἐλάττων ἐστίν.

Εἰ γὰρ ἐστί τις γωνία εὐθύγραμμος μείζων μὲν τῆς περιεχομένης ὑπό τε τῆς ΒΑ εὐθείας καὶ τῆς ΓΘΑ περιφερείας, ἐλάττων δὲ τῆς περιεχομένης ὑπό τε τῆς ΓΘΑ περιφερείας καὶ τῆς ΑΕ εὐθείας, εἰς τὸν μεταξὺ τόπον τῆς τε ΓΘΑ περιφερείας καὶ τῆς ΑΕ εὐθείας εὐθεῖα περεμπεσεῖται, ἥτις ποιήσει

μείζονα μὲν τῆς περιεχομένης ὑπό τε τῆς ΒΑ εὐθείας καὶ τῆς ΓΘΑ περιφερείας ὑπὸ εὐθειῶν περιεχομένην, ἐλάττονα δὲ τῆς περιεχομένης ὑπό τε τῆς ΓΘΑ περιφερείας καὶ τῆς ΑΕ εὐθείας. οὐ παρεμπίπτει δέ· οὐκ ἄρα τῆς περιεχομένης γωνίας ὑπό τε τῆς ΒΑ εὐθείας καὶ τῆς ΓΘΑ περιφερείας ἔσται μείζων ὀξεῖα ὑπὸ εὐθειῶν περιεχομένη, οὐδὲ μὴν ἐλάττων τῆς περιεχομένης ὑπό τε τῆς ΓΘΑ περιφερείας καὶ τῆς ΑΕ εὐθείας.

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι ἡ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ᾽ ἄκρας ἀγομένη ἐφάπτεται τοῦ κύκλου [καὶ ὅτι εὐθεῖα κύκλου καθ᾽ ἓν μόνον ἐφάπτεται σημεῖον, ἐπειδήπερ καὶ ἡ κατὰ δύο αὐτῷ συμβάλλουσα ἐντὸς αὐτοῦ πίπτουσα ἐδείχθη]. ὅπερ ἔδει δεῖξαι.

ιζ΄. Ἀπὸ τοῦ δοθέντος σημείου τοῦ δοθέντος κύκλου ἐφαπτομένην εὐθεῖαν γραμμὴν ἀγαγεῖν.

Ἔστω τὸ μὲν δοθὲν σημεῖον τὸ Α, ὁ δὲ δοθεὶς κύκλος ὁ ΒΓΔ· δεῖ δὴ ἀπὸ τοῦ Α σημείου τοῦ ΒΓΔ κύκλου ἐφαπτομένην εὐθεῖαν γραμμὴν ἀγαγεῖν.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ κύκλου τὸ Ε, καὶ ἐπεζεύχθω ἡ ΑΕ, καὶ κέντρῳ μὲν τῷ Ε διαστήματι δὲ τῷ ΕΑ κύκλος γεγράφθω ὁ ΑΖΗ, καὶ ἀπὸ τοῦ Δ τῇ ΕΑ πρὸς ὀρθὰς ἤχθω ἡ ΔΖ, καὶ ἐπεζεύχθωσαν αἱ ΕΖ, ΑΒ· λέγω, ὅτι ἀπὸ τοῦ Α σημείου τοῦ ΒΓΔ κύκλου ἐφαπτομένη ἦκται ἡ ΑΒ.

Ἐπεὶ γὰρ τὸ Ε κέντρον ἐστὶ τῶν ΒΓΔ, ΑΖΗ κύκλων, ἴση ἄρα ἐστὶν ἡ μὲν ΕΑ τῇ ΕΖ, ἡ δὲ ΕΔ τῇ ΕΒ· δύο δὴ αἱ ΑΕ, ΕΒ δύο ταῖς ΖΕ, ΕΔ ἴσαι εἰσίν· καὶ γωνίαν κοινὴν περιέχουσι τὴν πρὸς τῷ Ε· βάσις ἄρα ἡ ΔΖ βάσει τῇ ΑΒ ἴση ἐστίν, καὶ τὸ ΔΕΖ τρίγωνον τῷ ΕΒΑ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις· ἴση ἄρα ἡ ὑπὸ ΕΔΖ τῇ ὑπὸ ΕΒΑ. ὀρθὴ δὲ ἡ ὑπὸ ΕΔΖ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΕΒΑ. καί ἐστιν ἡ ΕΒ ἐκ τοῦ κέντρου· ἡ δὲ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ᾽ ἄκρας ἀγομένη ἐφάπτεται τοῦ κύκλου· ἡ ΑΒ ἄρα ἐφάπτεται τοῦ ΒΓΔ κύκλου.

Ἀπὸ τοῦ ἄρα δοθέντος σημείου τοῦ Α τοῦ δοθέντος κύκλου τοῦ ΒΓΔ ἐφαπτομένη εὐθεῖα γραμμὴ ἦκται ἡ ΑΒ· ὅπερ ἔδει ποιῆσαι.

ιη΄. Ἐὰν κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τοῦ κέντρου ἐπὶ τὴν ἁφὴν ἐπιζευχθῇ τις εὐθεῖα, ἡ ἐπιζευχθεῖσα κάθετος ἔσται ἐπὶ τὴν ἐφαπτομένην.

Κύκλου γὰρ τοῦ ΑΒΓ ἐφαπτέσθω τις εὐθεῖα ἡ ΔΕ κατὰ τὸ Γ σημεῖον, καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓ κύκλου τὸ Ζ, καὶ ἀπὸ τοῦ Ζ ἐπὶ τὸ Γ ἐπεζεύχθω ἡ ΖΓ· λέγω, ὅτι ἡ ΖΓ κάθετός ἐστιν ἐπὶ τὴν ΔΕ.

Εἰ γὰρ μή, ἤχθω ἀπὸ τοῦ Ζ ἐπὶ τὴν ΔΕ κάθετος ἡ ΖΗ.

Ἐπεὶ οὖν ἡ ὑπὸ ΖΗΓ γωνία ὀρθή ἐστιν, ὀξεῖα ἄρα ἐστὶν ἡ ὑπὸ ΖΓΗ· ὑπὸ δὲ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει· μείζων ἄρα ἡ ΖΓ τῆς ΖΗ· ἴση δὲ ἡ ΖΓ τῇ ΖΒ· μείζων ἄρα καὶ ἡ ΖΒ τῆς ΖΗ ἡ ἐλάττων τῆς μείζονος· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ΖΗ κάθετός ἐστιν ἐπὶ τὴν ΔΕ. ὁμοίως δὴ δείξομεν, ὅτι οὐδ’ ἄλλη τις πλὴν τῆς ΖΓ· ἡ ΖΓ ἄρα κάθετός ἐστιν ἐπὶ τὴν ΔΕ.

Ἐὰν ἄρα κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τοῦ κέντρου ἐπὶ τὴν ἁφὴν ἐπιζευχθῇ τις εὐθεῖα, ἡ ἐπιζευχθεῖσα κάθετος ἔσται ἐπὶ τὴν ἐφαπτομένην· ὅπερ ἔδει δεῖξαι.

ιθ΄. Ἐὰν κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τῆς ἁφῆς τῇ ἐφαπτομένῃ πρὸς ὀρθὰς [γωνίας] εὐθεῖα γραμμὴ ἀχθῇ, ἐπὶ τῆς ἀχθείσης ἔσται τὸ κέντρον τοῦ κύκλου.

Κύκλου γὰρ τοῦ ΑΒΓ ἐφαπτέσθω τις εὐθεῖα ἡ ΔΕ κατὰ τὸ Γ σημεῖον, καὶ ἀπὸ τοῦ Γ τῇ ΔΕ πρὸς ὀρθὰς ἤχθω ἡ ΓΑ· λέγω, ὅτι ἐπὶ τῆς ΑΓ ἐστι τὸ κέντρον τοῦ κύκλου.

Μὴ γάρ, ἀλλ’ εἰ δυνατόν, ἔστω τὸ Ζ, καὶ ἐπεζεύχθω ἡ ΓΖ

Ἐπεὶ [οὖν] κύκλου τοῦ ΑΒΓ ἐφάπτεταί τις εὐθεῖα ἡ ΔΕ, ἀπὸ δὲ τοῦ κέντρου ἐπὶ τὴν ἁφὴν ἐπέζευκται ἡ ΖΓ, ἡ ΖΓ ἄρα κάθετός ἐστιν ἐπὶ τὴν ΔΕ· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΖΓΕ. ἐστὶ δὲ καὶ ἡ ὑπὸ ΑΓΕ ὀρθή· ἴση ἄρα ἐστὶν ἡ ὑπὸ

ΖΓΕ τῇ ὑπὸ ΑΓΕ ἡ ἐλάττων τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὸ Ζ κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου. ὁμοίως δὴ δείξομεν, ὅτι οὐδ' ἄλλο τι πλὴν ἐπὶ τῆς ΑΓ.

Ἐὰν ἄρα κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τῆς ἁφῆς τῇ ἐφαπτομένῃ πρὸς ὀρθὰς εὐθεῖα γραμμὴ ἀχθῇ, ἐπὶ τῆς ἀχθείσης ἔσται τὸ κέντρον τοῦ κύκλου· ὅπερ ἔδει δεῖξαι.

κ΄. Ἐν κύκλῳ ἡ πρὸς τῷ κέντρῳ γωνία διπλασίων ἐστὶ τῆς πρὸς τῇ περιφερείᾳ, ὅταν τὴν αὐτὴν περιφέρειαν βάσιν ἔχωσιν αἱ γωνίαι..

Ἔστω κύκλος ὁ ΑΒΓ, καὶ πρὸς μὲν τῷ κέντρῳ αὐτοῦ γωνία ἔστω ἡ ὑπὸ ΒΕΓ, πρὸς δὲ τῇ περιφερείᾳ ἡ ὑπὸ ΒΑΓ, ἐχέτωσαν δὲ τὴν αὐτὴν περιφέρειαν βάσιν τὴν ΒΓ· λέγω, ὅτι διπλασίων ἐστὶν ἡ ὑπὸ ΒΕΓ γωνία τῆς ὑπὸ ΒΑΓ.

Ἐπιζευχθεῖσα γὰρ ἡ ΑΕ διήχθω ἐπὶ τὸ Ζ.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΕΑ τῇ ΕΒ, ἴση καὶ γωνία ἡ ὑπὸ ΕΑΒ τῇ ὑπὸ ΕΒΑ· αἱ ἄρα ὑπὸ ΕΑΒ, ΕΒΑ γωνίαι τῆς ὑπὸ ΕΑΒ διπλασίους εἰσίν. ἴση δὲ ἡ ὑπὸ ΒΕΖ ταῖς ὑπὸ ΕΑΒ, ΕΒΑ· καὶ ἡ ὑπὸ ΒΕΖ ἄρα τῆς ὑπὸ ΕΑΒ ἐστι διπλῆ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΖΕΓ τῆς ὑπὸ ΕΑΓ ἐστι διπλῆ. ὅλη ἄρα ἡ ὑπὸ ΒΕΓ ὅλης τῆς ὑπὸ ΒΑΓ ἐστι διπλῆ.

Κεκλάσθω δὴ πάλιν, καὶ ἔστω ἑτέρα γωνία ἡ ὑπὸ ΒΔΓ, καὶ ἐπιζευχθεῖσα ἡ ΔΕ ἐκβεβλήσθω ἐπὶ τὸ Η. ὁμοίως δὴ δείξομεν, ὅτι διπλῆ ἐστιν ἡ ὑπὸ ΗΕΓ γωνία τῆς ὑπὸ ΕΔΓ, ὧν ἡ ὑπὸ ΗΕΒ διπλῆ ἐστι τῆς ὑπὸ ΕΔΒ· λοιπὴ ἄρα ἡ ὑπὸ ΒΕΓ διπλῆ ἐστι τῆς ὑπὸ ΒΔΓ.

Ἐν κύκλῳ ἄρα ἡ πρὸς τῷ κέντρῳ γωνία διπλασίων ἐστὶ τῆς πρὸς τῇ περιφερείᾳ, ὅταν τὴν αὐτὴν περιφέρειαν βάσιν ἔχωσιν [αἱ γωνίαι]· ὅπερ ἔδει δεῖξαι.

κα΄. Ἐν κύκλῳ αἱ ἐν τῷ αὐτῷ τμήματι γωνίαι ἴσαι ἀλλήλαις εἰσίν.

Ἔστω κύκλος ὁ ΑΒΓΔ, καὶ ἐν τῷ αὐτῷ τμήματι τῷ ΒΑΕΔ γωνίαι ἔστωσαν αἱ ὑπὸ ΒΑΔ, ΒΕΔ· λέγω, ὅτι αἱ ὑπὸ ΒΑΔ, ΒΕΔ γωνίαι ἴσαι ἀλλήλαις εἰσίν.

Εἰλήφθω γὰρ τοῦ ΑΒΓΔ κύκλου τὸ κέντρον, καὶ ἔστω τὸ Ζ, καὶ ἐπεζεύχθωσαν αἱ ΒΖ, ΖΔ.

Καὶ ἐπεὶ ἡ μὲν ὑπὸ ΒΖΔ γωνία πρὸς τῷ κέντρῳ ἐστίν, ἡ δὲ ὑπὸ ΒΑΔ πρὸς τῇ περιφερείᾳ, καὶ ἔχουσι τὴν αὐτὴν περιφέρειαν βάσιν τὴν ΒΓΔ, ἡ ἄρα ὑπὸ ΒΖΔ γωνία διπλασίων ἐστὶ τῆς ὑπὸ ΒΑΔ. διὰ τὰ αὐτὰ δὴ ἡ ὑπὸ ΒΖΔ καὶ τῆς ὑπὸ ΒΕΔ ἐστι διπλασίων· ἴση ἄρα ἡ ὑπὸ ΒΑΔ τῇ ὑπὸ ΒΕΔ.

Ἐν κύκλῳ ἄρα αἱ ἐν τῷ αὐτῷ τμήματι γωνίαι ἴσαι ἀλλήλαις εἰσίν· ὅπερ ἔδει δεῖξαι.

κβ΄. Τῶν ἐν τοῖς κύκλοις τετραπλεύρων αἱ ἀπεναντίον γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Ἔστω κύκλος ὁ ΑΒΓΔ, καὶ ἐν αὐτῷ τετράπλευρον ἔστω τὸ ΑΒΓΔ· λέγω, ὅτι αἱ ἀπεναντίον γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Ἐπεζεύχθωσαν αἱ ΑΓ, ΒΔ.

Ἐπεὶ οὖν παντὸς τριγώνου αἱ τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν, τοῦ ΑΒΓ ἄρα τριγώνου αἱ τρεῖς γωνίαι αἱ ὑπὸ ΓΑΒ, ΑΒΓ, ΒΓΑ δυσὶν ὀρθαῖς ἴσαι εἰσίν. ἴση δὲ ἡ μὲν ὑπὸ ΓΑΒ τῇ ὑπὸ ΒΔΓ· ἐν γὰρ τῷ αὐτῷ τμήματί εἰσι τῷ ΒΑΔΓ· ἡ δὲ ὑπὸ ΑΓΒ τῇ ὑπὸ ΑΔΒ· ἐν γὰρ τῷ αὐτῷ τμήματί εἰσι τῷ ΑΔΓΒ· ὅλη ἄρα ἡ ὑπὸ ΑΔΓ ταῖς ὑπὸ ΒΑΓ, ΑΓΒ ἴση ἐστίν. κοινὴ προσκείσθω ἡ ὑπὸ ΑΒΓ· αἱ ἄρα ὑπὸ ΑΒΓ, ΒΑΓ, ΑΓΒ ταῖς ὑπὸ ΑΒΓ, ΑΔΓ ἴσαι εἰσίν. ἀλλ’ αἱ ὑπὸ ΑΒΓ, ΒΑΓ, ΑΓΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν. καὶ αἱ ὑπὸ ΑΒΓ, ΑΔΓ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσίν. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ ὑπὸ ΒΑΔ, ΔΓΒ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Τῶν ἄρα ἐν τοῖς κύκλοις τετραπλεύρων αἱ ἀπεναντίον γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν· ὅπερ ἔδει δεῖξαι.

κγ΄. Ἐπὶ τῆς αὐτῆς εὐθείας δύο τμήματα κύκλων ὅμοια καὶ ἄνισα οὐ συσταθήσεται ἐπὶ τὰ αὐτὰ μέρη.

Εἰ γὰρ δυνατόν, ἐπὶ τῆς αὐτῆς εὐθείας τῆς ΑΒ δύο τμήματα κύκλων ὅμοια καὶ ἄνισα συνεστάτω ἐπὶ τὰ αὐτὰ μέρη τὰ ΑΓΒ, ΑΔΒ, καὶ διήχθω ἡ ΑΓΔ, καὶ ἐπεζεύχθωσαν αἱ ΓΒ, ΔΒ.

Ἐπεὶ οὖν ὅμοιόν ἐστι τὸ ΑΓΒ τμῆμα τῷ ΑΔΒ τμήματι, ὅμοια δὲ τμήματα κύκλων ἐστὶ τὰ δεχόμενα γωνίας ἴσας, ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΓΒ γωνία τῇ ὑπὸ ΑΔΒ ἡ ἐκτὸς τῇ ἐντός· ὅπερ ἐστὶν ἀδύνατον.

Οὐκ ἄρα ἐπὶ τῆς αὐτῆς εὐθείας δύο τμήματα κύκλων ὅμοια καὶ ἄνισα συσταθήσεται ἐπὶ τὰ αὐτὰ μέρη· ὅπερ ἔδει δεῖξαι.

κδ΄. Τὰ ἐπὶ ἴσων εὐθειῶν ὅμοια τμήματα κύκλων ἴσα ἀλλήλοις ἐστίν.

Ἔστωσαν γὰρ ἐπὶ ἴσων εὐθειῶν τῶν ΑΒ, ΓΔ ὅμοια τμήματα κύκλων τὰ ΑΕΒ, ΓΖΔ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΕΒ τμῆμα τῷ ΓΖΔ τμήματι.

Ἐφαρμοζομένου γὰρ τοῦ ΑΕΒ τμήματος ἐπὶ τὸ ΓΖΔ καὶ τιθεμένου τοῦ μὲν Α σημείου ἐπὶ τὸ Γ τῆς δὲ ΑΒ εὐθείας ἐπὶ τὴν ΓΔ, ἐφαρμόσει καὶ τὸ Β σημεῖον ἐπὶ τὸ Δ σημεῖον διὰ τὸ ἴσην εἶναι τὴν ΑΒ τῇ ΓΔ· τῆς δὲ ΑΒ ἐπὶ τὴν ΓΔ ἐφαρμοσάσης ἐφαρμόσει καὶ τὸ ΑΕΒ τμῆμα ἐπὶ τὸ ΓΖΔ. εἰ γὰρ ἡ ΑΒ εὐθεῖα ἐπὶ τὴν ΓΔ ἐφαρμόσει, τὸ δὲ ΑΕΒ τμῆμα ἐπὶ τὸ ΓΖΔ μὴ ἐφαρμόσει, ἤτοι ἐντὸς αὐτοῦ πεσεῖται ἢ ἐκτὸς ἢ παραλλάξει ὡς τὸ ΓΗΔ, καὶ κύκλος κύκλον τέμνει κατὰ πλείονα σημεῖα ἢ δύο· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἐφαρμοζομένης τῆς ΑΒ εὐθείας ἐπὶ τὴν ΓΔ οὐκ ἐφαρμόσει καὶ τὸ ΑΕΒ τμῆμα ἐπὶ τὸ ΓΖΔ· ἐφαρμόσει ἄρα, καὶ ἴσον αὐτῷ ἔσται.

Τὰ ἄρα ἐπὶ ἴσων εὐθειῶν ὅμοια τμήματα κύκλων ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

κε΄. Κύκλου τμήματος δοθέντος προσαναγράψαι τὸν κύκλον, οὗπέρ ἐστι τμῆμα.

Ἔστω τὸ δοθὲν τμῆμα κύκλου τὸ ΑΒΓ· δεῖ δὴ τοῦ ΑΒΓ τμήματος προσαναγράψαι τὸν κύκλον, οὗπέρ ἐστι τμῆμα.

Τετμήσθω γὰρ ἡ ΑΓ δίχα κατὰ τὸ Δ, καὶ ἤχθω ἀπὸ τοῦ Δ σημείου τῇ ΑΓ πρὸς ὀρθὰς ἡ ΔΒ, καὶ ἐπεζεύχθω ἡ ΑΒ· ἡ ὑπὸ ΑΒΔ γωνία ἄρα τῆς ὑπὸ ΒΑΔ ἤτοι μείζων ἐστὶν ἢ ἴση ἢ ἐλάττων.

Ἔστω πρότερον μείζων, καὶ συνεστάτω πρὸς τῇ ΒΑ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ ὑπὸ ΑΒΔ γωνίᾳ ἴση ἡ ὑπὸ ΒΑΕ, καὶ διήχθω ἡ ΔΒ ἐπὶ τὸ Ε, καὶ ἐπεζεύχθω ἡ ΕΓ. ἐπεὶ οὖν ἴση ἐστὶν ἡ ὑπὸ ΑΒΕ γωνία τῇ ὑπὸ ΒΑΕ, ἴση ἄρα ἐστὶ καὶ ἡ ΕΒ εὐθεῖα τῇ ΕΑ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΔΓ, κοινὴ δὲ ἡ ΔΕ, δύο δὴ αἱ ΑΔ, ΔΕ δύο ταῖς ΓΔ, ΔΕ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΑΔΕ γωνίᾳ τῇ ὑπὸ ΓΔΕ ἐστιν ἴση· ὀρθὴ γὰρ ἑκατέρα· βάσις ἄρα ἡ ΑΕ βάσει τῇ ΓΕ ἐστιν ἴση. ἀλλὰ ἡ ΑΕ τῇ ΒΕ ἐδείχθη ἴση· καὶ ἡ ΒΕ ἄρα τῇ ΓΕ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΑΕ, ΕΒ, ΕΓ ἴσαι ἀλλήλαις εἰσίν· ὁ ἄρα κέντρῳ τῷ Ε διαστήματι δὲ ἑνὶ τῶν ΑΕ, ΕΒ, ΕΓ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἔσται προσαναγεγραμμένος. κύκλου ἄρα τμήματος δοθέντος προσαναγέγραπται ὁ κύκλος. καὶ δῆλον, ὡς τὸ ΑΒΓ τμῆμα ἔλαττόν ἐστιν ἡμικυκλίου διὰ τὸ τὸ Ε κέντρον ἐκτὸς αὐτοῦ τυγχάνειν.

Ὁμοίως [δὲ] κἂν ᾖ ἡ ὑπὸ ΑΒΔ γωνία ἴση τῇ ὑπὸ ΒΑΔ, τῆς ΑΔ ἴσης γενομένης ἑκατέρᾳ τῶν ΒΔ, ΔΓ αἱ τρεῖς αἱ ΔΑ, ΔΒ, ΔΓ ἴσαι ἀλλήλαις ἔσονται, καὶ ἔσται τὸ Δ κέντρον τοῦ προσαναπεπληρωμένου κύκλου, καὶ δηλαδὴ ἔσται τὸ ΑΒΓ ἡμικύκλιον.

Ἐὰν δὲ ἡ ὑπὸ ΑΒΔ ἐλάττων ᾖ τῆς ὑπὸ ΒΑΔ, καὶ συστησώμεθα πρὸς τῇ ΒΑ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ ὑπὸ ΑΒΔ γωνίᾳ ἴσην, ἐντὸς τοῦ ΑΒΓ τμήματος πεσεῖται τὸ κέντρον ἐπὶ τῆς ΔΒ, καὶ ἔσται δηλαδὴ τὸ ΑΒΓ τμῆμα μεῖζον ἡμικυκλίου.

Κύκλου ἄρα τμήματος δοθέντος προσαναγέγραπται ὁ κύκλος· ὅπερ ἔδει ποιῆσαι.

κς΄. Ἐν τοῖς ἴσοις κύκλοις αἱ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ὦσι βεβηκυῖαι.

Ἑστωσαν ἴσοι κύκλοι οἱ ΑΒΓ, ΔΕΖ καὶ ἐν αὐτοῖς ἴσαι γωνίαι ἑστωσαν πρὸς μὲν τοῖς κέντροις αἱ ὑπὸ ΒΗΓ, ΕΘΖ, πρὸς δὲ ταῖς περιφερείαις αἱ ὑπὸ ΒΑΓ, ΕΔΖ· λέγω, ὅτι ἴση ἐστὶν ἡ ΒΚΓ περιφέρεια τῇ ΕΛΖ περιφερείᾳ.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΓ, ΕΖ.

Καὶ ἐπεὶ ἴσοι εἰσὶν οἱ ΑΒΓ, ΔΕΖ κύκλοι, ἴσαι εἰσὶν αἱ ἐκ τῶν κέντρων· δύο δὴ αἱ ΒΗ, ΗΓ δύο ταῖς ΕΘ, ΘΖ ἴσαι· καὶ γωνία ἡ πρὸς τῷ

Η γωνίᾳ τῇ πρὸς τῷ Θ ἴση· βάσις ἄρα ἡ ΒΓ βάσει τῇ ΕΖ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ πρὸς τῷ Α γωνία τῇ πρὸς τῷ Δ, ὅμοιον ἄρα ἐστὶ τὸ ΒΑΓ τμῆμα τῷ ΕΔΖ τμήματι· καί εἰσιν ἐπὶ ἴσων εὐθειῶν [τῶν ΒΓ, ΕΖ]· τὰ δὲ ἐπὶ ἴσων εὐθειῶν ὅμοια τμήματα κύκλων ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα τὸ ΒΑΓ τμῆμα τῷ ΕΔΖ. ἔστι δὲ καὶ ὅλος ὁ ΑΒΓ κύκλος ὅλῳ τῷ ΔΕΖ κύκλῳ ἴσος· λοιπὴ ἄρα ἡ ΒΚΓ περιφέρεια τῇ ΕΛΖ περιφερείᾳ ἐστὶν ἴση.

Ἐν ἄρα τοῖς ἴσοις κύκλοις αἱ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ὦσι βεβηκυῖαι· ὅπερ ἔδει δεῖξαι.

κζ΄. Ἐν τοῖς ἴσοις κύκλοις αἱ ἐπὶ ἴσων περιφερειῶν βεβηκυῖαι γωνίαι ἴσαι ἀλλήλαις εἰσίν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ὦσι βεβηκυῖαι.

Ἐν γὰρ ἴσοις κύκλοις τοῖς ΑΒΓ, ΔΕΖ ἐπὶ ἴσων περιφερειῶν τῶν ΒΓ, ΕΖ πρὸς μὲν τοῖς Η, Θ κέντροις γωνίαι βεβηκέτωσαν αἱ ὑπὸ ΒΗΓ, ΕΘΖ, πρὸς δὲ ταῖς περιφερείαις αἱ ὑπὸ ΒΑΓ, ΕΔΖ· λέγω, ὅτι ἡ μὲν ὑπὸ ΒΗΓ γωνία τῇ ὑπὸ ΕΘΖ ἐστιν ἴση, ἡ δὲ ὑπὸ ΒΑΓ τῇ ὑπὸ ΕΔΖ ἐστιν ἴση.

Εἰ γὰρ ἄνισός ἐστιν ἡ ὑπὸ ΒΗΓ τῇ ὑπὸ ΕΘΖ, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ ὑπὸ ΒΗΓ, καὶ συνεστάτω πρὸς τῇ ΒΗ εὐθείᾳ καὶ τῷ πρὸς

αὐτῇ σημείῳ τῷ Η τῇ ὑπὸ ΕΘΖ γωνίᾳ ἴση ἡ ὑπὸ ΒΗΚ· αἱ δὲ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν, ὅταν πρὸς τοῖς κέντροις ὦσιν· ἴση ἄρα ἡ ΒΚ περιφέρεια τῇ ΕΖ περιφερείᾳ. ἀλλὰ ἡ ΕΖ τῇ ΒΓ ἐστιν ἴση· καὶ ἡ ΒΚ ἄρα τῇ ΒΓ ἐστιν ἴση ἡ ἐλάττων τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἄνισός ἐστιν ἡ ὑπὸ ΒΗΓ γωνία τῇ ὑπὸ ΕΘΖ· ἴση ἄρα. καί ἐστι τῆς μὲν ὑπὸ ΒΗΓ ἡμίσεια ἡ πρὸς τῷ Α, τῆς δὲ ὑπὸ ΕΘΖ ἡμίσεια ἡ πρὸς τῷ Δ· ἴση ἄρα καὶ ἡ πρὸς τῷ Α γωνία τῇ πρὸς τῷ Δ.

Ἐν ἄρα τοῖς ἴσοις κύκλοις αἱ ἐπὶ ἴσων περιφερειῶν βεβηκυῖαι γωνίαι ἴσαι ἀλλήλαις εἰσίν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ὦσι βεβηκυῖαι· ὅπερ ἔδει δεῖξαι.

κη΄. Ἐν τοῖς ἴσοις κύκλοις αἱ ἴσαι εὐθεῖαι ἴσας περιφερείας ἀφαιροῦσι τὴν μὲν μείζονα τῇ μείζονι τὴν δὲ ἐλάττονα τῇ ἐλάττονι.

Ἔστωσαν ἴσοι κύκλοι οἱ ΑΒΓ, ΔΕΖ, καὶ ἐν τοῖς κύκλοις ἴσαι εὐθεῖαι ἔστωσαν αἱ ΑΒ, ΔΕ τὰς μὲν ΑΓΒ, ΔΖΕ περιφερείας μείζονας ἀφαιροῦσαι τὰς δὲ ΑΗΒ, ΔΘΕ ἐλάττονας· λέγω, ὅτι ἡ μὲν ΑΓΒ μείζων περιφέρεια ἴση ἐστὶ τῇ ΔΖΕ μείζονι περιφερείᾳ, ἡ δὲ ΑΗΒ ἐλάττων περιφέρεια τῇ ΔΘΕ.

Εἰλήφθω γὰρ τὰ κέντρα τῶν κύκλων τὰ Κ, Λ, καὶ ἐπεζεύχθωσαν αἱ ΑΚ, ΚΒ, ΔΛ, ΛΕ.

Καὶ ἐπεὶ ἴσοι κύκλοι εἰσίν, ἴσαι εἰσὶ καὶ αἱ ἐκ τῶν κέντρων· δύο δὴ αἱ ΑΚ, ΚΒ δυσὶ ταῖς ΔΛ, ΛΕ ἴσαι εἰσίν· καὶ βάσις ἡ ΑΒ βάσει τῇ ΔΕ ἴση· γωνία ἄρα ἡ ὑπὸ ΑΚΒ γωνίᾳ τῇ ὑπὸ ΔΛΕ ἴση ἐστίν. αἱ δὲ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν, ὅταν πρὸς τοῖς κέντροις ὦσιν· ἴση ἄρα ἡ ΑΗΒ περιφέρεια τῇ ΔΘΕ. ἐστὶ δὲ καὶ ὅλος ὁ ΑΒΓ κύκλος ὅλῳ τῷ ΔΕΖ κύκλῳ ἴσος· καὶ λοιπὴ ἄρα ἡ ΑΓΒ περιφέρεια λοιπῇ τῇ ΔΖΕ περιφερείᾳ ἴση ἐστίν.

Ἐν ἄρα τοῖς ἴσοις κύκλοις αἱ ἴσαι εὐθεῖαι ἴσας περιφερείας ἀφαιροῦσι τὴν μὲν μείζονα τῇ μείζονι τὴν δὲ ἐλάττονα τῇ ἐλάττονι· ὅπερ ἔδει δεῖξαι.

κθ΄. Ἐν τοῖς ἴσοις κύκλοις τὰς ἴσας περιφερείας ἴσαι εὐθεῖαι ὑποτείνουσιν.

Ἔστωσαν ἴσοι κύκλοι οἱ ΑΒΓ, ΔΕΖ, καὶ ἐν αὐτοῖς ἴσαι περιφέρειαι ἀπειλήφθωσαν αἱ ΒΗΓ, ΕΘΖ, καὶ ἐπεζεύχθωσαν αἱ ΒΓ, ΕΖ εὐθεῖαι· λέγω, ὅτι ἴση ἐστὶν ἡ ΒΓ τῇ ΕΖ.

Εἰλήφθω γὰρ τὰ κέντρα τῶν κύκλων, καὶ ἔστω τὰ Κ, Λ, καὶ ἐπεζεύχθωσαν αἱ ΒΚ, ΚΓ, ΕΛ, ΛΖ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΗΓ περιφέρεια τῇ ΕΘΖ περιφερείᾳ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΒΚΓ τῇ ὑπὸ ΕΛΖ. καὶ ἐπεὶ ἴσοι εἰσὶν οἱ ΑΒΓ, ΔΕΖ κύκλοι, ἴσαι εἰσὶ καὶ αἱ ἐκ τῶν κέντρων· δύο δὴ αἱ ΒΚ, ΚΓ δυσὶ ταῖς ΕΛ, ΛΖ ἴσαι εἰσίν· καὶ γωνίας ἴσας περιέχουσιν· βάσις ἄρα ἡ ΒΓ βάσει τῇ ΕΖ ἴση ἐστίν.

Ἐν ἄρα τοῖς ἴσοις κύκλοις τὰς ἴσας περιφερείας ἴσαι εὐθεῖαι ὑποτείνουσιν· ὅπερ ἔδει δεῖξαι.

λ΄. Ἐν τοῖς ἴσοις κύκλοις τὰς ἴσας περιφερείας ἴσαι εὐθεῖαι ὑποτείνουσιν.

Ἔστω ἡ δοθεῖσα περιφέρεια ἡ ΑΔΒ· δεῖ δὴ τὴν ΑΔΒ περιφέρειαν δίχα τεμεῖν.

Ἐπεζεύχθω ἡ ΑΒ, καὶ τετμήσθω δίχα κατὰ τὸ Γ, καὶ ἀπὸ τοῦ Γ σημείου τῇ ΑΒ εὐθείᾳ πρὸς ὀρθὰς ἤχθω ἡ ΓΔ, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΒ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΒ, κοινὴ δὲ ἡ ΓΔ, δύο δὴ αἱ ΑΓ, ΓΔ δυσὶ ταῖς ΒΓ, ΓΔ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΑΓΔ γωνίᾳ τῇ ὑπὸ ΒΓΔ ἴση· ὀρθὴ γὰρ ἑκατέρα· βάσις ἄρα ἡ ΑΔ βάσει τῇ ΔΒ ἴση ἐστίν. αἱ δὲ ἴσαι εὐθεῖαι ἴσας περιφερείας ἀφαιροῦσι τὴν μὲν μείζονα τῇ μείζονι τὴν δὲ ἐλάττονα τῇ ἐλάττονι· καί ἐστιν ἑκατέρα τῶν ΑΔ, ΔΒ περιφερειῶν ἐλάττων ἡμικυκλίου· ἴση ἄρα ἡ ΑΔ περιφέρεια τῇ ΔΒ περιφερείᾳ.

Ἡ ἄρα δοθεῖσα περιφέρεια δίχα τέτμηται κατὰ τὸ Δ σημεῖον· ὅπερ ἔδει ποιῆσαι.

λα΄. Ἐν κύκλῳ ἡ μὲν ἐν τῷ ἡμικυκλίῳ γωνία ὀρθή ἐστιν, ἡ δὲ ἐν τῷ μείζονι τμήματι ἐλάττων ὀρθῆς, ἡ δὲ ἐν τῷ ἐλάττονι τμήματι μείζων ὀρθῆς· καὶ ἔτι ἡ μὲν τοῦ μείζονος τμήματος γωνία μείζων ἐστὶν ὀρθῆς, ἡ δὲ τοῦ ἐλάττονος τμήματος γωνία ἐλάττων ὀρθῆς.

Ἔστω κύκλος ὁ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἔστω ἡ ΒΓ, κέντρον δὲ τὸ Ε, καὶ ἐπεζεύχθωσαν αἱ ΒΑ, ΑΓ, ΑΔ, ΔΓ· λέγω, ὅτι ἡ μὲν ἐν τῷ ΒΑΓ ἡμικυκλίῳ γωνία ἡ ὑπὸ ΒΑΓ ὀρθή ἐστιν, ἡ δὲ ἐν τῷ ΑΒΓ μείζονι τοῦ ἡμικυκλίου τμήματι γωνία ἡ ὑπὸ ΑΒΓ ἐλάττων ἐστὶν ὀρθῆς, ἡ δὲ ἐν τῷ ΑΔΓ ἐλάττονι τοῦ ἡμικυκλίου τμήματι γωνία ἡ ὑπὸ ΑΔΓ μείζων ἐστὶν ὀρθῆς.

Ἐπεζεύχθω ἡ ΑΕ, καὶ διήχθω ἡ ΒΑ ἐπὶ τὸ Ζ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΕ τῇ ΕΑ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΑΒΕ τῇ ὑπὸ ΒΑΕ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΓΕ τῇ ΕΑ, ἴση ἐστὶ καὶ ἡ ὑπὸ ΑΓΕ τῇ ὑπὸ ΓΑΕ· ὅλη ἄρα ἡ ὑπὸ ΒΑΓ δυσὶ ταῖς ὑπὸ ΑΒΓ, ΑΓΒ ἴση ἐστίν. ἐστὶ δὲ καὶ ἡ ὑπὸ ΖΑΓ ἐκτὸς τοῦ ΑΒΓ τριγώνου δυσὶ ταῖς ὑπὸ ΑΒΓ, ΑΓΒ γωνίαις ἴση· ἴση ἄρα καὶ ἡ ὑπὸ ΒΑΓ γωνία τῇ ὑπὸ ΖΑΓ· ὀρθὴ ἄρα ἑκατέρα· ἡ ἄρα ἐν τῷ ΒΑΓ ἡμικυκλίῳ γωνία ἡ ὑπὸ ΒΑΓ ὀρθή ἐστιν.

Καὶ ἐπεὶ τοῦ ΑΒΓ τριγώνου δύο γωνίαι αἱ ὑπὸ ΑΒΓ, ΒΑΓ δύο ὀρθῶν ἐλάττονές εἰσιν, ὀρθὴ δὲ ἡ ὑπὸ ΒΑΓ, ἐλάττων ἄρα ὀρθῆς ἐστιν ἡ ὑπὸ ΑΒΓ γωνία· καί ἐστιν ἐν τῷ ΑΒΓ μείζονι τοῦ ἡμικυκλίου τμήματι.

Καὶ ἐπεὶ ἐν κύκλῳ τετράπλευρόν ἐστι τὸ ΑΒΓΔ, τῶν δὲ ἐν τοῖς κύκλοις τετραπλεύρων αἱ ἀπεναντίον γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν [αἱ ἄρα ὑπὸ ΑΒΓ, ΑΔΓ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν], καί ἐστιν ἡ ὑπὸ ΑΒΓ ἐλάττων ὀρθῆς· λοιπὴ ἄρα ἡ ὑπὸ ΑΔΓ γωνία μείζων ὀρθῆς ἐστιν· καί ἐστιν ἐν τῷ ΑΔΓ ἐλάττονι τοῦ ἡμικυκλίου τμήματι.

Λέγω, ὅτι καὶ ἡ μὲν τοῦ μείζονος τμήματος γωνία ἡ περιεχομένη ὑπό [τε] τῆς ΑΒΓ περιφερείας καὶ τῆς ΑΓ εὐθείας μείζων ἐστὶν ὀρθῆς, ἡ δὲ τοῦ ἐλάττονος τμήματος γωνία ἡ περιεχομένη ὑπό [τε] τῆς ΑΔ[Γ] περιφερείας καὶ τῆς ΑΓ εὐθείας ἐλάττων ἐστὶν ὀρθῆς. καί ἐστιν αὐτόθεν φανερόν. ἐπεὶ

γὰρ ἡ ὑπὸ τῶν ΒΑ, ΑΓ εὐθειῶν ὀρθή ἐστιν, ἡ ἄρα ὑπὸ τῆς ΑΒΓ περιφερείας καὶ τῆς ΑΓ εὐθείας περιεχομένη μείζων ἐστὶν ὀρθῆς. πάλιν, ἐπεὶ ἡ ὑπὸ τῶν ΑΓ, ΑΖ εὐθειῶν ὀρθή ἐστιν, ἡ ἄρα ὑπὸ τῆς ΓΑ εὐθείας καὶ τῆς ΑΔ[Γ] περιφερείας περιεχομένη ἐλάττων ἐστὶν ὀρθῆς.

Ἐν κύκλῳ ἄρα ἡ μὲν ἐν τῷ ἡμικυκλίῳ γωνία ὀρθή ἐστιν, ἡ δὲ ἐν τῷ μείζονι τμήματι ἐλάττων ὀρθῆς, ἡ δὲ ἐν τῷ ἐλάττονι [τμήματι] μείζων ὀρθῆς, καὶ ἔτι ἡ μὲν τοῦ μείζονος τμήματος [γωνία] μείζων [ἐστὶν] ὀρθῆς, ἡ δὲ τοῦ ἐλάττονος τμήματος [γωνία] ἐλάττων ὀρθῆς· ὅπερ ἔδει δεῖξαι.

[Πόρισμα: Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν [ἡ] μία γωνία τριγώνου ταῖς δυσὶν ἴση ᾖ, ὀρθή ἐστιν ἡ γωνία διὰ τὸ καὶ τὴν ἐκείνης ἐκτὸς ταῖς αὐταῖς ἴσην εἶναι· ἐὰν δὲ αἱ ἐφεξῆς ἴσαι ὦσιν, ὀρθαί εἰσιν.]

λβ΄. Ἐὰν κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τῆς ἁφῆς εἰς τὸν κύκλον διαχθῇ τις εὐθεῖα τέμνουσα τὸν κύκλον, ἃς ποιεῖ γωνίας πρὸς τῇ ἐφαπτομένῃ, ἴσαι ἔσονται ταῖς ἐν τοῖς ἐναλλὰξ τοῦ κύκλου τμήμασι γωνίαις.

Κύκλου γὰρ τοῦ ΑΒΓΔ ἐφαπτέσθω τις εὐθεῖα ἡ ΕΖ κατὰ τὸ Β σημεῖον, καὶ ἀπὸ τοῦ Β σημείου διήχθω τις εὐθεῖα εἰς τὸν ΑΒΓΔ κύκλον τέμνουσα αὐτὸν ἡ ΒΔ. λέγω, ὅτι ἃς ποιεῖ γωνίας ἡ ΒΔ μετὰ τῆς ΕΖ ἐφαπτομένης, ἴσαι ἔσονται ταῖς ἐν τοῖς ἐναλλὰξ τμήμασι τοῦ κύκλου γωνίαις, τουτέστιν, ὅτι ἡ μὲν ὑπὸ ΖΒΔ γωνία ἴση ἐστὶ τῇ ἐν τῷ ΒΑΔ τμήματι συνισταμένῃ γωνίᾳ, ἡ δὲ ὑπὸ ΕΒΔ γωνία ἴση ἐστὶ τῇ ἐν τῷ ΔΓΒ τμήματι συνισταμένῃ γωνίᾳ.

Ἤχθω γὰρ ἀπὸ τοῦ Β τῇ ΕΖ πρὸς ὀρθὰς ἡ ΒΑ, καὶ εἰλήφθω ἐπὶ τῆς ΒΔ περιφερείας τυχὸν σημεῖον τὸ Γ, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΓ, ΓΒ.

Καὶ ἐπεὶ κύκλου τοῦ ΑΒΓΔ ἐφάπτεταί τις εὐθεῖα ἡ ΕΖ κατὰ τὸ Β, καὶ ἀπὸ τῆς ἁφῆς ἦκται τῇ ἐφαπτομένῃ πρὸς ὀρθὰς ἡ ΒΑ, ἐπὶ τῆς ΒΑ ἄρα τὸ κέντρον ἐστὶ τοῦ ΑΒΓΔ κύκλου. ἡ ΒΑ ἄρα διάμετρός ἐστι τοῦ ΑΒΓΔ κύκλου· ἡ ἄρα ὑπὸ ΑΔΒ γωνία ἐν ἡμικυκλίῳ οὖσα ὀρθή ἐστιν. λοιπαὶ ἄρα αἱ ὑπὸ ΒΑΔ, ΑΒΔ μιᾷ ὀρθῇ ἴσαι εἰσίν. ἐστὶ δὲ καὶ ἡ ὑπὸ ΑΒΖ ὀρθή· ἡ ἄρα ὑπὸ ΑΒΖ ἴση

ἐστὶ ταῖς ὑπὸ ΒΑΔ, ΑΒΔ. κοινὴ ἀφῃρήσθω ἡ ὑπὸ ΑΒΔ· λοιπὴ ἄρα ἡ ὑπὸ ΔΒΖ γωνία ἴση ἐστὶ τῇ ἐν τῷ ἐναλλὰξ τμήματι τοῦ κύκλου γωνίᾳ τῇ ὑπὸ ΒΑΔ. καὶ ἐπεὶ ἐν κύκλῳ τετράπλευρόν ἐστι τὸ ΑΒΓΔ, αἱ ἀπεναντίον αὐτοῦ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. εἰσὶ δὲ καὶ αἱ ὑπὸ ΔΒΖ, ΔΒΕ δυσὶν ὀρθαῖς ἴσαι· αἱ ἄρα ὑπὸ ΔΒΖ, ΔΒΕ ταῖς ὑπὸ ΒΑΔ, ΒΓΔ ἴσαι εἰσίν, ὧν ἡ ὑπὸ ΒΑΔ τῇ ὑπὸ ΔΒΖ ἐδείχθη ἴση· λοιπὴ ἄρα ἡ ὑπὸ ΔΒΕ τῇ ἐν τῷ ἐναλλὰξ τοῦ κύκλου τμήματι τῷ ΔΓΒ τῇ ὑπὸ ΔΓΒ γωνίᾳ ἐστὶν ἴση.

Ἐὰν ἄρα κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τῆς ἁφῆς εἰς τὸν κύκλον διαχθῇ τις εὐθεῖα τέμνουσα τὸν κύκλον, ἃς ποιεῖ γωνίας πρὸς τῇ ἐφαπτομένῃ, ἴσαι ἔσονται ταῖς ἐν τοῖς ἐναλλὰξ τοῦ κύκλου τμήμασι γωνίαις· ὅπερ ἔδει δεῖξαι.

λγ΄. Ἐπὶ τῆς δοθείσης εὐθείας γράψαι τμῆμα κύκλου δεχόμενον γωνίαν ἴσην τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ.

Ἔστω ἡ δοθεῖσα εὐθεῖα ἡ ΑΒ, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ πρὸς τῷ Γ· δεῖ δὴ ἐπὶ τῆς δοθείσης εὐθείας τῆς ΑΒ γράψαι τμῆμα κύκλου δεχόμενον γωνίαν ἴσην τῇ πρὸς τῷ Γ.

Ἡ δὴ πρὸς τῷ Γ [γωνία] ἤτοι ὀξεῖά ἐστιν ἢ ὀρθὴ ἢ ἀμβλεῖα· ἔστω πρότερον ὀξεῖα, καὶ ὡς ἐπὶ τῆς πρώτης καταγραφῆς συνεστάτω πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ Α σημείῳ τῇ πρὸς τῷ Γ γωνίᾳ ἴση ἡ ὑπὸ ΒΑΔ· ὀξεῖα ἄρα ἐστὶ καὶ ἡ ὑπὸ ΒΑΔ. ἤχθω τῇ ΔΑ πρὸς ὀρθὰς ἡ ΑΕ, καὶ τετμήσθω ἡ ΑΒ δίχα κατὰ τὸ Ζ, καὶ ἤχθω ἀπὸ τοῦ Ζ σημείου τῇ ΑΒ πρὸς ὀρθὰς ἡ ΖΗ, καὶ ἐπεζεύχθω ἡ ΗΒ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΖ τῇ ΖΒ, κοινὴ δὲ ἡ ΖΗ, δύο δὴ αἱ ΑΖ, ΖΗ δύο ταῖς ΒΖ, ΖΗ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΑΖΗ [γωνίᾳ] τῇ ὑπὸ ΒΖΗ ἴση· βάσις ἄρα ἡ ΑΗ βάσει τῇ ΒΗ ἴση ἐστίν. ὁ ἄρα κέντρῳ μὲν τῷ Η διαστήματι δὲ τῷ ΗΑ κύκλος γραφόμενος ἥξει καὶ διὰ τοῦ Β. γεγράφθω καὶ ἔστω ὁ ΑΒΕ, καὶ ἐπεζεύχθω ἡ ΕΒ. ἐπεὶ οὖν ἀπ’ ἄκρας τῆς ΑΕ διαμέτρου ἀπὸ τοῦ Α τῇ ΑΕ

πρὸς ὀρθάς ἐστιν ἡ ΑΔ, ἡ ΑΔ ἄρα ἐφάπτεται τοῦ ΑΒΕ κύκλου· ἐπεὶ οὖν κύκλου τοῦ ΑΒΕ ἐφάπτεταί τις εὐθεῖα ἡ ΑΔ, καὶ ἀπὸ τῆς κατὰ τὸ Α ἁφῆς εἰς τὸν ΑΒΕ κύκλον διῆκταί τις εὐθεῖα ἡ ΑΒ, ἡ ἄρα ὑπὸ ΔΑΒ γωνία ἴση ἐστὶ τῇ ἐν τῷ ἐναλλὰξ τοῦ κύκλου τμήματι γωνίᾳ τῇ ὑπὸ ΑΕΒ. ἀλλ' ἡ ὑπὸ ΔΑΒ τῇ πρὸς τῷ Γ ἐστιν ἴση· καὶ ἡ πρὸς τῷ Γ ἄρα γωνία ἴση ἐστὶ τῇ ὑπὸ ΑΕΒ.

Ἐπὶ τῆς δοθείσης ἄρα εὐθείας τῆς ΑΒ τμῆμα κύκλου γέγραπται τὸ ΑΕΒ δεχόμενον γωνίαν τὴν ὑπὸ ΑΕΒ ἴσην τῇ δοθείσῃ τῇ πρὸς τῷ Γ.

Ἀλλὰ δὴ ὀρθὴ ἔστω ἡ πρὸς τῷ Γ· καὶ δέον πάλιν ἔστω ἐπὶ τῆς ΑΒ γράψαι τμῆμα κύκλου δεχόμενον γωνίαν ἴσην τῇ πρὸς τῷ Γ ὀρθῇ [γωνίᾳ]. συνεστάτω [πάλιν] τῇ πρὸς τῷ Γ ὀρθῇ γωνίᾳ ἴση ἡ ὑπὸ ΒΑΔ, ὡς ἔχει ἐπὶ τῆς δευτέρας καταγραφῆς, καὶ τετμήσθω ἡ ΑΒ δίχα κατὰ τὸ Ζ, καὶ κέντρῳ τῷ Ζ, διαστήματι δὲ ὁποτέρῳ τῶν ΖΑ, ΖΒ, κύκλος γεγράφθω ὁ ΑΕΒ.

Ἐφάπτεται ἄρα ἡ ΑΔ εὐθεῖα τοῦ ΑΒΕ κύκλου διὰ τὸ ὀρθὴν εἶναι τὴν πρὸς τῷ Α γωνίαν. καὶ ἴση ἐστὶν ἡ ὑπὸ ΒΑΔ γωνία τῇ ἐν τῷ ΑΕΒ τμήματι· ὀρθὴ γὰρ καὶ αὐτὴ ἐν ἡμικυκλίῳ οὖσα. ἀλλὰ καὶ ἡ ὑπὸ ΒΑΔ τῇ πρὸς τῷ Γ ἴση ἐστίν. καὶ ἡ ἐν τῷ ΑΕΒ ἄρα ἴση ἐστὶ τῇ πρὸς τῷ Γ. γέγραπται ἄρα πάλιν ἐπὶ τῆς ΑΒ τμῆμα κύκλου τὸ ΑΕΒ δεχόμενον γωνίαν ἴσην τῇ πρὸς τῷ Γ.

Ἀλλὰ δὴ ἡ πρὸς τῷ Γ ἀμβλεῖα ἔστω· καὶ συνεστάτω αὐτῇ ἴση πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ Α σημείῳ ἡ ὑπὸ ΒΑΔ, ὡς ἔχει ἐπὶ τῆς τρίτης καταγραφῆς, καὶ τῇ ΑΔ πρὸς ὀρθὰς ἤχθω ἡ ΑΕ, καὶ τετμήσθω πάλιν ἡ ΑΒ δίχα κατὰ τὸ Ζ, καὶ τῇ ΑΒ πρὸς ὀρθὰς ἤχθω ἡ ΖΗ, καὶ ἐπεζεύχθω ἡ ΗΒ.

Καὶ ἐπεὶ πάλιν ἴση ἐστὶν ἡ ΑΖ τῇ ΖΒ, καὶ κοινὴ ἡ ΖΗ, δύο δὴ αἱ ΑΖ, ΖΗ δύο ταῖς ΒΖ, ΖΗ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΑΖΗ γωνίᾳ τῇ ὑπὸ ΒΖΗ ἴση· βάσις ἄρα ἡ ΑΗ βάσει τῇ ΒΗ ἴση ἐστίν· ὁ ἄρα κέντρῳ μὲν τῷ Η διαστήματι δὲ τῷ ΗΑ κύκλος γραφόμενος ἥξει καὶ διὰ τοῦ Β. ἐρχέσθω ὡς ὁ ΑΕΒ. καὶ ἐπεὶ τῇ ΑΕ διαμέτρῳ ἀπ' ἄκρας πρὸς ὀρθάς ἐστιν ἡ ΑΔ, ἡ ΑΔ ἄρα ἐφάπτεται τοῦ ΑΕΒ κύκλου. καὶ ἀπὸ τῆς κατὰ τὸ Α ἐπαφῆς διῆκται ἡ ΑΒ· ἡ ἄρα ὑπὸ ΒΑΔ γωνία ἴση ἐστὶ τῇ ἐν τῷ ἐναλλὰξ τοῦ κύκλου τμήματι τῷ ΑΘΒ

συνισταμένῃ γωνίᾳ. ἀλλ᾽ ἡ ὑπὸ ΒΑΔ γωνία τῇ πρὸς τῷ Γ ἴση ἐστίν. καὶ ἡ ἐν τῷ ΑΘΒ ἄρα τμήματι γωνία ἴση ἐστὶ τῇ πρὸς τῷ Γ.

Ἐπὶ τῆς ἄρα δοθείσης εὐθείας τῆς ΑΒ γέγραπται τμῆμα κύκλου τὸ ΑΘΒ δεχόμενον γωνίαν ἴσην τῇ πρὸς τῷ Γ· ὅπερ ἔδει ποιῆσαι.

λδ΄. Ἀπὸ τοῦ δοθέντος κύκλου τμῆμα ἀφελεῖν δεχόμενον γωνίαν ἴσην τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ.

Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓ, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ πρὸς τῷ Δ· δεῖ δὴ ἀπὸ τοῦ ΑΒΓ κύκλου τμῆμα ἀφελεῖν δεχόμενον γωνίαν ἴσην τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ τῇ πρὸς τῷ Δ.

Ἤχθω τοῦ ΑΒΓ ἐφαπτομένη ἡ ΕΖ κατὰ τὸ Β σημεῖον, καὶ συνεστάτω πρὸς τῇ ΖΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Β τῇ πρὸς τῷ Δ γωνίᾳ ἴση ἡ ὑπὸ ΖΒΓ.

Ἐπεὶ οὖν κύκλου τοῦ ΑΒΓ ἐφάπτεταί τις εὐθεῖα ἡ ΕΖ, καὶ ἀπὸ τῆς κατὰ τὸ Β ἐπαφῆς διῆκται ἡ ΒΓ, ἡ ὑπὸ ΖΒΓ ἄρα γωνία ἴση ἐστὶ τῇ ἐν τῷ ΒΑΓ ἐναλλὰξ τμήματι συνισταμένῃ γωνίᾳ. ἀλλ᾽ ἡ ὑπὸ ΖΒΓ τῇ πρὸς τῷ Δ ἐστιν ἴση· καὶ ἡ ἐν τῷ ΒΑΓ ἄρα τμήματι ἴση ἐστὶ τῇ πρὸς τῷ Δ [γωνίᾳ].

Ἀπὸ τοῦ δοθέντος ἄρα κύκλου τοῦ ΑΒΓ τμῆμα ἀφῄρηται τὸ ΒΑΓ δεχόμενον γωνίαν ἴσην τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ τῇ πρὸς τῷ Δ· ὅπερ ἔδει ποιῆσαι.

λε΄. Ἐὰν ἐν κύκλῳ δύο εὐθεῖαι τέμνωσιν ἀλλήλας, τὸ ὑπὸ τῶν τῆς μιᾶς τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν τῆς ἑτέρας τμημάτων περιεχομένῳ ὀρθογωνίῳ.

Ἐν γὰρ κύκλῳ τῷ ΑΒΓΔ δύο εὐθεῖαι αἱ ΑΓ, ΒΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε σημεῖον· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΕ, ΕΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν ΔΕ, ΕΒ περιεχομένῳ ὀρθογωνίῳ.

Εἰ μὲν οὖν αἱ ΑΓ, ΒΔ διὰ τοῦ κέντρου εἰσὶν ὥστε τὸ Ε κέντρον εἶναι τοῦ ΑΒΓΔ κύκλου, φανερόν, ὅτι ἴσων οὐσῶν τῶν ΑΕ, ΕΓ, ΔΕ, ΕΒ καὶ τὸ ὑπὸ τῶν ΑΕ, ΕΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν ΔΕ, ΕΒ περιεχομένῳ ὀρθογωνίῳ.

Μὴ ἔστωσαν δὴ αἱ ΑΓ, ΔΒ διὰ τοῦ κέντρου, καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓΔ, καὶ ἔστω τὸ Ζ, καὶ ἀπὸ τοῦ Ζ ἐπὶ τὰς ΑΓ, ΔΒ εὐθείας κάθετοι ἤχθωσαν αἱ ΖΗ, ΖΘ, καὶ ἐπεζεύχθωσαν αἱ ΖΒ, ΖΓ, ΖΕ.

Καὶ ἐπεὶ εὐθεῖά τις διὰ τοῦ κέντρου ἡ ΗΖ εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου τὴν ΑΓ πρὸς ὀρθὰς τέμνει, καὶ δίχα αὐτὴν τέμνει· ἴση ἄρα ἡ ΑΗ τῇ ΗΓ. ἐπεὶ οὖν εὐθεῖα ἡ ΑΓ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Η, εἰς δὲ ἄνισα κατὰ τὸ Ε, τὸ ἄρα ὑπὸ τῶν ΑΕ, ΕΓ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΕΗ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΗΓ· [κοινὸν] προσκείσθω τὸ ἀπὸ τῆς ΗΖ· τὸ ἄρα ὑπὸ τῶν ΑΕ, ΕΓ μετὰ τῶν ἀπὸ τῶν ΗΕ, ΗΖ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΓΗ, ΗΖ. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΕΗ, ΗΖ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΖΕ, τοῖς δὲ ἀπὸ τῶν ΓΗ, ΗΖ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΖΓ· τὸ ἄρα ὑπὸ τῶν ΑΕ, ΕΓ μετὰ τοῦ ἀπὸ τῆς ΖΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΖΓ. ἴση δὲ ἡ ΖΓ τῇ ΖΒ· τὸ ἄρα ὑπὸ τῶν ΑΕ, ΕΓ μετὰ τοῦ ἀπὸ τῆς ΕΖ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΖΒ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ὑπὸ τῶν ΔΕ, ΕΒ μετὰ τοῦ ἀπὸ τῆς ΖΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΖΒ. ἐδείχθη δὲ καὶ τὸ ὑπὸ τῶν ΑΕ, ΕΓ μετὰ τοῦ ἀπὸ τῆς ΖΕ ἴσον τῷ ἀπὸ τῆς ΖΒ· τὸ ἄρα ὑπὸ τῶν ΑΕ, ΕΓ μετὰ τοῦ ἀπὸ τῆς ΖΕ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΔΕ, ΕΒ μετὰ τοῦ ἀπὸ τῆς ΖΕ. κοινὸν ἀφῃρήσθω τὸ ἀπὸ τῆς ΖΕ· λοιπὸν ἄρα τὸ ὑπὸ τῶν ΑΕ, ΕΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν ΔΕ, ΕΒ περιεχομένῳ ὀρθογωνίῳ.

Ἐὰν ἄρα ἐν κύκλῳ εὐθεῖαι δύο τέμνωσιν ἀλλήλας, τὸ ὑπὸ τῶν τῆς μιᾶς τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν τῆς ἑτέρας τμημάτων περιεχομένῳ ὀρθογωνίῳ· ὅπερ ἔδει δεῖξαι.

λς΄. Ἐὰν κύκλου ληφθῇ τι σημεῖον ἐκτός, καὶ ἀπ᾽ αὐτοῦ πρὸς τὸν κύκλον προσπίπτωσι δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν τέμνῃ τὸν κύκλον, ἡ δὲ ἐφάπτηται,

ἔσται τὸ ὑπὸ ὅλης τῆς τεμνούσης καὶ τῆς ἐκτὸς ἀπολαμβανομένης μεταξὺ τοῦ τε σημείου καὶ τῆς κυρτῆς περιφερείας ἴσον τῷ ἀπὸ τῆς ἐφαπτομένης τετραγώνῳ.

Κύκλου γὰρ τοῦ ΑΒΓ εἰλήφθω τι σημεῖον ἐκτὸς τὸ Δ, καὶ ἀπὸ τοῦ Δ πρὸς τὸν ΑΒΓ κύκλον προσπιπτέτωσαν δύο εὐθεῖαι αἱ ΔΓ[Α], ΔΒ· καὶ ἡ μὲν ΔΓΑ τεμνέτω τὸν ΑΒΓ κύκλον, ἡ δὲ ΒΔ ἐφαπτέσθω· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΔ, ΔΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΒ τετραγώνῳ.

Ἢ ἄρα [Δ]ΓΑ ἤτοι διὰ τοῦ κέντρου ἐστὶν ἢ οὔ. ἔστω πρότερον διὰ τοῦ κέντρου, καὶ ἔστω τὸ Ζ κέντρον τοῦ ΑΒΓ κύκλου, καὶ ἐπεζεύχθω ἡ ΖΒ· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΖΒΔ. καὶ ἐπεὶ εὐθεῖα ἡ ΑΓ δίχα τέτμηται κατὰ τὸ Ζ, πρόσκειται δὲ αὐτῇ ἡ ΓΔ, τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΖΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΖΔ. ἴση δὲ ἡ ΖΓ τῇ ΖΒ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΖΒ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΖΔ. τῷ δὲ ἀπὸ τῆς ΖΔ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΖΒ, ΒΔ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΖΒ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΖΒ, ΒΔ. κοινὸν ἀφῃρήσθω τὸ ἀπὸ τῆς ΖΒ· λοιπὸν ἄρα τὸ ὑπὸ τῶν ΑΔ, ΔΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΒ ἐφαπτομένης. ἀλλὰ δὴ ἡ ΔΓΑ μὴ ἔστω διὰ τοῦ κέντρου τοῦ ΑΒΓ κύκλου, καὶ εἰλήφθω τὸ κέντρον τὸ Ε, καὶ ἀπὸ τοῦ Ε ἐπὶ τὴν ΑΓ κάθετος ἤχθω ἡ ΕΖ, καὶ ἐπεζεύχθωσαν αἱ ΕΒ, ΕΓ, ΕΔ· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΕΒΔ. καὶ ἐπεὶ εὐθεῖά τις διὰ τοῦ κέντρου ἡ ΕΖ εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου τὴν ΑΓ πρὸς ὀρθὰς τέμνει, καὶ δίχα αὐτὴν τέμνει· ἡ ΑΖ ἄρα τῇ ΖΓ ἐστιν ἴση. καὶ ἐπεὶ εὐθεῖα ἡ ΑΓ τέτμηται δίχα κατὰ τὸ Ζ σημεῖον, πρόσκειται δὲ αὐτῇ ἡ ΓΔ, τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΖΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΖΔ. κοινὸν προσκείσθω τὸ ἀπὸ τῆς ΖΕ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τῶν ἀπὸ τῶν ΓΖ, ΖΕ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΖΔ, ΖΕ. τοῖς δὲ ἀπὸ τῶν ΓΖ, ΖΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΓ· ὀρθὴ γὰρ [ἐστιν] ἡ ὑπὸ ΕΖΓ [γωνία]· τοῖς δὲ ἀπὸ τῶν ΔΖ, ΖΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΔ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΕΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΔ. ἴση δὲ ἡ ΕΓ τῇ ΕΒ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΕΒ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΔ. τῷ δὲ ἀπὸ τῆς ΕΔ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΕΒ, ΒΔ· ὀρθὴ γὰρ ἡ ὑπὸ ΕΒΔ γωνία· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ

μετὰ τοῦ ἀπὸ τῆς ΕΒ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΕΒ, ΒΔ. κοινὸν ἀφῃρήσθω τὸ ἀπὸ τῆς ΕΒ· λοιπὸν ἄρα τὸ ὑπὸ τῶν ΑΔ, ΔΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΒ.

Ἐὰν ἄρα κύκλου ληφθῇ τι σημεῖον ἐκτός, καὶ ἀπ᾽ αὐτοῦ πρὸς τὸν κύκλον προσπίπτωσι δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν τέμνῃ τὸν κύκλον, ἡ δὲ ἐφάπτηται, ἔσται τὸ ὑπὸ ὅλης τῆς τεμνούσης καὶ τῆς ἐκτὸς ἀπολαμβανομένης μεταξὺ τοῦ τε σημείου καὶ τῆς κυρτῆς περιφερείας ἴσον τῷ ἀπὸ τῆς ἐφαπτομένης τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

λζ΄. Ἐὰν κύκλου ληφθῇ τι σημεῖον ἐκτός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσι δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν τέμνῃ τὸν κύκλον, ἡ δὲ προσπίπτῃ, ᾖ δὲ τὸ ὑπὸ [τῆς] ὅλης τῆς τεμνούσης καὶ τῆς ἐκτὸς ἀπολαμβανομένης μεταξὺ τοῦ τε σημείου καὶ τῆς κυρτῆς περιφερείας ἴσον τῷ ἀπὸ τῆς προσπιπτούσης, ἡ προσπίπτουσα ἐφάψεται τοῦ κύκλου.

κύκλου γὰρ τοῦ ΑΒΓ εἰλήφθω τι σημεῖον ἐκτὸς τὸ Δ, καὶ ἀπὸ τοῦ Δ πρὸς τὸν ΑΒΓ κύκλον προσπιπτέτωσαν δύο εὐθεῖαι αἱ ΔΓΑ, ΔΒ, καὶ ἡ μὲν ΔΓΑ τεμνέτω τὸν κύκλον, ἡ δὲ ΔΒ προσπιπτέτω, ἔστω δὲ τὸ ὑπὸ τῶν ΑΔ, ΔΓ ἴσον τῷ ἀπὸ τῆς ΔΒ. λέγω, ὅτι ἡ ΔΒ ἐφάπτεται τοῦ ΑΒΓ κύκλου.

Ἤχθω γὰρ τοῦ ΑΒΓ ἐφαπτομένη ἡ ΔΕ, καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓ κύκλου, καὶ ἔστω τὸ Ζ, καὶ ἐπεζεύχθωσαν αἱ ΖΕ, ΖΒ, ΖΔ. ἡ ἄρα ὑπὸ ΖΕΔ ὀρθή ἐστιν. καὶ ἐπεὶ ἡ ΔΕ ἐφάπτεται τοῦ ΑΒΓ κύκλου, τέμνει δὲ ἡ ΔΓΑ, τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΕ. ἦν δὲ καὶ τὸ ὑπὸ τῶν ΑΔ, ΔΓ ἴσον τῷ ἀπὸ τῆς ΔΒ· τὸ ἄρα ἀπὸ τῆς ΔΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΒ· ἴση ἄρα ἡ ΔΕ τῇ ΔΒ. ἐστὶ δὲ καὶ ἡ ΖΕ τῇ ΖΒ ἴση· δύο δὴ αἱ ΔΕ, ΕΖ δύο ταῖς ΔΒ, ΒΖ ἴσαι εἰσίν· καὶ βάσις αὐτῶν κοινὴ ἡ ΖΔ· γωνία ἄρα ἡ ὑπὸ ΔΕΖ γωνίᾳ τῇ ὑπὸ ΔΒΖ ἐστιν ἴση. ὀρθὴ δὲ ἡ ὑπὸ ΔΕΖ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΔΒΖ. καί ἐστιν ἡ ΖΒ ἐκβαλλομένη διάμετρος· ἡ δὲ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ᾽ ἄκρας ἀγομένη ἐφάπτεται τοῦ κύκλου· ἡ ΔΒ ἄρα ἐφάπτεται τοῦ ΑΒΓ κύκλου. ὁμοίως δὴ δειχθήσεται, κἂν τὸ κέντρον ἐπὶ τῆς ΑΓ τυγχάνῃ.

Ἐὰν ἄρα κύκλου ληφθῇ τι σημεῖον ἐκτός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσι δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν τέμνῃ τὸν κύκλον, ἡ δὲ προσπίπτῃ, ᾗ δὲ τὸ ὑπὸ ὅλης τῆς τεμνούσης καὶ τῆς ἐκτὸς ἀπολαμβανομένης μεταξὺ τοῦ τε σημείου καὶ τῆς κυρτῆς περιφερείας ἴσον τῷ ἀπὸ τῆς προσπιπτούσης, ἡ προσπίπτουσα ἐφάψεται τοῦ κύκλου· ὅπερ ἔδει δεῖξαι.

BOOK IV.

Ὅροι ζ΄.

α΄. Σχῆμα εὐθύγραμμον εἰς σχῆμα εὐθύγραμμον ἐγγράφεσθαι λέγεται, ὅταν ἑκάστη τῶν τοῦ ἐγγραφομένου σχήματος γωνιῶν ἑκάστης πλευρᾶς τοῦ, εἰς ὃ ἐγγράφεται, ἅπτηται.

β΄. Σχῆμα δὲ ὁμοίως περὶ σχῆμα περιγράφεσθαι λέγεται, ὅταν ἑκάστη πλευρὰ τοῦ περιγραφομένου ἑκάστης γωνίας τοῦ, περὶ ὃ περιγράφεται, ἅπτηται.

γ΄. Σχῆμα εὐθύγραμμον εἰς κύκλον ἐγγράφεσθαι λέγεται, ὅταν ἑκάστη γωνία τοῦ ἐγγραφομένου ἅπτηται τῆς τοῦ κύκλου περιφερείας.

δ΄. Σχῆμα δὲ εὐθύγραμμον περὶ κύκλον περιγράφεσθαι λέγεται, ὅταν ἑκάστη πλευρὰ τοῦ περιγραφομένου ἐφάπτηται τῆς τοῦ κύκλου περιφερείας.

ε΄. Κύκλος δὲ εἰς σχῆμα ὁμοίως ἐγγράφεσθαι λέγεται, ὅταν ἡ τοῦ κύκλου περιφέρεια ἑκάστης πλευρᾶς τοῦ, εἰς ὃ ἐγγράφεται, ἅπτηται.

ς΄. Κύκλος δὲ περὶ σχῆμα περιγράφεσθαι λέγεται, ὅταν ἡ τοῦ κύκλου περιφέρεια ἑκάστης γωνίας τοῦ, περὶ ὃ περιγράφεται, ἅπτηται.

ζ΄. Εὐθεῖα εἰς κύκλον ἐναρμόζεσθαι λέγεται, ὅταν τὰ πέρατα αὐτῆς ἐπὶ τῆς περιφερείας ᾖ τοῦ κύκλου.

Προτάσεις ιζ΄.

α΄. Εἰς τὸν δοθέντα κύκλον τῇ δοθείσῃ εὐθείᾳ μὴ μείζονι οὔσῃ τῆς τοῦ κύκλου διαμέτρου ἴσην εὐθεῖαν ἐναρμόσαι.

Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓ, ἡ δὲ δοθεῖσα εὐθεῖα μὴ μείζων τῆς τοῦ κύκλου διαμέτρου ἡ Δ. δεῖ δὴ εἰς τὸν ΑΒΓ κύκλον τῇ Δ εὐθείᾳ ἴσην εὐθεῖαν ἐναρμόσαι.

Ἤχθω τοῦ ΑΒΓ κύκλου διάμετρος ἡ ΒΓ. εἰ μὲν οὖν ἴση ἐστὶν ἡ ΒΓ τῇ Δ, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν· ἐνήρμοσται γὰρ εἰς τὸν ΑΒΓ κύκλον τῇ Δ εὐθείᾳ ἴση ἡ ΒΓ. εἰ δὲ μείζων ἐστὶν ἡ ΒΓ τῆς Δ, κείσθω τῇ Δ ἴση ἡ ΓΕ, καὶ κέντρῳ τῷ Γ διαστήματι δὲ τῷ ΓΕ κύκλος γεγράφθω ὁ ΕΑΖ, καὶ ἐπεζεύχθω ἡ ΓΑ.

Ἐπεὶ οὖν τὸ Γ σημεῖον κέντρον ἐστὶ τοῦ ΕΑΖ κύκλου, ἴση ἐστὶν ἡ ΓΑ τῇ ΓΕ. ἀλλὰ τῇ Δ ἡ ΓΕ ἐστιν ἴση· καὶ ἡ Δ ἄρα τῇ ΓΑ ἐστιν ἴση.

Εἰς ἄρα τὸν δοθέντα κύκλον τὸν ΑΒΓ τῇ δοθείσῃ εὐθείᾳ τῇ Δ ἴση ἐνήρμοσται ἡ ΓΑ· ὅπερ ἔδει ποιῆσαι.

β΄. Εἰς τὸν δοθέντα κύκλον τῷ δοθέντι τριγώνῳ ἰσογώνιον τρίγωνον ἐγγράψαι.

Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓ, τὸ δὲ δοθὲν τρίγωνον τὸ ΔΕΖ· δεῖ δὴ εἰς τὸν ΑΒΓ κύκλον τῷ ΔΕΖ τριγώνῳ ἰσογώνιον τρίγωνον ἐγγράψαι.

Ἤχθω τοῦ ΑΒΓ κύκλου ἐφαπτομένη ἡ ΗΘ κατὰ τὸ Α, καὶ συνεστάτω πρὸς τῇ ΑΘ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ ὑπὸ ΔΕΖ γωνίᾳ ἴση ἡ ὑπὸ ΘΑΓ, πρὸς δὲ τῇ ΑΗ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ ὑπὸ ΔΖΕ [γωνίᾳ] ἴση ἡ ὑπὸ ΗΑΒ, καὶ ἐπεζεύχθω ἡ ΒΓ.

Ἐπεὶ οὖν κύκλου τοῦ ΑΒΓ ἐφάπτεταί τις εὐθεῖα ἡ ΑΘ, καὶ ἀπὸ τῆς κατὰ τὸ Α ἐπαφῆς εἰς τὸν κύκλον διῆκται εὐθεῖα ἡ ΑΓ, ἡ ἄρα ὑπὸ ΘΑΓ ἴση ἐστὶ τῇ ἐν τῷ ἐναλλὰξ τοῦ κύκλου τμήματι γωνίᾳ τῇ ὑπὸ ΑΒΓ. ἀλλ’ ἡ ὑπὸ ΘΑΓ τῇ

ὑπὸ ΔΕΖ ἐστιν ἴση· καὶ ἡ ὑπὸ ΑΒΓ ἄρα γωνία τῇ ὑπὸ ΔΕΖ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΑΓΒ τῇ ὑπὸ ΔΖΕ ἐστιν ἴση· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΒΑΓ λοιπῇ τῇ ὑπὸ ΕΔΖ ἐστιν ἴση· [ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ, καὶ ἐγγέγραπται εἰς τὸν ΑΒΓ κύκλον].

Εἰς τὸν δοθέντα ἄρα κύκλον τῷ δοθέντι τριγώνῳ ἰσογώνιον τρίγωνον ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

γ΄. Περὶ τὸν δοθέντα κύκλον τῷ δοθέντι τριγώνῳ ἰσογώνιον τρίγωνον περιγράψαι.

Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓ, τὸ δὲ δοθὲν τρίγωνον τὸ ΔΕΖ· δεῖ δὴ περὶ τὸν ΑΒΓ κύκλον τῷ ΔΕΖ τριγώνῳ ἰσογώνιον τρίγωνον περιγράψαι.

Ἐκβεβλήσθω ἡ ΕΖ ἐφ' ἑκάτερα τὰ μέρη κατὰ τὰ Η, Θ σημεῖα, καὶ εἰλήφθω τοῦ ΑΒΓ κύκλου κέντρον τὸ Κ, καὶ διήχθω, ὡς ἔτυχεν, εὐθεῖα ἡ ΚΒ, καὶ συνεστάτω πρὸς τῇ ΚΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Κ τῇ μὲν ὑπὸ ΔΕΗ γωνίᾳ ἴση ἡ ὑπὸ ΒΚΑ, τῇ δὲ ὑπὸ ΔΖΘ ἴση ἡ ὑπὸ ΒΚΓ, καὶ διὰ τῶν Α, Β, Γ σημείων ἤχθωσαν ἐφαπτόμεναι τοῦ ΑΒΓ κύκλου αἱ ΛΑΜ, ΜΒΝ, ΝΓΛ.

Καὶ ἐπεὶ ἐφάπτονται τοῦ ΑΒΓ κύκλου αἱ ΛΜ, ΜΝ, ΝΛ κατὰ τὰ Α, Β, Γ σημεῖα, ἀπὸ δὲ τοῦ Κ κέντρου ἐπὶ τὰ Α, Β, Γ σημεῖα ἐπεζευγμέναι εἰσὶν αἱ ΚΑ, ΚΒ, ΚΓ, ὀρθαὶ ἄρα εἰσὶν αἱ πρὸς τοῖς Α, Β, Γ σημείοις γωνίαι. καὶ ἐπεὶ τοῦ ΑΜΒΚ τετραπλεύρου αἱ τέσσαρες γωνίαι τέτρασιν ὀρθαῖς ἴσαι εἰσίν, ἐπειδήπερ καὶ εἰς δύο τρίγωνα διαιρεῖται τὸ ΑΜΒΚ, καί εἰσιν ὀρθαὶ αἱ ὑπὸ ΚΑΜ, ΚΒΜ γωνίαι, λοιπαὶ ἄρα αἱ ὑπὸ ΑΚΒ, ΑΜΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν. εἰσὶ δὲ καὶ αἱ ὑπὸ ΔΕΗ, ΔΕΖ δυσὶν ὀρθαῖς ἴσαι· αἱ ἄρα ὑπὸ ΑΚΒ, ΑΜΒ ταῖς ὑπὸ ΔΕΗ, ΔΕΖ ἴσαι εἰσίν, ὧν ἡ ὑπὸ ΑΚΒ τῇ ὑπὸ ΔΕΗ ἐστιν ἴση· λοιπὴ ἄρα ἡ ὑπὸ ΑΜΒ λοιπῇ τῇ ὑπὸ ΔΕΖ ἐστιν ἴση. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἡ ὑπὸ ΛΝΒ τῇ ὑπὸ ΔΖΕ ἐστιν ἴση· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΜΛΝ [λοιπῇ] τῇ ὑπὸ ΕΔΖ ἐστιν ἴση. ἰσογώνιον ἄρα ἐστὶ τὸ ΛΜΝ τρίγωνον τῷ ΔΕΖ τριγώνῳ· καὶ περιγέγραπται περὶ τὸν ΑΒΓ κύκλον.

Περὶ τὸν δοθέντα ἄρα κύκλον τῷ δοθέντι τριγώνῳ ἰσογώνιον τρίγωνον περιγέγραπται· ὅπερ ἔδει ποιῆσαι.

δ΄. Εἰς τὸ δοθὲν τρίγωνον κύκλον ἐγγράψαι.

Ἔστω τὸ δοθὲν τρίγωνον τὸ ΑΒΓ· δεῖ δὴ εἰς τὸ ΑΒΓ τρίγωνον κύκλον ἐγγράψαι.

Τετμήσθωσαν αἱ ὑπὸ ΑΒΓ, ΑΓΒ γωνίαι δίχα ταῖς ΒΔ, ΓΔ εὐθείαις, καὶ συμβαλλέτωσαν ἀλλήλαις κατὰ τὸ Δ σημεῖον, καὶ ἤχθωσαν ἀπὸ τοῦ Δ ἐπὶ τὰς ΑΒ, ΒΓ, ΓΑ εὐθείας κάθετοι αἱ ΔΕ, ΔΖ, ΔΗ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΑΒΔ γωνία τῇ ὑπὸ ΓΒΔ, ἐστὶ δὲ καὶ ὀρθὴ ἡ ὑπὸ ΒΕΔ ὀρθῇ τῇ ὑπὸ ΒΖΔ ἴση, δύο δὴ τρίγωνά ἐστι τὰ ΕΒΔ, ΖΒΔ τὰς δύο γωνίας ταῖς δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν κοινὴν αὐτῶν τὴν ΒΔ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξουσιν· ἴση ἄρα ἡ ΔΕ τῇ ΔΖ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΔΗ τῇ ΔΖ ἐστιν ἴση. αἱ τρεῖς ἄρα εὐθεῖαι αἱ ΔΕ, ΔΖ, ΔΗ ἴσαι ἀλλήλαις εἰσίν· ὁ ἄρα κέντρῳ τῷ Δ καὶ διαστήματι ἑνὶ τῶν Ε, Ζ, Η κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἐφάψεται τῶν ΑΒ, ΒΓ, ΓΑ εὐθειῶν διὰ τὸ ὀρθὰς εἶναι τὰς πρὸς τοῖς Ε, Ζ, Η σημείοις γωνίας. εἰ γὰρ τεμεῖ αὐτάς, ἔσται ἡ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ᾽ ἄκρας ἀγομένη ἐντὸς πίπτουσα τοῦ κύκλου· ὅπερ ἄτοπον ἐδείχθη· οὐκ ἄρα ὁ κέντρῳ τῷ Δ διαστήματι δὲ ἑνὶ τῶν Ε, Ζ, Η γραφόμενος κύκλος τεμεῖ τὰς ΑΒ, ΒΓ, ΓΑ εὐθείας· ἐφάψεται ἄρα αὐτῶν, καὶ ἔσται ὁ κύκλος ἐγγεγραμμένος εἰς τὸ ΑΒΓ τρίγωνον. ἐγγεγράφθω ὡς ὁ ΖΗΕ.

Εἰς ἄρα τὸ δοθὲν τρίγωνον τὸ ΑΒΓ κύκλος ἐγγέγραπται ὁ ΕΖΗ· ὅπερ ἔδει ποιῆσαι.

ε΄. Περὶ τὸ δοθὲν τρίγωνον κύκλον περιγράψαι.

Ἔστω τὸ δοθὲν τρίγωνον τὸ ΑΒΓ· δεῖ δὴ περὶ τὸ δοθὲν τρίγωνον τὸ ΑΒΓ κύκλον περιγράψαι.

Τετμήσθωσαν αἱ ΑΒ, ΑΓ εὐθεῖαι δίχα κατὰ τὰ Δ, Ε σημεῖα, καὶ ἀπὸ τῶν Δ, Ε σημείων ταῖς ΑΒ, ΑΓ πρὸς ὀρθὰς ἤχθωσαν αἱ ΔΖ, ΕΖ· συμπεσοῦνται δὴ ἤτοι ἐντὸς τοῦ ΑΒΓ τριγώνου ἢ ἐπὶ τῆς ΒΓ εὐθείας ἢ ἐκτὸς τῆς ΒΓ.

Συμπιπτέτωσαν πρότερον ἐντὸς κατὰ τὸ Ζ, καὶ ἐπεζεύχθωσαν αἱ ΖΒ, ΖΓ, ΖΑ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΔΒ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΔΖ, βάσις ἄρα ἡ ΑΖ βάσει τῇ ΖΒ ἐστιν ἴση. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΓΖ τῇ ΑΖ ἐστιν ἴση· ὥστε καὶ ἡ ΖΒ τῇ ΖΓ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΖΑ, ΖΒ, ΖΓ ἴσαι ἀλλήλαις εἰσίν. ὁ ἄρα κέντρῳ τῷ Ζ διαστήματι δὲ ἑνὶ τῶν Α, Β, Γ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων, καὶ ἔσται περιγεγραμμένος ὁ κύκλος περὶ τὸ ΑΒΓ τρίγωνον. περιγεγράφθω ὡς ὁ ΑΒΓ.

Ἀλλὰ δὴ αἱ ΔΖ, ΕΖ συμπιπτέτωσαν ἐπὶ τῆς ΒΓ εὐθείας κατὰ τὸ Ζ, ὡς ἔχει ἐπὶ τῆς δευτέρας καταγραφῆς, καὶ ἐπεζεύχθω ἡ ΑΖ. ὁμοίως δὴ δείξομεν, ὅτι τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ περὶ τὸ ΑΒΓ τρίγωνον περιγραφομένου κύκλου.

Ἀλλὰ δὴ αἱ ΔΖ, ΕΖ συμπιπτέτωσαν ἐκτὸς τοῦ ΑΒΓ τριγώνου κατὰ τὸ Ζ πάλιν, ὡς ἔχει ἐπὶ τῆς τρίτης καταγραφῆς, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΒΖ, ΓΖ. καὶ ἐπεὶ πάλιν ἴση ἐστὶν ἡ ΑΔ τῇ ΔΒ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΔΖ, βάσις ἄρα ἡ ΑΖ βάσει τῇ ΒΖ ἐστιν ἴση. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΓΖ τῇ ΑΖ ἐστιν ἴση· ὥστε καὶ ἡ ΒΖ τῇ ΖΓ ἐστιν ἴση· ὁ ἄρα [πάλιν] κέντρῳ τῷ Ζ διαστήματι δὲ ἑνὶ τῶν ΖΑ, ΖΒ, ΖΓ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων, καὶ ἔσται περιγεγραμμένος περὶ τὸ ΑΒΓ τρίγωνον.

Περὶ τὸ δοθὲν ἄρα τρίγωνον κύκλος περιγέγραπται· ὅπερ ἔδει ποιῆσαι.

[Πόρισμα:

Καὶ φανερόν, ὅτι, ὅτε μὲν ἐντὸς τοῦ τριγώνου πίπτει τὸ κέντρον τοῦ κύκλου, ἡ ὑπὸ ΒΑΓ γωνία ἐν μείζονι τμήματι τοῦ ἡμικυκλίου τυγχάνουσα ἐλάττων ἐστὶν ὀρθῆς· ὅτε δὲ ἐπὶ τῆς ΒΓ εὐθείας τὸ κέντρον πίπτει, ἡ ὑπὸ

ΒΑΓ γωνία ἐν ἡμικυκλίῳ τυγχάνουσα ὀρθή ἐστιν· ὅτε δὲ τὸ κέντρον τοῦ κύκλου ἐκτὸς τοῦ τριγώνου πίπτει, ἡ ὑπὸ ΒΑΓ ἐν ἐλάττονι τμήματι τοῦ ἡμικυκλίου τυγχάνουσα μείζων ἐστὶν ὀρθῆς. [ὥστε καὶ ὅταν ἐλάττων ὀρθῆς τυγχάνῃ ἡ διδομένη γωνία, ἐντὸς τοῦ τριγώνου πεσοῦνται αἱ ΔΖ, ΕΖ, ὅταν δὲ ὀρθή, ἐπὶ τῆς ΒΓ, ὅταν δὲ μείζων ὀρθῆς, ἐκτὸς τῆς ΒΓ· ὅπερ ἔδει ποιῆσαι.]

ς΄. Εἰς τὸν δοθέντα κύκλον τετράγωνον ἐγγράψαι.

Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔ· δεῖ δὴ εἰς τὸν ΑΒΓΔ κύκλον τετράγωνον ἐγγράψαι.

Ἤχθωσαν τοῦ ΑΒΓΔ κύκλου δύο διάμετροι πρὸς ὀρθὰς ἀλλήλαις αἱ ΑΓ, ΒΔ, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΒΓ, ΓΔ, ΔΑ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΕ τῇ ΕΔ· κέντρον γὰρ τὸ Ε· κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΕΑ, βάσις ἄρα ἡ ΑΒ βάσει τῇ ΑΔ ἴση ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ΒΓ, ΓΔ ἑκατέρᾳ τῶν ΑΒ, ΑΔ ἴση ἐστίν· ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓΔ τετράπλευρον. λέγω δή, ὅτι καὶ ὀρθογώνιον. ἐπεὶ γὰρ ἡ ΒΔ εὐθεῖα διάμετρός ἐστι τοῦ ΑΒΓΔ κύκλου, ἡμικύκλιον ἄρα ἐστὶ τὸ ΒΑΔ· ὀρθὴ ἄρα ἡ ὑπὸ ΒΑΔ γωνία. διὰ τὰ αὐτὰ δὴ καὶ ἑκάστη τῶν ὑπὸ ΑΒΓ, ΒΓΔ, ΓΔΑ ὀρθή ἐστιν· ὀρθογώνιον ἄρα ἐστὶ τὸ ΑΒΓΔ τετράπλευρον. ἐδείχθη δὲ καὶ ἰσόπλευρον· τετράγωνον ἄρα ἐστίν. καὶ ἐγγέγραπται εἰς τὸν ΑΒΓΔ κύκλον.

Εἰς ἄρα τὸν δοθέντα κύκλον τετράγωνον ἐγγέγραπται τὸ ΑΒΓΔ· ὅπερ ἔδει ποιῆσαι.

ζ΄. Περὶ τὸν δοθέντα κύκλον τετράγωνον περιγράψαι.

Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔ· δεῖ δὴ περὶ τὸν ΑΒΓΔ κύκλον τετράγωνον περιγράψαι.

Ἤχθωσαν τοῦ ΑΒΓΔ κύκλου δύο διάμετροι πρὸς ὀρθὰς ἀλλήλαις αἱ ΑΓ, ΒΔ, καὶ διὰ τῶν Α, Β, Γ, Δ σημείων ἤχθωσαν ἐφαπτόμεναι τοῦ ΑΒΓΔ κύκλου αἱ ΖΗ, ΗΘ, ΘΚ, ΚΖ.

Ἐπεὶ οὖν ἐφάπτεται ἡ ΖΗ τοῦ ΑΒΓΔ κύκλου, ἀπὸ δὲ τοῦ Ε κέντρου ἐπὶ τὴν κατὰ τὸ Α ἐπαφὴν ἐπέζευκται ἡ ΕΑ, αἱ ἄρα πρὸς τῷ Α γωνίαι ὀρθαί εἰσιν. διὰ τὰ αὐτὰ δὴ καὶ αἱ πρὸς τοῖς Β, Γ, Δ σημείοις γωνίαι ὀρθαί εἰσιν. καὶ ἐπεὶ ὀρθή ἐστιν ἡ ὑπὸ ΑΕΒ γωνία, ἐστὶ δὲ ὀρθὴ καὶ ἡ ὑπὸ ΕΒΗ, παράλληλος ἄρα ἐστὶν ἡ ΗΘ τῇ ΑΓ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΑΓ τῇ ΖΚ ἐστι παράλληλος. ὥστε καὶ ἡ ΗΘ τῇ ΖΚ ἐστι παράλληλος. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἑκατέρα τῶν ΗΖ, ΘΚ τῇ ΒΕΔ ἐστι παράλληλος. παραλληλόγραμμα ἄρα ἐστὶ τὰ ΗΚ, ΗΓ, ΑΚ, ΖΒ, ΒΚ· ἴση ἄρα ἐστὶν ἡ μὲν ΗΖ τῇ ΘΚ, ἡ δὲ ΗΘ τῇ ΖΚ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΒΔ, ἀλλὰ καὶ ἡ μὲν ΑΓ ἑκατέρᾳ τῶν ΗΘ, ΖΚ, ἡ δὲ ΒΔ ἑκατέρᾳ τῶν ΗΖ, ΘΚ ἐστιν ἴση [καὶ ἑκατέρα ἄρα τῶν ΗΘ, ΖΚ ἑκατέρᾳ τῶν ΗΖ, ΘΚ ἐστιν ἴση], ἰσόπλευρον ἄρα ἐστὶ τὸ ΖΗΘΚ τετράπλευρον. λέγω δή, ὅτι καὶ ὀρθογώνιον. ἐπεὶ γὰρ παραλληλόγραμμόν ἐστι τὸ ΗΒΕΑ, καί ἐστιν ὀρθὴ ἡ ὑπὸ ΑΕΒ, ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΑΗΒ. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ πρὸς τοῖς Θ, Κ, Ζ γωνίαι ὀρθαί εἰσιν. ὀρθογώνιον ἄρα ἐστὶ τὸ ΖΗΘΚ. ἐδείχθη δὲ καὶ ἰσόπλευρον· τετράγωνον ἄρα ἐστίν. καὶ περιγέγραπται περὶ τὸν ΑΒΓΔ κύκλον.

Περὶ τὸν δοθέντα ἄρα κύκλον τετράγωνον περιγέγραπται· ὅπερ ἔδει ποιῆσαι.

η΄. Εἰς τὸ δοθὲν τετράγωνον κύκλον ἐγγράψαι.

Ἔστω τὸ δοθὲν τετράγωνον τὸ ΑΒΓΔ· δεῖ δὴ εἰς τὸ ΑΒΓΔ τετράγωνον κύκλον ἐγγράψαι.

Τετμήσθω ἑκατέρα τῶν ΑΔ, ΑΒ δίχα κατὰ τὰ Ε, Ζ σημεῖα, καὶ διὰ μὲν τοῦ Ε ὁποτέρᾳ τῶν ΑΒ, ΓΔ παράλληλος ἤχθω ἡ ΕΘ, διὰ δὲ τοῦ Ζ ὁποτέρᾳ τῶν ΑΔ, ΒΓ παράλληλος ἤχθω ἡ ΖΚ· παραλληλόγραμμον ἄρα ἐστὶν ἕκαστον τῶν ΑΚ, ΚΒ, ΑΘ, ΘΔ, ΑΗ, ΗΓ, ΒΗ, ΗΔ, καὶ αἱ ἀπεναντίον αὐτῶν πλευραὶ δηλονότι ἴσαι [εἰσίν]. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΑΒ, καί ἐστι τῆς μὲν ΑΔ ἡμίσεια ἡ ΑΕ, τῆς δὲ ΑΒ ἡμίσεια ἡ ΑΖ, ἴση ἄρα καὶ ἡ ΑΕ τῇ ΑΖ· ὥστε καὶ αἱ

ἀπεναντίον· ἴση ἄρα καὶ ἡ ΖΗ τῇ ΗΕ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἑκατέρα τῶν ΗΘ, ΗΚ ἑκατέρᾳ τῶν ΖΗ, ΗΕ ἐστιν ἴση· αἱ τέσσαρες ἄρα αἱ ΗΕ, ΗΖ, ΗΘ, ΗΚ ἴσαι ἀλλήλαις [εἰσίν]. ὁ ἄρα κέντρῳ μὲν τῷ Η διαστήματι δὲ ἑνὶ τῶν Ε, Ζ, Θ, Κ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων· καὶ ἐφάψεται τῶν ΑΒ, ΒΓ, ΓΔ, ΔΑ εὐθειῶν διὰ τὸ ὀρθὰς εἶναι τὰς πρὸς τοῖς Ε, Ζ, Θ, Κ γωνίας· εἰ γὰρ τεμεῖ ὁ κύκλος τὰς ΑΒ, ΒΓ, ΓΔ, ΔΑ, ἡ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ᾽ ἄκρας ἀγομένη ἐντὸς πεσεῖται τοῦ κύκλου· ὅπερ ἄτοπον ἐδείχθη. οὐκ ἄρα ὁ κέντρῳ τῷ Η διαστήματι δὲ ἑνὶ τῶν Ε, Ζ, Θ, Κ κύκλος γραφόμενος τεμεῖ τὰς ΑΒ, ΒΓ, ΓΔ, ΔΑ εὐθείας. ἐφάψεται ἄρα αὐτῶν καὶ ἔσται ἐγγεγραμμένος εἰς τὸ ΑΒΓΔ τετράγωνον.

Εἰς ἄρα τὸ δοθὲν τετράγωνον κύκλος ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

θ΄. Περὶ τὸ δοθὲν τετράγωνον κύκλον περιγράψαι.

Ἔστω τὸ δοθὲν τετράγωνον τὸ ΑΒΓΔ· δεῖ δὴ περὶ τὸ ΑΒΓΔ τετράγωνον κύκλον περιγράψαι.

Ἐπιζευχθεῖσαι γὰρ αἱ ΑΓ, ΒΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΑΒ, κοινὴ δὲ ἡ ΑΓ, δύο δὴ αἱ ΔΑ, ΑΓ δυσὶ ταῖς ΒΑ, ΑΓ ἴσαι εἰσίν· καὶ βάσις ἡ ΔΓ βάσει τῇ ΒΓ ἴση· γωνία ἄρα ἡ ὑπὸ ΔΑΓ γωνίᾳ τῇ ὑπὸ ΒΑΓ ἴση ἐστίν· ἡ ἄρα ὑπὸ ΔΑΒ γωνία δίχα τέτμηται ὑπὸ τῆς ΑΓ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἑκάστη τῶν ὑπὸ ΑΒΓ, ΒΓΔ, ΓΔΑ δίχα τέτμηται ὑπὸ τῶν ΑΓ, ΔΒ εὐθειῶν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΔΑΒ γωνία τῇ ὑπὸ ΑΒΓ, καί ἐστι τῆς μὲν ὑπὸ ΔΑΒ ἡμίσεια ἡ ὑπὸ ΕΑΒ, τῆς δὲ ὑπὸ ΑΒΓ ἡμίσεια ἡ ὑπὸ ΕΒΑ, καὶ ἡ ὑπὸ ΕΑΒ ἄρα τῇ ὑπὸ ΕΒΑ ἐστιν ἴση· ὥστε καὶ πλευρὰ ἡ ΕΑ τῇ ΕΒ ἐστιν ἴση. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἑκατέρα τῶν ΕΑ, ΕΒ [εὐθειῶν] ἑκατέρᾳ τῶν ΕΓ, ΕΔ ἴση ἐστίν. αἱ τέσσαρες ἄρα αἱ ΕΑ, ΕΒ, ΕΓ, ΕΔ ἴσαι ἀλλήλαις εἰσίν. ὁ ἄρα κέντρῳ τῷ Ε καὶ διαστήματι ἑνὶ τῶν Α, Β, Γ, Δ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἔσται περιγεγραμμένος περὶ τὸ ΑΒΓΔ τετράγωνον. περιγεγράφθω ὡς ὁ ΑΒΓΔ.

Περὶ τὸ δοθὲν ἄρα τετράγωνον κύκλος περιγέγραπται· ὅπερ ἔδει ποιῆσαι.

ι΄. Ἰσοσκελὲς τρίγωνον συστήσασθαι ἔχον ἑκατέραν τῶν πρὸς τῇ βάσει γωνιῶν διπλασίονα τῆς λοιπῆς.

Ἐκκείσθω τις εὐθεῖα ἡ ΑΒ, καὶ τετμήσθω κατὰ τὸ Γ σημεῖον, ὥστε τὸ ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον ἴσον εἶναι τῷ ἀπὸ τῆς ΓΑ τετραγώνῳ· καὶ κέντρῳ τῷ Α καὶ διαστήματι τῷ ΑΒ κύκλος γεγράφθω ὁ ΒΔΕ, καὶ ἐνηρμόσθω εἰς τὸν ΒΔΕ κύκλον τῇ ΑΓ εὐθείᾳ μὴ μείζονι οὔσῃ τῆς τοῦ ΒΔΕ κύκλου διαμέτρου ἴση εὐθεῖα ἡ ΒΔ· καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΓ, καὶ περιγεγράφθω περὶ τὸ ΑΓΔ τρίγωνον κύκλος ὁ ΑΓΔ.

Καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ, ἴση δὲ ἡ ΑΓ τῇ ΒΔ, τὸ ἄρα ὑπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΒΔ. καὶ ἐπεὶ κύκλου τοῦ ΑΓΔ εἴληπταί τι σημεῖον ἐκτὸς τὸ Β, καὶ ἀπὸ τοῦ Β πρὸς τὸν ΑΓΔ κύκλον προσπεπτώκασι δύο εὐθεῖαι αἱ ΒΑ, ΒΔ, καὶ ἡ μὲν αὐτῶν τέμνει, ἡ δὲ προσπίπτει, καί ἐστι τὸ ὑπὸ τῶν ΑΒ, ΒΓ ἴσον τῷ ἀπὸ τῆς ΒΔ, ἡ ΒΔ ἄρα ἐφάπτεται τοῦ ΑΓΔ κύκλου. ἐπεὶ οὖν ἐφάπτεται μὲν ἡ ΒΔ, ἀπὸ δὲ τῆς κατὰ τὸ Δ ἐπαφῆς διῆκται ἡ ΔΓ, ἡ ἄρα ὑπὸ ΒΔΓ γωνία ἴση ἐστὶ τῇ ἐν τῷ ἐναλλὰξ τοῦ κύκλου τμήματι γωνίᾳ τῇ ὑπὸ ΔΑΓ. ἐπεὶ οὖν ἴση ἐστὶν ἡ ὑπὸ ΒΔΓ τῇ ὑπὸ ΔΑΓ, κοινὴ προσκείσθω ἡ ὑπὸ ΓΔΑ· ὅλη ἄρα ἡ ὑπὸ ΒΔΑ ἴση ἐστὶ δυσὶ ταῖς ὑπὸ ΓΔΑ, ΔΑΓ. ἀλλὰ ταῖς ὑπὸ ΓΔΑ, ΔΑΓ ἴση ἐστὶν ἡ ἐκτὸς ἡ ὑπὸ ΒΓΔ· καὶ ἡ ὑπὸ ΒΔΑ ἄρα ἴση ἐστὶ τῇ ὑπὸ ΒΓΔ. ἀλλὰ ἡ ὑπὸ ΒΔΑ τῇ ὑπὸ ΓΒΔ ἐστιν ἴση, ἐπεὶ καὶ πλευρὰ ἡ ΑΔ τῇ ΑΒ ἐστιν ἴση· ὥστε καὶ ἡ ὑπὸ ΔΒΑ τῇ ὑπὸ ΒΓΔ ἐστιν ἴση. αἱ τρεῖς ἄρα αἱ ὑπὸ ΒΔΑ, ΔΒΑ, ΒΓΔ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΔΒΓ γωνία τῇ ὑπὸ ΒΓΔ, ἴση ἐστὶ καὶ πλευρὰ ἡ ΒΔ πλευρᾷ τῇ ΔΓ. ἀλλὰ ἡ ΒΔ τῇ ΓΑ ὑπόκειται ἴση· καὶ ἡ ΓΑ ἄρα τῇ ΓΔ ἐστιν ἴση· ὥστε καὶ γωνία ἡ ὑπὸ ΓΔΑ γωνίᾳ τῇ ὑπὸ ΔΑΓ ἐστιν ἴση· αἱ ἄρα ὑπὸ ΓΔΑ, ΔΑΓ τῆς ὑπὸ ΔΑΓ εἰσι διπλασίους. ἴση δὲ ἡ ὑπὸ ΒΓΔ ταῖς ὑπὸ ΓΔΑ, ΔΑΓ· καὶ ἡ ὑπὸ ΒΓΔ ἄρα τῆς ὑπὸ ΓΑΔ ἐστι διπλῆ. ἴση δὲ ἡ ὑπὸ ΒΓΔ

ἑκατέρᾳ τῶν ὑπὸ ΒΔΑ, ΔΒΑ· καὶ ἑκατέρα ἄρα τῶν ὑπὸ ΒΔΑ, ΔΒΑ τῆς ὑπὸ ΔΑΒ ἐστι διπλῆ.

Ἰσοσκελὲς ἄρα τρίγωνον συνέσταται τὸ ΑΒΔ ἔχον ἑκατέραν τῶν πρὸς τῇ ΔΒ βάσει γωνιῶν διπλασίονα τῆς λοιπῆς· ὅπερ ἔδει ποιῆσαι.

ια΄. Εἰς τὸν δοθέντα κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔΕ· δεῖ δὴ εἰς τὸν ΑΒΓΔΕ κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Ἐκκείσθω τρίγωνον ἰσοσκελὲς τὸ ΖΗΘ διπλασίονα ἔχον ἑκατέραν τῶν πρὸς τοῖς Η, Θ γωνιῶν τῆς πρὸς τῷ Ζ, καὶ ἐγγεγράφθω εἰς τὸν ΑΒΓΔΕ κύκλον τῷ ΖΗΘ τριγώνῳ ἰσογώνιον τρίγωνον τὸ ΑΓΔ, ὥστε τῇ μὲν πρὸς τῷ Ζ γωνίᾳ ἴσην εἶναι τὴν ὑπὸ ΓΑΔ, ἑκατέραν δὲ τῶν πρὸς τοῖς Η, Θ ἴσην ἑκατέρᾳ τῶν ὑπὸ ΑΓΔ, ΓΔΑ· καὶ ἑκατέρα ἄρα τῶν ὑπὸ ΑΓΔ, ΓΔΑ τῆς ὑπὸ ΓΑΔ ἐστι διπλῆ. τετμήσθω δὴ ἑκατέρα τῶν ὑπὸ ΑΓΔ, ΓΔΑ δίχα ὑπὸ ἑκατέρας τῶν ΓΕ, ΔΒ εὐθειῶν, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΒΓ, [ΓΔ], ΔΕ, ΕΑ.

Ἐπεὶ οὖν ἑκατέρα τῶν ὑπὸ ΑΓΔ, ΓΔΑ γωνιῶν διπλασίων ἐστὶ τῆς ὑπὸ ΓΑΔ, καὶ τετμημέναι εἰσὶ δίχα ὑπὸ τῶν ΓΕ, ΔΒ εὐθειῶν, αἱ πέντε ἄρα γωνίαι αἱ ὑπὸ ΔΑΓ, ΑΓΕ, ΕΓΔ, ΓΔΒ, ΒΔΑ ἴσαι ἀλλήλαις εἰσίν. αἱ δὲ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν· αἱ πέντε ἄρα περιφέρειαι αἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ ἴσαι ἀλλήλαις εἰσίν. ὑπὸ δὲ τὰς ἴσας περιφερείας ἴσαι εὐθεῖαι ὑποτείνουσιν· αἱ πέντε ἄρα εὐθεῖαι αἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ ἴσαι ἀλλήλαις εἰσίν· ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓΔΕ πεντάγωνον. λέγω δή, ὅτι καὶ ἰσογώνιον. ἐπεὶ γὰρ ἡ ΑΒ περιφέρεια τῇ ΔΕ περιφερείᾳ ἐστὶν ἴση, κοινὴ προσκείσθω ἡ ΒΓΔ· ὅλη ἄρα ἡ ΑΒΓΔ περιφέρεια ὅλῃ τῇ ΕΔΓΒ περιφερείᾳ ἐστὶν ἴση. καὶ βέβηκεν ἐπὶ μὲν τῆς ΑΒΓΔ περιφερείας γωνία ἡ ὑπὸ ΑΕΔ, ἐπὶ δὲ τῆς ΕΔΓΒ περιφερείας γωνία ἡ ὑπὸ ΒΑΕ· καὶ ἡ ὑπὸ ΒΑΕ ἄρα γωνία τῇ ὑπὸ ΑΕΔ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἑκάστη τῶν ὑπὸ ΑΒΓ, ΒΓΔ, ΓΔΕ

γωνιῶν ἑκατέρᾳ τῶν ὑπὸ ΒΑΕ, ΑΕΔ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓΔΕ πεντάγωνον. ἐδείχθη δὲ καὶ ἰσόπλευρον.

Εἰς ἄρα τὸν δοθέντα κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

ιβ΄. Περὶ τὸν δοθέντα κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον περιγράψαι.

Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔΕ· δεῖ δὴ περὶ τὸν ΑΒΓΔΕ κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον περιγράψαι.

Νενοήσθω τοῦ ἐγγεγραμμένου πενταγώνου τῶν γωνιῶν σημεῖα τὰ Α, Β, Γ, Δ, Ε, ὥστε ἴσας εἶναι τὰς ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ περιφερείας· καὶ διὰ τῶν Α, Β, Γ, Δ, Ε ἤχθωσαν τοῦ κύκλου ἐφαπτόμεναι αἱ ΗΘ, ΘΚ, ΚΛ, ΛΜ, ΜΗ, καὶ εἰλήφθω τοῦ ΑΒΓΔΕ κύκλου κέντρον τὸ Ζ, καὶ ἐπεζεύχθωσαν αἱ ΖΒ, ΖΚ, ΖΓ, ΖΛ, ΖΔ.

Καὶ ἐπεὶ ἡ μὲν ΚΛ εὐθεῖα ἐφάπτεται τοῦ ΑΒΓΔΕ κατὰ τὸ Γ, ἀπὸ δὲ τοῦ Ζ κέντρου ἐπὶ τὴν κατὰ τὸ Γ ἐπαφὴν ἐπέζευκται ἡ ΖΓ, ἡ ΖΓ ἄρα κάθετός ἐστιν ἐπὶ τὴν ΚΛ· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν πρὸς τῷ Γ γωνιῶν. διὰ τὰ αὐτὰ δὴ καὶ αἱ πρὸς τοῖς Β, Δ σημείοις γωνίαι ὀρθαί εἰσιν. καὶ ἐπεὶ ὀρθή ἐστιν ἡ ὑπὸ ΖΓΚ γωνία, τὸ ἄρα ἀπὸ τῆς ΖΚ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΖΓ, ΓΚ. διὰ τὰ αὐτὰ δὴ καὶ τοῖς ἀπὸ τῶν ΖΒ, ΒΚ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΖΚ· ὥστε τὰ ἀπὸ τῶν ΖΓ, ΓΚ τοῖς ἀπὸ τῶν ΖΒ, ΒΚ ἐστιν ἴσα, ὧν τὸ ἀπὸ τῆς ΖΓ τῷ ἀπὸ τῆς ΖΒ ἐστιν ἴσον· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΓΚ τῷ ἀπὸ τῆς ΒΚ ἐστιν ἴσον. ἴση ἄρα ἡ ΒΚ τῇ ΓΚ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΖΒ τῇ ΖΓ, καὶ κοινὴ ἡ ΖΚ, δύο δὴ αἱ ΒΖ, ΖΚ δυσὶ ταῖς ΓΖ, ΖΚ ἴσαι εἰσίν· καὶ βάσις ἡ ΒΚ βάσει τῇ ΓΚ [ἐστιν] ἴση· γωνία ἄρα ἡ μὲν ὑπὸ ΒΖΚ [γωνίᾳ] τῇ ὑπὸ ΚΖΓ ἐστιν ἴση· ἡ δὲ ὑπὸ ΒΚΖ τῇ ὑπὸ ΖΚΓ· διπλῆ ἄρα ἡ μὲν ὑπὸ ΒΖΓ τῆς ὑπὸ ΚΖΓ, ἡ δὲ ὑπὸ ΒΚΓ τῆς ὑπὸ ΖΚΓ. διὰ τὰ αὐτὰ δὴ καὶ ἡ μὲν ὑπὸ ΓΖΔ τῆς ὑπὸ ΓΖΛ ἐστι διπλῆ, ἡ δὲ ὑπὸ ΔΛΓ τῆς ὑπὸ ΖΛΓ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΓ περιφέρεια τῇ ΓΔ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ

ΒΖΓ τῇ ὑπὸ ΓΖΔ. καί ἐστιν ἡ μὲν ὑπὸ ΒΖΓ τῆς ὑπὸ ΚΖΓ διπλῆ, ἡ δὲ ὑπὸ ΔΖΓ τῆς ὑπὸ ΛΖΓ· ἴση ἄρα καὶ ἡ ὑπὸ ΚΖΓ τῇ ὑπὸ ΛΖΓ· ἐστὶ δὲ καὶ ἡ ὑπὸ ΖΓΚ γωνία τῇ ὑπὸ ΖΓΛ ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΖΚΓ, ΖΛΓ τὰς δύο γωνίας ταῖς δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην κοινὴν αὐτῶν τὴν ΖΓ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει καὶ τὴν λοιπὴν γωνίαν τῇ λοιπῇ γωνίᾳ· ἴση ἄρα ἡ μὲν ΚΓ εὐθεῖα τῇ ΓΛ, ἡ δὲ ὑπὸ ΖΚΓ γωνία τῇ ὑπὸ ΖΛΓ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΚΓ τῇ ΓΛ, διπλῆ ἄρα ἡ ΚΛ τῆς ΚΓ. διὰ τὰ αὐτὰ δὴ δειχθήσεται καὶ ἡ ΘΚ τῆς ΒΚ διπλῆ. καί ἐστιν ἡ ΒΚ τῇ ΚΓ ἴση· καὶ ἡ ΘΚ ἄρα τῇ ΚΛ ἐστιν ἴση. ὁμοίως δὴ δειχθήσεται καὶ ἑκάστη τῶν ΘΗ, ΗΜ, ΜΛ ἑκατέρᾳ τῶν ΘΚ, ΚΛ ἴση· ἰσόπλευρον ἄρα ἐστὶ τὸ ΗΘΚΛΜ πεντάγωνον. λέγω δή, ὅτι καὶ ἰσογώνιον. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ὑπὸ ΖΚΓ γωνία τῇ ὑπὸ ΖΛΓ, καὶ ἐδείχθη τῆς μὲν ὑπὸ ΖΚΓ διπλῆ ἡ ὑπὸ ΘΚΛ, τῆς δὲ ὑπὸ ΖΛΓ διπλῆ ἡ ὑπὸ ΚΛΜ, καὶ ἡ ὑπὸ ΘΚΛ ἄρα τῇ ὑπὸ ΚΛΜ ἐστιν ἴση. ὁμοίως δὴ δειχθήσεται καὶ ἑκάστη τῶν ὑπὸ ΚΘΗ, ΘΗΜ, ΗΜΛ ἑκατέρᾳ τῶν ὑπὸ ΘΚΛ, ΚΛΜ ἴση· αἱ πέντε ἄρα γωνίαι αἱ ὑπὸ ΗΘΚ, ΘΚΛ, ΚΛΜ, ΛΜΗ, ΜΗΘ ἴσαι ἀλλήλαις εἰσίν. ἰσογώνιον ἄρα ἐστὶ τὸ ΗΘΚΛΜ πεντάγωνον. ἐδείχθη δὲ καὶ ἰσόπλευρον, καὶ περιγέγραπται περὶ τὸν ΑΒΓΔΕ κύκλον.

[Περὶ τὸν δοθέντα ἄρα κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον περιγέγραπται]· ὅπερ ἔδει ποιῆσαι.

ιγ΄. Εἰς τὸ δοθὲν πεντάγωνον, ὅ ἐστιν ἰσόπλευρόν τε καὶ ἰσογώνιον, κύκλον ἐγγράψαι.

Ἔστω τὸ δοθὲν πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον τὸ ΑΒΓΔΕ· δεῖ δὴ εἰς τὸ ΑΒΓΔΕ πεντάγωνον κύκλον ἐγγράψαι.

Τετμήσθω γὰρ ἑκατέρα τῶν ὑπὸ ΒΓΔ, ΓΔΕ γωνιῶν δίχα ὑπὸ ἑκατέρας τῶν ΓΖ, ΔΖ εὐθειῶν· καὶ ἀπὸ τοῦ Ζ σημείου, καθ’ ὃ συμβάλλουσιν ἀλλήλαις αἱ ΓΖ, ΔΖ εὐθεῖαι, ἐπεζεύχθωσαν αἱ ΖΒ, ΖΑ, ΖΕ εὐθεῖαι. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΓ τῇ ΓΔ, κοινὴ δὲ ἡ ΓΖ, δύο δὴ αἱ ΒΓ, ΓΖ δυσὶ ταῖς ΔΓ, ΓΖ ἴσαι εἰσίν· καὶ

γωνία ἡ ὑπὸ ΒΓΖ γωνίᾳ τῇ ὑπὸ ΔΓΖ [ἐστιν] ἴση· βάσις ἄρα ἡ ΒΖ βάσει τῇ ΔΖ ἐστιν ἴση, καὶ τὸ ΒΓΖ τρίγωνον τῷ ΔΓΖ τριγώνῳ ἐστιν ἴσον, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὑφ' ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἡ ὑπὸ ΓΒΖ γωνία τῇ ὑπὸ ΓΔΖ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ὑπὸ ΓΔΕ τῆς ὑπὸ ΓΔΖ, ἴση δὲ ἡ μὲν ὑπὸ ΓΔΕ τῇ ὑπὸ ΑΒΓ, ἡ δὲ ὑπὸ ΓΔΖ τῇ ὑπὸ ΓΒΖ, καὶ ἡ ὑπὸ ΓΒΑ ἄρα τῆς ὑπὸ ΓΒΖ ἐστι διπλῆ· ἴση ἄρα ἡ ὑπὸ ΑΒΖ γωνία τῇ ὑπὸ ΖΒΓ· ἡ ἄρα ὑπὸ ΑΒΓ γωνία δίχα τέτμηται ὑπὸ τῆς ΒΖ εὐθείας. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἑκατέρα τῶν ὑπὸ ΒΑΕ, ΑΕΔ δίχα τέτμηται ὑπὸ ἑκατέρας τῶν ΖΑ, ΖΕ εὐθειῶν. ἤχθωσαν δὴ ἀπὸ τοῦ Ζ σημείου ἐπὶ τὰς ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ εὐθείας κάθετοι αἱ ΖΗ, ΖΘ, ΖΚ, ΖΛ, ΖΜ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΘΓΖ γωνία τῇ ὑπὸ ΚΓΖ, ἐστὶ δὲ καὶ ὀρθὴ ἡ ὑπὸ ΖΘΓ [ὀρθῇ] τῇ ὑπὸ ΖΚΓ ἴση, δύο δὴ τρίγωνά ἐστι τὰ ΖΘΓ, ΖΚΓ τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην κοινὴν αὐτῶν τὴν ΖΓ ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει· ἴση ἄρα ἡ ΖΘ κάθετος τῇ ΖΚ καθέτῳ. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἑκάστη τῶν ΖΛ, ΖΜ, ΖΗ ἑκατέρᾳ τῶν ΖΘ, ΖΚ ἴση ἐστίν· αἱ πέντε ἄρα εὐθεῖαι αἱ ΖΗ, ΖΘ, ΖΚ, ΖΛ, ΖΜ ἴσαι ἀλλήλαις εἰσίν. ὁ ἄρα κέντρῳ τῷ Ζ διαστήματι δὲ ἑνὶ τῶν Η, Θ, Κ, Λ, Μ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἐφάψεται τῶν ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ εὐθειῶν διὰ τὸ ὀρθὰς εἶναι τὰς πρὸς τοῖς Η, Θ, Κ, Λ, Μ σημείοις γωνίας. εἰ γὰρ οὐκ ἐφάψεται αὐτῶν, ἀλλὰ τεμεῖ αὐτάς, συμβήσεται τὴν τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ' ἄκρας ἀγομένην ἐντὸς πίπτειν τοῦ κύκλου· ὅπερ ἄτοπον ἐδείχθη. οὐκ ἄρα ὁ κέντρῳ τῷ Ζ διαστήματι δὲ ἑνὶ τῶν Η, Θ, Κ, Λ, Μ σημείων γραφόμενος κύκλος τεμεῖ τὰς ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ εὐθείας· ἐφάψεται ἄρα αὐτῶν. γεγράφθω ὡς ὁ ΗΘΚΛΜ.

Εἰς ἄρα τὸ δοθὲν πεντάγωνον, ὅ ἐστιν ἰσόπλευρόν τε καὶ ἰσογώνιον, κύκλος ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

ιδ΄. Περὶ τὸ δοθὲν πεντάγωνον, ὅ ἐστιν ἰσόπλευρόν τε καὶ ἰσογώνιον, κύκλον περιγράψαι.

Ἔστω τὸ δοθὲν πεντάγωνον, ὅ ἐστιν ἰσόπλευρόν τε καὶ ἰσογώνιον, τὸ ΑΒΓΔΕ· δεῖ δὴ περὶ τὸ ΑΒΓΔΕ πεντάγωνον κύκλον περιγράψαι.

Τετμήσθω δὴ ἑκατέρα τῶν ὑπὸ ΒΓΔ, ΓΔΕ γωνιῶν δίχα ὑπὸ ἑκατέρας τῶν ΓΖ, ΔΖ, καὶ ἀπὸ τοῦ Ζ σημείου, καθ’ ὃ συμβάλλουσιν αἱ εὐθεῖαι, ἐπὶ τὰ Β, Α, Ε σημεῖα ἐπεζεύχθωσαν εὐθεῖαι αἱ ΖΒ, ΖΑ, ΖΕ. ὁμοίως δὴ τῷ πρὸ τούτου δειχθήσεται, ὅτι καὶ ἑκάστη τῶν ὑπὸ ΓΒΑ, ΒΑΕ, ΑΕΔ γωνιῶν δίχα τέτμηται ὑπὸ ἑκάστης τῶν ΖΒ, ΖΑ, ΖΕ εὐθειῶν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΒΓΔ γωνία τῇ ὑπὸ ΓΔΕ, καί ἐστι τῆς μὲν ὑπὸ ΒΓΔ ἡμίσεια ἡ ὑπὸ ΖΓΔ, τῆς δὲ ὑπὸ ΓΔΕ ἡμίσεια ἡ ὑπὸ ΓΔΖ, καὶ ἡ ὑπὸ ΖΓΔ ἄρα τῇ ὑπὸ ΖΔΓ ἐστιν ἴση· ὥστε καὶ πλευρὰ ἡ ΖΓ πλευρᾷ τῇ ΖΔ ἐστιν ἴση. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἑκάστη τῶν ΖΒ, ΖΑ, ΖΕ ἑκατέρᾳ τῶν ΖΓ, ΖΔ ἐστιν ἴση· αἱ πέντε ἄρα εὐθεῖαι αἱ ΖΑ, ΖΒ, ΖΓ, ΖΔ, ΖΕ ἴσαι ἀλλήλαις εἰσίν. ὁ ἄρα κέντρῳ τῷ Ζ καὶ διαστήματι ἑνὶ τῶν ΖΑ, ΖΒ, ΖΓ, ΖΔ, ΖΕ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἔσται περιγεγραμμένος. περιγεγράφθω καὶ ἔστω ὁ ΑΒΓΔΕ.

Περὶ ἄρα τὸ δοθὲν πεντάγωνον, ὅ ἐστιν ἰσόπλευρόν τε καὶ ἰσογώνιον, κύκλος περιγέγραπται· ὅπερ ἔδει ποιῆσαι.

ιε΄. Εἰς τὸν δοθέντα κύκλον ἑξάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔΕΖ· δεῖ δὴ εἰς τὸν ΑΒΓΔΕΖ κύκλον ἑξάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Ἤχθω τοῦ ΑΒΓΔΕΖ κύκλου διάμετρος ἡ ΑΔ, καὶ εἰλήφθω τὸ κέντρον τοῦ κύκλου τὸ Η, καὶ κέντρῳ μὲν τῷ Δ διαστήματι δὲ τῷ ΔΗ κύκλος γεγράφθω ὁ ΕΗΓΘ, καὶ ἐπιζευχθεῖσαι αἱ ΕΗ, ΓΗ διήχθωσαν ἐπὶ τὰ Β, Ζ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΖ, ΖΑ· λέγω, ὅτι τὸ ΑΒΓΔΕΖ ἑξάγωνον ἰσόπλευρόν τέ ἐστι καὶ ἰσογώνιον.

Ἐπεὶ γὰρ τὸ Η σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓΔΕΖ κύκλου, ἴση ἐστὶν ἡ ΗΕ τῇ ΗΔ. πάλιν, ἐπεὶ τὸ Δ σημεῖον κέντρον ἐστὶ τοῦ ΗΓΘ κύκλου, ἴση ἐστὶν ἡ ΔΕ τῇ ΔΗ. ἀλλ’ ἡ ΗΕ τῇ ΗΔ ἐδείχθη ἴση· καὶ ἡ ΗΕ ἄρα τῇ ΕΔ ἴση ἐστίν· ἰσόπλευρον ἄρα ἐστὶ τὸ ΕΗΔ τρίγωνον· καὶ αἱ τρεῖς ἄρα αὐτοῦ γωνίαι αἱ ὑπὸ ΕΗΔ, ΗΔΕ, ΔΕΗ ἴσαι ἀλλήλαις εἰσίν, ἐπειδήπερ τῶν ἰσοσκελῶν τριγώνων αἱ πρὸς τῇ βάσει γωνίαι ἴσαι ἀλλήλαις εἰσίν· καί εἰσιν αἱ τρεῖς τοῦ τριγώνου γωνίαι δυσὶν ὀρθαῖς ἴσαι· ἡ ἄρα ὑπὸ ΕΗΔ γωνία τρίτον ἐστὶ δύο ὀρθῶν. ὁμοίως δὴ δειχθήσεται καὶ ἡ ὑπὸ ΔΗΓ τρίτον δύο ὀρθῶν. καὶ ἐπεὶ ἡ ΓΗ εὐθεῖα ἐπὶ τὴν ΕΒ σταθεῖσα τὰς ἐφεξῆς γωνίας τὰς ὑπὸ ΕΗΓ, ΓΗΒ δυσὶν ὀρθαῖς ἴσας ποιεῖ, καὶ λοιπὴ ἄρα ἡ ὑπὸ ΓΗΒ τρίτον ἐστὶ δύο ὀρθῶν· αἱ ἄρα ὑπὸ ΕΗΔ, ΔΗΓ, ΓΗΒ γωνίαι ἴσαι ἀλλήλαις εἰσίν· ὥστε καὶ αἱ κατὰ κορυφὴν αὐταῖς αἱ ὑπὸ ΒΗΑ, ΑΗΖ, ΖΗΕ ἴσαι εἰσίν [ταῖς ὑπὸ ΕΗΔ, ΔΗΓ, ΓΗΒ]. αἱ ἓξ ἄρα γωνίαι αἱ ὑπὸ ΕΗΔ, ΔΗΓ, ΓΗΒ, ΒΗΑ, ΑΗΖ, ΖΗΕ ἴσαι ἀλλήλαις εἰσίν. αἱ δὲ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν· αἱ ἓξ ἄρα περιφέρειαι αἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΖ, ΖΑ ἴσαι ἀλλήλαις εἰσίν. ὑπὸ δὲ τὰς ἴσας περιφερείας αἱ ἴσαι εὐθεῖαι ὑποτείνουσιν· αἱ ἓξ ἄρα εὐθεῖαι ἴσαι ἀλλήλαις εἰσίν· ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓΔΕΖ ἑξάγωνον. λέγω δή, ὅτι καὶ ἰσογώνιον. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΖΑ περιφέρεια τῇ ΕΔ περιφερείᾳ, κοινὴ προσκείσθω ἡ ΑΒΓΔ περιφέρεια· ὅλη ἄρα ἡ ΖΑΒΓΔ ὅλῃ τῇ ΕΔΓΒΑ ἐστιν ἴση· καὶ βέβηκεν ἐπὶ μὲν τῆς ΖΑΒΓΔ περιφερείας ἡ ὑπὸ ΖΕΔ γωνία, ἐπὶ δὲ τῆς ΕΔΓΒΑ περιφερείας ἡ ὑπὸ ΑΖΕ γωνία· ἴση ἄρα ἡ ὑπὸ ΑΖΕ γωνία τῇ ὑπὸ ΔΕΖ. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ αἱ λοιπαὶ γωνίαι τοῦ ΑΒΓΔΕΖ ἑξαγώνου κατὰ μίαν ἴσαι εἰσὶν ἑκατέρᾳ τῶν ὑπὸ ΑΖΕ, ΖΕΔ γωνιῶν· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓΔΕΖ ἑξάγωνον. ἐδείχθη δὲ καὶ ἰσόπλευρον· καὶ ἐγγέγραπται εἰς τὸν ΑΒΓΔΕΖ κύκλον.

Εἰς ἄρα τὸν δοθέντα κύκλον ἑξάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι ἡ τοῦ ἑξαγώνου πλευρὰ ἴση ἐστὶ τῇ ἐκ τοῦ κέντρου τοῦ κύκλου.

Ὁμοίως δὲ τοῖς ἐπὶ τοῦ πενταγώνου ἐὰν διὰ τῶν κατὰ τὸν κύκλον διαιρέσεων ἐφαπτομένας τοῦ κύκλου ἀγάγωμεν, περιγραφήσεται περὶ τὸν κύκλον ἑξάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἀκολούθως τοῖς ἐπὶ τοῦ πενταγώνου εἰρημένοις. καὶ ἔτι διὰ τῶν ὁμοίων τοῖς ἐπὶ τοῦ πενταγώνου εἰρημένοις εἰς τὸ δοθὲν ἑξάγωνον κύκλον ἐγγράψομέν τε καὶ περιγράψομεν· ὅπερ ἔδει ποιῆσαι.

ις΄. Εἰς τὸν δοθέντα κύκλον πεντεκαιδεκάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔ· δεῖ δὴ εἰς τὸν ΑΒΓΔ κύκλον πεντεκαιδεκάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Ἐγγεγράφθω εἰς τὸν ΑΒΓΔ κύκλον τριγώνου μὲν ἰσοπλεύρου τοῦ εἰς αὐτὸν ἐγγραφομένου πλευρὰ ἡ ΑΓ, πενταγώνου δὲ ἰσοπλεύρου ἡ ΑΒ· οἵων ἄρα ἐστὶν ὁ ΑΒΓΔ κύκλος ἴσων τμημάτων δεκαπέντε, τοιούτων ἡ μὲν ΑΒΓ περιφέρεια τρίτον οὖσα τοῦ κύκλου ἔσται πέντε, ἡ δὲ ΑΒ περιφέρεια πέμπτον οὖσα τοῦ κύκλου ἔσται τριῶν· λοιπὴ ἄρα ἡ ΒΓ τῶν ἴσων δύο. τετμήσθω ἡ ΒΓ δίχα κατὰ τὸ Ε· ἑκατέρα ἄρα τῶν ΒΕ, ΕΓ περιφερειῶν πεντεκαιδέκατόν ἐστι τοῦ ΑΒΓΔ κύκλου.

Ἐὰν ἄρα ἐπιζεύξαντες τὰς ΒΕ, ΕΓ ἴσας αὐταῖς κατὰ τὸ συνεχὲς εὐθείας ἐναρμόσωμεν εἰς τὸν ΑΒΓΔ[Ε] κύκλον, ἔσται εἰς αὐτὸν ἐγγεγραμμένον πεντεκαιδεκάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον· ὅπερ ἔδει ποιῆσαι.

Ὁμοίως δὲ τοῖς ἐπὶ τοῦ πενταγώνου ἐὰν διὰ τῶν κατὰ τὸν κύκλον διαιρέσεων ἐφαπτομένας τοῦ κύκλου ἀγάγωμεν, περιγραφήσεται περὶ τὸν κύκλον πεντεκαιδεκάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον. ἔτι δὲ διὰ τῶν ὁμοίων τοῖς ἐπὶ τοῦ πενταγώνου δείξεων καὶ εἰς τὸ δοθὲν πεντεκαιδεκάγωνον κύκλον ἐγγράψομέν τε καὶ περιγράψομεν· ὅπερ ἔδει ποιῆσαι.

BOOK V.

Ὅροι ιη΄.

α΄. Μέρος ἐστὶ μέγεθος μεγέθους τὸ ἔλασσον τοῦ μείζονος, ὅταν καταμετρῇ τὸ μεῖζον.

β΄. Πολλαπλάσιον δὲ τὸ μεῖζον τοῦ ἐλάττονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάττονος.

γ΄. Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητά ποια σχέσις.

δ΄. Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἃ δύναται πολλαπλασιαζόμενα ἀλλήλων ὑπερέχειν.

ε΄. Ἐν τῷ αὐτῷ λόγῳ μεγέθη λέγεται εἶναι πρῶτον πρὸς δεύτερον καὶ τρίτον πρὸς τέταρτον, ὅταν τὰ τοῦ πρώτου καὶ τρίτου ἰσάκις πολλαπλάσια τῶν τοῦ δευτέρου καὶ τετάρτου ἰσάκις πολλαπλασίων καθ' ὁποιονοῦνπολλαπλασιασμὸν ἑκάτερον ἑκατέρου ἢ ἅμα ὑπερέχῃ ἢ ἅμα ἴσα ᾖ ἢ ἅμα ἐλλείπῃ ληφθέντα κατάλληλα.

ς΄. Τὰ δὲ τὸν αὐτὸν ἔχοντα λόγον μεγέθη ἀνάλογον καλείσθω.

ζ΄. Ὅταν δὲ τῶν ἰσάκις πολλαπλασίων τὸ μὲν τοῦ πρώτου πολλαπλάσιον ὑπερέχῃ τοῦ τοῦ δευτέρου πολλαπλασίου, τὸ δὲ τοῦ τρίτου πολλαπλάσιον μὴ ὑπερέχῃ τοῦ τοῦ τετάρτου πολλαπλασίου, τότε τὸ πρῶτον πρὸς τὸδεύτερον μείζονα λόγον ἔχειν λέγεται, ἤπερ τὸ τρίτον πρὸς τὸ τέταρτον.

η΄. Ἀναλογία δὲ ἐν τρισὶν ὅροις ἐλαχίστη ἐστίν.

θ΄. Ὅταν δὲ τρία μεγέθη ἀνάλογον ᾖ, τὸ πρῶτον πρὸς τὸ τρίτον διπλασίονα λόγον ἔχειν λέγεται ἤπερ πρὸς τὸ δεύτερον.

ι΄. Ὅταν δὲ τέσσαρα μεγέθη ἀνάλογον ᾖ, τὸ πρῶτον πρὸς τὸ τέταρτον τριπλασίονα λόγον ἔχειν λέγεται ἤπερ πρὸς τὸ δεύτερον, καὶ ἀεὶ ἑξῆς ὁμοίως, ὡς ἂν ἡ ἀναλογία ὑπάρχῃ.

ια΄. Ὁμόλογα μεγέθη λέγεται τὰ μὲν ἡγούμενα τοῖς ἡγουμένοις τὰ δὲ ἑπόμενα τοῖς ἑπομένοις.

ιβ΄. Ἐναλλὰξ λόγος ἐστὶ λῆψις τοῦ ἡγουμένου πρὸς τὸ ἡγούμενον καὶ τοῦ ἑπομένου πρὸς τὸ ἑπόμενον.

ιγ΄. Ἀνάπαλιν λόγος ἐστὶ λῆψις τοῦ ἑπομένου ὡς ἡγουμένου πρὸς τὸ ἡγούμενον ὡς ἑπόμενον.

ιδ΄. Σύνθεσις λόγου ἐστὶ λῆψις τοῦ ἡγουμένου μετὰ τοῦ ἑπομένου ὡς ἑνὸς πρὸς αὐτὸ τὸ ἑπόμενον.

ιε΄. Διαίρεσις λόγου ἐστὶ λῆψις τῆς ὑπεροχῆς, ᾗ ὑπερέχει τὸ ἡγούμενον τοῦ ἑπομένου, πρὸς αὐτὸ τὸ ἑπόμενον.

ις΄. Ἀναστροφὴ λόγου ἐστὶ λῆψις τοῦ ἡγουμένου πρὸς τὴν ὑπεροχήν, ᾗ ὑπερέχει τὸ ἡγούμενον τοῦ ἑπομένου.

ιζ΄. Δι᾽ ἴσου λόγος ἐστὶ πλειόνων ὄντων μεγεθῶν καὶ ἄλλων αὐτοῖς ἴσων τὸ πλῆθος σύνδυο λαμβανομένων καὶ ἐν τῷ αὐτῷ λόγῳ, ὅταν ᾖ ὡς ἐν τοῖς πρώτοις μεγέθεσι τὸ πρῶτον πρὸς τὸ ἔσχατον, οὕτως ἐν τοῖς δευτέροις μεγέθεσι τὸ πρῶτον πρὸς τὸ ἔσχατον· ἢ ἄλλως· Λῆψις τῶν ἄκρων καθ᾽ ὑπεξαίρεσιν τῶν μέσων.

ιη΄. Τεταραγμένη δὲ ἀναλογία ἐστίν, ὅταν τριῶν ὄντων μεγεθῶν καὶ ἄλλων αὐτοῖς ἴσων τὸ πλῆθος γίνηται ὡς μὲν ἐν τοῖς πρώτοις μεγέθεσιν ἡγούμενον πρὸς ἑπόμενον, οὕτως ἐν τοῖς δευτέροις μεγέθεσιν ἡγούμενον πρὸς ἑπόμενον, ὡς δὲ ἐν τοῖς πρώτοις μεγέθεσιν ἑπόμενον πρὸς ἄλλο τι, οὕτως ἐν τοῖς δευτέροις ἄλλο τι πρὸς ἡγούμενον.

Προτάσεις κε΄.

α΄. Ἐὰν ᾖ ὁποσαοῦν μεγέθη ὁποσωνοῦν μεγεθῶν ἴσων τὸ πλῆθος ἕκαστον ἑκάστου ἰσάκις πολλαπλάσιον, ὁσαπλάσιόν ἐστιν ἓν τῶν μεγεθῶν ἑνός, τοσαυταπλάσια ἔσται καὶ τὰ πάντα τῶν πάντων.

Ἔστω ὁποσαοῦν μεγέθη τὰ ΑΒ, ΓΔ ὁποσωνοῦν μεγεθῶν τῶν Ε, Ζ ἴσων τὸ πλῆθος ἕκαστον ἑκάστου ἰσάκις πολλαπλάσιον· λέγω, ὅτι ὁσαπλάσιόν ἐστι τὸ ΑΒ τοῦ Ε, τοσαυταπλάσια ἔσται καὶ τὰ ΑΒ, ΓΔ τῶν Ε, Ζ.

Ἐπεὶ γὰρ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΑΒ τοῦ Ε καὶ τὸ ΓΔ τοῦ Ζ, ὅσα ἄρα ἐστὶν ἐν τῷ ΑΒ μεγέθη ἴσα τῷ Ε, τοσαῦτα καὶ ἐν τῷ ΓΔ ἴσα τῷ Ζ. διῃρήσθω τὸ μὲν ΑΒ εἰς τὰ τῷ Ε μεγέθη ἴσα τὰ ΑΗ, ΗΒ, τὸ δὲ ΓΔ εἰς τὰ τῷ Ζ ἴσα τὰ ΓΘ, ΘΔ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΑΗ, ΗΒ τῷ πλήθει τῶν ΓΘ, ΘΔ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ μὲν ΑΗ τῷ Ε, τὸ δὲ ΓΘ τῷ Ζ, ἴσον ἄρα τὸ ΑΗ τῷ Ε, καὶ τὰ ΑΗ, ΓΘ τοῖς Ε, Ζ. διὰ τὰ αὐτὰ δὴ ἴσον ἐστὶ τὸ ΗΒ τῷ Ε, καὶ τὰ ΗΒ, ΘΔ τοῖς Ε, Ζ· ὅσα ἄρα ἐστὶν ἐν τῷ ΑΒ ἴσα τῷ Ε, τοσαῦτα καὶ ἐν τοῖς ΑΒ, ΓΔ ἴσα τοῖς Ε, Ζ· ὁσαπλάσιον ἄρα ἐστὶ τὸ ΑΒ τοῦ Ε, τοσαυταπλάσια ἔσται καὶ τὰ ΑΒ, ΓΔ τῶν Ε, Ζ.

Ἐὰν ἄρα ᾖ ὁποσαοῦν μεγέθη ὁποσωνοῦν μεγεθῶν ἴσων τὸ πλῆθος ἕκαστον ἑκάστου ἰσάκις πολλαπλάσιον, ὁσαπλάσιόν ἐστιν ἓν τῶν μεγεθῶν ἑνός, τοσαυταπλάσια ἔσται καὶ τὰ πάντα τῶν πάντων· ὅπερ ἔδει δεῖξαι.

β΄. Ἐὰν πρῶτον δευτέρου ἰσάκις ᾖ πολλαπλάσιον καὶ τρίτον τετάρτου, ᾖ δὲ καὶ πέμπτον δευτέρου ἰσάκις πολλαπλάσιον καὶ ἕκτον τετάρτου, καὶ συντεθὲν πρῶτον καὶ πέμπτον δευτέρου ἰσάκις ἔσται πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τετάρτου.

Πρῶτον γὰρ τὸ ΑΒ δευτέρου τοῦ Γ ἰσάκις ἔστω πολλαπλάσιον καὶ τρίτον τὸ ΔΕ τετάρτου τοῦ Ζ, ἔστω δὲ καὶ πέμπτον τὸ ΒΗ δευτέρου τοῦ Γ ἰσάκις

πολλαπλάσιον καὶ ἔκτον τὸ ΕΘ τετάρτου τοῦ Ζ· λέγω, ὅτι καὶ συντεθὲν πρῶτον καὶ πέμπτον τὸ ΑΗ δευτέρου τοῦ Γ ἰσάκις ἔσται πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τὸ ΔΘ τετάρτου τοῦ Ζ.

Ἐπεὶ γὰρ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΑΒ τοῦ Γ καὶ τὸ ΔΕ τοῦ Ζ, ὅσα ἄρα ἐστὶν ἐν τῷ ΑΒ ἴσα τῷ Γ, τοσαῦτα καὶ ἐν τῷ ΔΕ ἴσα τῷ Ζ. διὰ τὰ αὐτὰ δὴ καὶ ὅσα ἐστὶν ἐν τῷ ΒΗ ἴσα τῷ Γ, τοσαῦτα καὶ ἐν τῷ ΕΘ ἴσα τῷ Ζ· ὅσα ἄρα ἐστὶν ἐν ὅλῳ τῷ ΑΗ ἴσα τῷ Γ, τοσαῦτα καὶ ἐν ὅλῳ τῷ ΔΘ ἴσα τῷ Ζ· ὁσαπλάσιον ἄρα ἐστὶ τὸ ΑΗ τοῦ Γ, τοσαυταπλάσιον ἔσται καὶ τὸ ΔΘ τοῦ Ζ. καὶ συντεθὲν ἄρα πρῶτον καὶ πέμπτον τὸ ΑΗ δευτέρου τοῦ Γ ἰσάκις ἔσται πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τὸ ΔΘ τετάρτου τοῦ Ζ.

Ἐὰν ἄρα πρῶτον δευτέρου ἰσάκις ᾖ πολλαπλάσιον καὶ τρίτον τετάρτου, ᾖ δὲ καὶ πέμπτον δευτέρου ἰσάκις πολλαπλάσιον καὶ ἕκτον τετάρτου, καὶ συντεθὲν πρῶτον καὶ πέμπτον δευτέρου ἰσάκις ἔσται πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τετάρτου· ὅπερ ἔδει δεῖξαι.

γ΄. Ἐὰν πρῶτον δευτέρου ἰσάκις ᾖ πολλαπλάσιον καὶ τρίτον τετάρτου, ληφθῇ δὲ ἰσάκις πολλαπλάσια τοῦ τε πρώτου καὶ τρίτου, καὶ δι᾽ ἴσου τῶν ληφθέντων ἑκάτερον ἑκατέρου ἰσάκις ἔσται πολλαπλάσιον τὸ μὲν τοῦ δευτέρου τὸ δὲ τοῦ τετάρτου.

Πρῶτον γὰρ τὸ Α δευτέρου τοῦ Β ἰσάκις ἔστω πολλαπλάσιον καὶ τρίτον τὸ Γ τετάρτου τοῦ Δ, καὶ εἰλήφθω τῶν Α, Γ ἰσάκις πολλαπλάσια τὰ ΕΖ, ΗΘ· λέγω, ὅτι ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΕΖ τοῦ Β καὶ τὸ ΗΘ τοῦ Δ.

Ἐπεὶ γὰρ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΕΖ τοῦ Α καὶ τὸ ΗΘ τοῦ Γ, ὅσα ἄρα ἐστὶν ἐν τῷ ΕΖ ἴσα τῷ Α, τοσαῦτα καὶ ἐν τῷ ΗΘ ἴσα τῷ Γ. διῃρήσθω τὸ μὲν ΕΖ εἰς τὰ τῷ Α μεγέθη ἴσα τὰ ΕΚ, ΚΖ, τὸ δὲ ΗΘ εἰς τὰ τῷ Γ ἴσα τὰ ΗΛ, ΛΘ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΕΚ, ΚΖ τῷ πλήθει τῶν ΗΛ, ΛΘ. καὶ ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ Α τοῦ Β καὶ τὸ Γ τοῦ Δ, ἴσον δὲ τὸ μὲν ΕΚ τῷ Α, τὸ δὲ ΗΛ τῷ Γ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΕΚ τοῦ Β καὶ τὸ ΗΛ τοῦ

Δ. διὰ τὰ αὐτὰ δὴ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΚΖ τοῦ Β καὶ τὸ ΛΘ τοῦ Δ. ἐπεὶ οὖν πρῶτον τὸ ΕΚ δευτέρου τοῦ Β ἰσάκις ἐστὶ πολλαπλάσιον καὶ τρίτον τὸ ΗΛ τετάρτου τοῦ Δ, ἔστι δὲ καὶ πέμπτον τὸ ΚΖ δευτέρου τοῦ Β ἰσάκις πολλαπλάσιον καὶ ἕκτον τὸ ΛΘ τετάρτου τοῦ Δ, καὶ συντεθὲν ἄρα πρῶτον καὶ πέμπτον τὸ ΕΖ δευτέρου τοῦ Β ἰσάκις ἐστὶ πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τὸ ΗΘ τετάρτου τοῦ Δ.

Ἐὰν ἄρα πρῶτον δευτέρου ἰσάκις ᾖ πολλαπλάσιον καὶ τρίτον τετάρτου, ληφθῇ δὲ τοῦ πρώτου καὶ τρίτου ἰσάκις πολλαπλάσια, καὶ δι' ἴσου τῶν ληφθέντων ἑκάτερον ἑκατέρου ἰσάκις ἔσται πολλαπλάσιον τὸ μὲν τοῦ δευτέρου τὸ δὲ τοῦ τετάρτου· ὅπερ ἔδει δεῖξαι.

δ΄. Ἐὰν πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, καὶ τὰ ἰσάκις πολλαπλάσια τοῦ τε πρώτου καὶ τρίτου πρὸς τὰ ἰσάκις πολλαπλάσια τοῦ δευτέρου καὶ τετάρτου καθ' ὁποιονοῦν πολλαπλασιασμὸν τὸν αὐτὸν ἕξει λόγον ληφθέντα κατάλληλα.

Πρῶτον γὰρ τὸ Α πρὸς δεύτερον τὸ Β τὸν αὐτὸν ἐχέτω λόγον καὶ τρίτον τὸ Γ πρὸς τέταρτον τὸ Δ, καὶ εἰλήφθω τῶν μὲν Α, Γ ἰσάκις πολλαπλάσια τὰ Ε, Ζ, τῶν δὲ Β, Δ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Η, Θ· λέγω, ὅτι ἐστὶν ὡς τὸ Ε πρὸς τὸ Η, οὕτως τὸ Ζ πρὸς τὸ Θ.

Εἰλήφθω γὰρ τῶν μὲν Ε, Ζ ἰσάκις πολλαπλάσια τὰ Κ, Λ, τῶν δὲ Η, Θ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Μ, Ν.

[Καὶ] ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ μὲν Ε τοῦ Α, τὸ δὲ Ζ τοῦ Γ, καὶ εἴληπται τῶν Ε, Ζ ἰσάκις πολλαπλάσια τὰ Κ, Λ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ Κ τοῦ Α καὶ τὸ Λ τοῦ Γ. διὰ τὰ αὐτὰ δὴ ἰσάκις ἐστὶ πολλαπλάσιον τὸ Μ τοῦ Β καὶ τὸ Ν τοῦ Λ. καὶ ἐπεί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ εἴληπται τῶν μὲν Α, Γ ἰσάκις πολλαπλάσια τὰ Κ, Λ, τῶν δὲ Β, Δ ἄλλα ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Μ, Ν, εἰ ἄρα ὑπερέχει τὸ Κ τοῦ Μ, ὑπερέχει καὶ τὸ Λ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον,

ἔλαττον. καί ἐστι τὰ μὲν Κ, Λ τῶν Ε, Ζ ἰσάκις πολλαπλάσια, τὰ δὲ Μ, Ν τῶν Η, Θ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια· ἔστιν ἄρα ὡς τὸ Ε πρὸς τὸ Η, οὕτως τὸ Ζ πρὸς τὸ Θ.

Ἐὰν ἄρα πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, καὶ τὰ ἰσάκις πολλαπλάσια τοῦ τε πρώτου καὶ τρίτου πρὸς τὰ ἰσάκις πολλαπλάσια τοῦ δευτέρου καὶ τετάρτου τὸν αὐτὸν ἕξει λόγον καθ' ὁποιονοῦν πολλαπλασιασμὸν ληφθέντα κατάλληλα· ὅπερ ἔδει δεῖξαι.

ε΄. Ἐὰν μέγεθος μεγέθους ἰσάκις ᾖ πολλαπλάσιον, ὅπερ ἀφαιρεθὲν ἀφαιρεθέντος, καὶ τὸ λοιπὸν τοῦ λοιποῦ ἰσάκις ἔσται πολλαπλάσιον, ὁσαπλάσιόν ἐστι τὸ ὅλον τοῦ ὅλου.

Μέγεθος γὰρ τὸ ΑΒ μεγέθους τοῦ ΓΔ ἰσάκις ἔστω πολλαπλάσιον, ὅπερ ἀφαιρεθὲν τὸ ΑΕ ἀφαιρεθέντος τοῦ ΓΖ· λέγω, ὅτι καὶ λοιπὸν τὸ ΕΒ λοιποῦ τοῦ ΖΔ ἰσάκις ἔσται πολλαπλάσιον, ὁσαπλάσιόν ἐστιν ὅλον τὸ ΑΒ ὅλου τοῦ ΓΔ.

Ὁσαπλάσιον γάρ ἐστι τὸ ΑΕ τοῦ ΓΖ, τοσαυταπλάσιον γεγονέτω καὶ τὸ ΕΒ τοῦ ΓΗ.

Καὶ ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΑΕ τοῦ ΓΖ καὶ τὸ ΕΒ τοῦ ΗΓ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΑΕ τοῦ ΓΖ καὶ τὸ ΑΒ τοῦ ΗΖ. κεῖται δὲ ἰσάκις πολλαπλάσιον τὸ ΑΕ τοῦ ΓΖ καὶ τὸ ΑΒ τοῦ ΓΔ. ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΑΒ ἑκατέρου τῶν ΗΖ, ΓΔ· ἴσον ἄρα τὸ ΗΖ τῷ ΓΔ. κοινὸν ἀφῃρήσθω τὸ ΓΖ· λοιπὸν ἄρα τὸ ΗΓ λοιπῷ τῷ ΖΔ ἴσον ἐστίν. καὶ ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΑΕ τοῦ ΓΖ καὶ τὸ ΕΒ τοῦ ΗΓ, ἴσον δὲ τὸ ΗΓ τῷ ΔΖ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΑΕ τοῦ ΓΖ καὶ τὸ ΕΒ τοῦ ΖΔ. ἰσάκις δὲ ὑπόκειται πολλαπλάσιον τὸ ΑΕ τοῦ ΓΖ καὶ τὸ ΑΒ τοῦ ΓΔ· ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΕΒ τοῦ ΖΔ καὶ τὸ ΑΒ τοῦ ΓΔ. καὶ λοιπὸν ἄρα τὸ ΕΒ λοιποῦ τοῦ ΖΔ ἰσάκις ἔσται πολλαπλάσιον, ὁσαπλάσιόν ἐστιν ὅλον τὸ ΑΒ ὅλου τοῦ ΓΔ.

Ἐὰν ἄρα μέγεθος μεγέθους ἰσάκις ᾖ πολλαπλάσιον, ὅπερ ἀφαιρεθὲν ἀφαιρεθέντος, καὶ τὸ λοιπὸν τοῦ λοιποῦ ἰσάκις ἔσται πολλαπλάσιον, ὁσαπλάσιόν ἐστι καὶ τὸ ὅλον τοῦ ὅλου· ὅπερ ἔδει δεῖξαι.

ς΄. Ἐὰν δύο μεγέθη δύο μεγεθῶν ἰσάκις ᾖ πολλαπλάσια, καὶ ἀφαιρεθέντα τινὰ τῶν αὐτῶν ἰσάκις ᾖ πολλαπλάσια, καὶ τὰ λοιπὰ τοῖς αὐτοῖς ἤτοι ἴσα ἐστὶν ἢ ἰσάκις αὐτῶν πολλαπλάσια.

Δύο γὰρ μεγέθη τὰ ΑΒ, ΓΔ δύο μεγεθῶν τῶν Ε, Ζ ἰσάκις ἔστω πολλαπλάσια, καὶ ἀφαιρεθέντα τὰ ΑΗ, ΓΘ τῶν αὐτῶν τῶν Ε, Ζ ἰσάκις ἔστω πολλαπλάσια· λέγω, ὅτι καὶ λοιπὰ τὰ ΗΒ, ΘΔ τοῖς Ε, Ζ ἤτοι ἴσα ἐστὶν ἢ ἰσάκις αὐτῶν πολλαπλάσια.

Ἔστω γὰρ πρότερον τὸ ΗΒ τῷ Ε ἴσον. λέγω, ὅτι καὶ τὸ ΘΔ τῷ Ζ ἴσον ἐστίν.

Κείσθω γὰρ τῷ Ζ ἴσον τὸ ΓΚ. ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΑΗ τοῦ Ε καὶ τὸ ΓΘ τοῦ Ζ, ἴσον δὲ τὸ μὲν ΗΒ τῷ Ε, τὸ δὲ ΚΓ τῷ Ζ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΑΒ τοῦ Ε καὶ τὸ ΚΘ τοῦ Ζ. ἰσάκις δὲ ὑπόκειται πολλαπλάσιον τὸ ΑΒ τοῦ Ε καὶ τὸ ΓΔ τοῦ Ζ· ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΚΘ τοῦ Ζ καὶ τὸ ΓΔ τοῦ Ζ. ἐπεὶ οὖν ἑκάτερον τῶν ΚΘ, ΓΔ τοῦ Ζ ἰσάκις ἐστὶ πολλαπλάσιον, ἴσον ἄρα ἐστὶ τὸ ΚΘ τῷ ΓΔ. κοινὸν ἀφῃρήσθω τὸ ΓΘ· λοιπὸν ἄρα τὸ ΚΓ λοιπῷ τῷ ΘΔ ἴσον ἐστίν. ἀλλὰ τὸ Ζ τῷ ΚΓ ἐστιν ἴσον· καὶ τὸ ΘΔ ἄρα τῷ Ζ ἴσον ἐστίν. ὥστε εἰ τὸ ΗΒ τῷ Ε ἴσον ἐστίν, καὶ τὸ ΘΔ ἴσον ἔσται τῷ Ζ.

Ὁμοίως δὴ δείξομεν, ὅτι, κἂν πολλαπλάσιον ᾖ τὸ ΗΒ τοῦ Ε, τοσαυταπλάσιον ἔσται καὶ τὸ ΘΔ τοῦ Ζ.

Ἐὰν ἄρα δύο μεγέθη δύο μεγεθῶν ἰσάκις ᾖ πολλαπλάσια, καὶ ἀφαιρεθέντα τινὰ τῶν αὐτῶν ἰσάκις ᾖ πολλαπλάσια, καὶ τὰ λοιπὰ τοῖς αὐτοῖς ἤτοι ἴσα ἐστὶν ἢ ἰσάκις αὐτῶν πολλαπλάσια· ὅπερ ἔδει δεῖξαι.

ζ΄. Τὰ ἴσα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον καὶ τὸ αὐτὸ πρὸς τὰ ἴσα.

Ἔστω ἴσα μεγέθη τὰ Α, Β, ἄλλο δέ τι, ὃ ἔτυχεν, μέγεθος τὸ Γ· λέγω, ὅτι ἑκάτερον τῶν Α, Β πρὸς τὸ Γ τὸν αὐτὸν ἔχει λόγον, καὶ τὸ Γ πρὸς ἑκάτερον τῶν Α, Β.

Εἰλήφθω γὰρ τῶν μὲν Α, Β ἰσάκις πολλαπλάσια τὰ Δ, Ε, τοῦ δὲ Γ ἄλλο, ὃ ἔτυχεν, πολλαπλάσιον τὸ Ζ.

Ἐπεὶ οὖν ἰσάκις ἐστὶ πολλαπλάσιον τὸ Δ τοῦ Α καὶ τὸ Ε τοῦ Β, ἴσον δὲ τὸ Α τῷ Β, ἴσον ἄρα καὶ τὸ Δ τῷ Ε. ἄλλο δέ, ὃ ἔτυχεν, τὸ Ζ. Εἰ ἄρα ὑπερέχει τὸ Δ τοῦ Ζ, ὑπερέχει καὶ τὸ Ε τοῦ Ζ, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καί ἐστι τὰ μὲν Δ, Ε τῶν Α, Β ἰσάκις πολλαπλάσια, τὸ δὲ Ζ τοῦ Γ ἄλλο, ὃ ἔτυχεν, πολλαπλάσιον· ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Γ, οὕτως τὸ Β πρὸς τὸ Γ.

Λέγω [δή], ὅτι καὶ τὸ Γ πρὸς ἑκάτερον τῶν Α, Β τὸν αὐτὸν ἔχει λόγον.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι ἴσον ἐστὶ τὸ Δ τῷ Ε· ἄλλο δέ τι τὸ Ζ· εἰ ἄρα ὑπερέχει τὸ Ζ τοῦ Δ, ὑπερέχει καὶ τοῦ Ε, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καί ἐστι τὸ μὲν Ζ τοῦ Γ πολλαπλάσιον, τὰ δὲ Δ, Ε τῶν Α, Β ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια· ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Α, οὕτως τὸ Γ πρὸς τὸ Β.

Τὰ ἴσα ἄρα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον καὶ τὸ αὐτὸ πρὸς τὰ ἴσα.

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν μεγέθη τινὰ ἀνάλογον ᾖ, καὶ ἀνάπαλιν ἀνάλογον ἔσται. ὅπερ ἔδει δεῖξαι.

η΄. Τῶν ἀνίσων μεγεθῶν τὸ μεῖζον πρὸς τὸ αὐτὸ μείζονα λόγον ἔχει ἤπερ τὸ ἔλαττον. καὶ τὸ αὐτὸ πρὸς τὸ ἔλαττον μείζονα λόγον ἔχει ἤπερ πρὸς τὸ μεῖζον.

Ἔστω ἄνισα μεγέθη τὰ ΑΒ, Γ, καὶ ἔστω μεῖζον τὸ ΑΒ, ἄλλο δέ, ὃ ἔτυχεν, τὸ Δ· λέγω, ὅτι τὸ ΑΒ πρὸς τὸ Δ μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ Δ, καὶ τὸ Δ πρὸς τὸ Γ μείζονα λόγον ἔχει ἤπερ πρὸς τὸ ΑΒ.

Ἐπεὶ γὰρ μεῖζόν ἐστι τὸ ΑΒ τοῦ Γ, κείσθω τῷ Γ ἴσον τὸ ΒΕ· τὸ δὴ ἔλασσον τῶν ΑΕ, ΕΒ πολλαπλασιαζόμενον ἔσται ποτὲ τοῦ Δ μεῖζον. ἔστω πρότερον τὸ ΑΕ ἔλαττον τοῦ ΕΒ, καὶ πεπολλαπλασιάσθω τὸ ΑΕ, καὶ ἔστω αὐτοῦ πολλαπλάσιον τὸ ΖΗ μεῖζον ὂν τοῦ Δ, καὶ ὁσαπλάσιόν ἐστι τὸ ΖΗ τοῦ ΑΕ, τοσαυταπλάσιον γεγονέτω καὶ τὸ μὲν ΗΘ τοῦ ΕΒ τὸ δὲ Κ τοῦ Γ· καὶ εἰλήφθω τοῦ Δ διπλάσιον μὲν τὸ Λ, τριπλάσιον δὲ τὸ Μ, καὶ ἑξῆς ἑνὶ πλεῖον, ἕως ἂν τὸ λαμβανόμενον πολλαπλάσιον μὲν γένηται τοῦ Δ, πρώτως δὲ μεῖζον τοῦ Κ. εἰλήφθω, καὶ ἔστω τὸ Ν τετραπλάσιον μὲν τοῦ Δ, πρώτως δὲ μεῖζον τοῦ Κ.

Ἐπεὶ οὖν τὸ Κ τοῦ Ν πρώτως ἐστὶν ἔλαττον, τὸ Κ ἄρα τοῦ Μ οὐκ ἐστιν ἔλαττον. καὶ ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΖΗ τοῦ ΑΕ καὶ τὸ ΗΘ τοῦ ΕΒ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΖΗ τοῦ ΑΕ καὶ τὸ ΖΘ τοῦ ΑΒ. ἰσάκις δέ ἐστι πολλαπλάσιον τὸ ΖΗ τοῦ ΑΕ καὶ τὸ Κ τοῦ Γ· ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΖΘ τοῦ ΑΒ καὶ τὸ Κ τοῦ Γ. τὰ ΖΘ, Κ ἄρα τῶν ΑΒ, Γ ἰσάκις ἐστὶ πολλαπλάσια. πάλιν, ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΗΘ τοῦ ΕΒ καὶ τὸ Κ τοῦ Γ, ἴσον δὲ τὸ ΕΒ τῷ Γ, ἴσον ἄρα καὶ τὸ ΗΘ τῷ Κ· τὸ δὲ Κ τοῦ Μ οὐκ ἐστιν ἔλαττον· οὐδ’ ἄρα τὸ ΗΘ τοῦ Μ ἔλαττόν ἐστιν. μεῖζον δὲ τὸ ΖΗ τοῦ Δ· ὅλον ἄρα τὸ ΖΘ συναμφοτέρων τῶν Δ, Μ μεῖζόν ἐστιν. ἀλλὰ συναμφότερα τὰ Δ, Μ τῷ Ν ἐστιν ἴσα, ἐπειδήπερ τὸ Μ τοῦ Δ τριπλάσιόν ἐστιν, συναμφότερα δὲ τὰ Μ, Δ τοῦ Δ ἐστι τετραπλάσια, ἔστι δὲ καὶ τὸ Ν τοῦ Δ τετραπλάσιον· συναμφότερα ἄρα τὰ Μ, Δ τῷ Ν ἴσα ἐστίν. ἀλλὰ τὸ ΖΘ τῶν Μ, Δ μεῖζόν ἐστιν· τὸ ΖΘ ἄρα τοῦ Ν ὑπερέχει· τὸ δὲ Κ τοῦ Ν οὐχ ὑπερέχει. καί ἐστι τὰ μὲν ΖΘ, Κ τῶν ΑΒ, Γ ἰσάκις πολλαπλάσια, τὸ δὲ Ν τοῦ Δ ἄλλο, ὃ ἔτυχεν, πολλαπλάσιον· τὸ ΑΒ ἄρα πρὸς τὸ Δ μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ Δ.

Λέγω δή, ὅτι καὶ τὸ Δ πρὸς τὸ Γ μείζονα λόγον ἔχει ἤπερ τὸ Δ πρὸς τὸ ΑΒ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι τὸ μὲν Ν τοῦ Κ ὑπερέχει, τὸ δὲ Ν τοῦ ΖΘ οὐχ ὑπερέχει. καί ἐστι τὸ μὲν Ν τοῦ Δ πολλαπλάσιον, τὰ δὲ ΖΘ, Κ τῶν ΑΒ, Γ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια· τὸ Δ ἄρα πρὸς τὸ Γ μείζονα λόγον ἔχει ἤπερ τὸ Δ πρὸς τὸ ΑΒ.

Ἀλλὰ δὴ τὸ ΑΕ τοῦ ΕΒ μεῖζον ἔστω. τὸ δὴ ἔλαττον τὸ ΕΒ πολλαπλασιαζόμενον ἔσται ποτὲ τοῦ Δ μεῖζον. πεπολλαπλασιάσθω, καὶ ἔστω τὸ ΗΘ πολλαπλάσιον μὲν τοῦ ΕΒ, μεῖζον δὲ τοῦ Δ· καὶ ὁσαπλάσιόν ἐστι τὸ ΗΘ τοῦ ΕΒ, τοσαυταπλάσιον γεγονέτω καὶ τὸ μὲν ΖΗ τοῦ ΑΕ, τὸ δὲ Κ τοῦ Γ. ὁμοίως δὴ δείξομεν, ὅτι τὰ ΖΘ, Κ τῶν ΑΒ, Γ ἰσάκις ἐστὶ πολλαπλάσια· καὶ εἰλήφθω ὁμοίως τὸ Ν πολλαπλάσιον μὲν τοῦ Δ, πρώτως δὲ μεῖζον τοῦ ΖΗ· ὥστε πάλιν τὸ ΖΗ τοῦ Μ οὐκ ἐστιν ἔλασσον. μεῖζον δὲ τὸ ΗΘ τοῦ Δ· ὅλον ἄρα τὸ ΖΘ τῶν Δ, Μ, τουτέστι τοῦ Ν, ὑπερέχει. τὸ δὲ Κ τοῦ Ν οὐχ ὑπερέχει, ἐπειδήπερ καὶ τὸ ΖΗ μεῖζον ὂν τοῦ ΗΘ, τουτέστι τοῦ Κ, τοῦ Ν οὐχ ὑπερέχει. καὶ ὡσαύτως κατακολουθοῦντες τοῖς ἐπάνω περαίνομεν τὴν ἀπόδειξιν.

Τῶν ἄρα ἀνίσων μεγεθῶν τὸ μεῖζον πρὸς τὸ αὐτὸ μείζονα λόγον ἔχει ἤπερ τὸ ἔλαττον· καὶ τὸ αὐτὸ πρὸς τὸ ἔλαττον μείζονα λόγον ἔχει ἤπερ πρὸς τὸ μεῖζον· ὅπερ ἔδει δεῖξαι.

θ΄. Τὰ πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχοντα λόγον ἴσα ἀλλήλοις ἐστίν· καὶ πρὸς ἃ τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον, ἐκεῖνα ἴσα ἐστίν.

Ἐχέτω γὰρ ἑκάτερον τῶν Α, Β πρὸς τὸ Γ τὸν αὐτὸν λόγον· λέγω, ὅτι ἴσον ἐστὶ τὸ Α τῷ Β.

Εἰ γὰρ μή, οὐκ ἂν ἑκάτερον τῶν Α, Β πρὸς τὸ Γ τὸν αὐτὸν εἶχε λόγον· ἔχει δέ· ἴσον ἄρα ἐστὶ τὸ Α τῷ Β.

Ἐχέτω δὴ πάλιν τὸ Γ πρὸς ἑκάτερον τῶν Α, Β τὸν αὐτὸν λόγον· λέγω, ὅτι ἴσον ἐστὶ τὸ Α τῷ Β.

Εἰ γὰρ μή, οὐκ ἂν τὸ Γ πρὸς ἑκάτερον τῶν Α, Β τὸν αὐτὸν εἶχε λόγον· ἔχει δέ· ἴσον ἄρα ἐστὶ τὸ Α τῷ Β.

Τὰ ἄρα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχοντα λόγον ἴσα ἀλλήλοις ἐστίν· καὶ πρὸς ἃ τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον, ἐκεῖνα ἴσα ἐστίν· ὅπερ ἔδει δεῖξαι.

ι΄. Τῶν πρὸς τὸ αὐτὸ λόγον ἐχόντων τὸ μείζονα λόγον ἔχον ἐκεῖνο μεῖζόν ἐστιν· πρὸς ὃ δὲ τὸ αὐτὸ μείζονα λόγον ἔχει, ἐκεῖνο ἔλαττόν ἐστιν.

Ἐχέτω γὰρ τὸ Α πρὸς τὸ Γ μείζονα λόγον ἥπερ τὸ Β πρὸς τὸ Γ· λέγω, ὅτι μεῖζόν ἐστι τὸ Α τοῦ Β.

Εἰ γὰρ μή, ἤτοι ἴσον ἐστὶ τὸ Α τῷ Β ἢ ἔλασσον. ἴσον μὲν οὖν οὐκ ἐστι τὸ Α τῷ Β· ἑκάτερον γὰρ ἂν τῶν Α, Β πρὸς τὸ Γ τὸν αὐτὸν εἶχε λόγον. οὐκ ἔχει δέ· οὐκ ἄρα ἴσον ἐστὶ τὸ Α τῷ Β. οὐδὲ μὴν ἔλασσόν ἐστι τὸ Α τοῦ Β· τὸ Α γὰρ ἂν πρὸς τὸ Γ ἐλάσσονα λόγον εἶχεν ἥπερ τὸ Β πρὸς τὸ Γ. οὐκ ἔχει δέ· οὐκ ἄρα ἔλασσόν ἐστι τὸ Α τοῦ Β. ἐδείχθη δὲ οὐδὲ ἴσον· μεῖζον ἄρα ἐστὶ τὸ Α τοῦ Β.

Ἐχέτω δὴ πάλιν τὸ Γ πρὸς τὸ Β μείζονα λόγον ἥπερ τὸ Γ πρὸς τὸ Α· λέγω, ὅτι ἔλασσόν ἐστι τὸ Β τοῦ Α.

Εἰ γὰρ μή, ἤτοι ἴσον ἐστὶν ἢ μεῖζον. ἴσον μὲν οὖν οὐκ ἐστι τὸ Β τῷ Α· τὸ Γ γὰρ ἂν πρὸς ἑκάτερον τῶν Α, Β τὸν αὐτὸν εἶχε λόγον. οὐκ ἔχει δέ· οὐκ ἄρα ἴσον ἐστὶ τὸ Α τῷ Β. οὐδὲ μὴν μεῖζόν ἐστι τὸ Β τοῦ Α· τὸ Γ γὰρ ἂν πρὸς τὸ Β ἐλάσσονα λόγον εἶχεν ἥπερ πρὸς τὸ Α. οὐκ ἔχει δέ· οὐκ ἄρα μεῖζόν ἐστι τὸ Β τοῦ Α. ἐδείχθη δέ, ὅτι οὐδὲ ἴσον· ἔλαττον ἄρα ἐστὶ τὸ Β τοῦ Α.

Τῶν ἄρα πρὸς τὸ αὐτὸ λόγον ἐχόντων τὸ μείζονα λόγον ἔχον μεῖζόν ἐστιν· καὶ πρὸς ὃ τὸ αὐτὸ μείζονα λόγον ἔχει, ἐκεῖνο ἔλαττόν ἐστιν· ὅπερ ἔδει δεῖξαι.

ια΄. Οἱ τῷ αὐτῷ λόγῳ οἱ αὐτοὶ καὶ ἀλλήλοις εἰσὶν οἱ αὐτοί.

Ἔστωσαν γὰρ ὡς μὲν τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, ὡς δὲ τὸ Γ πρὸς τὸ Δ, οὕτως τὸ Ε πρὸς τὸ Ζ· λέγω, ὅτι ἐστὶν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ.

Εἰλήφθω γὰρ τῶν Α, Γ, Ε ἰσάκις πολλαπλάσια τὰ Η, Θ, Κ, τῶν δὲ Β, Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Λ, Μ, Ν.

Καὶ ἐπεί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ εἴληπται τῶν μὲν Α, Γ ἰσάκις πολλαπλάσια τὰ Η, Θ, τῶν δὲ Β, Δ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Λ, Μ, εἰ ἄρα ὑπερέχει τὸ Η τοῦ Λ, ὑπερέχει καὶ τὸ Θ τοῦ Μ, καὶ εἰ ἴσον ἐστίν, ἴσον, καὶ εἰ ἐλλείπει, ἐλλείπει. πάλιν, ἐπεί ἐστιν ὡς τὸ Γ πρὸς τὸ Δ, οὕτως τὸ Ε πρὸς τὸ Ζ, καὶ εἴληπται τῶν Γ, Ε ἰσάκις πολλαπλάσια τὰ Θ, Κ, τῶν δὲ Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Μ, Ν, εἰ ἄρα ὑπερέχει τὸ Θ τοῦ Μ, ὑπερέχει καὶ τὸ Κ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. ἀλλὰ εἰ ὑπερεῖχε τὸ Θ τοῦ Μ, ὑπερεῖχε καὶ τὸ Η τοῦ Λ, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον· ὥστε καὶ εἰ ὑπερέχει τὸ Η τοῦ Λ, ὑπερέχει καὶ τὸ Κ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καί ἐστι τὰ μὲν Η, Κ τῶν Α, Ε ἰσάκις πολλαπλάσια, τὰ δὲ Λ, Ν τῶν Β, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια· ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ.

Οἱ ἄρα τῷ αὐτῷ λόγῳ οἱ αὐτοὶ καὶ ἀλλήλοις εἰσὶν οἱ αὐτοί· ὅπερ ἔδει δεῖξαι.

ιβ΄. Ἐὰν ᾖ ὁποσαοῦν μεγέθη ἀνάλογον, ἔσται ὡς ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα.

Ἔστωσαν ὁποσαοῦν μεγέθη ἀνάλογον τὰ Α, Β, Γ, Δ, Ε, Ζ, ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ τὸ Ε πρὸς τὸ Ζ· λέγω, ὅτι ἐστὶν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὰ Α, Γ, Ε πρὸς τὰ Β, Δ, Ζ.

Εἰλήφθω γὰρ τῶν μὲν Α, Γ, Ε ἰσάκις πολλαπλάσια τὰ Η, Θ, Κ, τῶν δὲ Β, Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Λ, Μ, Ν.

Καὶ ἐπεί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ τὸ Ε πρὸς τὸ Ζ, καὶ εἴληπται τῶν μὲν Α, Γ, Ε ἰσάκις πολλαπλάσια τὰ Η, Θ, Κ τῶν δὲ Β, Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Λ, Μ, Ν, εἰ ἄρα ὑπερέχει τὸ Η τοῦ

Λ, ὑπερέχει καὶ τὸ Θ τοῦ Μ, καὶ τὸ Κ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. ὥστε καὶ εἰ ὑπερέχει τὸ Η τοῦ Λ, ὑπερέχει καὶ τὰ Η, Θ, Κ τῶν Λ, Μ, Ν, καὶ εἰ ἴσον, ἴσα, καὶ εἰ ἔλαττον, ἐλάττονα. καί ἐστι τὸ μὲν Η καὶ τὰ Η, Θ, Κ τοῦ Α καὶ τῶν Α, Γ, Ε ἰσάκις πολλαπλάσια, ἐπειδήπερ ἐὰν ᾖ ὁποσαοῦν μεγέθη ὁποσωνοῦν μεγεθῶν ἴσων τὸ πλῆθος ἕκαστον ἑκάστου ἰσάκις πολλαπλάσιον, ὁσαπλάσιόν ἐστιν ἓν τῶν μεγεθῶν ἑνός, τοσαυταπλάσια ἔσται καὶ τὰ πάντα τῶν πάντων. διὰ τὰ αὐτὰ δὴ καὶ τὸ Λ καὶ τὰ Λ, Μ, Ν τοῦ Β καὶ τῶν Β, Δ, Ζ ἰσάκις ἐστὶ πολλαπλάσια· ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Β, οὕτως τὰ Α, Γ, Ε πρὸς τὰ Β, Δ, Ζ.

Ἐὰν ἄρα ᾖ ὁποσαοῦν μεγέθη ἀνάλογον, ἔσται ὡς ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα· ὅπερ ἔδει δεῖξαι.

ιγ΄. Ἐὰν πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, τρίτον δὲ πρὸς τέταρτον μείζονα λόγον ἔχῃ ἢ πέμπτον πρὸς ἕκτον, καὶ πρῶτον πρὸς δεύτερον μείζονα λόγον ἕξει ἢ πέμπτον πρὸς ἕκτον.

Πρῶτον γὰρ τὸ Α πρὸς δεύτερον τὸ Β τὸν αὐτὸν ἐχέτω λόγον καὶ τρίτον τὸ Γ πρὸς τέταρτον τὸ Δ, τρίτον δὲ τὸ Γ πρὸς τέταρτον τὸ Δ μείζονα λόγον ἐχέτω ἢ πέμπτον τὸ Ε πρὸς ἕκτον τὸ Ζ. λέγω, ὅτι καὶ πρῶτον τὸ Α πρὸς δεύτερον τὸ Β μείζονα λόγον ἕξει ἤπερ πέμπτον τὸ Ε πρὸς ἕκτον τὸ Ζ.

Ἐπεὶ γὰρ ἔστι τινὰ τῶν μὲν Γ, Ε ἰσάκις πολλαπλάσια, τῶν δὲ Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια, καὶ τὸ μὲν τοῦ Γ πολλαπλάσιον τοῦ τοῦ Δ πολλαπλασίου ὑπερέχει, τὸ δὲ τοῦ Ε πολλαπλάσιον τοῦ τοῦ Ζ πολλαπλασίου οὐχ ὑπερέχει, εἰλήφθω, καὶ ἔστω τῶν μὲν Γ, Ε ἰσάκις πολλαπλάσια τὰ Η, Θ, τῶν δὲ Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Κ, Λ, ὥστε τὸ μὲν Η τοῦ Κ ὑπερέχειν, τὸ δὲ Θ τοῦ Λ μὴ ὑπερέχειν· καὶ ὁσαπλάσιον μέν ἐστι τὸ Η τοῦ Γ, τοσαυταπλάσιον ἔστω καὶ τὸ Μ τοῦ Α, ὁσαπλάσιον δὲ τὸ Κ τοῦ Δ, τοσαυταπλάσιον ἔστω καὶ τὸ Ν τοῦ Β.

Καὶ ἐπεί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ εἴληπται τῶν μὲν Α, Γ ἰσάκις πολλαπλάσια τὰ Μ, Η, τῶν δὲ Β, Δ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Ν, Κ, εἰ ἄρα ὑπερέχει τὸ Μ τοῦ Ν, ὑπερέχει καὶ τὸ Η τοῦ Κ, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. ὑπερέχει δὲ τὸ Η τοῦ Κ· ὑπερέχει ἄρα καὶ τὸ Μ τοῦ Ν. τὸ δὲ Θ τοῦ Λ οὐχ ὑπερέχει· καί ἐστι τὰ μὲν Μ, Θ τῶν Α, Ε ἰσάκις πολλαπλάσια, τὰ δὲ Ν, Λ τῶν Β, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια· τὸ ἄρα Α πρὸς τὸ Β μείζονα λόγον ἔχει ἤπερ τὸ Ε πρὸς τὸ Ζ.

Ἐὰν ἄρα πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, τρίτον δὲ πρὸς τέταρτον μείζονα λόγον ἔχῃ ἢ πέμπτον πρὸς ἕκτον, καὶ πρῶτον πρὸς δεύτερον μείζονα λόγον ἕξει ἢ πέμπτον πρὸς ἕκτον· ὅπερ ἔδει δεῖξαι.

ιδ΄. Ἐὰν πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, τὸ δὲ πρῶτον τοῦ τρίτου μεῖζον ᾖ, καὶ τὸ δεύτερον τοῦ τετάρτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον.

Πρῶτον γὰρ τὸ Α πρὸς δεύτερον τὸ Β τὸν αὐτὸν ἐχέτω λόγον καὶ τρίτον τὸ Γ πρὸς τέταρτον τὸ Δ, μεῖζον δὲ ἔστω τὸ Α τοῦ Γ· λέγω, ὅτι καὶ τὸ Β τοῦ Δ μεῖζόν ἐστιν.

Ἐπεὶ γὰρ τὸ Α τοῦ Γ μεῖζόν ἐστιν, ἄλλο δέ, ὃ ἔτυχεν, [μέγεθος] τὸ Β, τὸ Α ἄρα πρὸς τὸ Β μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ Β. ὡς δὲ τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ· καὶ τὸ Γ ἄρα πρὸς τὸ Δ μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ Β. πρὸς ὃ δὲ τὸ αὐτὸ μείζονα λόγον ἔχει, ἐκεῖνο ἔλασσόν ἐστιν· ἔλασσον ἄρα τὸ Δ τοῦ Β· ὥστε μεῖζόν ἐστι τὸ Β τοῦ Δ.

Ὁμοίως δὴ δείξομεν, ὅτι κἂν ἴσον ᾖ τὸ Α τῷ Γ, ἴσον ἔσται καὶ τὸ Β τῷ Δ, κἂν ἔλασσον ᾖ τὸ Α τοῦ Γ, ἔλασσον ἔσται καὶ τὸ Β τοῦ Δ.

Ἐὰν ἄρα πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, τὸ δὲ πρῶτον τοῦ τρίτου μεῖζον ᾖ, καὶ τὸ δεύτερον τοῦ τετάρτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον· ὅπερ ἔδει δεῖξαι.

ιε΄. Τὰ μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον ληφθέντα κατάλληλα.

Ἔστω γὰρ ἰσάκις πολλαπλάσιον τὸ ΑΒ τοῦ Γ καὶ τὸ ΔΕ τοῦ Ζ· λέγω, ὅτι ἐστὶν ὡς τὸ Γ πρὸς τὸ Ζ, οὕτως τὸ ΑΒ πρὸς τὸ ΔΕ.

Ἐπεὶ γὰρ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΑΒ τοῦ Γ καὶ τὸ ΔΕ τοῦ Ζ, ὅσα ἄρα ἐστὶν ἐν τῷ ΑΒ μεγέθη ἴσα τῷ Γ, τοσαῦτα καὶ ἐν τῷ ΔΕ ἴσα τῷ Ζ. διῃρήσθω τὸ μὲν ΑΒ εἰς τὰ τῷ Γ ἴσα τὰ ΑΗ, ΗΘ, ΘΒ, τὸ δὲ ΔΕ εἰς τὰ τῷ Ζ ἴσα τὰ ΔΚ, ΚΛ, ΛΕ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΑΗ, ΗΘ, ΘΒ τῷ πλήθει τῶν ΔΚ, ΚΛ, ΛΕ. καὶ ἐπεὶ ἴσα ἐστὶ τὰ ΑΗ, ΗΘ, ΘΒ ἀλλήλοις, ἔστι δὲ καὶ τὰ ΔΚ, ΚΛ, ΛΕ ἴσα ἀλλήλοις, ἔστιν ἄρα ὡς τὸ ΑΗ πρὸς τὸ ΔΚ, οὕτως τὸ ΗΘ πρὸς τὸ ΚΛ, καὶ τὸ ΘΒ πρὸς τὸ ΛΕ. ἔσται ἄρα καὶ ὡς ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα· ἔστιν ἄρα ὡς τὸ ΑΗ πρὸς τὸ ΔΚ, οὕτως τὸ ΑΒ πρὸς τὸ ΔΕ. ἴσον δὲ τὸ μὲν ΑΗ τῷ Γ, τὸ δὲ ΔΚ τῷ Ζ· ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Ζ οὕτως τὸ ΑΒ πρὸς τὸ ΔΕ.

Τὰ ἄρα μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον ληφθέντα κατάλληλα· ὅπερ ἔδει δεῖξαι.

ις΄. Ἐὰν τέσσαρα μεγέθη ἀνάλογον ᾖ, καὶ ἐναλλὰξ ἀνάλογον ἔσται.

Ἔστω τέσσαρα μεγέθη ἀνάλογον τὰ Α, Β, Γ, Δ, ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ· λέγω, ὅτι καὶ ἐναλλὰξ [ἀνάλογον] ἔσται, ὡς τὸ Α πρὸς τὸ Γ, οὕτως τὸ Β πρὸς τὸ Δ.

Εἰλήφθω γὰρ τῶν μὲν Α, Β ἰσάκις πολλαπλάσια τὰ Ε, Ζ, τῶν δὲ Γ, Δ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Η, Θ.

Καὶ ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ Ε τοῦ Α καὶ τὸ Ζ τοῦ Β, τὰ δὲ μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ. ὡς δὲ τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ· καὶ

ὡς ἄρα τὸ Γ πρὸς τὸ Δ, οὕτως τὸ Ε πρὸς τὸ Ζ. πάλιν, ἐπεὶ τὰ Η, Θ τῶν Γ, Δ ἰσάκις ἐστὶ πολλαπλάσια, ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Δ, οὕτως τὸ Η πρὸς τὸ Θ. ὡς δὲ τὸ Γ πρὸς τὸ Δ, [οὕτως] τὸ Ε πρὸς τὸ Ζ· καὶ ὡς ἄρα τὸ Ε πρὸς τὸ Ζ, οὕτως τὸ Η πρὸς τὸ Θ. ἐὰν δὲ τέσσαρα μεγέθη ἀνάλογον ᾖ, τὸ δὲ πρῶτον τοῦ τρίτου μεῖζον ᾖ, καὶ τὸ δεύτερον τοῦ τετάρτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον. εἰ ἄρα ὑπερέχει τὸ Ε τοῦ Η, ὑπερέχει καὶ τὸ Ζ τοῦ Θ, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καί ἐστι τὰ μὲν Ε, Ζ τῶν Α, Β ἰσάκις πολλαπλάσια, τὰ δὲ Η, Θ τῶν Γ, Δ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια· ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Γ, οὕτως τὸ Β πρὸς τὸ Δ.

Ἐὰν ἄρα τέσσαρα μεγέθη ἀνάλογον ᾖ, καὶ ἐναλλὰξ ἀνάλογον ἔσται· ὅπερ ἔδει δεῖξαι.

ιζ΄. Ἐὰν συγκείμενα μεγέθη ἀνάλογον ᾖ, καὶ διαιρεθέντα ἀνάλογον ἔσται.

Ἔστω συγκείμενα μεγέθη ἀνάλογον τὰ ΑΒ, ΒΕ, ΓΔ, ΔΖ, ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς τὸ ΔΖ· λέγω, ὅτι καὶ διαιρεθέντα ἀνάλογον ἔσται, ὡς τὸ ΑΕ πρὸς τὸ ΕΒ, οὕτως τὸ ΓΖ πρὸς τὸ ΔΖ.

Εἰλήφθω γὰρ τῶν μὲν ΑΕ, ΕΒ, ΓΖ, ΖΔ ἰσάκις πολλαπλάσια τὰ ΗΘ, ΘΚ, ΛΜ, ΜΝ, τῶν δὲ ΕΒ, ΖΔ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ ΚΞ, ΝΠ.

Καὶ ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΗΘ τοῦ ΑΕ καὶ τὸ ΘΚ τοῦ ΕΒ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΗΘ τοῦ ΑΕ καὶ τὸ ΗΚ τοῦ ΑΒ. ἰσάκις δέ ἐστι πολλαπλάσιον τὸ ΗΘ τοῦ ΑΕ καὶ τὸ ΛΜ τοῦ ΓΖ· ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΗΚ τοῦ ΑΒ καὶ τὸ ΛΜ τοῦ ΓΖ. πάλιν, ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΛΜ τοῦ ΓΖ καὶ τὸ ΜΝ τοῦ ΖΔ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΛΜ τοῦ ΓΖ καὶ τὸ ΛΝ τοῦ ΓΔ. ἰσάκις δὲ ἦν πολλαπλάσιον τὸ ΛΜ τοῦ ΓΖ καὶ τὸ ΗΚ τοῦ ΑΒ· ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΗΚ τοῦ ΑΒ καὶ τὸ ΛΝ τοῦ ΓΔ. τὰ ΗΚ, ΛΝ ἄρα τῶν ΑΒ, ΓΔ ἰσάκις ἐστὶ πολλαπλάσια. πάλιν, ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΘΚ τοῦ ΕΒ καὶ τὸ ΜΝ τοῦ ΖΔ, ἔστι δὲ καὶ τὸ ΚΞ τοῦ ΕΒ ἰσάκις πολλαπλάσιον καὶ τὸ ΝΠ τοῦ

ΖΔ, καὶ συντεθὲν τὸ ΘΞ τοῦ ΕΒ ἰσάκις ἐστὶ πολλαπλάσιον καὶ τὸ ΜΠ τοῦ ΖΔ. Καὶ ἐπεί ἐστιν ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς τὸ ΔΖ, καὶ εἴληπται τῶν μὲν ΑΒ, ΓΔ ἰσάκις πολλαπλάσια τὰ ΗΚ, ΛΝ, τῶν δὲ ΕΒ, ΖΔ ἰσάκις πολλαπλάσια τὰ ΘΞ, ΜΠ, εἰ ἄρα ὑπερέχει τὸ ΗΚ τοῦ ΘΞ, ὑπερέχει καὶ τὸ ΛΝ τοῦ ΜΠ, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. ὑπερεχέτω δὴ τὸ ΗΚ τοῦ ΘΞ, καὶ κοινοῦ ἀφαιρεθέντος τοῦ ΘΚ ὑπερέχει ἄρα καὶ τὸ ΗΘ τοῦ ΚΞ. ἀλλὰ εἰ ὑπερεῖχε τὸ ΗΚ τοῦ ΘΞ, ὑπερεῖχε καὶ τὸ ΛΝ τοῦ ΜΠ· ὑπερέχει ἄρα καὶ τὸ ΛΝ τοῦ ΜΠ, καὶ κοινοῦ ἀφαιρεθέντος τοῦ ΜΝ ὑπερέχει καὶ τὸ ΛΜ τοῦ ΝΠ· ὥστε εἰ ὑπερέχει τὸ ΗΘ τοῦ ΚΞ, ὑπερέχει καὶ τὸ ΛΜ τοῦ ΝΠ. ὁμοίως δὴ δείξομεν, ὅτι κἂν ἴσον ᾖ τὸ ΗΘ τῷ ΚΞ, ἴσον ἔσται καὶ τὸ ΛΜ τῷ ΝΠ, κἂν ἔλαττον, ἔλαττον. καί ἐστι τὰ μὲν ΗΘ, ΛΜ τῶν ΑΕ, ΓΖ ἰσάκις πολλαπλάσια, τὰ δὲ ΚΞ, ΝΠ τῶν ΕΒ, ΖΔ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια· ἔστιν ἄρα ὡς τὸ ΑΕ πρὸς τὸ ΕΒ, οὕτως τὸ ΓΖ πρὸς τὸ ΖΔ.

Ἐὰν ἄρα συγκείμενα μεγέθη ἀνάλογον ᾖ, καὶ διαιρεθέντα ἀνάλογον ἔσται· ὅπερ ἔδει δεῖξαι.

ιη΄. Ἐὰν διῃρημένα μεγέθη ἀνάλογον ᾖ, καὶ συντεθέντα ἀνάλογον ἔσται.

Ἔστω διῃρημένα μεγέθη ἀνάλογον τὰ ΑΕ, ΕΒ, ΓΖ, ΖΔ, ὡς τὸ ΑΕ πρὸς τὸ ΕΒ, οὕτως τὸ ΓΖ πρὸς τὸ ΖΔ· λέγω, ὅτι καὶ συντεθέντα ἀνάλογον ἔσται, ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς τὸ ΖΔ.

Εἰ γὰρ μή ἐστιν ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς τὸ ΔΖ, ἔσται ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ ἤτοι πρὸς ἔλασσόν τι τοῦ ΔΖ ἢ πρὸς μεῖζον.

Ἔστω πρότερον πρὸς ἔλασσον τὸ ΔΗ. καὶ ἐπεί ἐστιν ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς τὸ ΔΗ, συγκείμενα μεγέθη ἀνάλογόν ἐστιν· ὥστε καὶ διαιρεθέντα ἀνάλογον ἔσται. ἔστιν ἄρα ὡς τὸ ΑΕ πρὸς τὸ ΕΒ, οὕτως τὸ ΓΗ πρὸς τὸ ΗΔ. ὑπόκειται δὲ καὶ ὡς τὸ ΑΕ πρὸς τὸ ΕΒ, οὕτως τὸ ΓΖ πρὸς τὸ ΖΔ. καὶ ὡς ἄρα τὸ ΓΗ πρὸς τὸ ΗΔ, οὕτως τὸ ΓΖ πρὸς τὸ ΖΔ. μεῖζον δὲ τὸ πρῶτον τὸ ΓΗ τοῦ τρίτου τοῦ ΓΖ· μεῖζον ἄρα καὶ τὸ δεύτερον τὸ ΗΔ τοῦ τετάρτου

τοῦ ΖΔ. ἀλλὰ καὶ ἔλαττον· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα ἐστὶν ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς ἔλασσον τοῦ ΖΔ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ πρὸς μεῖζον· πρὸς αὐτὸ ἄρα.

Ἐὰν ἄρα διῃρημένα μεγέθη ἀνάλογον ᾖ, καὶ συντεθέντα ἀνάλογον ἔσται· ὅπερ ἔδει δεῖξαι.

ιθ΄. Ἐὰν ᾖ ὡς ὅλον πρὸς ὅλον, οὕτως ἀφαιρεθὲν πρὸς ἀφαιρεθέν, καὶ τὸ λοιπὸν πρὸς τὸ λοιπὸν ἔσται ὡς ὅλον πρὸς ὅλον.

Ἔστω γὰρ ὡς ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ, οὕτως ἀφαιρεθὲν τὸ ΑΕ πρὸς ἀφαιρεθὲν τὸ ΓΖ· λέγω, ὅτι καὶ λοιπὸν τὸ ΕΒ πρὸς λοιπὸν τὸ ΖΔ ἔσται ὡς ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ.

Ἐπεὶ γάρ ἐστιν ὡς τὸ ΑΒ πρὸς τὸ ΓΔ, οὕτως τὸ ΑΕ πρὸς τὸ ΓΖ, καὶ ἐναλλὰξ ὡς τὸ ΒΑ πρὸς τὸ ΑΕ, οὕτως τὸ ΔΓ πρὸς τὸ ΓΖ. καὶ ἐπεὶ συγκείμενα μεγέθη ἀνάλογόν ἐστιν, καὶ διαιρεθέντα ἀνάλογον ἔσται, ὡς τὸ ΒΕ πρὸς τὸ ΕΑ, οὕτως τὸ ΔΖ πρὸς τὸ ΓΖ· καὶ ἐναλλάξ, ὡς τὸ ΒΕ πρὸς τὸ ΔΖ, οὕτως τὸ ΕΑ πρὸς τὸ ΖΓ. ὡς δὲ τὸ ΑΕ πρὸς τὸ ΓΖ, οὕτως ὑπόκειται ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ. καὶ λοιπὸν ἄρα τὸ ΕΒ πρὸς λοιπὸν τὸ ΖΔ ἔσται ὡς ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ.

Ἐὰν ἄρα ᾖ ὡς ὅλον πρὸς ὅλον, οὕτως ἀφαιρεθὲν πρὸς ἀφαιρεθέν, καὶ τὸ λοιπὸν πρὸς τὸ λοιπὸν ἔσται ὡς ὅλον πρὸς ὅλον [ὅπερ ἔδει δεῖξαι].

[Καὶ ἐπεὶ ἐδείχθη ὡς τὸ ΑΒ πρὸς τὸ ΓΔ, οὕτως τὸ ΕΒ πρὸς τὸ ΖΔ, καὶ ἐναλλὰξ ὡς τὸ ΑΒ πρὸς τὸ ΒΕ οὕτως τὸ ΓΔ πρὸς τὸ ΖΔ, συγκείμενα ἄρα μεγέθη ἀνάλογόν ἐστιν· ἐδείχθη δὲ ὡς τὸ ΒΑ πρὸς τὸ ΑΕ, οὕτως τὸ ΔΓ πρὸς τὸ ΓΖ· καί ἐστιν ἀναστρέψαντι].

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν συγκείμενα μεγέθη ἀνάλογον ᾖ, καὶ ἀναστρέψαντι ἀνάλογον ἔσται· ὅπερ ἔδει δεῖξαι.

κ΄. Ἐὰν ᾖ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, δι᾽ ἴσου δὲ τὸ πρῶτον τοῦ τρίτου μεῖζον ᾖ, καὶ τὸ τέταρτον τοῦ ἕκτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον.

Ἔστω τρία μεγέθη τὰ Α, Β, Γ, καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Δ, Ε, Ζ, σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, ὡς μὲν τὸ Α πρὸς τὸ Β, οὕτως τὸ Δ πρὸς τὸ Ε, ὡς δὲ τὸ Β πρὸς τὸ Γ, οὕτως τὸ Ε πρὸς τὸ Ζ, δι᾽ ἴσου δὲ μεῖζον ἔστω τὸ Α τοῦ Γ· λέγω, ὅτι καὶ τὸ Δ τοῦ Ζ μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον.

Ἐπεὶ γὰρ μεῖζόν ἐστι τὸ Α τοῦ Γ, ἄλλο δέ τι τὸ Β, τὸ δὲ μεῖζον πρὸς τὸ αὐτὸ μείζονα λόγον ἔχει ἤπερ τὸ ἔλαττον, τὸ Α ἄρα πρὸς τὸ Β μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ Β. ἀλλ᾽ ὡς μὲν τὸ Α πρὸς τὸ Β, [οὕτως] τὸ Δ πρὸς τὸ Ε, ὡς δὲ τὸ Γ πρὸς τὸ Β, ἀνάπαλιν οὕτως τὸ Ζ πρὸς τὸ Ε· καὶ τὸ Δ ἄρα πρὸς τὸ Ε μείζονα λόγον ἔχει ἤπερ τὸ Ζ πρὸς τὸ Ε. τῶν δὲ πρὸς τὸ αὐτὸ λόγον ἐχόντων τὸ μείζονα λόγον ἔχον μεῖζόν ἐστιν. μεῖζον ἄρα τὸ Δ τοῦ Ζ. ὁμοίως δὴ δείξομεν, ὅτι κἂν ἴσον ᾖ τὸ Α τῷ Γ, ἴσον ἔσται καὶ τὸ Δ τῷ Ζ, κἂν ἔλαττον, ἔλαττον.

Ἐὰν ἄρα ᾖ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, δι᾽ ἴσου δὲ τὸ πρῶτον τοῦ τρίτου μεῖζον ᾖ, καὶ τὸ τέταρτον τοῦ ἕκτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον· ὅπερ ἔδει δεῖξαι.

κα΄. Ἐὰν ᾖ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, ᾖ δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, δι᾽ ἴσου δὲ τὸ πρῶτον τοῦ τρίτου μεῖζον ᾖ, καὶ τὸ τέταρτον τοῦ ἕκτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον.

Ἔστω τρία μεγέθη τὰ Α, Β, Γ καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Δ, Ε, Ζ, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, ἔστω δὲ τεταραγμένη αὐτῶν ἡ

ἀναλογία, ὡς μὲν τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ, ὡς δὲ τὸ Β πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Ε, δι' ἴσου δὲ τὸ Α τοῦ Γ μεῖζον ἔστω· λέγω, ὅτι καὶ τὸ Δ τοῦ Ζ μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον.

Ἐπεὶ γὰρ μεῖζόν ἐστι τὸ Α τοῦ Γ, ἄλλο δέ τι τὸ Β, τὸ Α ἄρα πρὸς τὸ Β μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ Β. ἀλλ' ὡς μὲν τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ, ὡς δὲ τὸ Γ πρὸς τὸ Β, ἀνάπαλιν οὕτως τὸ Ε πρὸς τὸ Δ. καὶ τὸ Ε ἄρα πρὸς τὸ Ζ μείζονα λόγον ἔχει ἤπερ τὸ Ε πρὸς τὸ Δ. πρὸς ὃ δὲ τὸ αὐτὸ μείζονα λόγον ἔχει, ἐκεῖνο ἔλασσόν ἐστιν· ἔλασσον ἄρα ἐστὶ τὸ Ζ τοῦ Δ· μεῖζον ἄρα ἐστὶ τὸ Δ τοῦ Ζ. ὁμοίως δὴ δείξομεν, ὅτι κἂν ἴσον ᾖ τὸ Α τῷ Γ, ἴσον ἔσται καὶ τὸ Δ τῷ Ζ, κἂν ἔλαττον, ἔλαττον.

Ἐὰν ἄρα ᾖ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, ᾖ δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, δι' ἴσου δὲ τὸ πρῶτον τοῦ τρίτου μεῖζον ᾖ, καὶ τὸ τέταρτον τοῦ ἕκτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον· ὅπερ ἔδει δεῖξαι.

κβ΄. Ἐὰν ᾖ ὁποσαοῦν μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ δι' ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται.

Ἔστω ὁποσαοῦν μεγέθη τὰ Α, Β, Γ καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Δ, Ε, Ζ, σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, ὡς μὲν τὸ Α πρὸς τὸ Β, οὕτως τὸ Δ πρὸς τὸ Ε, ὡς δὲ τὸ Β πρὸς τὸ Γ, οὕτως τὸ Ε πρὸς τὸ Ζ· λέγω, ὅτι καὶ δι' ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται.

Εἰλήφθω γὰρ τῶν μὲν Α, Δ ἰσάκις πολλαπλάσια τὰ Η, Θ, τῶν δὲ Β, Ε ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Κ, Λ, καὶ ἔτι τῶν Γ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Μ, Ν.

Καὶ ἐπεί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Δ πρὸς τὸ Ε, καὶ εἴληπται τῶν μὲν Α, Δ ἰσάκις πολλαπλάσια τὰ Η, Θ, τῶν δὲ Β, Ε ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Κ, Λ, ἔστιν ἄρα ὡς τὸ Η πρὸς τὸ Κ, οὕτως τὸ Θ πρὸς τὸ Λ. διὰ τὰ αὐτὰ δὴ καὶ ὡς τὸ Κ πρὸς τὸ Μ, οὕτως τὸ Λ πρὸς τὸ Ν. ἐπεὶ οὖν τρία

μεγέθη ἐστὶ τὰ Η, Κ, Μ, καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Θ, Λ, Ν, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, δι' ἴσου ἄρα, εἰ ὑπερέχει τὸ Η τοῦ Μ, ὑπερέχει καὶ τὸ Θ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καί ἐστι τὰ μὲν Η, Θ τῶν Α, Δ ἰσάκις πολλαπλάσια, τὰ δὲ Μ, Ν τῶν Γ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια. ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Ζ.

Ἐὰν ἄρα ᾖ ὁποσαοῦν μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, καὶ δι' ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται· ὅπερ ἔδει δεῖξαι.

κγ΄. Ἐὰν ᾖ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, ᾖ δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, καὶ δι' ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται.

Ἔστω τρία μεγέθη τὰ Α, Β, Γ καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ τὰ Δ, Ε, Ζ, ἔστω δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, ὡς μὲν τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ, ὡς δὲ τὸ Β πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Ε· λέγω, ὅτι ἐστὶν ὡς τὸ Α πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Ζ.

Εἰλήφθω τῶν μὲν Α, Β, Δ ἰσάκις πολλαπλάσια τὰ Η, Θ, Κ, τῶν δὲ Γ, Ε, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Λ, Μ, Ν.

Καὶ ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσια τὰ Η, Θ τῶν Α, Β, τὰ δὲ μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Η πρὸς τὸ Θ. διὰ τὰ αὐτὰ δὴ καὶ ὡς τὸ Ε πρὸς τὸ Ζ, οὕτως τὸ Μ πρὸς τὸ Ν· καί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ· καὶ ὡς ἄρα τὸ Η πρὸς τὸ Θ, οὕτως τὸ Μ πρὸς τὸ Ν. καὶ ἐπεί ἐστιν ὡς τὸ Β πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Ε, καὶ ἐναλλὰξ ὡς τὸ Β πρὸς τὸ Δ, οὕτως τὸ Γ πρὸς τὸ Ε. καὶ ἐπεὶ τὰ Θ, Κ τῶν Β, Δ ἰσάκις ἐστὶ πολλαπλάσια, τὰ δὲ μέρη τοῖς ἰσάκις πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ Β πρὸς τὸ Δ, οὕτως τὸ

Θ πρὸς τὸ Κ. ἀλλ’ ὡς τὸ Β πρὸς τὸ Δ, οὕτως τὸ Γ πρὸς τὸ Ε· καὶ ὡς ἄρα τὸ Θ πρὸς τὸ Κ, οὕτως τὸ Γ πρὸς τὸ Ε. πάλιν, ἐπεὶ τὰ Λ, Μ τῶν Γ, Ε ἰσάκις ἐστι πολλαπλάσια, ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Ε, οὕτως τὸ Λ πρὸς τὸ Μ. ἀλλ’ ὡς τὸ Γ πρὸς τὸ Ε, οὕτως τὸ Θ πρὸς τὸ Κ· καὶ ὡς ἄρα τὸ Θ πρὸς τὸ Κ, οὕτως τὸ Λ πρὸς τὸ Μ, καὶ ἐναλλὰξ ὡς τὸ Θ πρὸς τὸ Λ, τὸ Κ πρὸς τὸ Μ. ἐδείχθη δὲ καὶ ὡς τὸ Η πρὸς τὸ Θ, οὕτως τὸ Μ πρὸς τὸ Ν. ἐπεὶ οὖν τρία μεγέθη ἐστὶ τὰ Η, Θ, Λ, καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Κ, Μ, Ν σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, καί ἐστιν αὐτῶν τεταραγμένη ἡ ἀναλογία, δι’ ἴσου ἄρα, εἰ ὑπερέχει τὸ Η τοῦ Λ, ὑπερέχει καὶ τὸ Κ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καί ἐστι τὰ μὲν Η, Κ τῶν Α, Δ ἰσάκις πολλαπλάσια, τὰ δὲ Λ, Ν τῶν Γ, Ζ. ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Ζ.

Ἐὰν ἄρα ᾖ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, ᾖ δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, καὶ δι’ ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται· ὅπερ ἔδει δεῖξαι.

κδ΄. Ἐὰν πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, ἔχῃ δὲ καὶ πέμπτον πρὸς δεύτερον τὸν αὐτὸν λόγον καὶ ἕκτον πρὸς τέταρτον, καὶ συντεθὲν πρῶτον καὶ πέμπτον πρὸς δεύτερον τὸν αὐτὸν ἕξει λόγον καὶ τρίτον καὶ ἕκτον πρὸς τέταρτον.

Πρῶτον γὰρ τὸ ΑΒ πρὸς δεύτερον τὸ Γ τὸν αὐτὸν ἐχέτω λόγον καὶ τρίτον τὸ ΔΕ πρὸς τέταρτον τὸ Ζ, ἐχέτω δὲ καὶ πέμπτον τὸ ΒΗ πρὸς δεύτερον τὸ Γ τὸν αὐτὸν λόγον καὶ ἕκτον τὸ ΕΘ πρὸς τέταρτον τὸ Ζ· λέγω, ὅτι καὶ συντεθὲν πρῶτον καὶ πέμπτον τὸ ΑΗ πρὸς δεύτερον τὸ Γ τὸν αὐτὸν ἕξει λόγον, καὶ τρίτον καὶ ἕκτον τὸ ΔΘ πρὸς τέταρτον τὸ Ζ.

Ἐπεὶ γάρ ἐστιν ὡς τὸ ΒΗ πρὸς τὸ Γ, οὕτως τὸ ΕΘ πρὸς τὸ Ζ, ἀνάπαλιν ἄρα ὡς τὸ Γ πρὸς τὸ ΒΗ, οὕτως τὸ Ζ πρὸς τὸ ΕΘ. ἐπεὶ οὖν ἐστιν ὡς τὸ ΑΒ πρὸς τὸ Γ, οὕτως τὸ ΔΕ πρὸς τὸ Ζ, ὡς δὲ τὸ Γ πρὸς τὸ ΒΗ, οὕτως τὸ Ζ πρὸς τὸ ΕΘ, δι’ ἴσου ἄρα ἐστὶν ὡς τὸ ΑΒ πρὸς τὸ ΒΗ, οὕτως τὸ ΔΕ πρὸς τὸ ΕΘ.

καὶ ἐπεὶ διῃρημένα μεγέθη ἀνάλογόν ἐστιν, καὶ συντεθέντα ἀνάλογον ἔσται· ἔστιν ἄρα ὡς τὸ ΑΗ πρὸς τὸ ΗΒ, οὕτως τὸ ΔΘ πρὸς τὸ ΘΕ. ἔστι δὲ καὶ ὡς τὸ ΒΗ πρὸς τὸ Γ, οὕτως τὸ ΕΘ πρὸς τὸ Ζ· δι᾽ ἴσου ἄρα ἐστὶν ὡς τὸ ΑΗ πρὸς τὸ Γ, οὕτως τὸ ΔΘ πρὸς τὸ Ζ.

Ἐὰν ἄρα πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, ἔχῃ δὲ καὶ πέμπτον πρὸς δεύτερον τὸν αὐτὸν λόγον καὶ ἕκτον πρὸς τέταρτον, καὶ συντεθὲν πρῶτον καὶ πέμπτον πρὸς δεύτερον τὸν αὐτὸν ἕξει λόγον καὶ τρίτον καὶ ἕκτον πρὸς τέταρτον· ὅπερ ἔδει δεῖξαι.

κε΄. Ἐὰν τέσσαρα μεγέθη ἀνάλογον ᾖ, τὸ μέγιστον [αὐτῶν] καὶ τὸ ἐλάχιστον δύο τῶν λοιπῶν μείζονά ἐστιν.

Ἔστω τέσσαρα μεγέθη ἀνάλογον τὰ ΑΒ, ΓΔ, Ε, Ζ, ὡς τὸ ΑΒ πρὸς τὸ ΓΔ, οὕτως τὸ Ε πρὸς τὸ Ζ, ἔστω δὲ μέγιστον μὲν αὐτῶν τὸ ΑΒ, ἐλάχιστον δὲ τὸ Ζ· λέγω, ὅτι τὰ ΑΒ, Ζ τῶν ΓΔ, Ε μείζονά ἐστιν.

Κείσθω γὰρ τῷ μὲν Ε ἴσον τὸ ΑΗ, τῷ δὲ Ζ ἴσον τὸ ΓΘ.

Ἐπεὶ [οὖν] ἐστιν ὡς τὸ ΑΒ πρὸς τὸ ΓΔ, οὕτως τὸ Ε πρὸς τὸ Ζ, ἴσον δὲ τὸ μὲν Ε τῷ ΑΗ, τὸ δὲ Ζ τῷ ΓΘ, ἔστιν ἄρα ὡς τὸ ΑΒ πρὸς τὸ ΓΔ, οὕτως τὸ ΑΗ πρὸς τὸ ΓΘ. καὶ ἐπεί ἐστιν ὡς ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ, οὕτως ἀφαιρεθὲν τὸ ΑΗ πρὸς ἀφαιρεθὲν τὸ ΓΘ, καὶ λοιπὸν ἄρα τὸ ΗΒ πρὸς λοιπὸν τὸ ΘΔ ἔσται ὡς ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ. μεῖζον δὲ τὸ ΑΒ τοῦ ΓΔ· μεῖζον ἄρα καὶ τὸ ΗΒ τοῦ ΘΔ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ μὲν ΑΗ τῷ Ε, τὸ δὲ ΓΘ τῷ Ζ, τὰ ἄρα ΑΗ, Ζ ἴσα ἐστὶ τοῖς ΓΘ, Ε. Καὶ [ἐπεὶ] ἐὰν [ἀνίσοις ἴσα προστεθῇ, τὰ ὅλα ἄνισά ἐστιν, ἐὰν ἄρα] τῶν ΗΒ, ΘΔ ἀνίσων ὄντων καὶ μείζονος τοῦ ΗΒ τῷ μὲν ΗΒ προστεθῇ τὰ ΑΗ, Ζ, τῷ δὲ ΘΔ προστεθῇ τὰ ΓΘ, Ε, συνάγεται τὰ ΑΒ, Ζ μείζονα τῶν ΓΔ, Ε.

Ἐὰν ἄρα τέσσαρα μεγέθη ἀνάλογον ᾖ, τὸ μέγιστον αὐτῶν καὶ τὸ ἐλάχιστον δύο τῶν λοιπῶν μείζονά ἐστιν· ὅπερ ἔδει δεῖξαι.

BOOK VI.

Ὅροι ε΄.

α΄. Ὅμοια σχήματα εὐθύγραμμά ἐστιν, ὅσα τάς τε γωνίας ἴσας ἔχει κατὰ μίαν καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον.

β΄. [Ἀντιπεπονθότα δὲ σχήματά ἐστιν, ὅταν ἐν ἑκατέρῳ τῶν σχημάτων ἡγούμενοί τε καὶ ἑπόμενοι λόγοι ὦσιν.]

γ΄. Ἄκρον καὶ μέσον λόγον εὐθεῖα τετμῆσθαι λέγεται, ὅταν ᾖ ὡς ἡ ὅλη πρὸς τὸ μεῖζον τμῆμα, οὕτως τὸ μεῖζον πρὸς τὸ ἔλαττον.

δ΄.Ὕψος ἐστὶ παντὸς σχήματος ἡ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν βάσιν κάθετος ἀγομένη.

ε΄. [Λόγος ἐκ λόγων συγκεῖσθαι λέγεται, ὅταν αἱ τῶν λόγων πηλικότητες ἐφ᾽ ἑαυτὰς πολλαπλασιασθεῖσαι ποιῶσί τινα.]

Προτάσεις λγ΄.

α΄. Τὰ τρίγωνα καὶ τὰ παραλληλόγραμμα, τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις.

Ἔστω τρίγωνα μὲν τὰ ΑΒΓ, ΑΓΔ, παραλληλόγραμμα δὲ τὰ ΕΓ, ΓΖ ὑπὸ τὸ αὐτὸ ὕψος τὸ ΑΓ· λέγω, ὅτι ἐστὶν ὡς ἡ ΒΓ βάσις πρὸς τὴν ΓΔ βάσιν, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον, καὶ τὸ ΕΓ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον.

Ἐκβεβλήσθω γὰρ ἡ ΒΔ ἐφ' ἑκάτερα τὰ μέρη ἐπὶ τὰ Θ, Λ σημεῖα, καὶ κείσθωσαν τῇ μὲν ΒΓ βάσει ἴσαι [ὁσαιδηποτοῦν] αἱ ΒΗ, ΗΘ, τῇ δὲ ΓΔ βάσει ἴσαι ὁσαιδηποτοῦν αἱ ΔΚ, ΚΛ, καὶ ἐπεζεύχθωσαν αἱ ΑΗ, ΑΘ, ΑΚ, ΑΛ.

Καὶ ἐπεὶ ἴσαι εἰσὶν αἱ ΓΒ, ΒΗ, ΗΘ ἀλλήλαις, ἴσα ἐστὶ καὶ τὰ ΑΘΗ, ΑΗΒ, ΑΒΓ τρίγωνα ἀλλήλοις. ὁσαπλασίων ἄρα ἐστὶν ἡ ΘΓ βάσις τῆς ΒΓ βάσεως, τοσαυταπλάσιόν ἐστι καὶ τὸ ΑΘΓ τρίγωνον τοῦ ΑΒΓ τριγώνου. διὰ τὰ αὐτὰ δὴ ὁσαπλασίων ἐστὶν ἡ ΛΓ βάσις τῆς ΓΔ βάσεως, τοσαυταπλάσιόν ἐστι καὶ τὸ ΑΛΓ τρίγωνον τοῦ ΑΓΔ τριγώνου· καὶ εἰ ἴση ἐστὶν ἡ ΘΓ βάσις τῇ ΓΛ βάσει, ἴσον ἐστὶ καὶ τὸ ΑΘΓ τρίγωνον τῷ ΑΓΛ τριγώνῳ, καὶ εἰ ὑπερέχει ἡ ΘΓ βάσις τῆς ΓΛ βάσεως, ὑπερέχει καὶ τὸ ΑΘΓ τρίγωνον τοῦ ΑΓΛ τριγώνου, καὶ εἰ ἐλάσσων, ἔλασσον. τεσσάρων δὴ ὄντων μεγεθῶν δύο μὲν βάσεων τῶν ΒΓ, ΓΔ, δύο δὲ τριγώνων τῶν ΑΒΓ, ΑΓΔ εἴληπται ἰσάκις πολλαπλάσια τῆς μὲν ΒΓ βάσεως καὶ τοῦ ΑΒΓ τριγώνου ἥ τε ΘΓ βάσις καὶ τὸ ΑΘΓ τρίγωνον, τῆς δὲ ΓΔ βάσεως καὶ τοῦ ΑΔΓ τριγώνου ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια ἥ τε ΛΓ βάσις καὶ τὸ ΑΛΓ τρίγωνον· καὶ δέδεικται, ὅτι, εἰ ὑπερέχει ἡ ΘΓ βάσις τῆς ΓΛ βάσεως, ὑπερέχει καὶ τὸ ΑΘΓ τρίγωνον τοῦ ΑΛΓ τριγώνου, καὶ εἰ ἴση, ἴσον, καὶ εἰ ἐλάσσων, ἔλασσον· ἔστιν ἄρα ὡς ἡ ΒΓ βάσις πρὸς τὴν ΓΔ βάσιν, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον.

Καὶ ἐπεὶ τοῦ μὲν ΑΒΓ τριγώνου διπλάσιόν ἐστι τὸ ΕΓ παραλληλόγραμμον, τοῦ δὲ ΑΓΔ τριγώνου διπλάσιόν ἐστι τὸ ΖΓ

παραλληλόγραμμον, τὰ δὲ μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον, οὕτως τὸ ΕΓ παραλληλόγραμμον πρὸς τὸ ΖΓ παραλληλόγραμμον. ἐπεὶ οὖν ἐδείχθη, ὡς μὲν ἡ ΒΓ βάσις πρὸς τὴν ΓΔ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον, ὡς δὲ τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον, οὕτως τὸ ΕΓ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον, καὶ ὡς ἄρα ἡ ΒΓ βάσις πρὸς τὴν ΓΔ βάσιν, οὕτως τὸ ΕΓ παραλληλόγραμμον πρὸς τὸ ΖΓ παραλληλόγραμμον.

Τὰ ἄρα τρίγωνα καὶ τὰ παραλληλόγραμμα τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις· ὅπερ ἔδει δεῖξαι.

β΄. Ἐὰν τριγώνου παρὰ μίαν τῶν πλευρῶν ἀχθῇ τις εὐθεῖα, ἀνάλογον τεμεῖ τὰς τοῦ τριγώνου πλευράς· καὶ ἐὰν αἱ τοῦ τριγώνου πλευραὶ ἀνάλογον τμηθῶσιν, ἡ ἐπὶ τὰς τομὰς ἐπιζευγνυμένη εὐθεῖα παρὰ τὴν λοιπὴν ἔσται τοῦ τριγώνου πλευράν.

Τριγώνου γὰρ τοῦ ΑΒΓ παράλληλος μιᾷ τῶν πλευρῶν τῇ ΒΓ ἤχθω ἡ ΔΕ· λέγω, ὅτι ἐστὶν ὡς ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΑ.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΕ, ΓΔ.

Ἴσον ἄρα ἐστὶ ΒΔΕ τρίγωνον τῷ ΓΔΕ τριγώνῳ· ἐπὶ γὰρ τῆς αὐτῆς βάσεώς ἐστι τῆς ΔΕ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΔΕ, ΒΓ· ἄλλο δέ τι τὸ ΑΔΕ τρίγωνον. τὰ δὲ ἴσα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον· ἔστιν ἄρα ὡς τὸ ΒΔΕ τρίγωνον πρὸς τὸ ΑΔΕ [τρίγωνον], οὕτως τὸ ΓΔΕ τρίγωνον πρὸς τὸ ΑΔΕ τρίγωνον. ἀλλ’ ὡς μὲν τὸ ΒΔΕ τρίγωνον πρὸς τὸ ΑΔΕ, οὕτως ἡ ΒΔ πρὸς τὴν ΔΑ· ὑπὸ γὰρ τὸ αὐτὸ ὕψος ὄντα τὴν ἀπὸ τοῦ Ε ἐπὶ τὴν ΑΒ κάθετον ἀγομένην πρὸς ἄλληλά εἰσιν ὡς αἱ βάσεις. διὰ τὰ αὐτὰ δὴ ὡς τὸ ΓΔΕ τρίγωνον πρὸς τὸ ΑΔΕ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΑ· καὶ ὡς ἄρα ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΑ.

Ἀλλὰ δὴ αἱ τοῦ ΑΒΓ τριγώνου πλευραὶ αἱ ΑΒ, ΑΓ ἀνάλογον τετμήσθωσαν, ὡς ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΑ, καὶ ἐπεζεύχθω ἡ ΔΕ· λέγω, ὅτι παράλληλός ἐστιν ἡ ΔΕ τῇ ΒΓ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεί ἐστιν ὡς ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΑ, ἀλλ' ὡς μὲν ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως τὸ ΒΔΕ τρίγωνον πρὸς τὸ ΑΔΕ τρίγωνον, ὡς δὲ ἡ ΓΕ πρὸς τὴν ΕΑ, οὕτως τὸ ΓΔΕ τρίγωνον πρὸς τὸ ΑΔΕ τρίγωνον, καὶ ὡς ἄρα τὸ ΒΔΕ τρίγωνον πρὸς τὸ ΑΔΕ τρίγωνον, οὕτως τὸ ΓΔΕ τρίγωνον πρὸς τὸ ΑΔΕ τρίγωνον. ἑκάτερον ἄρα τῶν ΒΔΕ, ΓΔΕ τριγώνων πρὸς τὸ ΑΔΕ τὸν αὐτὸν ἔχει λόγον. ἴσον ἄρα ἐστὶ τὸ ΒΔΕ τρίγωνον τῷ ΓΔΕ τριγώνῳ· καί εἰσιν ἐπὶ τῆς αὐτῆς βάσεως τῆς ΔΕ. τὰ δὲ ἴσα τρίγωνα καὶ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν. παράλληλος ἄρα ἐστὶν ἡ ΔΕ τῇ ΒΓ.

Ἐὰν ἄρα τριγώνου παρὰ μίαν τῶν πλευρῶν ἀχθῇ τις εὐθεῖα, ἀνάλογον τεμεῖ τὰς τοῦ τριγώνου πλευράς· καὶ ἐὰν αἱ τοῦ τριγώνου πλευραὶ ἀνάλογον τμηθῶσιν, ἡ ἐπὶ τὰς τομὰς ἐπιζευγνυμένη εὐθεῖα παρὰ τὴν λοιπὴν ἔσται τοῦ τριγώνου πλευράν· ὅπερ ἔδει δεῖξαι.

γ΄. Ἐὰν τριγώνου ἡ γωνία δίχα τμηθῇ, ἡ δὲ τέμνουσα τὴν γωνίαν εὐθεῖα τέμνῃ καὶ τὴν βάσιν, τὰ τῆς βάσεως τμήματα τὸν αὐτὸν ἕξει λόγον ταῖς λοιπαῖς τοῦ τριγώνου πλευραῖς· καὶ ἐὰν τὰ τῆς βάσεως τμήματα τὸν αὐτὸν ἔχῃ λόγον ταῖς λοιπαῖς τοῦ τριγώνου πλευραῖς, ἡ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν τομὴν ἐπιζευγνυμένη εὐθεῖα δίχα τεμεῖ τὴν τοῦ τριγώνου γωνίαν.

Ἔστω τρίγωνον τὸ ΑΒΓ, καὶ τετμήσθω ἡ ὑπὸ ΒΑΓ γωνία δίχα ὑπὸ τῆς ΑΔ εὐθείας· λέγω, ὅτι ἐστὶν ὡς ἡ ΒΔ πρὸς τὴν ΓΔ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΓ.

Ἤχθω γὰρ διὰ τοῦ Γ τῇ ΔΑ παράλληλος ἡ ΓΕ καὶ διαχθεῖσα ἡ ΒΑ συμπιπτέτω αὐτῇ κατὰ τὸ Ε.

Καὶ ἐπεὶ εἰς παραλλήλους τὰς ΑΔ, ΕΓ εὐθεῖα ἐνέπεσεν ἡ ΑΓ, ἡ ἄρα ὑπὸ ΑΓΕ γωνία ἴση ἐστὶ τῇ ὑπὸ ΓΑΔ. ἀλλ' ἡ ὑπὸ ΓΑΔ τῇ ὑπὸ ΒΑΔ ὑπόκειται

ἴση· καὶ ἡ ὑπὸ ΒΑΔ ἄρα τῇ ὑπὸ ΑΓΕ ἐστιν ἴση. πάλιν, ἐπεὶ εἰς παραλλήλους τὰς ΑΔ, ΕΓ εὐθεῖα ἐνέπεσεν ἡ ΒΑΕ, ἡ ἐκτὸς γωνία ἡ ὑπὸ ΒΑΔ ἴση ἐστὶ τῇ ἐντὸς τῇ ὑπὸ ΑΕΓ. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΑΓΕ τῇ ὑπὸ ΒΑΔ ἴση· καὶ ἡ ὑπὸ ΑΓΕ ἄρα γωνία τῇ ὑπὸ ΑΕΓ ἐστιν ἴση· ὥστε καὶ πλευρὰ ἡ ΑΕ πλευρᾷ τῇ ΑΓ ἐστιν ἴση. καὶ ἐπεὶ τριγώνου τοῦ ΒΓΕ παρὰ μίαν τῶν πλευρῶν τὴν ΕΓ ἦκται ἡ ΑΔ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΕ. ἴση δὲ ἡ ΑΕ τῇ ΑΓ· ὡς ἄρα ἡ ΒΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΓ.

Ἀλλὰ δὴ ἔστω ὡς ἡ ΒΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΓ, καὶ ἐπεζεύχθω ἡ ΑΔ· λέγω, ὅτι δίχα τέτμηται ἡ ὑπὸ ΒΑΓ γωνία ὑπὸ τῆς ΑΔ εὐθείας.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεί ἐστιν ὡς ἡ ΒΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΓ, ἀλλὰ καὶ ὡς ἡ ΒΔ πρὸς τὴν ΔΓ, οὕτως ἐστὶν ἡ ΒΑ πρὸς τὴν ΑΕ· τριγώνου γὰρ τοῦ ΒΓΕ παρὰ μίαν τὴν ΕΓ ἦκται ἡ ΑΔ· καὶ ὡς ἄρα ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΕ. ἴση ἄρα ἡ ΑΓ τῇ ΑΕ· ὥστε καὶ γωνία ἡ ὑπὸ ΑΕΓ τῇ ὑπὸ ΑΓΕ ἐστιν ἴση. ἀλλ’ ἡ μὲν ὑπὸ ΑΕΓ τῇ ἐκτὸς τῇ ὑπὸ ΒΑΔ [ἐστιν] ἴση, ἡ δὲ ὑπὸ ΑΓΕ τῇ ἐναλλὰξ τῇ ὑπὸ ΓΑΔ ἐστιν ἴση· καὶ ἡ ὑπὸ ΒΑΔ ἄρα τῇ ὑπὸ ΓΑΔ ἐστιν ἴση. ἡ ἄρα ὑπὸ ΒΑΓ γωνία δίχα τέτμηται ὑπὸ τῆς ΑΔ εὐθείας.

Ἐὰν ἄρα τριγώνου ἡ γωνία δίχα τμηθῇ, ἡ δὲ τέμνουσα τὴν γωνίαν εὐθεῖα τέμνῃ καὶ τὴν βάσιν, τὰ τῆς βάσεως τμήματα τὸν αὐτὸν ἕξει λόγον ταῖς λοιπαῖς τοῦ τριγώνου πλευραῖς· καὶ ἐὰν τὰ τῆς βάσεως τμήματα τὸν αὐτὸν ἔχῃ λόγον ταῖς λοιπαῖς τοῦ τριγώνου πλευραῖς, ἡ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν τομὴν ἐπιζευγνυμένη εὐθεῖα δίχα τέμνει τὴν τοῦ τριγώνου γωνίαν· ὅπερ ἔδει δεῖξαι.

δ΄. Τῶν ἰσογωνίων τριγώνων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας καὶ ὁμόλογοι αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι.

Ἔστω ἰσογώνια τρίγωνα τὰ ΑΒΓ, ΔΓΕ ἴσην ἔχοντα τὴν μὲν ὑπὸ ΑΒΓ γωνίαν τῇ ὑπὸ ΔΓΕ, τὴν δὲ ὑπὸ ΒΑΓ τῇ ὑπὸ ΓΔΕ καὶ ἔτι τὴν ὑπὸ ΑΓΒ τῇ ὑπὸ ΓΕΔ· λέγω, ὅτι τῶν ΑΒΓ, ΔΓΕ τριγώνων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας καὶ ὁμόλογοι αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι.

Κείσθω γὰρ ἐπ᾽ εὐθείας ἡ ΒΓ τῇ ΓΕ. καὶ ἐπεὶ αἱ ὑπὸ ΑΒΓ, ΑΓΒ γωνίαι δύο ὀρθῶν ἐλάττονές εἰσιν, ἴση δὲ ἡ ὑπὸ ΑΓΒ τῇ ὑπὸ ΔΕΓ, αἱ ἄρα ὑπὸ ΑΒΓ, ΔΕΓ δύο ὀρθῶν ἐλάττονές εἰσιν· αἱ ΒΑ, ΕΔ ἄρα ἐκβαλλόμεναι συμπεσοῦνται. ἐκβεβλήσθωσαν καὶ συμπιπτέτωσαν κατὰ τὸ Ζ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΔΓΕ γωνία τῇ ὑπὸ ΑΒΓ, παράλληλός ἐστιν ἡ ΒΖ τῇ ΓΔ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΑΓΒ τῇ ὑπὸ ΔΕΓ, παράλληλός ἐστιν ἡ ΑΓ τῇ ΖΕ. παραλληλόγραμμον ἄρα ἐστὶ τὸ ΖΑΓΔ· ἴση ἄρα ἡ μὲν ΖΑ τῇ ΔΓ, ἡ δὲ ΑΓ τῇ ΖΔ. καὶ ἐπεὶ τριγώνου τοῦ ΖΒΕ παρὰ μίαν τὴν ΖΕ ἦκται ἡ ΑΓ, ἔστιν ἄρα ὡς ἡ ΒΑ πρὸς τὴν ΑΖ, οὕτως ἡ ΒΓ πρὸς τὴν ΓΕ. ἴση δὲ ἡ ΑΖ τῇ ΓΔ· ὡς ἄρα ἡ ΒΑ πρὸς τὴν ΓΔ, οὕτως ἡ ΒΓ πρὸς τὴν ΓΕ, καὶ ἐναλλὰξ ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως ἡ ΔΓ πρὸς τὴν ΓΕ. πάλιν, ἐπεὶ παράλληλός ἐστιν ἡ ΓΔ τῇ ΒΖ, ἔστιν ἄρα ὡς ἡ ΒΓ πρὸς τὴν ΓΕ, οὕτως ἡ ΖΔ πρὸς τὴν ΔΕ. ἴση δὲ ἡ ΖΔ τῇ ΑΓ· ὡς ἄρα ἡ ΒΓ πρὸς τὴν ΓΕ, οὕτως ἡ ΑΓ πρὸς τὴν ΔΕ, καὶ ἐναλλὰξ ὡς ἡ ΒΓ πρὸς τὴν ΓΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΔ. ἐπεὶ οὖν ἐδείχθη ὡς μὲν ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως ἡ ΔΓ πρὸς τὴν ΓΕ, ὡς δὲ ἡ ΒΓ πρὸς τὴν ΓΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΔ, δι᾽ ἴσου ἄρα ὡς ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΓΔ πρὸς τὴν ΔΕ.

Τῶν ἄρα ἰσογωνίων τριγώνων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας καὶ ὁμόλογοι αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι· ὅπερ ἔδει δεῖξαι.

ε΄. Ἐὰν δύο τρίγωνα τὰς πλευρὰς ἀνάλογον ἔχῃ, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἕξει τὰς γωνίας, ὑφ᾽ ἃς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν.

Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ τὰς πλευρὰς ἀνάλογον ἔχοντα, ὡς μὲν τὴν ΑΒ πρὸς τὴν ΒΓ, οὕτως τὴν ΔΕ πρὸς τὴν ΕΖ, ὡς δὲ τὴν ΒΓ πρὸς τὴν ΓΑ, οὕτως τὴν ΕΖ πρὸς τὴν ΖΔ, καὶ ἔτι ὡς τὴν ΒΑ πρὸς τὴν ΑΓ, οὕτως τὴν ΕΔ

πρὸς τὴν ΔΖ. λέγω, ὅτι ἰσογώνιόν ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ καὶ ἴσας ἕξουσι τὰς γωνίας, ὑφ᾽ ἃς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν, τὴν μὲν ὑπὸ ΑΒΓ τῇ ὑπὸ ΔΕΖ, τὴν δὲ ὑπὸ ΒΓΑ τῇ ὑπὸ ΕΖΔ καὶ ἔτι τὴν ὑπὸ ΒΑΓ τῇ ὑπὸ ΕΔΖ.

Συνεστάτω γὰρ πρὸς τῇ ΕΖ εὐθείᾳ καὶ τοῖς πρὸς αὐτῇ σημείοις τοῖς Ε, Ζ τῇ μὲν ὑπὸ ΑΒΓ γωνίᾳ ἴση ἡ ὑπὸ ΖΕΗ, τῇ δὲ ὑπὸ ΑΓΒ ἴση ἡ ὑπὸ ΕΖΗ· λοιπὴ ἄρα ἡ πρὸς τῷ Α λοιπῇ τῇ πρὸς τῷ Η ἐστιν ἴση. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΕΗΖ [τριγώνῳ]. τῶν ἄρα ΑΒΓ, ΕΗΖ τριγώνων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας καὶ ὁμόλογοι αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι· ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, [οὕτως] ἡ ΗΕ πρὸς τὴν ΕΖ. ἀλλ᾽ ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως ὑπόκειται ἡ ΔΕ πρὸς τὴν ΕΖ· ὡς ἄρα ἡ ΔΕ πρὸς τὴν ΕΖ, οὕτως ἡ ΗΕ πρὸς τὴν ΕΖ. ἑκατέρα ἄρα τῶν ΔΕ, ΗΕ πρὸς τὴν ΕΖ τὸν αὐτὸν ἔχει λόγον· ἴση ἄρα ἐστὶν ἡ ΔΕ τῇ ΗΕ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΔΖ τῇ ΗΖ ἐστιν ἴση. ἐπεὶ οὖν ἴση ἐστὶν ἡ ΔΕ τῇ ΕΗ, κοινὴ δὲ ἡ ΕΖ, δύο δὴ αἱ ΔΕ, ΕΖ δυσὶ ταῖς ΗΕ, ΕΖ ἴσαι εἰσίν· καὶ βάσις ἡ ΔΖ βάσει τῇ ΖΗ [ἐστιν] ἴση· γωνία ἄρα ἡ ὑπὸ ΔΕΖ γωνίᾳ τῇ ὑπὸ ΗΕΖ ἐστιν ἴση, καὶ τὸ ΔΕΖ τρίγωνον τῷ ΗΕΖ τριγώνῳ ἴσον, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν. ἴση ἄρα ἐστὶ καὶ ἡ μὲν ὑπὸ ΔΖΕ γωνία τῇ ὑπὸ ΗΖΕ, ἡ δὲ ὑπὸ ΕΔΖ τῇ ὑπὸ ΕΗΖ. καὶ ἐπεὶ ἡ μὲν ὑπὸ ΖΕΔ τῇ ὑπὸ ΗΕΖ ἐστιν ἴση, ἀλλ᾽ ἡ ὑπὸ ΗΕΖ τῇ ὑπὸ ΑΒΓ, καὶ ἡ ὑπὸ ΑΒΓ ἄρα γωνία τῇ ὑπὸ ΔΕΖ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΑΓΒ τῇ ὑπὸ ΔΖΕ ἐστιν ἴση, καὶ ἔτι ἡ πρὸς τῷ Α τῇ πρὸς τῷ Δ· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Ἐὰν ἄρα δύο τρίγωνα τὰς πλευρὰς ἀνάλογον ἔχῃ, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἕξει τὰς γωνίας, ὑφ᾽ ἃς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν· ὅπερ ἔδει δεῖξαι.

ς΄. Ἐὰν δύο τρίγωνα μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχῃ, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἕξει τὰς γωνίας,

ὑφ᾽ ἃς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν.

Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ μίαν γωνίαν τὴν ὑπὸ ΒΑΓ μιᾷ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἴσην ἔχοντα, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ὡς τὴν ΒΑ πρὸς τὴν ΑΓ, οὕτως τὴν ΕΔ πρὸς τὴν ΔΖ· λέγω, ὅτι ἰσογώνιόν ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ καὶ ἴσην ἕξει τὴν ὑπὸ ΑΒΓ γωνίαν τῇ ὑπὸ ΔΕΖ, τὴν δὲ ὑπὸ ΑΓΒ τῇ ὑπὸ ΔΖΕ.

Συνεστάτω γὰρ πρὸς τῇ ΔΖ εὐθείᾳ καὶ τοῖς πρὸς αὐτῇ σημείοις τοῖς Δ, Ζ ὁποτέρᾳ μὲν τῶν ὑπὸ ΒΑΓ, ΕΔΖ ἴση ἡ ὑπὸ ΖΔΗ, τῇ δὲ ὑπὸ ΑΓΒ ἴση ἡ ὑπὸ ΔΖΗ· λοιπὴ ἄρα ἡ πρὸς τῷ Β γωνία λοιπῇ τῇ πρὸς τῷ Η ἴση ἐστίν.

Ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΗΖ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΗΔ πρὸς τὴν ΔΖ. ὑπόκειται δὲ καὶ ὡς ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΕΔ πρὸς τὴν ΔΖ· καὶ ὡς ἄρα ἡ ΕΔ πρὸς τὴν ΔΖ, οὕτως ἡ ΗΔ πρὸς τὴν ΔΖ. ἴση ἄρα ἡ ΕΔ τῇ ΔΗ· καὶ κοινὴ ἡ ΔΖ· δύο δὴ αἱ ΕΔ, ΔΖ δυσὶ ταῖς ΗΔ, ΔΖ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΕΔΖ γωνίᾳ τῇ ὑπὸ ΗΔΖ [ἐστιν] ἴση· βάσις ἄρα ἡ ΕΖ βάσει τῇ ΗΖ ἐστιν ἴση, καὶ τὸ ΔΕΖ τρίγωνον τῷ ΗΔΖ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν. ἴση ἄρα ἐστὶν ἡ μὲν ὑπὸ ΔΖΗ τῇ ὑπὸ ΔΖΕ, ἡ δὲ ὑπὸ ΔΗΖ τῇ ὑπὸ ΔΕΖ. ἀλλ᾽ ἡ ὑπὸ ΔΖΗ τῇ ὑπὸ ΑΓΒ ἐστιν ἴση· καὶ ἡ ὑπὸ ΑΓΒ ἄρα τῇ ὑπὸ ΔΖΕ ἐστιν ἴση. ὑπόκειται δὲ καὶ ἡ ὑπὸ ΒΑΓ τῇ ὑπὸ ΕΔΖ ἴση· καὶ λοιπὴ ἄρα ἡ πρὸς τῷ Β λοιπῇ τῇ πρὸς τῷ Ε ἴση ἐστίν· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Ἐὰν ἄρα δύο τρίγωνα μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχῃ, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἕξει τὰς γωνίας, ὑφ᾽ ἃς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν· ὅπερ ἔδει δεῖξαι.

ζ΄. Ἐὰν δύο τρίγωνα μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχῃ, περὶ δὲ ἄλλας γωνίας τὰς πλευρὰς ἀνάλογον, τῶν δὲ λοιπῶν ἑκατέραν ἅμα ἤτοι ἐλάσσονα ἢ μὴ ἐλάσσονα

ὀρθῆς, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἕξει τὰς γωνίας, περὶ ἃς ἀνάλογόν εἰσιν αἱ πλευραί.

Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχοντα τὴν ὑπὸ ΒΑΓ τῇ ὑπὸ ΕΔΖ, περὶ δὲ ἄλλας γωνίας τὰς ὑπὸ ΑΒΓ, ΔΕΖ τὰς πλευρὰς ἀνάλογον, ὡς τὴν ΑΒ πρὸς τὴν ΒΓ, οὕτως τὴν ΔΕ πρὸς τὴν ΕΖ, τῶν δὲ λοιπῶν τῶν πρὸς τοῖς Γ, Ζ πρότερον ἑκατέραν ἅμα ἐλάσσονα ὀρθῆς· λέγω, ὅτι ἰσογώνιόν ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ, καὶ ἴση ἔσται ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΕΖ, καὶ λοιπὴ δηλονότι ἡ πρὸς τῷ Γ λοιπῇ τῇ πρὸς τῷ Ζ ἴση.

Εἰ γὰρ ἄνισός ἐστιν ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΕΖ, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ ὑπὸ ΑΒΓ. καὶ συνεστάτω πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Β τῇ ὑπὸ ΔΕΖ γωνίᾳ ἴση ἡ ὑπὸ ΑΒΗ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν Α γωνία τῇ Δ, ἡ δὲ ὑπὸ ΑΒΗ τῇ ὑπὸ ΔΕΖ, λοιπὴ ἄρα ἡ ὑπὸ ΑΗΒ λοιπῇ τῇ ὑπὸ ΔΖΕ ἐστιν ἴση. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΗ τρίγωνον τῷ ΔΕΖ τριγώνῳ. ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΒΗ, οὕτως ἡ ΔΕ πρὸς τὴν ΕΖ. ὡς δὲ ἡ ΔΕ πρὸς τὴν ΕΖ, [οὕτως] ὑπόκειται ἡ ΑΒ πρὸς τὴν ΒΓ· ἡ ΑΒ ἄρα πρὸς ἑκατέραν τῶν ΒΓ, ΒΗ τὸν αὐτὸν ἔχει λόγον· ἴση ἄρα ἡ ΒΓ τῇ ΒΗ. ὥστε καὶ γωνία ἡ πρὸς τῷ Γ γωνίᾳ τῇ ὑπὸ ΒΗΓ ἐστιν ἴση. ἐλάττων δὲ ὀρθῆς ὑπόκειται ἡ πρὸς τῷ Γ· ἐλάττων ἄρα ἐστὶν ὀρθῆς καὶ ἡ ὑπὸ ΒΗΓ· ὥστε ἡ ἐφεξῆς αὐτῇ γωνία ἡ ὑπὸ ΑΗΒ μείζων ἐστὶν ὀρθῆς. καὶ ἐδείχθη ἴση οὖσα τῇ πρὸς τῷ Ζ· καὶ ἡ πρὸς τῷ Ζ ἄρα μείζων ἐστὶν ὀρθῆς. ὑπόκειται δὲ ἐλάσσων ὀρθῆς· ὅπερ ἐστὶν ἄτοπον. οὐκ ἄρα ἄνισός ἐστιν ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΕΖ· ἴση ἄρα. ἔστι δὲ καὶ ἡ πρὸς τῷ Α ἴση τῇ πρὸς τῷ Δ· καὶ λοιπὴ ἄρα ἡ πρὸς τῷ Γ λοιπῇ τῇ πρὸς τῷ Ζ ἴση ἐστίν. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Ἀλλὰ δὴ πάλιν ὑποκείσθω ἑκατέρα τῶν πρὸς τοῖς Γ, Ζ μὴ ἐλάσσων ὀρθῆς· λέγω πάλιν, ὅτι καὶ οὕτως ἐστὶν ἰσογώνιον τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι ἴση ἐστὶν ἡ ΒΓ τῇ ΒΗ· ὥστε καὶ γωνία ἡ πρὸς τῷ Γ τῇ ὑπὸ ΒΗΓ ἴση ἐστίν. οὐκ ἐλάττων δὲ ὀρθῆς ἡ πρὸς τῷ Γ· οὐκ ἐλάττων ἄρα ὀρθῆς οὐδὲ ἡ ὑπὸ ΒΗΓ. τριγώνου δὴ τοῦ ΒΗΓ αἱ δύο γωνίαι δύο ὀρθῶν οὔκ εἰσιν ἐλάττονες· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα πάλιν ἄνισός ἐστιν ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΕΖ· ἴση ἄρα. ἔστι δὲ καὶ ἡ πρὸς τῷ Α τῇ πρὸς τῷ Δ ἴση· λοιπὴ ἄρα ἡ πρὸς τῷ Γ λοιπῇ τῇ πρὸς τῷ Ζ ἴση ἐστίν. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Ἐὰν ἄρα δύο τρίγωνα μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχῃ, περὶ δὲ ἄλλας γωνίας τὰς πλευρὰς ἀνάλογον, τῶν δὲ λοιπῶν ἑκατέραν ἅμα ἐλάττονα ἢ μὴ ἐλάττονα ὀρθῆς, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἕξει τὰς γωνίας, περὶ ἃς ἀνάλογόν εἰσιν αἱ πλευραί· ὅπερ ἔδει δεῖξαι.

η΄. Ἐὰν ἐν ὀρθογωνίῳ τριγώνῳ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἀχθῇ, τὰ πρὸς τῇ καθέτῳ τρίγωνα ὅμοιά ἐστι τῷ τε ὅλῳ καὶ ἀλλήλοις.

Ἔστω τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν ἔχον τὴν ὑπὸ ΒΑΓ γωνίαν, καὶ ἤχθω ἀπὸ τοῦ Α ἐπὶ τὴν ΒΓ κάθετος ἡ ΑΔ· λέγω, ὅτι ὅμοιόν ἐστιν ἑκάτερον τῶν ΑΒΔ, ΑΔΓ τριγώνων ὅλῳ τῷ ΑΒΓ καὶ ἔτι ἀλλήλοις.

Ἐπεὶ γὰρ ἴση ἐστὶν ἡ ὑπὸ ΒΑΓ τῇ ὑπὸ ΑΔΒ· ὀρθὴ γὰρ ἑκατέρα· καὶ κοινὴ τῶν δύο τριγώνων τοῦ τε ΑΒΓ καὶ τοῦ ΑΒΔ ἡ πρὸς τῷ Β, λοιπὴ ἄρα ἡ ὑπὸ ΑΓΒ λοιπῇ τῇ ὑπὸ ΒΑΔ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΒΔ τριγώνῳ. ἔστιν ἄρα ὡς ἡ ΒΓ ὑποτείνουσα τὴν ὀρθὴν τοῦ ΑΒΓ τριγώνου πρὸς τὴν ΒΑ ὑποτείνουσαν τὴν ὀρθὴν τοῦ ΑΒΔ τριγώνου, οὕτως αὐτὴ ἡ ΑΒ ὑποτείνουσα τὴν πρὸς τῷ Γ γωνίαν τοῦ ΑΒΓ τριγώνου πρὸς τὴν ΒΔ ὑποτείνουσαν τὴν ἴσην τὴν ὑπὸ ΒΑΔ τοῦ ΑΒΔ τριγώνου, καὶ ἔτι ἡ ΑΓ πρὸς τὴν ΑΔ ὑποτείνουσαν τὴν πρὸς τῷ Β γωνίαν κοινὴν τῶν δύο τριγώνων. τὸ ΑΒΓ ἄρα τρίγωνον τῷ ΑΒΔ τριγώνῳ ἰσογώνιόν τέ ἐστι καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει. ὅμοιον ἄρα [ἐστὶ] τὸ ΑΒΓ τρίγωνον τῷ ΑΒΔ τριγώνῳ. ὁμοίως δὴ δείξομεν, ὅτι καὶ τῷ ΑΔΓ τριγώνῳ ὅμοιόν ἐστι τὸ ΑΒΓ

τρίγωνον· ἑκάτερον ἄρα τῶν ΑΒΔ, ΑΔΓ [τριγώνων] ὅμοιόν ἐστιν ὅλῳ τῷ ΑΒΓ.

Λέγω δή, ὅτι καὶ ἀλλήλοις ἐστὶν ὅμοια τὰ ΑΒΔ, ΑΔΓ τρίγωνα.

Ἐπεὶ γὰρ ὀρθὴ ἡ ὑπὸ ΒΔΑ ὀρθῇ τῇ ὑπὸ ΑΔΓ ἐστιν ἴση, ἀλλὰ μὴν καὶ ἡ ὑπὸ ΒΑΔ τῇ πρὸς τῷ Γ ἐδείχθη ἴση, καὶ λοιπὴ ἄρα ἡ πρὸς τῷ Β λοιπῇ τῇ ὑπὸ ΔΑΓ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΔ τρίγωνον τῷ ΑΔΓ τριγώνῳ. ἔστιν ἄρα ὡς ἡ ΒΔ τοῦ ΑΒΔ τριγώνου ὑποτείνουσα τὴν ὑπὸ ΒΑΔ πρὸς τὴν ΔΑ τοῦ ΑΔΓ τριγώνου ὑποτείνουσαν τὴν πρὸς τῷ Γ ἴσην τῇ ὑπὸ ΒΑΔ, οὕτως αὐτὴ ἡ ΑΔ τοῦ ΑΒΔ τριγώνου ὑποτείνουσα τὴν πρὸς τῷ Β γωνίαν πρὸς τὴν ΔΓ ὑποτείνουσαν τὴν ὑπὸ ΔΑΓ τοῦ ΑΔΓ τριγώνου ἴσην τῇ πρὸς τῷ Β, καὶ ἔτι ἡ ΒΑ πρὸς τὴν ΑΓ ὑποτείνουσαι τὰς ὀρθάς· ὅμοιον ἄρα ἐστὶ τὸ ΑΒΔ τρίγωνον τῷ ΑΔΓ τριγώνῳ.

Ἐὰν ἄρα ἐν ὀρθογωνίῳ τριγώνῳ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἀχθῇ, τὰ πρὸς τῇ καθέτῳ τρίγωνα ὅμοιά ἐστι τῷ τε ὅλῳ καὶ ἀλλήλοις [ὅπερ ἔδει δεῖξαι].

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν ἐν ὀρθογωνίῳ τριγώνῳ ἀπὸ τῆς ὀρθῆς ἐπὶ τὴν βάσιν κάθετος ἀχθῇ, ἡ ἀχθεῖσα τῶν τῆς βάσεως τμημάτων μέση ἀνάλογόν ἐστιν· ὅπερ ἔδει δεῖξαι [καὶ ἔτι τῆς βάσεως καὶ ἑνὸς ὁποιουοῦν τῶν τμημάτων ἡ πρὸς τῷ τμήματι πλευρὰ μέση ἀνάλογόν ἐστιν].

θ΄. Τῆς δοθείσης εὐθείας τὸ προσταχθὲν μέρος ἀφελεῖν.

Ἔστω ἡ δοθεῖσα εὐθεῖα ἡ ΑΒ· δεῖ δὴ τῆς ΑΒ τὸ προσταχθὲν μέρος ἀφελεῖν.

Ἐπιτετάχθω δὴ τὸ τρίτον. [καὶ] διήχθω τις ἀπὸ τοῦ Α εὐθεῖα ἡ ΑΓ γωνίαν περιέχουσα μετὰ τῆς ΑΒ τυχοῦσαν· καὶ εἰλήφθω τυχὸν σημεῖον ἐπὶ τῆς ΑΓ τὸ Δ, καὶ κείσθωσαν τῇ ΑΔ ἴσαι αἱ ΔΕ, ΕΓ. καὶ ἐπεζεύχθω ἡ ΒΓ, καὶ διὰ τοῦ Δ παράλληλος αὐτῇ ἤχθω ἡ ΔΖ.

Ἐπεὶ οὖν τριγώνου τοῦ ΑΒΓ παρὰ μίαν τῶν πλευρῶν τὴν ΒΓ ἦκται ἡ ΖΔ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΓΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΒΖ πρὸς τὴν ΖΑ. διπλῆ δὲ ἡ ΓΔ τῆς ΔΑ· διπλῆ ἄρα καὶ ἡ ΒΖ τῆς ΖΑ· τριπλῆ ἄρα ἡ ΒΑ τῆς ΑΖ.

Τῆς ἄρα δοθείσης εὐθείας τῆς ΑΒ τὸ ἐπιταχθὲν τρίτον μέρος ἀφῄρηται τὸ ΑΖ· ὅπερ ἔδει ποιῆσαι.

ι΄. Τὴν δοθεῖσαν εὐθεῖαν ἄτμητον τῇ δοθείσῃ τετμημένῃ ὁμοίως τεμεῖν.

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἄτμητος ἡ ΑΒ, ἡ δὲ τετμημένη ἡ ΑΓ κατὰ τὰ Δ, Ε σημεῖα, καὶ κείσθωσαν ὥστε γωνίαν τυχοῦσαν περιέχειν, καὶ ἐπεζεύχθω ἡ ΓΒ, καὶ διὰ τῶν Δ, Ε τῇ ΒΓ παράλληλοι ἤχθωσαν αἱ ΔΖ, ΕΗ, διὰ δὲ τοῦ Δ τῇ ΑΒ παράλληλος ἤχθω ἡ ΔΘΚ.

Παραλληλόγραμμον ἄρα ἐστὶν ἑκάτερον τῶν ΖΘ, ΘΒ· ἴση ἄρα ἡ μὲν ΔΘ τῇ ΖΗ, ἡ δὲ ΘΚ τῇ ΗΒ. καὶ ἐπεὶ τριγώνου τοῦ ΔΚΓ παρὰ μίαν τῶν πλευρῶν τὴν ΚΓ εὐθεῖα ἦκται ἡ ΘΕ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΓΕ πρὸς τὴν ΕΔ, οὕτως ἡ ΚΘ πρὸς τὴν ΘΔ. ἴση δὲ ἡ μὲν ΚΘ τῇ ΒΗ, ἡ δὲ ΘΔ τῇ ΗΖ. ἔστιν ἄρα ὡς ἡ ΓΕ πρὸς τὴν ΕΔ, οὕτως ἡ ΒΗ πρὸς τὴν ΗΖ. πάλιν, ἐπεὶ τριγώνου τοῦ ΑΗΕ παρὰ μίαν τῶν πλευρῶν τὴν ΗΕ ἦκται ἡ ΖΔ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΕΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΑ. ἐδείχθη δὲ καὶ ὡς ἡ ΓΕ πρὸς τὴν ΕΔ, οὕτως ἡ ΒΗ πρὸς τὴν ΗΖ· ἔστιν ἄρα ὡς μὲν ἡ ΓΕ πρὸς τὴν ΕΔ, οὕτως ἡ ΒΗ πρὸς τὴν ΗΖ, ὡς δὲ ἡ ΕΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΑ.

Ἡ ἄρα δοθεῖσα εὐθεῖα ἄτμητος ἡ ΑΒ τῇ δοθείσῃ εὐθείᾳ τετμημένῃ τῇ ΑΓ ὁμοίως τέτμηται· ὅπερ ἔδει ποιῆσαι.

ια΄. Δύο δοθεισῶν εὐθειῶν τρίτην ἀνάλογον προσευρεῖν.

Ἔστωσαν αἱ δοθεῖσαι [δύο εὐθεῖαι] αἱ ΒΑ, ΑΓ καὶ κείσθωσαν γωνίαν περιέχουσαι τυχοῦσαν. δεῖ δὴ τῶν ΒΑ, ΑΓ τρίτην ἀνάλογον προσευρεῖν. ἐκβεβλήσθωσαν γὰρ ἐπὶ τὰ Δ, Ε σημεῖα, καὶ κείσθω τῇ ΑΓ ἴση ἡ ΒΔ, καὶ ἐπεζεύχθω ἡ ΒΓ, καὶ διὰ τοῦ Δ παράλληλος αὐτῇ ἤχθω ἡ ΔΕ.

Ἐπεὶ οὖν τριγώνου τοῦ ΑΔΕ παρὰ μίαν τῶν πλευρῶν τὴν ΔΕ ἦκται ἡ ΒΓ, ἀνάλογόν ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΔ, οὕτως ἡ ΑΓ πρὸς τὴν ΓΕ. ἴση δὲ ἡ ΒΔ τῇ ΑΓ. ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΑΓ, οὕτως ἡ ΑΓ πρὸς τὴν ΓΕ.

Δύο ἄρα δοθεισῶν εὐθειῶν τῶν ΑΒ, ΑΓ τρίτη ἀνάλογον αὐταῖς προσεύρηται ἡ ΓΕ· ὅπερ ἔδει ποιῆσαι.

ιβ΄. Τριῶν δοθεισῶν εὐθειῶν τετάρτην ἀνάλογον προσευρεῖν.

Ἔστωσαν αἱ δοθεῖσαι τρεῖς εὐθεῖαι αἱ Α, Β, Γ· δεῖ δὴ τῶν Α, Β, Γ τετάρτην ἀνάλογον προσευρεῖν.

Ἐκκείσθωσαν δύο εὐθεῖαι αἱ ΔΕ, ΔΖ γωνίαν περιέχουσαι [τυχοῦσαν] τὴν ὑπὸ ΕΔΖ· καὶ κείσθω τῇ μὲν Α ἴση ἡ ΔΗ, τῇ δὲ Β ἴση ἡ ΗΕ, καὶ ἔτι τῇ Γ ἴση ἡ ΔΘ· καὶ ἐπιζευχθείσης τῆς ΗΘ παράλληλος αὐτῇ ἤχθω διὰ τοῦ Ε ἡ ΕΖ.

Ἐπεὶ οὖν τριγώνου τοῦ ΔΕΖ παρὰ μίαν τὴν ΕΖ ἦκται ἡ ΗΘ, ἔστιν ἄρα ὡς ἡ ΔΗ πρὸς τὴν ΗΕ, οὕτως ἡ ΔΘ πρὸς τὴν ΘΖ. ἴση δὲ ἡ μὲν ΔΗ τῇ Α, ἡ δὲ ΗΕ τῇ Β, ἡ δὲ ΔΘ τῇ Γ· ἔστιν ἄρα ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν ΘΖ.

Τριῶν ἄρα δοθεισῶν εὐθειῶν τῶν Α, Β, Γ τετάρτη ἀνάλογον προσεύρηται ἡ ΘΖ· ὅπερ ἔδει ποιῆσαι.

ιγ΄. Δύο δοθεισῶν εὐθειῶν μέσην ἀνάλογον προσευρεῖν.

Ἔστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι αἱ ΑΒ, ΒΓ· δεῖ δὴ τῶν ΑΒ, ΒΓ μέσην ἀνάλογον προσευρεῖν.

Κείσθωσαν ἐπ’ εὐθείας, καὶ γεγράφθω ἐπὶ τῆς ΑΓ ἡμικύκλιον τὸ ΑΔΓ, καὶ ἤχθω ἀπὸ τοῦ Β σημείου τῇ ΑΓ εὐθείᾳ πρὸς ὀρθὰς ἡ ΒΔ, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΓ.

Ἐπεὶ ἐν ἡμικυκλίῳ γωνία ἐστὶν ἡ ὑπὸ ΑΔΓ, ὀρθή ἐστιν. καὶ ἐπεὶ ἐν ὀρθογωνίῳ τριγώνῳ τῷ ΑΔΓ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἦκται ἡ ΔΒ, ἡ ΔΒ ἄρα τῶν τῆς βάσεως τμημάτων τῶν ΑΒ, ΒΓ μέση ἀνάλογόν ἐστιν.

Δύο ἄρα δοθεισῶν εὐθειῶν τῶν ΑΒ, ΒΓ μέση ἀνάλογον προσεύρηται ἡ ΔΒ· ὅπερ ἔδει ποιῆσαι.

ιδ΄. Τῶν ἴσων τε καὶ ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· καὶ ὧν ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἴσα ἐστὶν ἐκεῖνα.

Ἔστω ἴσα τε καὶ ἰσογώνια παραλληλόγραμμα τὰ ΑΒ, ΒΓ ἴσας ἔχοντα τὰς πρὸς τῷ Β γωνίας, καὶ κείσθωσαν ἐπ᾽ εὐθείας αἱ ΔΒ, ΒΕ· ἐπ᾽ εὐθείας ἄρα εἰσὶ καὶ αἱ ΖΒ, ΒΗ. λέγω, ὅτι τῶν ΑΒ, ΒΓ ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, τουτέστιν, ὅτι ἐστὶν ὡς ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΗΒ πρὸς τὴν ΒΖ.

Συμπεπληρώσθω γὰρ τὸ ΖΕ παραλληλόγραμμον. ἐπεὶ οὖν ἴσον ἐστὶ τὸ ΑΒ παραλληλόγραμμον τῷ ΒΓ παραλληλογράμμῳ, ἄλλο δέ τι τὸ ΖΕ, ἔστιν ἄρα ὡς τὸ ΑΒ πρὸς τὸ ΖΕ, οὕτως τὸ ΒΓ πρὸς τὸ ΖΕ. ἀλλ᾽ ὡς μὲν τὸ ΑΒ πρὸς τὸ ΖΕ, οὕτως ἡ ΔΒ πρὸς τὴν ΒΕ, ὡς δὲ τὸ ΒΓ πρὸς τὸ ΖΕ, οὕτως ἡ ΗΒ πρὸς τὴν ΒΖ· καὶ ὡς ἄρα ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΗΒ πρὸς τὴν ΒΖ. τῶν ἄρα ΑΒ, ΒΓ παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας.

Ἀλλὰ δὴ ἔστω ὡς ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΗΒ πρὸς τὴν ΒΖ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒ παραλληλόγραμμον τῷ ΒΓ παραλληλογράμμῳ.

Ἐπεὶ γάρ ἐστιν ὡς ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΗΒ πρὸς τὴν ΒΖ, ἀλλ᾽ ὡς μὲν ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως τὸ ΑΒ παραλληλόγραμμον πρὸς τὸ ΖΕ παραλληλόγραμμον, ὡς δὲ ἡ ΗΒ πρὸς τὴν ΒΖ, οὕτως τὸ ΒΓ παραλληλόγραμμον πρὸς τὸ ΖΕ παραλληλόγραμμον, καὶ ὡς ἄρα τὸ ΑΒ πρὸς τὸ ΖΕ, οὕτως τὸ ΒΓ πρὸς τὸ ΖΕ· ἴσον ἄρα ἐστὶ τὸ ΑΒ παραλληλόγραμμον τῷ ΒΓ παραλληλογράμμῳ.

Τῶν ἄρα ἴσων τε καὶ ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· καὶ ὧν ἰσογωνίων παραλληλογράμμων

ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἴσα ἐστὶν ἐκεῖνα· ὅπερ ἔδει δεῖξαι.

ιε΄. Τῶν ἴσων καὶ μίαν μιᾷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· καὶ ὧν μίαν μιᾷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἴσα ἐστὶν ἐκεῖνα.

Ἔστω ἴσα τρίγωνα τὰ ΑΒΓ, ΑΔΕ μίαν μιᾷ ἴσην ἔχοντα γωνίαν τὴν ὑπὸ ΒΑΓ τῇ ὑπὸ ΔΑΕ· λέγω, ὅτι τῶν ΑΒΓ, ΑΔΕ τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, τουτέστιν, ὅτι ἐστὶν ὡς ἡ ΓΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΒ.

Κείσθω γὰρ ὥστε ἐπ᾽ εὐθείας εἶναι τὴν ΓΑ τῇ ΑΔ· ἐπ᾽ εὐθείας ἄρα ἐστὶ καὶ ἡ ΕΑ τῇ ΑΒ. καὶ ἐπεζεύχθω ἡ ΒΔ.

Ἐπεὶ οὖν ἴσον ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΔΕ τριγώνῳ, ἄλλο δέ τι τὸ ΒΑΔ, ἔστιν ἄρα ὡς τὸ ΓΑΒ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον, οὕτως τὸ ΕΑΔ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον. ἀλλ᾽ ὡς μὲν τὸ ΓΑΒ πρὸς τὸ ΒΑΔ, οὕτως ἡ ΓΑ πρὸς τὴν ΑΔ, ὡς δὲ τὸ ΕΑΔ πρὸς τὸ ΒΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΒ. καὶ ὡς ἄρα ἡ ΓΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΒ. τῶν ΑΒΓ, ΑΔΕ ἄρα τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας.

Ἀλλὰ δὴ ἀντιπεπονθέτωσαν αἱ πλευραὶ τῶν ΑΒΓ, ΑΔΕ τριγώνων, καὶ ἔστω ὡς ἡ ΓΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΒ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΔΕ τριγώνῳ.

Ἐπιζευχθείσης γὰρ πάλιν τῆς ΒΔ, ἐπεί ἐστιν ὡς ἡ ΓΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΒ, ἀλλ᾽ ὡς μὲν ἡ ΓΑ πρὸς τὴν ΑΔ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον, ὡς δὲ ἡ ΕΑ πρὸς τὴν ΑΒ, οὕτως τὸ ΕΑΔ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον, ὡς ἄρα τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον, οὕτως τὸ ΕΑΔ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον. ἑκάτερον ἄρα τῶν ΑΒΓ, ΕΑΔ πρὸς τὸ ΒΑΔ τὸν αὐτὸν ἔχει λόγον. ἴσον ἄρα ἐστὶ τὸ ΑΒΓ [τρίγωνον] τῷ ΕΑΔ τριγώνῳ.

Τῶν ἄρα ἴσων καὶ μίαν μιᾷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· καὶ ὧν μίαν μιᾷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἐκεῖνα ἴσα ἐστίν· ὅπερ ἔδει δεῖξαι.

ις΄. Ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν μέσων περιεχομένῳ ὀρθογωνίῳ· κἂν τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ᾖ τῷ ὑπὸ τῶν μέσων περιεχομένῳ ὀρθογωνίῳ, αἱ τέσσαρες εὐθεῖαι ἀνάλογον ἔσονται.

Ἔστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἱ ΑΒ, ΓΔ, Ε, Ζ, ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ Ε πρὸς τὴν Ζ· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΒ, Ζ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν ΓΔ, Ε περιεχομένῳ ὀρθογωνίῳ.

Ἤχθωσαν [γὰρ] ἀπὸ τῶν Α, Γ σημείων ταῖς ΑΒ, ΓΔ εὐθείαις πρὸς ὀρθὰς αἱ ΑΗ, ΓΘ, καὶ κείσθω τῇ μὲν Ζ ἴση ἡ ΑΓ, τῇ δὲ Ε ἴση ἡ ΓΘ. καὶ συμπεπληρώσθω τὰ ΒΗ, ΔΘ παραλληλόγραμμα.

Καὶ ἐπεί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ Ε πρὸς τὴν Ζ, ἴση δὲ ἡ μὲν Ε τῇ ΓΘ, ἡ δὲ Ζ τῇ ΑΗ, ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΓΘ πρὸς τὴν ΑΗ. τῶν ΒΗ, ΔΘ ἄρα παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας. ὧν δὲ ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἴσα ἐστὶν ἐκεῖνα· ἴσον ἄρα ἐστὶ τὸ ΒΗ παραλληλόγραμμον τῷ ΔΘ παραλληλογράμμῳ. καί ἐστι τὸ μὲν ΒΗ τὸ ὑπὸ τῶν ΑΒ, Ζ· ἴση γὰρ ἡ ΑΗ τῇ Ζ· τὸ δὲ ΔΘ τὸ ὑπὸ τῶν ΓΔ, Ε· ἴση γὰρ ἡ Ε τῇ ΓΘ· τὸ ἄρα ὑπὸ τῶν ΑΒ, Ζ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν ΓΔ, Ε περιεχομένῳ ὀρθογωνίῳ.

Ἀλλὰ δὴ τὸ ὑπὸ τῶν ΑΒ, Ζ περιεχόμενον ὀρθογώνιον ἴσον ἔστω τῷ ὑπὸ τῶν ΓΔ, Ε περιεχομένῳ ὀρθογωνίῳ· λέγω, ὅτι αἱ τέσσαρες εὐθεῖαι ἀνάλογον ἔσονται, ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ Ε πρὸς τὴν Ζ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ τὸ ὑπὸ τῶν ΑΒ, Ζ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΓΔ, Ε, καί ἐστι τὸ μὲν ὑπὸ τῶν ΑΒ, Ζ τὸ ΒΗ· ἴση γάρ ἐστιν ἡ ΑΗ τῇ Ζ· τὸ δὲ ὑπὸ τῶν ΓΔ, Ε τὸ ΔΘ· ἴση γὰρ ἡ ΓΘ τῇ Ε· τὸ ἄρα ΒΗ ἴσον ἐστὶ τῷ ΔΘ. καί ἐστιν ἰσογώνια. τῶν δὲ ἴσων καὶ ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας. ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΓΘ πρὸς τὴν ΑΗ. ἴση δὲ ἡ μὲν ΓΘ τῇ Ε, ἡ δὲ ΑΗ τῇ Ζ· ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ Ε πρὸς τὴν Ζ.

Ἐὰν ἄρα τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν μέσων περιεχομένῳ ὀρθογωνίῳ· κἂν τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ᾖ τῷ ὑπὸ τῶν μέσων περιεχομένῳ ὀρθογωνίῳ, αἱ τέσσαρες εὐθεῖαι ἀνάλογον ἔσονται· ὅπερ ἔδει δεῖξαι.

ιζ΄. Ἐὰν τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς μέσης τετραγώνῳ· κἂν τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ᾖ τῷ ἀπὸ τῆς μέσης τετραγώνῳ, αἱ τρεῖς εὐθεῖαι ἀνάλογον ἔσονται.

Ἔστωσαν τρεῖς εὐθεῖαι ἀνάλογον αἱ Α, Β, Γ, ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Γ· λέγω, ὅτι τὸ ὑπὸ τῶν Α, Γ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς Β τετραγώνῳ.

Κείσθω τῇ Β ἴση ἡ Δ.

Καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Γ, ἴση δὲ ἡ Β τῇ Δ, ἔστιν ἄρα ὡς ἡ Α πρὸς τὴν Β, ἡ Δ πρὸς τὴν Γ. ἐὰν δὲ τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, τὸ ὑπὸ τῶν ἄκρων περιεχόμενον [ὀρθογώνιον] ἴσον ἐστὶ τῷ ὑπὸ τῶν μέσων περιεχομένῳ ὀρθογωνίῳ. τὸ ἄρα ὑπὸ τῶν Α, Γ ἴσον ἐστὶ τῷ ὑπὸ τῶν Β, Δ. ἀλλὰ τὸ ὑπὸ τῶν Β, Δ τὸ ἀπὸ τῆς Β ἐστιν· ἴση γὰρ ἡ Β τῇ Δ· τὸ ἄρα ὑπὸ τῶν Α, Γ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς Β τετραγώνῳ.

Ἀλλὰ δὴ τὸ ὑπὸ τῶν Α, Γ ἴσον ἔστω τῷ ἀπὸ τῆς Β· λέγω, ὅτι ἐστὶν ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Γ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ τὸ ὑπὸ τῶν Α, Γ ἴσον ἐστὶ τῷ ἀπὸ τῆς Β, ἀλλὰ τὸ ἀπὸ τῆς Β τὸ ὑπὸ τῶν Β, Δ ἐστιν· ἴση γὰρ ἡ Β τῇ Δ· τὸ ἄρα ὑπὸ τῶν Α, Γ ἴσον ἐστὶ τῷ ὑπὸ τῶν Β, Δ. ἐὰν δὲ τὸ ὑπὸ τῶν ἄκρων ἴσον ᾖ τῷ ὑπὸ τῶν μέσων, αἱ τέσσαρες εὐθεῖαι ἀνάλογόν εἰσιν. ἔστιν ἄρα ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Δ πρὸς τὴν Γ. ἴση δὲ ἡ Β τῇ Δ· ὡς ἄρα ἡ Α πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Γ.

Ἐὰν ἄρα τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς μέσης τετραγώνῳ· κἂν τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ᾖ τῷ ἀπὸ τῆς μέσης τετραγώνῳ, αἱ τρεῖς εὐθεῖαι ἀνάλογον ἔσονται· ὅπερ ἔδει δεῖξαι.

ιη΄. Ἀπὸ τῆς δοθείσης εὐθείας τῷ δοθέντι εὐθυγράμμῳ ὁμοιόν τε καὶ ὁμοίως κείμενον εὐθύγραμμον ἀναγράψαι.

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ δοθὲν εὐθύγραμμον τὸ ΓΕ· δεῖ δὴ ἀπὸ τῆς ΑΒ εὐθείας τῷ ΓΕ εὐθυγράμμῳ ὁμοιόν τε καὶ ὁμοίως κείμενον εὐθύγραμμον ἀναγράψαι.

Ἐπεζεύχθω ἡ ΔΖ, καὶ συνεστάτω πρὸς τῇ ΑΒ εὐθείᾳ καὶ τοῖς πρὸς αὐτῇ σημείοις τοῖς Α, Β τῇ μὲν πρὸς τῷ Γ γωνίᾳ ἴση ἡ ὑπὸ ΗΑΒ, τῇ δὲ ὑπὸ ΓΔΖ ἴση ἡ ὑπὸ ΑΒΗ. λοιπὴ ἄρα ἡ ὑπὸ ΓΖΔ τῇ ὑπὸ ΑΗΒ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΖΓΔ τρίγωνον τῷ ΗΑΒ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΖΔ πρὸς τὴν ΗΒ, οὕτως ἡ ΖΓ πρὸς τὴν ΗΑ, καὶ ἡ ΓΔ πρὸς τὴν ΑΒ. πάλιν συνεστάτω πρὸς τῇ ΒΗ εὐθείᾳ καὶ τοῖς πρὸς αὐτῇ σημείοις τοῖς Β, Η τῇ μὲν ὑπὸ ΔΖΕ γωνίᾳ ἴση ἡ ὑπὸ ΒΗΘ, τῇ δὲ ὑπὸ ΖΔΕ ἴση ἡ ὑπὸ ΗΒΘ. λοιπὴ ἄρα ἡ πρὸς τῷ Ε λοιπῇ τῇ πρὸς τῷ Θ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΖΔΕ τρίγωνον τῷ ΗΘΒ τριγώνῳ· ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΖΔ πρὸς τὴν ΗΒ, οὕτως ἡ ΖΕ πρὸς τὴν ΗΘ καὶ ἡ ΕΔ πρὸς τὴν ΘΒ. ἐδείχθη δὲ καὶ ὡς ἡ ΖΔ πρὸς τὴν ΗΒ, οὕτως

ἡ ΖΓ πρὸς τὴν ΗΑ καὶ ἡ ΓΔ πρὸς τὴν ΑΒ· καὶ ὡς ἄρα ἡ ΖΓ πρὸς τὴν ΑΗ, οὕτως ἥ τε ΓΔ πρὸς τὴν ΑΒ καὶ ἡ ΖΕ πρὸς τὴν ΗΘ καὶ ἔτι ἡ ΕΔ πρὸς τὴν ΘΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ὑπὸ ΓΖΔ γωνία τῇ ὑπὸ ΑΗΒ, ἡ δὲ ὑπὸ ΔΖΕ τῇ ὑπὸ ΒΗΘ, ὅλη ἄρα ἡ ὑπὸ ΓΖΕ ὅλῃ τῇ ὑπὸ ΑΗΘ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΓΔΕ τῇ ὑπὸ ΑΒΘ ἐστιν ἴση. ἔστι δὲ καὶ ἡ μὲν πρὸς τῷ Γ τῇ πρὸς τῷ Α ἴση, ἡ δὲ πρὸς τῷ Ε τῇ πρὸς τῷ Θ. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΘ τῷ ΓΕ· καὶ τὰς περὶ τὰς ἴσας γωνίας αὐτῶν πλευρὰς ἀνάλογον ἔχει· ὅμοιον ἄρα ἐστὶ τὸ ΑΘ εὐθύγραμμον τῷ ΓΕ εὐθυγράμμῳ.

Ἀπὸ τῆς δοθείσης ἄρα εὐθείας τῆς ΑΒ τῷ δοθέντι εὐθυγράμμῳ τῷ ΓΕ ὁμοιόν τε καὶ ὁμοίως κείμενον εὐθύγραμμον ἀναγέγραπται τὸ ΑΘ· ὅπερ ἔδει ποιῆσαι.

ιθ΄. Τὰ ὅμοια τρίγωνα πρὸς ἄλληλα ἐν διπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν.

Ἔστω ὅμοια τρίγωνα τὰ ΑΒΓ, ΔΕΖ ἴσην ἔχοντα τὴν πρὸς τῷ Β γωνίαν τῇ πρὸς τῷ Ε, ὡς δὲ τὴν ΑΒ πρὸς τὴν ΒΓ, οὕτως τὴν ΔΕ πρὸς τὴν ΕΖ, ὥστε ὁμόλογον εἶναι τὴν ΒΓ τῇ ΕΖ· λέγω, ὅτι τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΔΕΖ τρίγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ.

Εἰλήφθω γὰρ τῶν ΒΓ, ΕΖ τρίτη ἀνάλογον ἡ ΒΗ, ὥστε εἶναι ὡς τὴν ΒΓ πρὸς τὴν ΕΖ, οὕτως τὴν ΕΖ πρὸς τὴν ΒΗ· καὶ ἐπεζεύχθω ἡ ΑΗ.

Ἐπεὶ οὖν ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως ἡ ΔΕ πρὸς τὴν ΕΖ, ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν ΔΕ, οὕτως ἡ ΒΓ πρὸς τὴν ΕΖ. ἀλλ’ ὡς ἡ ΒΓ πρὸς ΕΖ, οὕτως ἐστὶν ἡ ΕΖ πρὸς ΒΗ. καὶ ὡς ἄρα ἡ ΑΒ πρὸς ΔΕ, οὕτως ἡ ΕΖ πρὸς ΒΗ· τῶν ΑΒΗ, ΔΕΖ ἄρα τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας. ὧν δὲ μίαν μιᾷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἴσα ἐστὶν ἐκεῖνα. ἴσον ἄρα ἐστὶ τὸ ΑΒΗ τρίγωνον τῷ ΔΕΖ τριγώνῳ. καὶ ἐπεί ἐστιν ὡς ἡ ΒΓ πρὸς τὴν ΕΖ, οὕτως ἡ ΕΖ πρὸς τὴν ΒΗ, ἐὰν δὲ τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, ἡ

πρώτη πρὸς τὴν τρίτην διπλασίονα λόγον ἔχει ἥπερ πρὸς τὴν δευτέραν, ἡ ΒΓ ἄρα πρὸς τὴν ΒΗ διπλασίονα λόγον ἔχει ἥπερ ἡ ΓΒ πρὸς τὴν ΕΖ. ὡς δὲ ἡ ΓΒ πρὸς τὴν ΒΗ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΒΗ τρίγωνον· καὶ τὸ ΑΒΓ ἄρα τρίγωνον πρὸς τὸ ΑΒΗ διπλασίονα λόγον ἔχει ἥπερ ἡ ΒΓ πρὸς τὴν ΕΖ. ἴσον δὲ τὸ ΑΒΗ τρίγωνον τῷ ΔΕΖ τριγώνῳ· καὶ τὸ ΑΒΓ ἄρα τρίγωνον πρὸς τὸ ΔΕΖ τρίγωνον διπλασίονα λόγον ἔχει ἥπερ ἡ ΒΓ πρὸς τὴν ΕΖ.

Τὰ ἄρα ὅμοια τρίγωνα πρὸς ἄλληλα ἐν διπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν· [ὅπερ ἔδει δεῖξαι].

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι, ἐὰν τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, ἔστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης εἶδος πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον [ἐπείπερ ἐδείχθη, ὡς ἡ ΓΒ πρὸς ΒΗ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΒΗ τρίγωνον, τουτέστι τὸ ΔΕΖ]· ὅπερ ἔδει δεῖξαι.

κ΄. Τὰ ὅμοια πολύγωνα εἴς τε ὅμοια τρίγωνα διαιρεῖται καὶ εἰς ἴσα τὸ πλῆθος καὶ ὁμόλογα τοῖς ὅλοις, καὶ τὸ πολύγωνον πρὸς τὸ πολύγωνον διπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν.

Ἔστω ὅμοια πολύγωνα τὰ ΑΒΓΔΕ, ΖΗΘΚΛ, ὁμόλογος δὲ ἔστω ἡ ΑΒ τῇ ΖΗ· λέγω, ὅτι τὰ ΑΒΓΔΕ, ΖΗΘΚΛ πολύγωνα εἴς τε ὅμοια τρίγωνα διαιρεῖται καὶ εἰς ἴσα τὸ πλῆθος καὶ ὁμόλογα τοῖς ὅλοις, καὶ τὸ ΑΒΓΔΕ πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον διπλασίονα λόγον ἔχει ἥπερ ἡ ΑΒ πρὸς τὴν ΖΗ.

Ἐπεζεύχθωσαν αἱ ΒΕ, ΕΓ, ΗΛ, ΛΘ.

Καὶ ἐπεὶ ὅμοιόν ἐστι τὸ ΑΒΓΔΕ πολύγωνον τῷ ΖΗΘΚΛ πολυγώνῳ, ἴση ἐστὶν ἡ ὑπὸ ΒΑΕ γωνία τῇ ὑπὸ ΗΖΛ. καί ἐστιν ὡς ἡ ΒΑ πρὸς ΑΕ, οὕτως ἡ ΗΖ πρὸς ΖΛ. ἐπεὶ οὖν δύο τρίγωνά ἐστι τὰ ΑΒΕ, ΖΗΛ μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχοντα, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΕ τρίγωνον τῷ ΖΗΛ τριγώνῳ· ὥστε καὶ ὅμοιον· ἴση ἄρα ἐστὶν ἡ

ὑπὸ ΑΒΕ γωνία τῇ ὑπὸ ΖΗΛ. ἔστι δὲ καὶ ὅλη ἡ ὑπὸ ΑΒΓ ὅλῃ τῇ ὑπὸ ΖΗΘ ἴση διὰ τὴν ὁμοιότητα τῶν πολυγώνων· λοιπὴ ἄρα ἡ ὑπὸ ΕΒΓ γωνία τῇ ὑπὸ ΛΗΘ ἐστιν ἴση. καὶ ἐπεὶ διὰ τὴν ὁμοιότητα τῶν ΑΒΕ, ΖΗΛ τριγώνων ἐστὶν ὡς ἡ ΕΒ πρὸς ΒΑ, οὕτως ἡ ΛΗ πρὸς ΗΖ, ἀλλὰ μὴν καὶ διὰ τὴν ὁμοιότητα τῶν πολυγώνων ἐστὶν ὡς ἡ ΑΒ πρὸς ΒΓ, οὕτως ἡ ΖΗ πρὸς ΗΘ, δι’ ἴσου ἄρα ἐστὶν ὡς ἡ ΕΒ πρὸς ΒΓ, οὕτως ἡ ΛΗ πρὸς ΗΘ, καὶ περὶ τὰς ἴσας γωνίας τὰς ὑπὸ ΕΒΓ, ΛΗΘ αἱ πλευραὶ ἀνάλογόν εἰσιν· ἰσογώνιον ἄρα ἐστὶ τὸ ΕΒΓ τρίγωνον τῷ ΛΗΘ τριγώνῳ· ὥστε καὶ ὅμοιόν ἐστι τὸ ΕΒΓ τρίγωνον τῷ ΛΗΘ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΕΓΔ τρίγωνον ὅμοιόν ἐστι τῷ ΛΘΚ τριγώνῳ. τὰ ἄρα ὅμοια πολύγωνα τὰ ΑΒΓΔΕ, ΖΗΘΚΛ εἴς τε ὅμοια τρίγωνα διῄρηται καὶ εἰς ἴσα τὸ πλῆθος.

Λέγω, ὅτι καὶ ὁμόλογα τοῖς ὅλοις, τουτέστιν ὥστε ἀνάλογον εἶναι τὰ τρίγωνα, καὶ ἡγούμενα μὲν εἶναι τὰ ΑΒΕ, ΕΒΓ, ΕΓΔ, ἑπόμενα δὲ αὐτῶν τὰ ΖΗΛ, ΛΗΘ, ΛΘΚ, καὶ ὅτι τὸ ΑΒΓΔΕ πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἡ ΑΒ πρὸς τὴν ΖΗ.

Ἐπεζεύχθωσαν γὰρ αἱ ΑΓ, ΖΘ. καὶ ἐπεὶ διὰ τὴν ὁμοιότητα τῶν πολυγώνων ἴση ἐστὶν ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΖΗΘ, καί ἐστιν ὡς ἡ ΑΒ πρὸς ΒΓ, οὕτως ἡ ΖΗ πρὸς ΗΘ, ἰσογώνιόν ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΖΗΘ τριγώνῳ· ἴση ἄρα ἐστὶν ἡ μὲν ὑπὸ ΒΑΓ γωνία τῇ ὑπὸ ΗΖΘ, ἡ δὲ ὑπὸ ΒΓΑ τῇ ὑπὸ ΗΘΖ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΒΑΜ γωνία τῇ ὑπὸ ΗΖΝ, ἔστι δὲ καὶ ἡ ὑπὸ ΑΒΜ τῇ ὑπὸ ΖΗΝ ἴση, καὶ λοιπὴ ἄρα ἡ ὑπὸ ΑΜΒ λοιπῇ τῇ ὑπὸ ΖΝΗ ἴση ἐστίν· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΜ τρίγωνον τῷ ΖΗΝ τριγώνῳ. ὁμοίως δὴ δείξομεν, ὅτι καὶ τὸ ΒΜΓ τρίγωνον ἰσογώνιόν ἐστι τῷ ΗΝΘ τριγώνῳ. ἀνάλογον ἄρα ἐστίν, ὡς μὲν ἡ ΑΜ πρὸς ΜΒ, οὕτως ἡ ΖΝ πρὸς ΝΗ, ὡς δὲ ἡ ΒΜ πρὸς ΜΓ, οὕτως ἡ ΗΝ πρὸς ΝΘ· ὥστε καὶ δι’ ἴσου, ὡς ἡ ΑΜ πρὸς ΜΓ, οὕτως ἡ ΖΝ πρὸς ΝΘ. ἀλλ’ ὡς ἡ ΑΜ πρὸς ΜΓ, οὕτως τὸ ΑΒΜ [τρίγωνον] πρὸς τὸ ΜΒΓ, καὶ τὸ ΑΜΕ πρὸς τὸ ΕΜΓ· πρὸς ἄλληλα γάρ εἰσιν ὡς αἱ

βάσεις. καὶ ὡς ἄρα ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα· ὡς ἄρα τὸ ΑΜΒ τρίγωνον πρὸς τὸ ΒΜΓ, οὕτως τὸ ΑΒΕ πρὸς τὸ ΓΒΕ.ἀλλ’ ὡς τὸ ΑΜΒ πρὸς τὸ ΒΜΓ, οὕτως ἡ ΑΜ πρὸς ΜΓ· καὶ ὡς ἄρα ἡ ΑΜ πρὸς ΜΓ, οὕτως τὸ ΑΒΕ τρίγωνον πρὸς τὸ ΕΒΓ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ὡς ἡ ΖΝ πρὸς ΝΘ, οὕτως τὸ ΖΗΛ τρίγωνον πρὸς τὸ ΗΛΘ τρίγωνον. καί ἐστιν ὡς ἡ ΑΜ πρὸς ΜΓ, οὕτως ἡ ΖΝ πρὸς ΝΘ· καὶ ὡς ἄρα τὸ ΑΒΕ τρίγωνον πρὸς τὸ ΒΕΓ τρίγωνον, οὕτως τὸ ΖΗΛ τρίγωνον πρὸς τὸ ΗΛΘ τρίγωνον, καὶ ἐναλλὰξ, ὡς τὸ ΑΒΕ τρίγωνον πρὸς τὸ ΖΗΛ τρίγωνον, οὕτως τὸ ΒΕΓ τρίγωνον πρὸς τὸ ΗΛΘ τρίγωνον. ὁμοίως δὴ δείξομεν ἐπιζευχθεισῶν τῶν ΒΔ, ΗΚ, ὅτι καὶ ὡς τὸ ΒΕΓ τρίγωνον πρὸς τὸ ΛΗΘ τρίγωνον, οὕτως τὸ ΕΓΔ τρίγωνον πρὸς τὸ ΛΘΚ τρίγωνον. καὶ ἐπεί ἐστιν ὡς τὸ ΑΒΕ τρίγωνον πρὸς τὸ ΖΗΛ τρίγωνον, οὕτως τὸ ΕΒΓ πρὸς τὸ ΛΗΘ, καὶ ἔτι τὸ ΕΓΔ πρὸς τὸ ΛΘΚ, καὶ ὡς ἄρα ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα· ἔστιν ἄρα ὡς τὸ ΑΒΕ τρίγωνον πρὸς τὸ ΖΗΛ τρίγωνον, οὕτως τὸ ΑΒΓΔΕ πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον. ἀλλὰ τὸ ΑΒΕ τρίγωνον πρὸς τὸ ΖΗΛ τρίγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ΑΒ ὁμόλογος πλευρὰ πρὸς τὴν ΖΗ ὁμόλογον πλευράν· τὰ γὰρ ὅμοια τρίγωνα ἐν διπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν. καὶ τὸ ΑΒΓΔΕ ἄρα πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ΑΒ ὁμόλογος πλευρὰ πρὸς τὴν ΖΗ ὁμόλογον πλευράν.

Τὰ ἄρα ὅμοια πολύγωνα εἴς τε ὅμοια τρίγωνα διαιρεῖται καὶ εἰς ἴσα τὸ πλῆθος καὶ ὁμόλογα τοῖς ὅλοις, καὶ τὸ πολύγωνον πρὸς τὸ πολύγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν· [ὅπερ ἔδει δεῖξαι].

Πόρισμα α΄.

Ὡσαύτως δὲ καὶ ἐπὶ τῶν [ὁμοίων] τετραπλεύρων δειχθήσεται, ὅτι ἐν διπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν. ἐδείχθη δὲ καὶ ἐπὶ τῶν

τριγώνων· ὥστε καὶ καθόλου τὰ ὅμοια εὐθύγραμμα σχήματα πρὸς ἄλληλα ἐν διπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν. ὅπερ ἔδει δεῖξαι.

[Πόρισμα β΄.

Καὶ ἐὰν τῶν ΑΒ, ΖΗ τρίτην ἀνάλογον λάβωμεν τὴν Ξ, ἡ ΒΑ πρὸς τὴν Ξ διπλασίονα λόγον ἔχει ἥπερ ἡ ΑΒ πρὸς τὴν ΖΗ. ἔχει δὲ καὶ τὸ πολύγωνον πρὸς τὸ πολύγωνον ἢ τὸ τετράπλευρον πρὸς τὸ τετράπλευρον διπλασίονα λόγον ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἡ ΑΒ πρὸς τὴν ΖΗ· ἐδείχθη δὲ τοῦτο καὶ ἐπὶ τῶν τριγώνων· ὥστε καὶ καθόλου φανερόν, ὅτι, ἐὰν τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, ἔσται ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης εἶδος πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον].

κα΄. Τὰ τῷ αὐτῷ εὐθυγράμμῳ ὅμοια καὶ ἀλλήλοις ἐστὶν ὅμοια.

Ἔστω γὰρ ἑκάτερον τῶν Α, Β εὐθυγράμμων τῷ Γ ὅμοιον· λέγω, ὅτι καὶ τὸ Α τῷ Β ἐστιν ὅμοιον.

Ἐπεὶ γὰρ ὅμοιόν ἐστι τὸ Α τῷ Γ, ἰσογώνιόν τέ ἐστιν αὐτῷ καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει. πάλιν, ἐπεὶ ὅμοιόν ἐστι τὸ Β τῷ Γ, ἰσογώνιόν τέ ἐστιν αὐτῷ καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει. ἑκάτερον ἄρα τῶν Α, Β τῷ Γ ἰσογώνιόν τέ ἐστι καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει [ὥστε καὶ τὸ Α τῷ Β ἰσογώνιόν τέ ἐστι καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει]. ὅμοιον ἄρα ἐστὶ τὸ Α τῷ Β· ὅπερ ἔδει δεῖξαι.

κβ΄. Ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, καὶ τὰ ἀπ᾽ αὐτῶν εὐθύγραμμα ὅμοιά τε καὶ ὁμοίως ἀναγεγραμμένα ἀνάλογον ἔσται· κἂν τὰ ἀπ᾽ αὐτῶν εὐθύγραμμα ὅμοιά τε καὶ ὁμοίως ἀναγεγραμμένα ἀνάλογον ᾖ, καὶ αὐταὶ αἱ εὐθεῖαι ἀνάλογον ἔσονται.

Ἔστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἱ ΑΒ, ΓΔ, ΕΖ, ΗΘ, ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ, καὶ ἀναγεγράφθωσαν ἀπὸ μὲν τῶν ΑΒ, ΓΔ ὅμοιά τε καὶ ὁμοίως κείμενα εὐθύγραμμα τὰ ΚΑΒ, ΛΓΔ, ἀπὸ δὲ τῶν ΕΖ, ΗΘ ὅμοιά τε καὶ ὁμοίως κείμενα εὐθύγραμμα τὰ ΜΖ, ΝΘ· λέγω, ὅτι ἐστὶν ὡς τὸ ΚΑΒ πρὸς τὸ ΛΓΔ, οὕτως τὸ ΜΖ πρὸς τὸ ΝΘ.

Εἰλήφθω γὰρ τῶν μὲν ΑΒ, ΓΔ τρίτη ἀνάλογον ἡ Ξ, τῶν δὲ ΕΖ, ΗΘ τρίτη ἀνάλογον ἡ Ο. καὶ ἐπεί ἐστιν ὡς μὲν ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ, ὡς δὲ ἡ ΓΔ πρὸς τὴν Ξ, οὕτως ἡ ΗΘ πρὸς τὴν Ο, δι’ ἴσου ἄρα ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν Ξ, οὕτως ἡ ΕΖ πρὸς τὴν Ο. ἀλλ’ ὡς μὲν ἡ ΑΒ πρὸς τὴν Ξ, οὕτως [καὶ] τὸ ΚΑΒ πρὸς τὸ ΛΓΔ, ὡς δὲ ἡ ΕΖ πρὸς τὴν Ο, οὕτως τὸ ΜΖ πρὸς τὸ ΝΘ· καὶ ὡς ἄρα τὸ ΚΑΒ πρὸς τὸ ΛΓΔ, οὕτως τὸ ΜΖ πρὸς τὸ ΝΘ.

Ἀλλὰ δὴ ἔστω ὡς τὸ ΚΑΒ πρὸς τὸ ΛΓΔ, οὕτως τὸ ΜΖ πρὸς τὸ ΝΘ· λέγω, ὅτι ἐστὶ καὶ ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ. εἰ γὰρ μή ἐστιν, ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ, ἔστω ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΠΡ, καὶ ἀναγεγράφθω ἀπὸ τῆς ΠΡ ὁποτέρῳ τῶν ΜΖ, ΝΘ ὅμοιόν τε καὶ ὁμοίως κείμενον εὐθύγραμμον τὸ ΣΡ.

Ἐπεὶ οὖν ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΠΡ, καὶ ἀναγέγραπται ἀπὸ μὲν τῶν ΑΒ, ΓΔ ὅμοιά τε καὶ ὁμοίως κείμενα τὰ ΚΑΒ, ΛΓΔ, ἀπὸ δὲ τῶν ΕΖ, ΠΡ ὅμοιά τε καὶ ὁμοίως κείμενα τὰ ΜΖ, ΣΡ, ἔστιν ἄρα ὡς τὸ ΚΑΒ πρὸς τὸ ΛΓΔ, οὕτως τὸ ΜΖ πρὸς τὸ ΣΡ. ὑπόκειται δὲ καὶ ὡς τὸ ΚΑΒ πρὸς τὸ ΛΓΔ, οὕτως τὸ ΜΖ πρὸς τὸ ΝΘ· καὶ ὡς ἄρα τὸ ΜΖ πρὸς τὸ ΣΡ, οὕτως τὸ ΜΖ πρὸς τὸ ΝΘ. τὸ ΜΖ ἄρα πρὸς ἑκάτερον τῶν ΝΘ, ΣΡ τὸν αὐτὸν ἔχει λόγον· ἴσον ἄρα ἐστὶ τὸ ΝΘ τῷ ΣΡ. ἔστι δὲ αὐτῷ καὶ ὅμοιον καὶ ὁμοίως κείμενον· ἴση ἄρα ἡ ΗΘ τῇ ΠΡ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΠΡ, ἴση δὲ ἡ ΠΡ τῇ ΗΘ, ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ.

Ἐὰν ἄρα τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, καὶ τὰ ἀπ’ αὐτῶν εὐθύγραμμα ὅμοιά τε καὶ ὁμοίως ἀναγεγραμμένα ἀνάλογον ἔσται· κἂν τὰ ἀπ’ αὐτῶν

εὐθύγραμμα ὅμοιά τε καὶ ὁμοίως ἀναγεγραμμένα ἀνάλογον ᾖ, καὶ αὐταὶ αἱ εὐθεῖαι ἀνάλογον ἔσονται· ὅπερ ἔδει δεῖξαι.

[Λῆμμα]

[Ὅτι δέ, ἐὰν εὐθύγραμμα ἴσα ᾖ καὶ ὅμοια, αἱ ὁμόλογοι αὐτῶν πλευραὶ ἴσαι ἀλλήλαις εἰσίν, δείξομεν οὕτως.

Ἔστω ἴσα καὶ ὅμοια εὐθύγραμμα τὰ ΝΘ, ΣΡ, καὶ ἔστω ὡς ἡ ΘΗ πρὸς τὴν ΗΝ, οὕτως ἡ ΡΠ πρὸς τὴν ΠΣ· λέγω, ὅτι ἴση ἐστὶν ἡ ΡΠ τῇ ΘΗ.

Εἰ γὰρ ἄνισοί εἰσιν, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ ΡΠ τῆς ΘΗ. καὶ ἐπεί ἐστιν ὡς ἡ ΡΠ πρὸς ΠΣ, οὕτως ἡ ΘΗ πρὸς τὴν ΗΝ, καὶ ἐναλλάξ, ὡς ἡ ΡΠ πρὸς τὴν ΘΗ, οὕτως ἡ ΠΣ πρὸς τὴν ΗΝ, μείζων δὲ ἡ ΠΡ τῆς ΘΗ, μείζων ἄρα καὶ ἡ ΠΣ τῆς ΗΝ· ὥστε καὶ τὸ ΡΣ μεῖζόν ἐστι τοῦ ΘΝ. ἀλλὰ καὶ ἴσον· ὅπερ ἀδύνατον. οὐκ ἄρα ἄνισός ἐστιν ἡ ΠΡ τῇ ΗΘ· ἴση ἄρα· ὅπερ ἔδει δεῖξαι.]

κγ΄. Τὰ ἰσογώνια παραλληλόγραμμα πρὸς ἄλληλα λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν.

Ἔστω ἰσογώνια παραλληλόγραμμα τὰ ΑΓ, ΓΖ ἴσην ἔχοντα τὴν ὑπὸ ΒΓΔ γωνίαν τῇ ὑπὸ ΕΓΗ· λέγω, ὅτι τὸ ΑΓ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν.

Κείσθω γὰρ ὥστε ἐπ’ εὐθείας εἶναι τὴν ΒΓ τῇ ΓΗ· ἐπ’ εὐθείας ἄρα ἐστὶ καὶ ἡ ΔΓ τῇ ΓΕ. καὶ συμπεπληρώσθω τὸ ΔΗ παραλληλόγραμμον, καὶ ἐκκείσθω τις εὐθεῖα ἡ Κ, καὶ γεγονέτω ὡς μὲν ἡ ΒΓ πρὸς τὴν ΓΗ, οὕτως ἡ Κ πρὸς τὴν Λ, ὡς δὲ ἡ ΔΓ πρὸς τὴν ΓΕ, οὕτως ἡ Λ πρὸς τὴν Μ.

Οἱ ἄρα λόγοι τῆς τε Κ πρὸς τὴν Λ καὶ τῆς Λ πρὸς τὴν Μ οἱ αὐτοί εἰσι τοῖς λόγοις τῶν πλευρῶν, τῆς τε ΒΓ πρὸς τὴν ΓΗ καὶ τῆς ΔΓ πρὸς τὴν ΓΕ. ἀλλ’ ὁ τῆς Κ πρὸς Μ λόγος σύγκειται ἔκ τε τοῦ τῆς Κ πρὸς Λ λόγου καὶ τοῦ τῆς Λ πρὸς Μ· ὥστε καὶ ἡ Κ πρὸς τὴν Μ λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν. καὶ ἐπεί ἐστιν ὡς ἡ ΒΓ πρὸς τὴν ΓΗ, οὕτως τὸ ΑΓ

παραλληλόγραμμον πρὸς τὸ ΓΘ, ἀλλ’ ὡς ἡ ΒΓ πρὸς τὴν ΓΗ, οὕτως ἡ Κ πρὸς τὴν Λ, καὶ ὡς ἄρα ἡ Κ πρὸς τὴν Λ, οὕτως τὸ ΑΓ πρὸς τὸ ΓΘ. πάλιν, ἐπεί ἐστιν ὡς ἡ ΔΓ πρὸς τὴν ΓΕ, οὕτως τὸ ΓΘ παραλληλόγραμμον πρὸς τὸ ΓΖ, ἀλλ’ ὡς ἡ ΔΓ πρὸς τὴν ΓΕ, οὕτως ἡ Λ πρὸς τὴν Μ, καὶ ὡς ἄρα ἡ Λ πρὸς τὴν Μ, οὕτως τὸ ΓΘ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον. ἐπεὶ οὖν ἐδείχθη, ὡς μὲν ἡ Κ πρὸς τὴν Λ, οὕτως τὸ ΑΓ παραλληλόγραμμον πρὸς τὸ ΓΘ παραλληλόγραμμον, ὡς δὲ ἡ Λ πρὸς τὴν Μ, οὕτως τὸ ΓΘ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον, δι’ ἴσου ἄρα ἐστὶν ὡς ἡ Κ πρὸς τὴν Μ, οὕτως τὸ ΑΓ πρὸς τὸ ΓΖ παραλληλόγραμμον. ἡ δὲ Κ πρὸς τὴν Μ λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν· καὶ τὸ ΑΓ ἄρα πρὸς τὸ ΓΖ λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν.

Τὰ ἄρα ἰσογώνια παραλληλόγραμμα πρὸς ἄλληλα λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν· ὅπερ ἔδει δεῖξαι.

κδ΄. Παντὸς παραλληλογράμμου τὰ περὶ τὴν διάμετρον παραλληλόγραμμα ὅμοιά ἐστι τῷ τε ὅλῳ καὶ ἀλλήλοις.

Ἔστω παραλληλόγραμμον τὸ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἡ ΑΓ, περὶ δὲ τὴν ΑΓ παραλληλόγραμμα ἔστω τὰ ΕΗ, ΘΚ· λέγω, ὅτι ἑκάτερον τῶν ΕΗ, ΘΚ παραλληλογράμμων ὅμοιόν ἐστι ὅλῳ τῷ ΑΒΓΔ καὶ ἀλλήλοις.

Ἐπεὶ γὰρ τριγώνου τοῦ ΑΒΓ παρὰ μίαν τῶν πλευρῶν τὴν ΒΓ ἦκται ἡ ΕΖ, ἀνάλογόν ἐστιν ὡς ἡ ΒΕ πρὸς τὴν ΕΑ, οὕτως, ἡ ΓΖ πρὸς τὴν ΖΑ. πάλιν, ἐπεὶ τριγώνου τοῦ ΑΓΔ παρὰ μίαν τὴν ΓΔ ἦκται ἡ ΖΗ, ἀνάλογόν ἐστιν ὡς ἡ ΓΖ πρὸς τὴν ΖΑ, οὕτως ἡ ΔΗ πρὸς τὴν ΗΑ. ἀλλ’ ὡς ἡ ΓΖ πρὸς τὴν ΖΑ, οὕτως ἐδείχθη καὶ ἡ ΒΕ πρὸς τὴν ΕΑ· καὶ ὡς ἄρα ἡ ΒΕ πρὸς τὴν ΕΑ, οὕτως ἡ ΔΗ πρὸς τὴν ΗΑ, καὶ συνθέντι ἄρα ὡς ἡ ΒΑ πρὸς ΑΕ, οὕτως ἡ ΔΑ πρὸς ΑΗ, καὶ ἐναλλὰξ ὡς ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΗ. τῶν ἄρα ΑΒΓΔ, ΕΗ παραλληλογράμμων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὴν κοινὴν γωνίαν τὴν ὑπὸ ΒΑΔ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΗΖ τῇ ΔΓ, ἴση ἐστὶν ἡ μὲν ὑπὸ

ΑΖΗ γωνία τῇ ὑπὸ ΔΓΑ· καὶ κοινὴ τῶν δύο τριγώνων τῶν ΑΔΓ, ΑΗΖ ἡ ὑπὸ ΔΑΓ γωνία· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΔΓ τρίγωνον τῷ ΑΗΖ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΑΓΒ τρίγωνον ἰσογώνιόν ἐστι τῷ ΑΖΕ τριγώνῳ, καὶ ὅλον τὸ ΑΒΓΔ παραλληλόγραμμον τῷ ΕΗ παραλληλογράμμῳ ἰσογώνιόν ἐστιν. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΑΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΑΗ πρὸς τὴν ΗΖ, ὡς δὲ ἡ ΔΓ πρὸς τὴν ΓΑ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΑ, ὡς δὲ ἡ ΑΓ πρὸς τὴν ΓΒ, οὕτως ἡ ΑΖ πρὸς τὴν ΖΕ, καὶ ἔτι ὡς ἡ ΓΒ πρὸς τὴν ΒΑ, οὕτως ἡ ΖΕ πρὸς τὴν ΕΑ. καὶ ἐπεὶ ἐδείχθη ὡς μὲν ἡ ΔΓ πρὸς τὴν ΓΑ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΑ, ὡς δὲ ἡ ΑΓ πρὸς τὴν ΓΒ, οὕτως ἡ ΑΖ πρὸς τὴν ΖΕ, δι’ ἴσου ἄρα ἐστὶν ὡς ἡ ΔΓ πρὸς τὴν ΓΒ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΕ. τῶν ἄρα ΑΒΓΔ, ΕΗ παραλληλογράμμων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· ὅμοιον ἄρα ἐστὶ τὸ ΑΒΓΔ παραλληλόγραμμον τῷ ΕΗ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ τὸ ΑΒΓΔ παραλληλόγραμμον καὶ τῷ ΚΘ παραλληλογράμμῳ ὅμοιόν ἐστιν· ἑκάτερον ἄρα τῶν ΕΗ, ΘΚ παραλληλογράμμων τῷ ΑΒΓΔ [παραλληλογράμμῳ] ὅμοιόν ἐστιν. τὰ δὲ τῷ αὐτῷ εὐθυγράμμῳ ὅμοια καὶ ἀλλήλοις ἐστὶν ὅμοια· καὶ τὸ ΕΗ ἄρα παραλληλόγραμμον τῷ ΘΚ παραλληλογράμμῳ ὅμοιόν ἐστιν.

Παντὸς ἄρα παραλληλογράμμου τὰ περὶ τὴν διάμετρον παραλληλόγραμμα ὅμοιά ἐστι τῷ τε ὅλῳ καὶ ἀλλήλοις· ὅπερ ἔδει δεῖξαι.

κε΄. Τῷ δοθέντι εὐθυγράμμῳ ὅμοιον καὶ ἄλλῳ τῷ δοθέντι ἴσον τὸ αὐτὸ συστήσασθαι.

Ἔστω τὸ μὲν δοθὲν εὐθύγραμμον, ᾧ δεῖ ὅμοιον συστήσασθαι, τὸ ΑΒΓ, ᾧ δὲ δεῖ ἴσον, τὸ Δ· δεῖ δὴ τῷ μὲν ΑΒΓ ὅμοιον, τῷ δὲ Δ ἴσον τὸ αὐτὸ συστήσασθαι.

Παραβεβλήσθω γὰρ παρὰ μὲν τὴν ΒΓ τῷ ΑΒΓ τριγώνῳ ἴσον παραλληλόγραμμον τὸ ΒΕ, παρὰ δὲ τὴν ΓΕ τῷ Δ ἴσον παραλληλόγραμμον τὸ ΓΜ ἐν γωνίᾳ τῇ ὑπὸ ΖΓΕ, ἥ ἐστιν ἴση τῇ ὑπὸ ΓΒΛ. ἐπ’ εὐθείας ἄρα ἐστὶν ἡ μὲν ΒΓ τῇ ΓΖ, ἡ δὲ ΛΕ τῇ ΕΜ. καὶ εἰλήφθω τῶν ΒΓ, ΓΖ μέση ἀνάλογον ἡ

ΗΘ, καὶ ἀναγεγράφθω ἀπὸ τῆς ΗΘ τῷ ΑΒΓ ὅμοιόν τε καὶ ὁμοίως κείμενον τὸ ΚΗΘ.

Καὶ ἐπεί ἐστιν ὡς ἡ ΒΓ πρὸς τὴν ΗΘ, οὕτως ἡ ΗΘ πρὸς τὴν ΓΖ, ἐὰν δὲ τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, ἔστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης εἶδος πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον, ἔστιν ἄρα ὡς ἡ ΒΓ πρὸς τὴν ΓΖ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΚΗΘ τρίγωνον. ἀλλὰ καὶ ὡς ἡ ΒΓ πρὸς τὴν ΓΖ, οὕτως τὸ ΒΕ παραλληλόγραμμον πρὸς τὸ ΕΖ παραλληλόγραμμον. καὶ ὡς ἄρα τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΚΗΘ τρίγωνον, οὕτως τὸ ΒΕ παραλληλόγραμμον πρὸς τὸ ΕΖ παραλληλόγραμμον· ἐναλλὰξ ἄρα ὡς τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΒΕ παραλληλόγραμμον, οὕτως τὸ ΚΗΘ τρίγωνον πρὸς τὸ ΕΖ παραλληλόγραμμον. ἴσον δὲ τὸ ΑΒΓ τρίγωνον τῷ ΒΕ παραλληλογράμμῳ· ἴσον ἄρα καὶ τὸ ΚΗΘ τρίγωνον τῷ ΕΖ παραλληλογράμμῳ. ἀλλὰ τὸ ΕΖ παραλληλόγραμμον τῷ Δ ἐστιν ἴσον· καὶ τὸ ΚΗΘ ἄρα τῷ Δ ἐστιν ἴσον. ἔστι δὲ τὸ ΚΗΘ καὶ τῷ ΑΒΓ ὅμοιον.

Τῷ ἄρα δοθέντι εὐθυγράμμῳ τῷ ΑΒΓ ὅμοιον καὶ ἄλλῳ τῷ δοθέντι τῷ Δ ἴσον τὸ αὐτὸ συνέσταται τὸ ΚΗΘ· ὅπερ ἔδει ποιῆσαι.

κς΄. Ἐὰν ἀπὸ παραλληλογράμμου παραλληλόγραμμον ἀφαιρεθῇ ὅμοιόν τε τῷ ὅλῳ καὶ ὁμοίως κείμενον κοινὴν γωνίαν ἔχον αὐτῷ, περὶ τὴν αὐτὴν διάμετρόν ἐστι τῷ ὅλῳ.

Ἀπὸ γὰρ παραλληλογράμμου τοῦ ΑΒΓΔ παραλληλόγραμμον ἀφῃρήσθω τὸ ΑΖ ὅμοιον τῷ ΑΒΓΔ καὶ ὁμοίως κείμενον κοινὴν γωνίαν ἔχον αὐτῷ τὴν ὑπὸ ΔΑΒ· λέγω, ὅτι περὶ τὴν αὐτὴν διάμετρόν ἐστι τὸ ΑΒΓΔ τῷ ΑΖ.

Μὴ γάρ, ἀλλ’ εἰ δυνατόν, ἔστω [αὐτῶν] διάμετρος ἡ ΑΘΓ, καὶ ἐκβληθεῖσα ἡ ΗΖ διήχθω ἐπὶ τὸ Θ, καὶ ἤχθω διὰ τοῦ Θ ὁποτέρᾳ τῶν ΑΔ, ΒΓ παράλληλος ἡ ΘΚ.

Ἐπεὶ οὖν περὶ τὴν αὐτὴν διάμετρόν ἐστι τὸ ΑΒΓΔ τῷ ΚΗ, ἔστιν ἄρα ὡς ἡ ΔΑ πρὸς τὴν ΑΒ, οὕτως ἡ ΗΑ πρὸς τὴν ΑΚ. ἔστι δὲ καὶ διὰ τὴν ὁμοιότητα τῶν ΑΒΓΔ, ΕΗ καὶ ὡς ἡ ΔΑ πρὸς τὴν ΑΒ, οὕτως ἡ ΗΑ πρὸς τὴν ΑΕ· καὶ ὡς ἄρα ἡ ΗΑ πρὸς τὴν ΑΚ, οὕτως ἡ ΗΑ πρὸς τὴν ΑΕ. ἡ ΗΑ ἄρα πρὸς ἑκατέραν τῶν ΑΚ, ΑΕ τὸν αὐτὸν ἔχει λόγον. ἴση ἄρα ἐστὶν ἡ ΑΕ τῇ ΑΚ ἡ ἐλάττων τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οὔκ ἐστι περὶ τὴν αὐτὴν διάμετρον τὸ ΑΒΓΔ τῷ ΑΖ· περὶ τὴν αὐτὴν ἄρα ἐστὶ διάμετρον τὸ ΑΒΓΔ παραλληλόγραμμον τῷ ΑΖ παραλληλογράμμῳ.

Ἐὰν ἄρα ἀπὸ παραλληλογράμμου παραλληλόγραμμον ἀφαιρεθῇ ὅμοιόν τε τῷ ὅλῳ καὶ ὁμοίως κείμενον κοινὴν γωνίαν ἔχον αὐτῷ, περὶ τὴν αὐτὴν διάμετρόν ἐστι τῷ ὅλῳ· ὅπερ ἔδει δεῖξαι.

κζ΄. Πάντων τῶν παρὰ τὴν αὐτὴν εὐθεῖαν παραβαλλομένων παραλληλογράμμων καὶ ἐλλειπόντων εἴδεσι παραλληλογράμμοις ὁμοίοις τε καὶ ὁμοίως κειμένοις τῷ ἀπὸ τῆς ἡμισείας ἀναγραφομένῳ μέγιστόν ἐστι τὸ ἀπὸ τῆς ἡμισείας παραβαλλόμενον [παραλληλόγραμμον] ὅμοιον ὂν τῷ ἐλλείμματι.

Ἔστω εὐθεῖα ἡ ΑΒ καὶ τετμήσθω δίχα κατὰ τὸ Γ, καὶ παραβεβλήσθω παρὰ τὴν ΑΒ εὐθεῖαν τὸ ΑΔ παραλληλόγραμμον ἐλλεῖπον εἴδει παραλληλογράμμῳ τῷ ΔΒ ἀναγραφέντι ἀπὸ τῆς ἡμισείας τῆς ΑΒ, τουτέστι τῆς ΓΒ· λέγω, ὅτι πάντων τῶν παρὰ τὴν ΑΒ παραβαλλομένων παραλληλογράμμων καὶ ἐλλειπόντων εἴδεσι [παραλληλογράμμοις] ὁμοίοις τε καὶ ὁμοίως κειμένοις τῷ ΔΒ μέγιστόν ἐστι τὸ ΑΔ. παραβεβλήσθω γὰρ παρὰ τὴν ΑΒ εὐθεῖαν τὸ ΑΖ παραλληλόγραμμον ἐλλεῖπον εἴδει παραλληλογράμμῳ τῷ ΖΒ ὁμοίῳ τε καὶ ὁμοίως κειμένῳ τῷ ΔΒ· λέγω, ὅτι μεῖζόν ἐστι τὸ ΑΔ τοῦ ΑΖ.

Ἐπεὶ γὰρ ὅμοιόν ἐστι τὸ ΔΒ παραλληλόγραμμον τῷ ΖΒ παραλληλογράμμῳ, περὶ τὴν αὐτήν εἰσι διάμετρον. ἤχθω αὐτῶν διάμετρος ἡ

ΔΒ, καὶ καταγεγράφθω τὸ σχῆμα.

Ἐπεὶ οὖν ἴσον ἐστὶ τὸ ΓΖ τῷ ΖΕ, κοινὸν δὲ τὸ ΖΒ, ὅλον ἄρα τὸ ΓΘ ὅλῳ τῷ ΚΕ ἐστιν ἴσον. ἀλλὰ τὸ ΓΘ τῷ ΓΗ ἐστιν ἴσον, ἐπεὶ καὶ ἡ ΑΓ τῇ ΓΒ. καὶ τὸ ΗΓ ἄρα τῷ ΕΚ ἐστιν ἴσον. κοινὸν προσκείσθω τὸ ΓΖ· ὅλον ἄρα τὸ ΑΖ τῷ ΛΜΝ γνώμονί ἐστιν ἴσον· ὥστε τὸ ΔΒ παραλληλόγραμμον, τουτέστι τὸ ΑΔ, τοῦ ΑΖ παραλληλογράμμου μεῖζόν ἐστιν.

Πάντων ἄρα τῶν παρὰ τὴν αὐτὴν εὐθεῖαν παραβαλλομένων παραλληλογράμμων καὶ ἐλλειπόντων εἴδεσι παραλληλογράμμοις ὁμοίοις τε καὶ ὁμοίως κειμένοις τῷ ἀπὸ τῆς ἡμισείας ἀναγραφομένῳ μέγιστόν ἐστι τὸ ἀπὸ τῆς ἡμισείας παραβληθέν· ὅπερ ἔδει δεῖξαι.

κη΄. Παρὰ τὴν δοθεῖσαν εὐθεῖαν τῷ δοθέντι εὐθυγράμμῳ ἴσον παραλληλόγραμμον παραβαλεῖν ἐλλεῖπον εἴδει παραλληλογράμμῳ ὁμοίῳ τῷ δοθέντι· δεῖ δὲ τὸ διδόμενον εὐθύγραμμον [ᾧ δεῖ ἴσον παραβαλεῖν] μὴ μεῖζον εἶναι τοῦ ἀπὸ τῆς ἡμισείας ἀναγραφομένου ὁμοίου τῷ ἐλλείμματι [τοῦ τε ἀπὸ τῆς ἡμισείας καὶ ᾧ δεῖ ὅμοιον ἐλλείπειν].

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ δοθὲν εὐθύγραμμον, ᾧ δεῖ ἴσον παρὰ τὴν ΑΒ παραβαλεῖν, τὸ Γ μὴ μεῖζον [ὂν] τοῦ ἀπὸ τῆς ἡμισείας τῆς ΑΒ ἀναγραφομένου ὁμοίου τῷ ἐλλείμματι, ᾧ δὲ δεῖ ὅμοιον ἐλλείπειν, τὸ Δ· δεῖ δὴ παρὰ τὴν δοθεῖσαν εὐθεῖαν τὴν ΑΒ τῷ δοθέντι εὐθυγράμμῳ τῷ Γ ἴσον παραλληλόγραμμον παραβαλεῖν ἐλλεῖπον εἴδει παραλληλογράμμῳ ὁμοίῳ ὄντι τῷ Δ.

Τετμήσθω ἡ ΑΒ δίχα κατὰ τὸ Ε σημεῖον, καὶ ἀναγεγράφθω ἀπὸ τῆς ΕΒ τῷ Δ ὅμοιον καὶ ὁμοίως κείμενον τὸ ΕΒΖΗ, καὶ συμπεπληρώσθω τὸ ΑΗ παραλληλόγραμμον.

Εἰ μὲν οὖν ἴσον ἐστὶ τὸ ΑΗ τῷ Γ, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν· παραβέβληται γὰρ παρὰ τὴν δοθεῖσαν εὐθεῖαν τὴν ΑΒ τῷ δοθέντι εὐθυγράμμῳ τῷ Γ ἴσον παραλληλόγραμμον τὸ ΑΗ ἐλλεῖπον εἴδει

παραλληλογράμμῳ τῷ ΗΒ ὁμοίῳ ὄντι τῷ Δ. εἰ δὲ οὔ, μεῖζον ἔστω τὸ ΘΕ τοῦ Γ. ἴσον δὲ τὸ ΘΕ τῷ ΗΒ· μεῖζον ἄρα καὶ τὸ ΗΒ τοῦ Γ. ᾧ δὴ μεῖζόν ἐστι τὸ ΗΒ τοῦ Γ, ταύτῃ τῇ ὑπεροχῇ ἴσον, τῷ δὲ Δ ὅμοιον καὶ ὁμοίως κείμενον τὸ αὐτὸ συνεστάτω τὸ ΚΛΜΝ. ἀλλὰ τὸ Δ τῷ ΗΒ [ἐστιν] ὅμοιον· καὶ τὸ ΚΜ ἄρα τῷ ΗΒ ἐστιν ὅμοιον. ἔστω οὖν ὁμόλογος ἡ μὲν ΚΛ τῇ ΗΕ, ἡ δὲ ΛΜ τῇ ΗΖ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΗΒ τοῖς Γ, ΚΜ, μεῖζον ἄρα ἐστὶ τὸ ΗΒ τοῦ ΚΜ· μείζων ἄρα ἐστὶ καὶ ἡ μὲν ΗΕ τῆς ΚΛ, ἡ δὲ ΗΖ τῆς ΛΜ. κείσθω τῇ μὲν ΚΛ ἴση ἡ ΗΞ, τῇ δὲ ΛΜ ἴση ἡ ΗΟ, καὶ συμπεπληρώσθω τὸ ΞΗΟΠ παραλληλόγραμμον· ἴσον ἄρα καὶ ὅμοιόν ἐστι [τὸ ΗΠ] τῷ ΚΜ [ἀλλὰ τὸ ΚΜ τῷ ΗΒ ὅμοιόν ἐστιν]. καὶ τὸ ΗΠ ἄρα τῷ ΗΒ ὅμοιόν ἐστιν· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὸ ΗΠ τῷ ΗΒ. ἔστω αὐτῶν διάμετρος ἡ ΗΠΒ, καὶ καταγεγράφθω τὸ σχῆμα.

Ἐπεὶ οὖν ἴσον ἐστὶ τὸ ΒΗ τοῖς Γ, ΚΜ, ὧν τὸ ΗΠ τῷ ΚΜ ἐστιν ἴσον, λοιπὸς ἄρα ὁ ΥΧΦ γνώμων λοιπῷ τῷ Γ ἴσος ἐστίν. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΟΡ τῷ ΞΣ, κοινὸν προσκείσθω τὸ ΠΒ· ὅλον ἄρα τὸ ΟΒ ὅλῳ τῷ ΞΒ ἴσον ἐστίν. ἀλλὰ τὸ ΞΒ τῷ ΤΕ ἐστιν ἴσον, ἐπεὶ καὶ πλευρὰ ἡ ΑΕ πλευρᾷ τῇ ΕΒ ἐστιν ἴση· καὶ τὸ ΤΕ ἄρα τῷ ΟΒ ἐστιν ἴσον. κοινὸν προσκείσθω τὸ ΞΣ· ὅλον ἄρα τὸ ΤΣ ὅλῳ τῷ ΦΧΥ γνώμονί ἐστιν ἴσον. ἀλλ’ ὁ ΦΧΥ γνώμων τῷ Γ ἐδείχθη ἴσος· καὶ τὸ ΤΣ ἄρα τῷ Γ ἐστιν ἴσον.

Παρὰ τὴν δοθεῖσαν ἄρα εὐθεῖαν τὴν ΑΒ τῷ δοθέντι εὐθυγράμμῳ τῷ Γ ἴσον παραλληλόγραμμον παραβέβληται τὸ ΣΤ ἐλλεῖπον εἴδει παραλληλογράμμῳ τῷ ΠΒ ὁμοίῳ ὄντι τῷ Δ [ἐπειδήπερ τὸ ΠΒ τῷ ΗΠ ὅμοιόν ἐστιν]· ὅπερ ἔδει ποιῆσαι.

κθ΄. Παρὰ τὴν δοθεῖσαν εὐθεῖαν τῷ δοθέντι εὐθυγράμμῳ ἴσον παραλληλόγραμμον παραβαλεῖν ὑπερβάλλον εἴδει παραλληλογράμμῳ ὁμοίῳ τῷ δοθέντι.

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ δοθὲν εὐθύγραμμον, ᾧ δεῖ ἴσον παρὰ τὴν ΑΒ παραβαλεῖν, τὸ Γ, ᾧ δὲ δεῖ ὅμοιον ὑπερβάλλειν, τὸ Δ· δεῖ δὴ παρὰ τὴν ΑΒ εὐθεῖαν τῷ Γ εὐθυγράμμῳ ἴσον παραλληλόγραμμον παραβαλεῖν ὑπερβάλλον εἴδει παραλληλογράμμῳ ὁμοίῳ τῷ Δ.

Τετμήσθω ἡ ΑΒ δίχα κατὰ τὸ Ε, καὶ ἀναγεγράφθω ἀπὸ τῆς ΕΒ τῷ Δ ὅμοιον καὶ ὁμοίως κείμενον παραλληλόγραμμον τὸ ΒΖ, καὶ συναμφοτέροις μὲν τοῖς ΒΖ, Γ ἴσον, τῷ δὲ Δ ὅμοιον καὶ ὁμοίως κείμενον τὸ αὐτὸ συνεστάτω τὸ ΗΘ. ὁμόλογος δὲ ἔστω ἡ μὲν ΚΘ τῇ ΖΛ, ἡ δὲ ΚΗ τῇ ΖΕ. καὶ ἐπεὶ μεῖζόν ἐστι τὸ ΗΘ τοῦ ΖΒ, μείζων ἄρα ἐστὶ καὶ ἡ μὲν ΚΘ τῆς ΖΛ, ἡ δὲ ΚΗ τῆς ΖΕ. ἐκβεβλήσθωσαν αἱ ΖΛ, ΖΕ, καὶ τῇ μὲν ΚΘ ἴση ἔστω ἡ ΖΛΜ, τῇ δὲ ΚΗ ἴση ἡ ΖΕΝ, καὶ συμπεπληρώσθω τὸ ΜΝ· τὸ ΜΝ ἄρα τῷ ΗΘ ἴσον τέ ἐστι καὶ ὅμοιον. ἀλλὰ τὸ ΗΘ τῷ ΕΛ ἐστιν ὅμοιον· καὶ τὸ ΜΝ ἄρα τῷ ΕΛ ὅμοιόν ἐστιν· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὸ ΕΛ τῷ ΜΝ. ἤχθω αὐτῶν διάμετρος ἡ ΖΞ, καὶ καταγεγράφθω τὸ σχῆμα.

Ἐπεὶ ἴσον ἐστὶ τὸ ΗΘ τοῖς ΕΛ, Γ, ἀλλὰ τὸ ΗΘ τῷ ΜΝ ἴσον ἐστίν, καὶ τὸ ΜΝ ἄρα τοῖς ΕΛ, Γ ἴσον ἐστίν. κοινὸν ἀφῃρήσθω τὸ ΕΛ· λοιπὸς ἄρα ὁ ΨΧΦ γνώμων τῷ Γ ἐστιν ἴσος. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΕ τῇ ΕΒ, ἴσον ἐστὶ καὶ τὸ ΑΝ τῷ ΝΒ, τουτέστι τῷ ΛΟ. κοινὸν προσκείσθω τὸ ΕΞ· ὅλον ἄρα τὸ ΑΞ ἴσον ἐστὶ τῷ ΦΧΨ γνώμονι. ἀλλὰ ὁ ΦΧΨ γνώμων τῷ Γ ἴσος ἐστίν· καὶ τὸ ΑΞ ἄρα τῷ Γ ἴσον ἐστίν.

Παρὰ τὴν δοθεῖσαν ἄρα εὐθεῖαν τὴν ΑΒ τῷ δοθέντι εὐθυγράμμῳ τῷ Γ ἴσον παραλληλόγραμμον παραβέβληται τὸ ΑΞ ὑπερβάλλον εἴδει παραλληλογράμμῳ τῷ ΠΟ ὁμοίῳ ὄντι τῷ Δ, ἐπεὶ καὶ τῷ ΕΛ ἐστιν ὅμοιον τὸ ΟΠ· ὅπερ ἔδει ποιῆσαι.

λ΄. Τὴν δοθεῖσαν εὐθεῖαν πεπερασμένην ἄκρον καὶ μέσον λόγον τεμεῖν.

Ἔστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ· δεῖ δὴ τὴν ΑΒ εὐθεῖαν ἄκρον καὶ μέσον λόγον τεμεῖν.

Ἀναγεγράφθω ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΒΓ, καὶ παραβεβλήσθω παρὰ τὴν ΑΓ τῇ ΒΓ ἴσον παραλληλόγραμμον τὸ ΓΔ ὑπερβάλλον εἴδει τῷ ΑΔ ὁμοίῳ τῷ ΒΓ.

Τετράγωνον δέ ἐστι τὸ ΒΓ· τετράγωνον ἄρα ἐστὶ καὶ τὸ ΑΔ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΒΓ τῷ ΓΔ, κοινὸν ἀφῃρήσθω τὸ ΓΕ· λοιπὸν ἄρα τὸ ΒΖ λοιπῷ τῷ ΑΔ ἐστιν ἴσον. ἔστι δὲ αὐτῷ καὶ ἰσογώνιον· τῶν ΒΖ, ΑΔ ἄρα ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· ἔστιν ἄρα ὡς ἡ ΖΕ πρὸς τὴν ΕΔ, οὕτως ἡ ΑΕ πρὸς τὴν ΕΒ. ἴση δὲ ἡ μὲν ΖΕ τῇ ΑΒ, ἡ δὲ ΕΔ τῇ ΑΕ. ἔστιν ἄρα ὡς ἡ ΒΑ πρὸς τὴν ΑΕ, οὕτως ἡ ΑΕ πρὸς τὴν ΕΒ. μείζων δὲ ἡ ΑΒ τῆς ΑΕ· μείζων ἄρα καὶ ἡ ΑΕ τῆς ΕΒ.

Ἡ ἄρα ΑΒ εὐθεῖα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ε, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστι τὸ ΑΕ· ὅπερ ἔδει ποιῆσαι.

λα΄. Ἐν τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς εἶδος ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν εἴδεσι τοῖς ὁμοίοις τε καὶ ὁμοίως ἀναγραφομένοις.

Ἔστω τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν ἔχον τὴν ὑπὸ ΒΑΓ γωνίαν· λέγω, ὅτι τὸ ἀπὸ τῆς ΒΓ εἶδος ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ εἴδεσι τοῖς ὁμοίοις τε καὶ ὁμοίως ἀναγραφομένοις.

Ἤχθω κάθετος ἡ ΑΔ.

Ἐπεὶ οὖν ἐν ὀρθογωνίῳ τριγώνῳ τῷ ΑΒΓ ἀπὸ τῆς πρὸς τῷ Α ὀρθῆς γωνίας ἐπὶ τὴν ΒΓ βάσιν κάθετος ἦκται ἡ ΑΔ, τὰ ΑΒΔ, ΑΔΓ πρὸς τῇ καθέτῳ τρίγωνα ὅμοιά ἐστι τῷ τε ὅλῳ τῷ ΑΒΓ καὶ ἀλλήλοις. καὶ ἐπεὶ ὅμοιόν ἐστι τὸ ΑΒΓ τῷ ΑΒΔ, ἔστιν ἄρα ὡς ἡ ΓΒ πρὸς τὴν ΒΑ, οὕτως ἡ ΑΒ πρὸς τὴν ΒΔ. καὶ ἐπεὶ τρεῖς εὐθεῖαι ἀνάλογόν εἰσιν, ἔστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης εἶδος πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον. ὡς ἄρα ἡ ΓΒ πρὸς τὴν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΓΒ εἶδος πρὸς τὸ ἀπὸ τῆς ΒΑ τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον. διὰ τὰ αὐτὰ

δὴ καὶ ὡς ἡ ΒΓ πρὸς τὴν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΒΓ εἶδος πρὸς τὸ ἀπὸ τῆς ΓΑ. ὥστε καὶ ὡς ἡ ΒΓ πρὸς τὰς ΒΔ, ΔΓ, οὕτως τὸ ἀπὸ τῆς ΒΓ εἶδος πρὸς τὰ ἀπὸ τῶν ΒΑ, ΑΓ τὰ ὅμοια καὶ ὁμοίως ἀναγραφόμενα. ἴση δὲ ἡ ΒΓ ταῖς ΒΔ, ΔΓ· ἴσον ἄρα καὶ τὸ ἀπὸ τῆς ΒΓ εἶδος τοῖς ἀπὸ τῶν ΒΑ, ΑΓ εἴδεσι τοῖς ὁμοίοις τε καὶ ὁμοίως ἀναγραφομένοις.

Ἐν ἄρα τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς εἶδος ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν εἴδεσι τοῖς ὁμοίοις τε καὶ ὁμοίως ἀναγραφομένοις· ὅπερ ἔδει δεῖξαι.

λβ΄. Ἐὰν δύο τρίγωνα συντεθῇ κατὰ μίαν γωνίαν τὰς δύο πλευρὰς ταῖς δυσὶ πλευραῖς ἀνάλογον ἔχοντα ὥστε τὰς ὁμολόγους αὐτῶν πλευρὰς καὶ παραλλήλους εἶναι, αἱ λοιπαὶ τῶν τριγώνων πλευραὶ ἐπ᾽ εὐθείας ἔσονται.

Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΓΕ τὰς δύο πλευρὰς τὰς ΒΑ, ΑΓ ταῖς δυσὶ πλευραῖς ταῖς ΔΓ, ΔΕ ἀνάλογον ἔχοντα, ὡς μὲν τὴν ΑΒ πρὸς τὴν ΑΓ, οὕτως τὴν ΔΓ πρὸς τὴν ΔΕ, παράλληλον δὲ τὴν μὲν ΑΒ τῇ ΔΓ, τὴν δὲ ΑΓ τῇ ΔΕ· λέγω, ὅτι ἐπ᾽ εὐθείας ἐστὶν ἡ ΒΓ τῇ ΓΕ.

Ἐπεὶ γὰρ παράλληλός ἐστιν ἡ ΑΒ τῇ ΔΓ, καὶ εἰς αὐτὰς ἐμπέπτωκεν εὐθεῖα ἡ ΑΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΒΑΓ, ΑΓΔ ἴσαι ἀλλήλαις εἰσίν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΓΔΕ τῇ ὑπὸ ΑΓΔ ἴση ἐστίν. ὥστε καὶ ἡ ὑπὸ ΒΑΓ τῇ ὑπὸ ΓΔΕ ἐστιν ἴση. καὶ ἐπεὶ δύο τρίγωνά ἐστι τὰ ΑΒΓ, ΔΓΕ μίαν γωνίαν τὴν πρὸς τῷ Α μιᾷ γωνίᾳ τῇ πρὸς τῷ Δ ἴσην ἔχοντα, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ὡς τὴν ΒΑ πρὸς τὴν ΑΓ, οὕτως τὴν ΓΔ πρὸς τὴν ΔΕ, ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΓΕ τριγώνῳ· ἴση ἄρα ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΓΕ. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΑΓΔ τῇ ὑπὸ ΒΑΓ ἴση· ὅλη ἄρα ἡ ὑπὸ ΑΓΕ δυσὶ ταῖς ὑπὸ ΑΒΓ, ΒΑΓ ἴση ἐστίν. κοινὴ προσκείσθω ἡ ὑπὸ ΑΓΒ· αἱ ἄρα ὑπὸ ΑΓΕ, ΑΓΒ ταῖς ὑπὸ ΒΑΓ, ΑΓΒ, ΓΒΑ ἴσαι εἰσίν. ἀλλ᾽ αἱ ὑπὸ ΒΑΓ, ΑΒΓ, ΑΓΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν· καὶ αἱ ὑπὸ ΑΓΕ, ΑΓΒ ἄρα δυσὶν ὀρθαῖς ἴσαι

εἰσίν. πρὸς δή τινι εὐθείᾳ τῇ ΑΓ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Γ δύο εὐθεῖαι αἱ ΒΓ, ΓΕ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας τὰς ὑπὸ ΑΓΕ, ΑΓΒ δυσὶν ὀρθαῖς ἴσας ποιοῦσιν· ἐπ᾽ εὐθείας ἄρα ἐστὶν ἡ ΒΓ τῇ ΓΕ.

Ἐὰν ἄρα δύο τρίγωνα συντεθῇ κατὰ μίαν γωνίαν τὰς δύο πλευρὰς ταῖς δυσὶ πλευραῖς ἀνάλογον ἔχοντα ὥστε τὰς ὁμολόγους αὐτῶν πλευρὰς καὶ παραλλήλους εἶναι, αἱ λοιπαὶ τῶν τριγώνων πλευραὶ ἐπ᾽ εὐθείας ἔσονται· ὅπερ ἔδει δεῖξαι.

λγ΄. Ἐν τοῖς ἴσοις κύκλοις αἱ γωνίαι τὸν αὐτὸν ἔχουσι λόγον ταῖς περιφερείαις, ἐφ᾽ ὧν βεβήκασιν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ὦσι βεβηκυῖαι.

Ἔστωσαν ἴσοι κύκλοι οἱ ΑΒΓ, ΔΕΖ, καὶ πρὸς μὲν τοῖς κέντροις αὐτῶν τοῖς Η, Θ γωνίαι ἔστωσαν αἱ ὑπὸ ΒΗΓ, ΕΘΖ, πρὸς δὲ ταῖς περιφερείαις αἱ ὑπὸ ΒΑΓ, ΕΔΖ· λέγω, ὅτι ἐστὶν ὡς ἡ ΒΓ περιφέρεια πρὸς τὴν ΕΖ περιφέρειαν, οὕτως ἥ τε ὑπὸ ΒΗΓ γωνία πρὸς τὴν ὑπὸ ΕΘΖ καὶ ἡ ὑπὸ ΒΑΓ πρὸς τὴν ὑπὸ ΕΔΖ.

Κείσθωσαν γὰρ τῇ μὲν ΒΓ περιφερείᾳ ἴσαι κατὰ τὸ ἑξῆς ὁσαιδηποτοῦν αἱ ΓΚ, ΚΛ, τῇ δὲ ΕΖ περιφερείᾳ ἴσαι ὁσαιδηποτοῦν αἱ ΖΜ, ΜΝ, καὶ ἐπεζεύχθωσαν αἱ ΗΚ, ΗΛ, ΘΜ, ΘΝ.

Ἐπεὶ οὖν ἴσαι εἰσὶν αἱ ΒΓ, ΓΚ, ΚΛ περιφέρειαι ἀλλήλαις, ἴσαι εἰσὶ καὶ αἱ ὑπὸ ΒΗΓ, ΓΗΚ, ΚΗΛ γωνίαι ἀλλήλαις· ὁσαπλασίων ἄρα ἐστὶν ἡ ΒΛ περιφέρεια τῆς ΒΓ, τοσαυταπλασίων ἐστὶ καὶ ἡ ὑπὸ ΒΗΛ γωνία τῆς ὑπὸ ΒΗΓ. διὰ τὰ αὐτὰ δὴ καὶ ὁσαπλασίων ἐστὶν ἡ ΝΕ περιφέρεια τῆς ΕΖ, τοσαυταπλασίων ἐστὶ καὶ ἡ ὑπὸ ΝΘΕ γωνία τῆς ὑπὸ ΕΘΖ. εἰ ἄρα ἴση ἐστὶν ἡ ΒΛ περιφέρεια τῇ ΕΝ περιφερείᾳ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΒΗΛ τῇ ὑπὸ ΕΘΝ, καὶ εἰ μείζων ἐστὶν ἡ ΒΛ περιφέρεια τῆς ΕΝ περιφερείας, μείζων ἐστὶ καὶ ἡ ὑπὸ ΒΗΛ γωνία τῆς ὑπὸ ΕΘΝ, καὶ εἰ ἐλάσσων, ἐλάσσων. τεσσάρων δὴ ὄντων μεγεθῶν, δύο μὲν περιφερειῶν τῶν ΒΓ, ΕΖ, δύο δὲ γωνιῶν τῶν ὑπὸ

ΒΗΓ, ΕΘΖ, εἴληπται τῆς μὲν ΒΓ περιφερείας καὶ τῆς ὑπὸ ΒΗΓ γωνίας ἰσάκις πολλαπλασίων ἥ τε ΒΛ περιφέρεια καὶ ἡ ὑπὸ ΒΗΛ γωνία, τῆς δὲ ΕΖ περιφερείας καὶ τῆς ὑπὸ ΕΘΖ γωνίας ἥ τε ΕΝ περιφέρεια καὶ ἡ ὑπὸ ΕΘΝ γωνία. καὶ δέδεικται, ὅτι εἰ ὑπερέχει ἡ ΒΛ περιφέρεια τῆς ΕΝ περιφερείας, ὑπερέχει καὶ ἡ ὑπὸ ΒΗΛ γωνία τῆς ὑπὸ ΕΘΝ γωνίας, καὶ εἰ ἴση, ἴση, καὶ εἰ ἐλάσσων, ἐλάσσων. ἔστιν ἄρα, ὡς ἡ ΒΓ περιφέρεια πρὸς τὴν ΕΖ, οὕτως ἡ ὑπὸ ΒΗΓ γωνία πρὸς τὴν ὑπὸ ΕΘΖ. ἀλλ’ ὡς ἡ ὑπὸ ΒΗΓ γωνία πρὸς τὴν ὑπὸ ΕΘΖ, οὕτως ἡ ὑπὸ ΒΑΓ πρὸς τὴν ὑπὸ ΕΔΖ· διπλασία γὰρ ἑκατέρα ἑκατέρας. καὶ ὡς ἄρα ἡ ΒΓ περιφέρεια πρὸς τὴν ΕΖ περιφέρειαν, οὕτως ἥ τε ὑπὸ ΒΗΓ γωνία πρὸς τὴν ὑπὸ ΕΘΖ καὶ ἡ ὑπὸ ΒΑΓ πρὸς τὴν ὑπὸ ΕΔΖ.

Ἐν ἄρα τοῖς ἴσοις κύκλοις αἱ γωνίαι τὸν αὐτὸν ἔχουσι λόγον ταῖς περιφερείαις, ἐφ’ ᾗν βεβήκασιν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ὦσι βεβηκυῖαι· ὅπερ ἔδει δεῖξαι.

BOOK VII.

Ὅροι κγ΄.

α΄. Μονάς ἐστιν, καθ' ἣν ἕκαστον τῶν ὄντων ἓν λέγεται.

β΄. Ἀριθμὸς δὲ τὸ ἐκ μονάδων συγκείμενον πλῆθος.

γ΄. Μέρος ἐστὶν ἀριθμὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος, ὅταν καταμετρῇ τὸν μείζονα.

δ΄. Μέρη δέ, ὅταν μὴ καταμετρῇ.

ε΄.Πολλαπλάσιος δὲ ὁ μείζων τοῦ ἐλάσσονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάσσονος.

ς΄. Ἄρτιος ἀριθμός ἐστιν ὁ δίχα διαιρούμενος.

ζ΄. Περισσὸς δὲ ὁ μὴ διαιρούμενος δίχα ἢ [ὁ] μονάδι διαφέρων ἀρτίου ἀριθμοῦ.

η΄. Ἀρτιάκις ἄρτιος ἀριθμός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν.

θ΄. Ἀρτιάκις δὲ περισσός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν. [Περισσάκις ἀρτιός ἐστιν ὁ ὑπὸ περισσοῦ ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν].

[ι΄. Περισσάκις ἄρτιός ἐστιν ὁ ὑπὸ περισσοῦ ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν.]

ια΄. Περισσάκις δὲ περισσὸς ἀριθμός ἐστιν ὁ ὑπὸ περισσοῦ ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν.

ιβ΄. Πρῶτος ἀριθμός ἐστιν ὁ μονάδι μόνῃ μετρούμενος.

ιγ΄. Πρῶτοι πρὸς ἀλλήλους ἀριθμοί εἰσιν οἱ μονάδι μόνῃ μετρούμενοι κοινῷ μέτρῳ.

ιδ΄. Σύνθετος ἀριθμός ἐστιν ὁ ἀριθμῷ τινι μετρούμενος.

ιε΄. Σύνθετοι δὲ πρὸς ἀλλήλους ἀριθμοί εἰσιν οἱ ἀριθμῷ τινι μετρούμενοι κοινῷ μέτρῳ.

ις΄. Ἀριθμὸς ἀριθμὸν πολλαπλασιάζειν λέγεται, ὅταν, ὅσαι εἰσὶν ἐν αὐτῷ μονάδες, τοσαυτάκις συντεθῇ ὁ πολλαπλασιαζόμενος, καὶ γένηταί τις.

ιζ΄. Ὅταν δὲ δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος ἐπίπεδος καλεῖται, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί.

ιη΄. Ὅταν δὲ τρεῖς ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος στερεός ἐστιν, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί.

ιθ΄. Τετράγωνος ἀριθμός ἐστιν ὁ ἰσάκις ἴσος ἢ [ὁ] ὑπὸ δύο ἴσων ἀριθμῶν περιεχόμενος.

κ΄. Κύβος δὲ ὁ ἰσάκις ἴσος ἰσάκις ἢ [ὁ] ὑπὸ τριῶν ἴσων ἀριθμῶν περιεχόμενος.

κα΄. Ἀριθμοὶ ἀνάλογόν εἰσιν, ὅταν ὁ πρῶτος τοῦ δευτέρου καὶ ὁ τρίτος τοῦ τετάρτου ἰσάκις ᾖ πολλαπλάσιος ἢ τὸ αὐτὸ μέρος ἢ τὰ αὐτὰ μέρη ὦσιν.

κβ΄. Ὅμοιοι ἐπίπεδοι καὶ στερεοὶ ἀριθμοί εἰσιν οἱ ἀνάλογον ἔχοντες τὰς πλευράς.

κγ΄. Τέλειος ἀριθμός ἐστιν ὁ τοῖς ἑαυτοῦ μέρεσιν ἴσος ὤν.

Προτάσεις λθ΄.

α΄. Δύο ἀριθμῶν ἀνίσων ἐκκειμένων, ἀνθυφαιρουμένου δὲ ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος, ἐὰν ὁ λειπόμενος μηδέποτε καταμετρῇ τὸν πρὸ ἑαυτοῦ, ἕως οὗ λειφθῇ μονάς, οἱ ἐξ ἀρχῆς ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ἔσονται.

Δύο γὰρ [ἀνίσων] ἀριθμῶν τῶν ΑΒ, ΓΔ ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος ὁ λειπόμενος μηδέποτε καταμετρείτω τὸν πρὸ ἑαυτοῦ, ἕως οὗ λειφθῇ μονάς· λέγω, ὅτι οἱ ΑΒ, ΓΔ πρῶτοι πρὸς ἀλλήλους εἰσίν, τουτέστιν ὅτι τοὺς ΑΒ, ΓΔ μονὰς μόνη μετρεῖ.

Εἰ γὰρ μή εἰσιν οἱ ΑΒ, ΓΔ πρῶτοι πρὸς ἀλλήλους, μετρήσει τις αὐτοὺς ἀριθμός. μετρείτω, καὶ ἔστω ὁ Ε· καὶ ὁ μὲν ΓΔ τὸν ΒΖ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ΖΑ, ὁ δὲ ΑΖ τὸν ΔΗ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ΗΓ, ὁ δὲ ΗΓ τὸν ΖΘ μετρῶν λειπέτω μονάδα τὴν ΘΑ.

Ἐπεὶ οὖν ὁ Ε τὸν ΓΔ μετρεῖ, ὁ δὲ ΓΔ τὸν ΒΖ μετρεῖ καὶ ὁ Ε ἄρα τὸν ΒΖ μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν ΒΑ· καὶ λοιπὸν ἄρα τὸν ΑΖ μετρήσει. ὁ δὲ ΑΖ τὸν ΔΗ μετρεῖ· καὶ ὁ Ε ἄρα τὸν ΔΗ μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν ΔΓ· καὶ λοιπὸν ἄρα τὸν ΓΗ μετρήσει. ὁ δὲ ΓΗ τὸν ΖΘ μετρεῖ· καὶ ὁ Ε ἄρα τὸν ΖΘ μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν ΖΑ· καὶ λοιπὴν ἄρα τὴν ΑΘ μονάδα μετρήσει ἀριθμὸς ὤν· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς ΑΒ, ΓΔ ἀριθμοὺς μετρήσει τις ἀριθμός· οἱ ΑΒ, ΓΔ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

β΄. Δύο ἀριθμῶν δοθέντων μὴ πρώτων πρὸς ἀλλήλους τὸ μέγιστον αὐτῶν κοινὸν μέτρον εὑρεῖν.

Ἔστωσαν οἱ δοθέντες δύο ἀριθμοὶ μὴ πρῶτοι πρὸς ἀλλήλους οἱ ΑΒ, ΓΔ. δεῖ δὴ τῶν ΑΒ, ΓΔ τὸ μέγιστον κοινὸν μέτρον εὑρεῖν.

Εἰ μὲν οὖν ὁ ΓΔ τὸν ΑΒ μετρεῖ, μετρεῖ δὲ καὶ ἑαυτόν, ὁ ΓΔ ἄρα τῶν ΓΔ, ΑΒ κοινὸν μέτρον ἐστίν. καὶ φανερόν, ὅτι καὶ μέγιστον· οὐδεὶς γὰρ μείζων

τοῦ ΓΔ τὸν ΓΔ μετρήσει.

Εἰ δὲ οὐ μετρεῖ ὁ ΓΔ τὸν ΑΒ, τῶν ΑΒ, ΓΔ ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος λειφθήσεταί τις ἀριθμός, ὃς μετρήσει τὸν πρὸ ἑαυτοῦ. μονὰς μὲν γὰρ οὐ λειφθήσεται· εἰ δὲ μή, ἔσονται οἱ ΑΒ, ΓΔ πρῶτοι πρὸς ἀλλήλους· ὅπερ οὐχ ὑπόκειται. λειφθήσεταί τις ἄρα ἀριθμός, ὃς μετρήσει τὸν πρὸ ἑαυτοῦ. καὶ ὁ μὲν ΓΔ τὸν ΒΕ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ΕΑ, ὁ δὲ ΕΑ τὸν ΔΖ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ΖΓ, ὁ δὲ ΓΖ τὸν ΑΕ μετρείτω. ἐπεὶ οὖν ὁ ΓΖ τὸν ΑΕ μετρεῖ, ὁ δὲ ΑΕ τὸν ΔΖ μετρεῖ, καὶ ὁ ΓΖ ἄρα τὸν ΔΖ μετρήσει· μετρεῖ δὲ καὶ ἑαυτόν· καὶ ὅλον ἄρα τὸν ΓΔ μετρήσει. ὁ δὲ ΓΔ τὸν ΒΕ μετρεῖ· καὶ ὁ ΓΖ ἄρα τὸν ΒΕ μετρεῖ· μετρεῖ δὲ καὶ τὸν ΕΑ· καὶ ὅλον ἄρα τὸν ΒΑ μετρήσει· μετρεῖ δὲ καὶ τὸν ΓΔ· ὁ ΓΖ ἄρα τοὺς ΑΒ, ΓΔ μετρεῖ. ὁ ΓΖ ἄρα τῶν ΑΒ, ΓΔ κοινὸν μέτρον ἐστίν. λέγω δή, ὅτι καὶ μέγιστον. εἰ γὰρ μή ἐστιν ὁ ΓΖ τῶν ΑΒ, ΓΔ μέγιστον κοινὸν μέτρον, μετρήσει τις τοὺς ΑΒ, ΓΔ ἀριθμοὺς ἀριθμὸς μείζων ὢν τοῦ ΓΖ. μετρείτω, καὶ ἔστω ὁ Η. καὶ ἐπεὶ ὁ Η τὸν ΓΔ μετρεῖ, ὁ δὲ ΓΔ τὸν ΒΕ μετρεῖ, καὶ ὁ Η ἄρα τὸν ΒΕ μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν ΒΑ· καὶ λοιπὸν ἄρα τὸν ΑΕ μετρήσει. ὁ δὲ ΑΕ τὸν ΔΖ μετρεῖ· καὶ ὁ Η ἄρα τὸν ΔΖ μετρήσει· μετρεῖ δὲ καὶ ὅλον τὸν ΔΓ· καὶ λοιπὸν ἄρα τὸν ΓΖ μετρήσει ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα τοὺς ΑΒ, ΓΔ ἀριθμοὺς ἀριθμός τις μετρήσει μείζων ὢν τοῦ ΓΖ· ὁ ΓΖ ἄρα τῶν ΑΒ, ΓΔ μέγιστόν ἐστι κοινὸν μέτρον· [ὅπερ ἔδει δεῖξαι].

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν ἀριθμὸς δύο ἀριθμοὺς μετρῇ, καὶ τὸ μέγιστον αὐτῶν κοινὸν μέτρον μετρήσει· ὅπερ ἔδει δεῖξαι.

γ΄. Τριῶν ἀριθμῶν δοθέντων μὴ πρώτων πρὸς ἀλλήλους τὸ μέγιστον αὐτῶν κοινὸν μέτρον εὑρεῖν.

Ἔστωσαν οἱ δοθέντες τρεῖς ἀριθμοὶ μὴ πρῶτοι πρὸς ἀλλήλους οἱ Α, Β, Γ· δεῖ δὴ τῶν Α, Β, Γ τὸ μέγιστον κοινὸν μέτρον εὑρεῖν.

Εἰλήφθω γὰρ δύο τῶν Α, Β τὸ μέγιστον κοινὸν μέτρον ὁ Δ· ὁ δὴ Δ τὸν Γ ἤτοι μετρεῖ ἢ οὐ μετρεῖ. μετρείτω πρότερον· μετρεῖ δὲ καὶ τοὺς Α, Β· ὁ Δ ἄρα τοὺς Α, Β, Γ μετρεῖ· ὁ Δ ἄρα τῶν Α, Β, Γ κοινὸν μέτρον ἐστίν. λέγω δή, ὅτι καὶ μέγιστον. εἰ γὰρ μή ἐστιν ὁ Δ τῶν Α, Β, Γ μέγιστον κοινὸν μέτρον, μετρήσει τις τοὺς Α, Β, Γ ἀριθμοὺς ἀριθμὸς μείζων ὢν τοῦ Δ. μετρείτω, καὶ ἔστω ὁ Ε. ἐπεὶ οὖν ὁ Ε τοὺς Α, Β, Γ μετρεῖ, καὶ τοὺς Α, Β ἄρα μετρήσει· καὶ τὸ τῶν Α, Β ἄρα μέγιστον κοινὸν μέτρον μετρήσει. τὸ δὲ τῶν Α, Β μέγιστον κοινὸν μέτρον ἐστὶν ὁ Δ· ὁ Ε ἄρα τὸν Δ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Α, Β, Γ ἀριθμοὺς ἀριθμός τις μετρήσει μείζων ὢν τοῦ Δ· ὁ Δ ἄρα τῶν Α, Β, Γ μέγιστόν ἐστι κοινὸν μέτρον.

Μὴ μετρείτω δὴ ὁ Δ τὸν Γ· λέγω πρῶτον, ὅτι οἱ Γ, Δ οὐκ εἰσὶ πρῶτοι πρὸς ἀλλήλους. ἐπεὶ γὰρ οἱ Α, Β, Γ οὐκ εἰσὶ πρῶτοι πρὸς ἀλλήλους, μετρήσει τις αὐτοὺς ἀριθμός. ὁ δὴ τοὺς Α, Β, Γ μετρῶν καὶ τοὺς Α, Β μετρήσει, καὶ τὸ τῶν Α, Β μέγιστον κοινὸν μέτρον τὸν Δ μετρήσει· μετρεῖ δὲ καὶ τὸν Γ· τοὺς Δ, Γ ἄρα ἀριθμοὺς ἀριθμός τις μετρήσει· οἱ Δ, Γ ἄρα οὐκ εἰσὶ πρῶτοι πρὸς ἀλλήλους. εἰλήφθω οὖν αὐτῶν τὸ μέγιστον κοινὸν μέτρον ὁ Ε. καὶ ἐπεὶ ὁ Ε τὸν Δ μετρεῖ, ὁ δὲ Δ τοὺς Α, Β μετρεῖ, καὶ ὁ Ε ἄρα τοὺς Α, Β μετρεῖ· μετρεῖ δὲ καὶ τὸν Γ· ὁ Ε ἄρα τοὺς Α, Β, Γ μετρεῖ· ὁ Ε ἄρα τῶν Α, Β, Γ κοινόν ἐστι μέτρον. λέγω δή, ὅτι καὶ μέγιστον. εἰ γὰρ μή ἐστιν ὁ Ε τῶν Α, Β, Γ τὸ μέγιστον κοινὸν μέτρον, μετρήσει τις τοὺς Α, Β, Γ ἀριθμοὺς ἀριθμὸς μείζων ὢν τοῦ Ε. μετρείτω, καὶ ἔστω ὁ Ζ. καὶ ἐπεὶ ὁ Ζ τοὺς Α, Β, Γ μετρεῖ, καὶ τοὺς Α, Β μετρεῖ· καὶ τὸ τῶν Α, Β ἄρα μέγιστον κοινὸν μέτρον μετρήσει. τὸ δὲ τῶν Α, Β μέγιστον κοινὸν μέτρον ἐστὶν ὁ Δ· ὁ Ζ ἄρα τὸν Δ μετρεῖ· μετρεῖ δὲ καὶ τὸν Γ· ὁ Ζ ἄρα τοὺς Δ, Γ μετρεῖ· καὶ τὸ τῶν Δ, Γ ἄρα μέγιστον κοινὸν μέτρον μετρήσει. τὸ δὲ τῶν Δ, Γ μέγιστον κοινὸν μέτρον ἐστὶν ὁ Ε· ὁ Ζ ἄρα τὸν Ε μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Α,

Β, Γ ἀριθμοὺς ἀριθμός τις μετρήσει μείζων ὢν τοῦ Ε· ὁ Ε ἄρα τῶν Α, Β, Γ μέγιστόν ἐστι κοινὸν μέτρον· ὅπερ ἔδει δεῖξαι.

δ΄. Ἅπας ἀριθμὸς παντὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος ἤτοι μέρος ἐστὶν ἢ μέρη.

Ἔστωσαν δύο ἀριθμοὶ οἱ Α, ΒΓ, καὶ ἔστω ἐλάσσων ὁ ΒΓ· λέγω, ὅτι ὁ ΒΓ τοῦ Α ἤτοι μέρος ἐστὶν ἢ μέρη.

Οἱ Α, ΒΓ γὰρ ἤτοι πρῶτοι πρὸς ἀλλήλους εἰσὶν ἢ οὔ. ἔστωσαν πρότερον οἱ Α, ΒΓ πρῶτοι πρὸς ἀλλήλους. διαιρεθέντος δὴ τοῦ ΒΓ εἰς τὰς ἐν αὐτῷ μονάδας ἔσται ἑκάστη μονὰς τῶν ἐν τῷ ΒΓ μέρος τι τοῦ Α· ὥστε μέρη ἐστὶν ὁ ΒΓ τοῦ Α.

Μὴ ἔστωσαν δὴ οἱ Α, ΒΓ πρῶτοι πρὸς ἀλλήλους· ὁ δὴ ΒΓ τὸν Α ἤτοι μετρεῖ ἢ οὐ μετρεῖ. εἰ μὲν οὖν ὁ ΒΓ τὸν Α μετρεῖ, μέρος ἐστὶν ὁ ΒΓ τοῦ Α. εἰ δὲ οὔ, εἰλήφθω τῶν Α, ΒΓ μέγιστον κοινὸν μέτρον ὁ Δ, καὶ διῃρήσθω ὁ ΒΓ εἰς τοὺς τῷ Δ ἴσους τοὺς ΒΕ, ΕΖ, ΖΓ. καὶ ἐπεὶ ὁ Δ τὸν Α μετρεῖ, μέρος ἐστὶν ὁ Δ τοῦ Α· ἴσος δὲ ὁ Δ ἑκάστῳ τῶν ΒΕ, ΕΖ, ΖΓ· καὶ ἕκαστος ἄρα τῶν ΒΕ, ΕΖ, ΖΓ τοῦ Α μέρος ἐστίν· ὥστε μέρη ἐστὶν ὁ ΒΓ τοῦ Α.

Ἅπας ἄρα ἀριθμὸς παντὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος ἤτοι μέρος ἐστὶν ἢ μέρη· ὅπερ ἔδει δεῖξαι.

ε΄. Ἐὰν ἀριθμός ἀριθμοῦ μέρος ᾖ, καὶ ἕτερος ἑτέρου τὸ αὐτὸ μέρος ᾖ, καὶ συναμφότερος συναμφοτέρου τὸ αὐτὸ μέρος ἔσται, ὅπερ ὁ εἷς τοῦ ἑνός.

Ἀριθμὸς γὰρ ὁ Α [ἀριθμοῦ] τοῦ ΒΓ μέρος ἔστω, καὶ ἕτερος ὁ Δ ἑτέρου τοῦ ΕΖ τὸ αὐτὸ μέρος, ὅπερ ὁ Α τοῦ ΒΓ· λέγω, ὅτι καὶ συναμφότερος ὁ Α, Δ συναμφοτέρου τοῦ ΒΓ, ΕΖ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὁ Α τοῦ ΒΓ.

Ἐπεὶ γάρ, ὃ μέρος ἐστὶν ὁ Α τοῦ ΒΓ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Δ τοῦ ΕΖ, ὅσοι ἄρα εἰσὶν ἐν τῷ ΒΓ ἀριθμοὶ ἴσοι τῷ Α, τοσοῦτοί εἰσι καὶ ἐν τῷ ΕΖ ἀριθμοὶ ἴσοι τῷ Δ. διῃρήσθω ὁ μὲν ΒΓ εἰς τοὺς τῷ Α ἴσους τοὺς ΒΗ, ΗΓ, ὁ δὲ

ΕΖ εἰς τοὺς τῷ Δ ἴσους τοὺς ΕΘ, ΘΖ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΒΗ, ΗΓ τῷ πλήθει τῶν ΕΘ, ΘΖ. καὶ ἐπεὶ ἴσος ἐστὶν ὁ μὲν ΒΗ τῷ Α, ὁ δὲ ΕΘ τῷ Δ, καὶ οἱ ΒΗ, ΕΘ ἄρα τοῖς Α, Δ ἴσοι. διὰ τὰ αὐτὰ δὴ καὶ οἱ ΗΓ, ΘΖ τοῖς Α, Δ. ὅσοι ἄρα [εἰσὶν] ἐν τῷ ΒΓ ἀριθμοὶ ἴσοι τῷ Α, τοσοῦτοί εἰσι καὶ ἐν τοῖς ΒΓ, ΕΖ ἴσοι τοῖς Α, Δ. ὁσαπλασίων ἄρα ἐστὶν ὁ ΒΓ τοῦ Α, τοσαυταπλασίων ἐστὶ καὶ συναμφότερος ὁ ΒΓ, ΕΖ συναμφοτέρου τοῦ Α, Δ. ὃ ἄρα μέρος ἐστὶν ὁ Α τοῦ ΒΓ, τὸ αὐτὸ μέρος ἐστὶ καὶ συναμφότερος ὁ Α, Δ συναμφοτέρου τοῦ ΒΓ, ΕΖ· ὅπερ ἔδει δεῖξαι.

ς΄. Ἐὰν ἀριθμὸς ἀριθμοῦ μέρη ᾖ, καὶ ἕτερος ἑτέρου τὰ αὐτὰ μέρη ᾖ, καὶ συναμφότερος συναμφοτέρου τὰ αὐτὰ μέρη ἔσται, ὅπερ ὁ εἷς τοῦ ἑνός.

Ἀριθμὸς γὰρ ὁ ΑΒ ἀριθμοῦ τοῦ Γ μέρη ἔστω, καὶ ἕτερος ὁ ΔΕ ἑτέρου τοῦ Ζ τὰ αὐτὰ μέρη, ἅπερ ὁ ΑΒ τοῦ Γ· λέγω, ὅτι καὶ συναμφότερος ὁ ΑΒ, ΔΕ συναμφοτέρου τοῦ Γ, Ζ τὰ αὐτὰ μέρη ἐστίν, ἅπερ ὁ ΑΒ τοῦ Γ.

Ἐπεὶ γάρ, ἃ μέρη ἐστὶν ὁ ΑΒ τοῦ Γ, τὰ αὐτὰ μέρη καὶ ὁ ΔΕ τοῦ Ζ, ὅσα ἄρα ἐστὶν ἐν τῷ ΑΒ μέρη τοῦ Γ, τοσαῦτά ἐστι καὶ ἐν τῷ ΔΕ μέρη τοῦ Ζ. διῃρήσθω ὁ μὲν ΑΒ εἰς τὰ τοῦ Γ μέρη τὰ ΑΗ, ΗΒ, ὁ δὲ ΔΕ εἰς τὰ τοῦ Ζ μέρη τὰ ΔΘ, ΘΕ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΑΗ, ΗΒ τῷ πλήθει τῶν ΔΘ, ΘΕ. καὶ ἐπεί, ὃ μέρος ἐστὶν ὁ ΑΗ τοῦ Γ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΔΘ τοῦ Ζ, ὃ ἄρα μέρος ἐστὶν ὁ ΑΗ τοῦ Γ, τὸ αὐτὸ μέρος ἐστὶ καὶ συναμφότερος ὁ ΑΗ, ΔΘ συναμφοτέρου τοῦ Γ, Ζ. διὰ τὰ αὐτὰ δὴ καὶ ὃ μέρος ἐστὶν ὁ ΗΒ τοῦ Γ, τὸ αὐτὸ μέρος ἐστὶ καὶ συναμφότερος ὁ ΗΒ, ΘΕ συναμφοτέρου τοῦ Γ, Ζ. ἃ ἄρα μέρη ἐστὶν ὁ ΑΒ τοῦ Γ, τὰ αὐτὰ μέρη ἐστὶ καὶ συναμφότερος ὁ ΑΒ, ΔΕ συναμφοτέρου τοῦ Γ, Ζ· ὅπερ ἔδει δεῖξαι.

ζ΄. Ἐὰν ἀριθμὸς ἀριθμοῦ μέρος ᾖ, ὅπερ ἀφαιρεθεὶς ἀφαιρεθέντος, καὶ ὁ λοιπὸς τοῦ λοιποῦ τὸ αὐτὸ μέρος ἔσται, ὅπερ ὁ ὅλος τοῦ ὅλου.

Ἀριθμὸς γὰρ ὁ ΑΒ ἀριθμοῦ τοῦ ΓΔ μέρος ἔστω, ὅπερ ἀφαιρεθεὶς ὁ ΑΕ ἀφαιρεθέντος τοῦ ΓΖ· λέγω, ὅτι καὶ λοιπὸς ὁ ΕΒ λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὅλος ὁ ΑΒ ὅλου τοῦ ΓΔ.

Ὅ γὰρ μέρος ἐστὶν ὁ ΑΕ τοῦ ΓΖ, τὸ αὐτὸ μέρος ἔστω καὶ ὁ ΕΒ τοῦ ΓΗ. καὶ ἐπεί, ὃ μέρος ἐστὶν ὁ ΑΕ τοῦ ΓΖ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΕΒ τοῦ ΓΗ, ὃ ἄρα μέρος ἐστὶν ὁ ΑΕ τοῦ ΓΖ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΑΒ τοῦ ΗΖ. ὃ δὲ μέρος ἐστὶν ὁ ΑΕ τοῦ ΓΖ, τὸ αὐτὸ μέρος ὑπόκειται καὶ ὁ ΑΒ τοῦ ΓΔ· ὃ ἄρα μέρος ἐστὶ καὶ ὁ ΑΒ τοῦ ΗΖ, τὸ αὐτὸ μέρος ἐστὶ καὶ τοῦ ΓΔ· ἴσος ἄρα ἐστὶν ὁ ΗΖ τῷ ΓΔ. κοινὸς ἀφῃρήσθω ὁ ΓΖ· λοιπὸς ἄρα ὁ ΗΓ λοιπῷ τῷ ΖΔ ἐστιν ἴσος. καὶ ἐπεί, ὃ μέρος ἐστὶν ὁ ΑΕ τοῦ ΓΖ, τὸ αὐτὸ μέρος [ἐστὶ] καὶ ὁ ΕΒ τοῦ ΗΓ, ἴσος δὲ ὁ ΗΓ τῷ ΖΔ, ὃ ἄρα μέρος ἐστὶν ὁ ΑΕ τοῦ ΓΖ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΕΒ τοῦ ΖΔ. ἀλλὰ ὃ μέρος ἐστὶν ὁ ΑΕ τοῦ ΓΖ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΑΒ τοῦ ΓΔ· καὶ λοιπὸς ἄρα ὁ ΕΒ λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὅλος ὁ ΑΒ ὅλου τοῦ ΓΔ· ὅπερ ἔδει δεῖξαι.

η΄. Ἐὰν ἀριθμὸς ἀριθμοῦ μέρη ᾖ, ἅπερ ἀφαιρεθεὶς ἀφαιρεθέντος, καὶ ὁ λοιπὸς τοῦ λοιποῦ τὰ αὐτὰ μέρη ἔσται, ἅπερ ὁ ὅλος τοῦ ὅλου.

Ἀριθμὸς γὰρ ὁ ΑΒ ἀριθμοῦ τοῦ ΓΔ μέρη ἔστω, ἅπερ ἀφαιρεθεὶς ὁ ΑΕ ἀφαιρεθέντος τοῦ ΓΖ· λέγω, ὅτι καὶ λοιπὸς ὁ ΕΒ λοιποῦ τοῦ ΖΔ τὰ αὐτὰ μέρη ἐστίν, ἅπερ ὅλος ὁ ΑΒ ὅλου τοῦ ΓΔ.

Κείσθω γὰρ τῷ ΑΒ ἴσος ὁ ΗΘ. ἃ ἄρα μέρη ἐστὶν ὁ ΗΘ τοῦ ΓΔ, τὰ αὐτὰ μέρη ἐστὶ καὶ ὁ ΑΕ τοῦ ΓΖ. διῃρήσθω ὁ μὲν ΗΘ εἰς τὰ τοῦ ΓΔ μέρη τὰ ΗΚ, ΚΘ, ὁ δὲ ΑΕ εἰς τὰ τοῦ ΓΖ μέρη τὰ ΑΛ, ΛΕ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΗΚ, ΚΘ τῷ πλήθει τῶν ΑΛ, ΛΕ. καὶ ἐπεί, ὃ μέρος ἐστὶν ὁ ΗΚ τοῦ ΓΔ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΑΛ τοῦ ΓΖ, μείζων δὲ ὁ ΓΔ τοῦ ΓΖ, μείζων ἄρα καὶ ὁ ΗΚ τοῦ ΑΛ. κείσθω τῷ ΑΛ ἴσος ὁ ΗΜ. ὃ ἄρα μέρος ἐστὶν ὁ ΗΚ τοῦ ΓΔ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΗΜ τοῦ ΓΖ· καὶ λοιπὸς ἄρα ὁ ΜΚ λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὅλος ὁ ΗΚ ὅλου τοῦ ΓΔ. πάλιν ἐπεί, ὃ μέρος ἐστὶν ὁ

ΚΘ τοῦ ΓΔ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΕΛ τοῦ ΓΖ, μείζων δὲ ὁ ΓΔ τοῦ ΓΖ, μείζων ἄρα καὶ ὁ ΘΚ τοῦ ΕΛ. κείσθω τῷ ΕΛ ἴσος ὁ ΚΝ. ὃ ἄρα μέρος ἐστὶν ὁ ΚΘ τοῦ ΓΔ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΚΝ τοῦ ΓΖ· καὶ λοιπὸς ἄρα ὁ ΝΘ λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὅλος ὁ ΚΘ ὅλου τοῦ ΓΔ. ἐδείχθη δὲ καὶ λοιπὸς ὁ ΜΚ λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ὤν, ὅπερ ὅλος ὁ ΗΚ ὅλου τοῦ ΓΔ· καὶ συναμφότερος ἄρα ὁ ΜΚ, ΝΘ τοῦ ΔΖ τὰ αὐτὰ μέρη ἐστίν, ἅπερ ὅλος ὁ ΘΗ ὅλου τοῦ ΓΔ. ἴσος δὲ συναμφότερος μὲν ὁ ΜΚ, ΝΘ τῷ ΕΒ, ὁ δὲ ΘΗ τῷ ΒΑ· καὶ λοιπὸς ἄρα ὁ ΕΒ λοιποῦ τοῦ ΖΔ τὰ αὐτὰ μέρη ἐστίν, ἅπερ ὅλος ὁ ΑΒ ὅλου τοῦ ΓΔ· ὅπερ ἔδει δεῖξαι.

θ΄. Ἐὰν ἀριθμὸς ἀριθμοῦ μέρος ᾖ, καὶ ἕτερος ἑτέρου τὸ αὐτὸ μέρος ᾖ, καὶ ἐναλλάξ, ὃ μέρος ἐστὶν ἢ μέρη ὁ πρῶτος τοῦ τρίτου, τὸ αὐτὸ μέρος ἔσται ἢ τὰ αὐτὰ μέρη καὶ ὁ δεύτερος τοῦ τετάρτου

Ἀριθμὸς γὰρ ὁ Α ἀριθμοῦ τοῦ ΒΓ μέρος ἔστω, καὶ ἕτερος ὁ Δ ἑτέρου τοῦ ΕΖ τὸ αὐτὸ μέρος, ὅπερ ὁ Α τοῦ ΒΓ· λέγω, ὅτι καὶ ἐναλλάξ, ὃ μέρος ἐστὶν ὁ Α τοῦ Δ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΒΓ τοῦ ΕΖ ἢ μέρη.

Ἐπεὶ γὰρ ὃ μέρος ἐστὶν ὁ Α τοῦ ΒΓ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Δ τοῦ ΕΖ, ὅσοι ἄρα εἰσὶν ἐν τῷ ΒΓ ἀριθμοὶ ἴσοι τῷ Α, τοσοῦτοί εἰσι καὶ ἐν τῷ ΕΖ ἴσοι τῷ Δ. διῃρήσθω ὁ μὲν ΒΓ εἰς τοὺς τῷ Α ἴσους τοὺς ΒΗ, ΗΓ, ὁ δὲ ΕΖ εἰς τοὺς τῷ Δ ἴσους τοὺς ΕΘ, ΘΖ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΒΗ, ΗΓ τῷ πλήθει τῶν ΕΘ, ΘΖ.

Καὶ ἐπεὶ ἴσοι εἰσὶν οἱ ΒΗ, ΗΓ ἀριθμοὶ ἀλλήλοις, εἰσὶ δὲ καὶ οἱ ΕΘ, ΘΖ ἀριθμοὶ ἴσοι ἀλλήλοις, καί ἐστιν ἴσον τὸ πλῆθος τῶν ΒΗ, ΗΓ τῷ πλήθει τῶν ΕΘ, ΘΖ, ὃ ἄρα μέρος ἐστὶν ὁ ΒΗ τοῦ ΕΘ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΗΓ τοῦ ΘΖ ἢ τὰ αὐτὰ μέρη· ὥστε καὶ ὃ μέρος ἐστὶν ὁ ΒΗ τοῦ ΕΘ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ συναμφότερος ὁ ΒΓ συναμφοτέρου τοῦ ΕΖ ἢ τὰ αὐτὰ μέρη. ἴσος δὲ ὁ μὲν ΒΗ τῷ Α, ὁ δὲ ΕΘ τῷ Δ· ὃ ἄρα μέρος ἐστὶν ὁ Α τοῦ Δ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΒΓ τοῦ ΕΖ ἢ τὰ αὐτὰ μέρη· ὅπερ ἔδει δεῖξαι.

ι΄. Ἐὰν ἀριθμὸς ἀριθμοῦ μέρη ᾖ, καὶ ἕτερος ἑτέρου τὰ αὐτὰ μέρη ᾖ, καὶ ἐναλλάξ, ἃ μέρη ἐστὶν ὁ πρῶτος τοῦ τρίτου ἢ μέρος, τὰ αὐτὰ μέρη ἔσται καὶ ὁ δεύτερος τοῦ τετάρτου ἢ τὸ αὐτὸ μέρος.

Ἀριθμὸς γὰρ ὁ ΑΒ ἀριθμοῦ τοῦ Γ μέρη ἔστω, καὶ ἕτερος ὁ ΔΕ ἑτέρου τοῦ Ζ τὰ αὐτὰ μέρη· λέγω, ὅτι καὶ ἐναλλάξ, ἃ μέρη ἐστὶν ὁ ΑΒ τοῦ ΔΕ ἢ μέρος, τὰ αὐτὰ μέρη ἐστὶ καὶ ὁ Γ τοῦ Ζ ἢ τὸ αὐτὸ μέρος.

Ἐπεὶ γάρ, ἃ μέρη ἐστὶν ὁ ΑΒ τοῦ Γ, τὰ αὐτὰ μέρη ἐστὶ καὶ ὁ ΔΕ τοῦ Ζ, ὅσα ἄρα ἐστὶν ἐν τῷ ΑΒ μέρη τοῦ Γ, τοσαῦτα καὶ ἐν τῷ ΔΕ μέρη τοῦ Ζ. διῃρήσθω ὁ μὲν ΑΒ εἰς τὰ τοῦ Γ μέρη τὰ ΑΗ, ΗΒ, ὁ δὲ ΔΕ εἰς τὰ τοῦ Ζ μέρη τὰ ΔΘ, ΘΕ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΑΗ, ΗΒ τῷ πλήθει τῶν ΔΘ, ΘΕ. καὶ ἐπεί, ὃ μέρος ἐστὶν ὁ ΑΗ τοῦ Γ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΔΘ τοῦ Ζ, καὶ ἐναλλάξ, ὃ μέρος ἐστὶν ὁ ΑΗ τοῦ ΔΘ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ Ζ ἢ τὰ αὐτὰ μέρη. διὰ τὰ αὐτὰ δὴ καί, ὃ μέρος ἐστὶν ὁ ΗΒ τοῦ ΘΕ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ Ζ ἢ τὰ αὐτὰ μέρη· ὥστε καί [ὃ μέρος ἐστὶν ὁ ΑΗ τοῦ ΔΘ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΗΒ τοῦ ΘΕ ἢ τὰ αὐτὰ μέρη· καὶ ὃ ἄρα μέρος ἐστὶν ὁ ΑΗ τοῦ ΔΘ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΑΒ τοῦ ΔΕ ἢ τὰ αὐτὰ μέρη· ἀλλ᾽ ὃ μέρος ἐστὶν ὁ ΑΗ τοῦ ΔΘ ἢ μέρη, τὸ αὐτὸ μέρος ἐδείχθη καὶ ὁ Γ τοῦ Ζ ἢ τὰ αὐτὰ μέρη, καὶ] ἃ [ἄρα] μέρη ἐστὶν ὁ ΑΒ τοῦ ΔΕ ἢ μέρος, τὰ αὐτὰ μέρη ἐστὶ καὶ ὁ Γ τοῦ Ζ ἢ τὸ αὐτὸ μέρος· ὅπερ ἔδει δεῖξαι.

ια΄. Ἐὰν ᾖ ὡς ὅλος πρὸς ὅλον, οὕτως ἀφαιρεθεὶς πρὸς ἀφαιρεθέντα, καὶ ὁ λοιπὸς πρὸς τὸν λοιπὸν ἔσται, ὡς ὅλος πρὸς ὅλον.

Ἔστω ὡς ὅλος ὁ ΑΒ πρὸς ὅλον τὸν ΓΔ, οὕτως ἀφαιρεθεὶς ὁ ΑΕ πρὸς ἀφαιρεθέντα τὸν ΓΖ· λέγω, ὅτι καὶ λοιπὸς ὁ ΕΒ πρὸς λοιπὸν τὸν ΖΔ ἐστιν, ὡς ὅλος ὁ ΑΒ πρὸς ὅλον τὸν ΓΔ.

Ἐπεί ἐστιν ὡς ὁ ΑΒ πρὸς τὸν ΓΔ, οὕτως ὁ ΑΕ πρὸς τὸν ΓΖ, ὃ ἄρα μέρος ἐστὶν ὁ ΑΒ τοῦ ΓΔ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΑΕ τοῦ ΓΖ ἢ τὰ αὐτὰ

μέρη. καὶ λοιπὸς ἄρα ὁ ΕΒ λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ἐστὶν ἢ μέρη, ἅπερ ὁ ΑΒ τοῦ ΓΔ. ἔστιν ἄρα ὡς ὁ ΕΒ πρὸς τὸν ΖΔ, οὕτως ὁ ΑΒ πρὸς τὸν ΓΔ· ὅπερ ἔδει δεῖξαι.

ιβ΄. Ἐὰν ὦσιν ὁποσοιοῦν ἀριθμοὶ ἀνάλογον, ἔσται ὡς εἷς τῶν ἡγουμένων πρὸς ἕνα τῶν ἑπομένων, οὕτως ἅπαντες οἱ ἡγούμενοι πρὸς ἅπαντας τοὺς ἑπομένους.

Ἔστωσαν ὁποσοιοῦν ἀριθμοὶ ἀνάλογον οἱ Α, Β, Γ, Δ, ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Γ πρὸς τὸν Δ· λέγω, ὅτι ἐστὶν ὡς ὁ Α πρὸς τὸν Β, οὕτως οἱ Α, Γ πρὸς τοὺς Β, Δ.

Ἐπεὶ γάρ ἐστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Γ πρὸς τὸν Δ, ὃ ἄρα μέρος ἐστὶν ὁ Α τοῦ Β ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ Δ ἢ μέρη. καὶ συναμφότερος ἄρα ὁ Α, Γ συναμφοτέρου τοῦ Β, Δ τὸ αὐτὸ μέρος ἐστὶν ἢ τὰ αὐτὰ μέρη, ἅπερ ὁ Α τοῦ Β. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως οἱ Α, Γ πρὸς τοὺς Β, Δ· ὅπερ ἔδει δεῖξαι.

ιγ΄. Ἐὰν τέσσαρες ἀριθμοὶ ἀνάλογον ὦσιν, καὶ ἐναλλὰξ ἀνάλογον ἔσονται.

Ἔστωσαν τέσσαρες ἀριθμοὶ ἀνάλογον οἱ Α, Β, Γ, Δ, ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Γ πρὸς τὸν Δ· λέγω, ὅτι καὶ ἐναλλὰξ ἀνάλογον ἔσονται, ὡς ὁ Α πρὸς τὸν Γ, οὕτως ὁ Β πρὸς τὸν Δ.

Ἐπεὶ γάρ ἐστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Γ πρὸς τὸν Δ, ὃ ἄρα μέρος ἐστὶν ὁ Α τοῦ Β ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ Δ ἢ τὰ αὐτὰ μέρη. ἐναλλὰξ ἄρα, ὃ μέρος ἐστὶν ὁ Α τοῦ Γ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Β τοῦ Δ ἢ τὰ αὐτὰ μέρη. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Γ, οὕτως ὁ Β πρὸς τὸν Δ· ὅπερ ἔδει δεῖξαι.

ιδ΄. Ἐὰν ὦσιν ὁποσοιοῦν ἀριθμοὶ καὶ ἄλλοι αὐτοῖς ἴσοι τὸ πλῆθος σύνδυο λαμβανόμενοι καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ δι᾽ ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσονται.

Ἔστωσαν ὁποσοιοῦν ἀριθμοὶ οἱ Α, Β, Γ καὶ ἄλλοι αὐτοῖς ἴσοι τὸ πλῆθος σύνδυο λαμβανόμενοι ἐν τῷ αὐτῷ λόγῳ οἱ Δ, Ε, Ζ, ὡς μὲν ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε, ὡς δὲ ὁ Β πρὸς τὸν Γ, οὕτως ὁ Ε πρὸς τὸν Ζ· λέγω, ὅτι καὶ δι’ ἴσου ἐστὶν ὡς ὁ Α πρὸς τὸν Γ, οὕτως ὁ Δ πρὸς τὸν Ζ.

Ἐπεὶ γάρ ἐστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε, ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Δ, οὕτως ὁ Β πρὸς τὸν Ε. πάλιν, ἐπεί ἐστιν ὡς ὁ Β πρὸς τὸν Γ, οὕτως ὁ Ε πρὸς τὸν Ζ, ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Β πρὸς τὸν Ε, οὕτως ὁ Γ πρὸς τὸν Ζ. ὡς δὲ ὁ Β πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Δ· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Δ, οὕτως ὁ Γ πρὸς τὸν Ζ· ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Γ, οὕτως ὁ Δ πρὸς τὸν Ζ· ὅπερ ἔδει δεῖξαι.

ιε΄. Ἐὰν μονὰς ἀριθμόν τινα μετρῇ, ἰσάκις δὲ ἕτερος ἀριθμὸς ἄλλον τινὰ ἀριθμὸν μετρῇ, καὶ ἐναλλὰξ ἰσάκις ἡ μονὰς τὸν τρίτον ἀριθμὸν μετρήσει καὶ ὁ δεύτερος τὸν τέταρτον.

Μονὰς γὰρ ἡ Α ἀριθμόν τινα τὸν ΒΓ μετρείτω, ἰσάκις δὲ ἕτερος ἀριθμὸς ὁ Δ ἄλλον τινὰ ἀριθμὸν τὸν ΕΖ μετρείτω· λέγω, ὅτι καὶ ἐναλλὰξ ἰσάκις ἡ Α μονὰς τὸν Δ ἀριθμὸν μετρεῖ καὶ ὁ ΒΓ τὸν ΕΖ.

Ἐπεὶ γὰρ ἰσάκις ἡ Α μονὰς τὸν ΒΓ ἀριθμὸν μετρεῖ καὶ ὁ Δ τὸν ΕΖ, ὅσαι ἄρα εἰσὶν ἐν τῷ ΒΓ μονάδες, τοσοῦτοί εἰσι καὶ ἐν τῷ ΕΖ ἀριθμοὶ ἴσοι τῷ Δ. διῃρήσθω ὁ μὲν ΒΓ εἰς τὰς ἐν ἑαυτῷ μονάδας τὰς ΒΗ, ΗΘ, ΘΓ, ὁ δὲ ΕΖ εἰς τοὺς τῷ Δ ἴσους τοὺς ΕΚ, ΚΛ, ΛΖ. ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΒΗ, ΗΘ, ΘΓ τῷ πλήθει τῶν ΕΚ, ΚΛ, ΛΖ. καὶ ἐπεὶ ἴσαι εἰσὶν αἱ ΒΗ, ΗΘ, ΘΓ μονάδες ἀλλήλαις, εἰσὶ δὲ καὶ οἱ ΕΚ, ΚΛ, ΛΖ ἀριθμοὶ ἴσοι ἀλλήλοις, καί ἐστιν ἴσον τὸ πλῆθος τῶν ΒΗ, ΗΘ, ΘΓ μονάδων τῷ πλήθει τῶν ΕΚ, ΚΛ, ΛΖ ἀριθμῶν, ἔσται ἄρα ὡς ἡ ΒΗ μονὰς πρὸς τὸν ΕΚ ἀριθμόν, οὕτως ἡ ΗΘ μονὰς πρὸς τὸν ΚΛ ἀριθμὸν καὶ ἡ ΘΓ μονὰς πρὸς τὸν ΛΖ ἀριθμόν. ἔσται ἄρα καὶ ὡς εἷς τῶν ἡγουμένων πρὸς ἕνα τῶν ἑπομένων, οὕτως ἅπαντες οἱ ἡγούμενοι πρὸς ἅπαντας τοὺς ἑπομένους· ἔστιν ἄρα ὡς ἡ ΒΗ μονὰς πρὸς τὸν ΕΚ ἀριθμόν,

οὕτως ὁ ΒΓ πρὸς τὸν ΕΖ. ἴση δὲ ἡ ΒΗ μονὰς τῇ Α μονάδι, ὁ δὲ ΕΚ ἀριθμὸς τῷ Δ ἀριθμῷ. ἔστιν ἄρα ὡς ἡ Α μονὰς πρὸς τὸν Δ ἀριθμόν, οὕτως ὁ ΒΓ πρὸς τὸν ΕΖ. ἰσάκις ἄρα ἡ Α μονὰς τὸν Δ ἀριθμὸν μετρεῖ καὶ ὁ ΒΓ τὸν ΕΖ· ὅπερ ἔδει δεῖξαι.

ις΄. Ἐὰν δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινας, οἱ γενόμενοι ἐξ αὐτῶν ἴσοι ἀλλήλοις ἔσονται.

Ἔστωσαν δύο ἀριθμοὶ οἱ Α, Β, καὶ ὁ μὲν Α τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω, ὁ δὲ Β τὸν Α πολλαπλασιάσας τὸν Δ ποιείτω· λέγω, ὅτι ἴσος ἐστὶν ὁ Γ τῷ Δ.

Ἐπεὶ γὰρ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ Β ἄρα τὸν Γ μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας. μετρεῖ δὲ καὶ ἡ Ε μονὰς τὸν Α ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκις ἄρα ἡ Ε μονὰς τὸν Α ἀριθμὸν μετρεῖ καὶ ὁ Β τὸν Γ. ἐναλλὰξ ἄρα ἰσάκις ἡ Ε μονὰς τὸν Β ἀριθμὸν μετρεῖ καὶ ὁ Α τὸν Γ. πάλιν, ἐπεὶ ὁ Β τὸν Α πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ Α ἄρα τὸν Δ μετρεῖ κατὰ τὰς ἐν τῷ Β μονάδας. μετρεῖ δὲ καὶ ἡ Ε μονὰς τὸν Β κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκις ἄρα ἡ Ε μονὰς τὸν Β ἀριθμὸν μετρεῖ καὶ ὁ Α τὸν Δ. ἰσάκις δὲ ἡ Ε μονὰς τὸν Β ἀριθμὸν ἐμέτρει καὶ ὁ Α τὸν Γ· ἰσάκις ἄρα ὁ Α ἑκάτερον τῶν Γ, Δ μετρεῖ. ἴσος ἄρα ἐστὶν ὁ Γ τῷ Δ· ὅπερ ἔδει δεῖξαι.

ιζ΄. Ἐὰν ἀριθμὸς δύο ἀριθμοὺς πολλαπλασιάσας ποιῇ τινας, οἱ γενόμενοι ἐξ αὐτῶν τὸν αὐτὸν ἕξουσι λόγον τοῖς πολλαπλασιασθεῖσιν.

Ἀριθμὸς γὰρ ὁ Α δύο ἀριθμοὺς τοὺς Β, Γ πολλαπλασιάσας τοὺς Δ, Ε ποιείτω· λέγω, ὅτι ἐστὶν ὡς ὁ Β πρὸς τὸν Γ, οὕτως ὁ Δ πρὸς τὸν Ε.

Ἐπεὶ γὰρ ὁ Α τὸν Β πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ Β ἄρα τὸν Δ μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας. μετρεῖ δὲ καὶ ἡ Ζ μονὰς τὸν Α ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκις ἄρα ἡ Ζ μονὰς τὸν Α ἀριθμὸν μετρεῖ καὶ ὁ Β τὸν Δ. ἔστιν ἄρα ὡς ἡ Ζ μονὰς πρὸς τὸν Α ἀριθμόν, οὕτως ὁ Β πρὸς τὸν Δ.

διὰ τὰ αὐτὰ δὴ καὶ ὡς ἡ Ζ μονὰς πρὸς τὸν Α ἀριθμόν, οὕτως ὁ Γ πρὸς τὸν Ε· καὶ ὡς ἄρα ὁ Β πρὸς τὸν Δ, οὕτως ὁ Γ πρὸς τὸν Ε. ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Β πρὸς τὸν Γ, οὕτως ὁ Δ πρὸς τὸν Ε· ὅπερ ἔδει δεῖξαι.

ιη΄. Ἐὰν δύο ἀριθμοὶ ἀριθμόν τινα πολλαπλασιάσαντες ποιῶσί τινας, οἱ γενόμενοι ἐξ αὐτῶν τὸν αὐτὸν ἕξουσι λόγον τοῖς πολλαπλασιάσασιν.

Δύο γὰρ ἀριθμοὶ οἱ Α, Β ἀριθμόν τινα τὸν Γ πολλαπλασιάσαντες τοὺς Δ, Ε ποιείτωσαν· λέγω, ὅτι ἐστὶν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε.

Ἐπεὶ γὰρ ὁ Α τὸν Γ πολλαπλασιάσας τὸν Δ πεποίηκεν, καὶ ὁ Γ ἄρα τὸν Α πολλαπλασιάσας τὸν Δ πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Γ τὸν Β πολλαπλασιάσας τὸν Ε πεποίηκεν. ἀριθμὸς δὴ ὁ Γ δύο ἀριθμοὺς τοὺς Α, Β πολλαπλασιάσας τοὺς Δ, Ε πεποίηκεν. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε· ὅπερ ἔδει δεῖξαι.

ιθ΄. Ἐὰν τέσσαρες ἀριθμοὶ ἀνάλογον ὦσιν, ὁ ἐκ πρώτου καὶ τετάρτου γενόμενος ἀριθμὸς ἴσος ἔσται τῷ ἐκ δευτέρου καὶ τρίτου γενομένῳ ἀριθμῷ· καὶ ἐὰν ὁ ἐκ πρώτου καὶ τετάρτου γενόμενος ἀριθμὸς ἴσος ᾖ τῷ ἐκ δευτέρου καὶ τρίτου, οἱ τέσσαρες ἀριθμοὶ ἀνάλογον ἔσονται.

Ἔστωσαν τέσσαρες ἀριθμοὶ ἀνάλογον οἱ Α, Β, Γ, Δ, ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Γ πρὸς τὸν Δ, καὶ ὁ μὲν Α τὸν Δ πολλαπλασιάσας τὸν Ε ποιείτω, ὁ δὲ Β τὸν Γ πολλαπλασιάσας τὸν Ζ ποιείτω· λέγω, ὅτι ἴσος ἐστὶν ὁ Ε τῷ Ζ.

Ὁ γὰρ Α τὸν Γ πολλαπλασιάσας τὸν Η ποιείτω. ἐπεὶ οὖν ὁ Α τὸν Γ πολλαπλασιάσας τὸν Η πεποίηκεν, τὸν δὲ Δ πολλαπλασιάσας τὸν Ε πεποίηκεν, ἀριθμὸς δὴ ὁ Α δύο ἀριθμοὺς τοὺς Γ, Δ πολλαπλασιάσας τοὺς Η, Ε πεποίηκεν. ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Η πρὸς τὸν Ε. ἀλλ’ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Β· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, οὕτως ὁ Η πρὸς τὸν Ε. πάλιν, ἐπεὶ ὁ Α τὸν Γ πολλαπλασιάσας τὸν Η πεποίηκεν, ἀλλὰ μὴν καὶ ὁ Β τὸν Γ πολλαπλασιάσας τὸν Ζ πεποίηκεν, δύο δὴ ἀριθμοὶ οἱ Α, Β

ἀριθμόν τινα τὸν Γ πολλαπλασιάσαντες τοὺς Η, Ζ πεποιήκασιν. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Η πρὸς τὸν Ζ. ἀλλὰ μὴν καὶ ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Η πρὸς τὸν Ε· καὶ ὡς ἄρα ὁ Η πρὸς τὸν Ε, οὕτως ὁ Η πρὸς τὸν Ζ. ὁ Η ἄρα πρὸς ἑκάτερον τῶν Ε, Ζ τὸν αὐτὸν ἔχει λόγον· ἴσος ἄρα ἐστὶν ὁ Ε τῷ Ζ.

Ἔστω δὴ πάλιν ἴσος ὁ Ε τῷ Ζ· λέγω, ὅτι ἐστὶν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Γ πρὸς τὸν Δ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ ἴσος ἐστὶν ὁ Ε τῷ Ζ, ἔστιν ἄρα ὡς ὁ Η πρὸς τὸν Ε, οὕτως ὁ Η πρὸς τὸν Ζ. ἀλλ' ὡς μὲν ὁ Η πρὸς τὸν Ε, οὕτως ὁ Γ πρὸς τὸν Δ, ὡς δὲ ὁ Η πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Β. καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, οὕτως ὁ Γ πρὸς τὸν Δ· ὅπερ ἔδει δεῖξαι

κ΄. Οἱ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα.

Ἔστωσαν γὰρ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β οἱ ΓΔ, ΕΖ· λέγω, ὅτι ἰσάκις ὁ ΓΔ τὸν Α μετρεῖ καὶ ὁ ΕΖ τὸν Β.

Ὁ ΓΔ γὰρ τοῦ Α οὐκ ἐστι μέρη. εἰ γὰρ δυνατόν, ἔστω· καὶ ὁ ΕΖ ἄρα τοῦ Β τὰ αὐτὰ μέρη ἐστίν, ἅπερ ὁ ΓΔ τοῦ Α. ὅσα ἄρα ἐστὶν ἐν τῷ ΓΔ μέρη τοῦ Α, τοσαῦτά ἐστι καὶ ἐν τῷ ΕΖ μέρη τοῦ Β. διῃρήσθω ὁ μὲν ΓΔ εἰς τὰ τοῦ Α μέρη τὰ ΓΗ, ΗΔ, ὁ δὲ ΕΖ εἰς τὰ τοῦ Β μέρη τὰ ΕΘ, ΘΖ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΓΗ, ΗΔ τῷ πλήθει τῶν ΕΘ, ΘΖ. καὶ ἐπεὶ ἴσοι εἰσὶν οἱ ΓΗ, ΗΔ ἀριθμοὶ ἀλλήλοις, εἰσὶ δὲ καὶ οἱ ΕΘ, ΘΖ ἀριθμοὶ ἴσοι ἀλλήλοις, καί ἐστιν ἴσον τὸ πλῆθος τῶν ΓΗ, ΗΔ τῷ πλήθει τῶν ΕΘ, ΘΖ, ἔστιν ἄρα ὡς ὁ ΓΗ πρὸς τὸν ΕΘ, οὕτως ὁ ΗΔ πρὸς τὸν ΘΖ. ἔσται ἄρα καὶ ὡς εἷς τῶν ἡγουμένων πρὸς ἕνα τῶν ἑπομένων, οὕτως ἅπαντες οἱ ἡγούμενοι πρὸς ἅπαντας τοὺς ἑπομένους. ἔστιν ἄρα ὡς ὁ ΓΗ πρὸς τὸν ΕΘ, οὕτως ὁ ΓΔ πρὸς τὸν ΕΖ· οἱ ΓΗ, ΕΘ ἄρα τοῖς ΓΔ, ΕΖ ἐν τῷ αὐτῷ λόγῳ εἰσὶν ἐλάσσονες ὄντες αὐτῶν· ὅπερ ἐστὶν ἀδύνατον·

ὑπόκεινται γὰρ οἱ ΓΔ, ΕΖ ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς. οὐκ ἄρα μέρη ἐστὶν ὁ ΓΔ τοῦ Α· μέρος ἄρα. καὶ ὁ ΕΖ τοῦ Β τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὁ ΓΔ τοῦ Α· ἰσάκις ἄρα ὁ ΓΔ τὸν Α μετρεῖ καὶ ὁ ΕΖ τὸν Β· ὅπερ ἔδει δεῖξαι.

κα΄. Οἱ πρῶτοι πρὸς ἀλλήλους ἀριθμοὶ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Ἔστωσαν πρῶτοι πρὸς ἀλλήλους ἀριθμοὶ οἱ Α, Β· λέγω, ὅτι οἱ Α, Β ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Εἰ γὰρ μή, ἔσονταί τινες τῶν Α, Β ἐλάσσονες ἀριθμοὶ ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς Α, Β. ἔστωσαν οἱ Γ, Δ.

Ἐπεὶ οὖν οἱ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάττων τὸν ἐλάττονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον, ἰσάκις ἄρα ὁ Γ τὸν Α μετρεῖ καὶ ὁ Δ τὸν Β. ὁσάκις δὴ ὁ Γ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε. καὶ ὁ Δ ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας. καὶ ἐπεὶ ὁ Γ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας, καὶ ὁ Ε ἄρα τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδας. διὰ τὰ αὐτὰ δὴ ὁ Ε καὶ τὸν Β μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας. ὁ Ε ἄρα τοὺς Α, Β μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἔσονταί τινες τῶν Α, Β ἐλάσσονες ἀριθμοὶ ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς Α, Β. οἱ Α, Β ἄρα ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς· ὅπερ ἔδει δεῖξαι.

κβ΄. Οἱ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς πρῶτοι πρὸς ἀλλήλους εἰσίν.

Ἔστωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς οἱ Α, Β· λέγω, ὅτι οἱ Α, Β πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ γὰρ μή εἰσι πρῶτοι πρὸς ἀλλήλους, μετρήσει τις αὐτοὺς ἀριθμός. μετρείτω, καὶ ἔστω ὁ Γ. καὶ ὁσάκις μὲν ὁ Γ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Δ, ὁσάκις δὲ ὁ Γ τὸν Β μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε.

Ἐπεὶ ὁ Γ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας, ὁ Γ ἄρα τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Γ τὸν Ε πολλαπλασιάσας τὸν Β πεποίηκεν. ἀριθμὸς δὴ ὁ Γ δύο ἀριθμοὺς τοὺς Δ, Ε πολλαπλασιάσας τοὺς Α, Β πεποίηκεν· ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Β· οἱ Δ, Ε ἄρα τοῖς Α, Β ἐν τῷ αὐτῷ λόγῳ εἰσὶν ἐλάσσονες ὄντες αὐτῶν· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Α, Β ἀριθμοὺς ἀριθμός τις μετρήσει. οἱ Α, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

κγ΄. Ἐὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, ὁ τὸν ἕνα αὐτῶν μετρῶν ἀριθμὸς πρὸς τὸν λοιπὸν πρῶτος ἔσται.

Ἔστωσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ Α, Β, τὸν δὲ Α μετρείτω τις ἀριθμὸς ὁ Γ· λέγω, ὅτι καὶ οἱ Γ, Β πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ γὰρ μή εἰσιν οἱ Γ, Β πρῶτοι πρὸς ἀλλήλους, μετρήσει [τις] τοὺς Γ, Β ἀριθμός. μετρείτω, καὶ ἔστω ὁ Δ. ἐπεὶ ὁ Δ τὸν Γ μετρεῖ, ὁ δὲ Γ τὸν Α μετρεῖ, καὶ ὁ Δ ἄρα τὸν Α μετρεῖ. μετρεῖ δὲ καὶ τὸν Β· ὁ Δ ἄρα τοὺς Α, Β μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Γ, Β ἀριθμοὺς ἀριθμός τις μετρήσει. οἱ Γ, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

κδ΄. Ἐὰν δύο ἀριθμοὶ πρός τινα ἀριθμὸν πρῶτοι ὦσιν, καὶ ὁ ἐξ αὐτῶν γενόμενος πρὸς τὸν αὐτὸν πρῶτος ἔσται.

Δύο γὰρ ἀριθμοὶ οἱ Α, Β πρός τινα ἀριθμὸν τὸν Γ πρῶτοι ἔστωσαν, καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Δ ποιείτω· λέγω, ὅτι οἱ Γ, Δ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ γὰρ μή εἰσιν οἱ Γ, Δ πρῶτοι πρὸς ἀλλήλους, μετρήσει [τις] τοὺς Γ, Δ ἀριθμός. μετρείτω, καὶ ἔστω ὁ Ε. καὶ ἐπεὶ οἱ Γ, Α πρῶτοι πρὸς ἀλλήλους εἰσίν, τὸν δὲ Γ μετρεῖ τις ἀριθμὸς ὁ Ε, οἱ Α, Ε ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. ὁσάκις δὴ ὁ Ε τὸν Δ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ζ· καὶ ὁ Ζ ἄρα τὸν Δ μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας. ὁ Ε ἄρα τὸν Ζ πολλαπλασιάσας τὸν Δ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Δ πεποίηκεν· ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν Ε, Ζ τῷ ἐκ τῶν Α, Β. ἐὰν δὲ ὁ ὑπὸ τῶν ἄκρων ἴσος ᾖ τῷ ὑπὸ τῶν μέσων, οἱ τέσσαρες ἀριθμοὶ ἀνάλογόν εἰσιν· ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Α, οὕτως ὁ Β πρὸς τὸν Ζ. οἱ δὲ Α, Ε πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον· ὁ Ε ἄρα τὸν Β μετρεῖ. μετρεῖ δὲ καὶ τὸν Γ· ὁ Ε ἄρα τοὺς Β, Γ μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Γ, Δ ἀριθμοὺς ἀριθμός τις μετρήσει. οἱ Γ, Δ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

κε΄. Ἐὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, ὁ ἐκ τοῦ ἑνὸς αὐτῶν γενόμενος πρὸς τὸν λοιπὸν πρῶτος ἔσται.

Ἔστωσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ Α, Β, καὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι οἱ Β, Γ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Κείσθω γὰρ τῷ Α ἴσος ὁ Δ. ἐπεὶ οἱ Α, Β πρῶτοι πρὸς ἀλλήλους εἰσίν, ἴσος δὲ ὁ Α τῷ Δ, καὶ οἱ Δ, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. ἑκάτερος ἄρα τῶν Δ, Α πρὸς τὸν Β πρῶτός ἐστιν· καὶ ὁ ἐκ τῶν Δ, Α ἄρα γενόμενος πρὸς τὸν Β πρῶτος ἔσται. ὁ δὲ ἐκ τῶν Δ, Α γενόμενος ἀριθμός ἐστιν ὁ Γ. οἱ Γ, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

κς΄. Ἐὰν δύο ἀριθμοὶ πρὸς δύο ἀριθμοὺς ἀμφότεροι πρὸς ἑκάτερον πρῶτοι ὦσιν, καὶ οἱ ἐξ αὐτῶν γενόμενοι πρῶτοι πρὸς ἀλλήλους ἔσονται.

Δύο γὰρ ἀριθμοὶ οἱ Α, Β πρὸς δύο ἀριθμοὺς τοὺς Γ, Δ ἀμφότεροι πρὸς ἑκάτερον πρῶτοι ἔστωσαν, καὶ ὁ μὲν Α τὸν Β πολλαπλασιάσας τὸν Ε ποιείτω, ὁ δὲ Γ τὸν Δ πολλαπλασιάσας τὸν Ζ ποιείτω· λέγω, ὅτι οἱ Ε, Ζ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Ἐπεὶ γὰρ ἑκάτερος τῶν Α, Β πρὸς τὸν Γ πρῶτός ἐστιν, καὶ ὁ ἐκ τῶν Α, Β ἄρα γενόμενος πρὸς τὸν Γ πρῶτος ἔσται. ὁ δὲ ἐκ τῶν Α, Β γενόμενός ἐστιν ὁ Ε· οἱ Ε, Γ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. διὰ τὰ αὐτὰ δὴ καὶ οἱ Δ, Ε πρῶτοι πρὸς ἀλλήλους εἰσίν. ἑκάτερος ἄρα τῶν Γ, Δ πρὸς τὸν Ε πρῶτός ἐστιν. καὶ ὁ ἐκ τῶν Γ, Δ ἄρα γενόμενος πρὸς τὸν Ε πρῶτος ἔσται. ὁ δὲ ἐκ τῶν Γ, Δ γενόμενός ἐστιν ὁ Ζ. οἱ Ε, Ζ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

κζ΄. Ἐὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, καὶ πολλαπλασιάσας ἑκάτερος ἑαυτὸν ποιῇ τινα, οἱ γενόμενοι ἐξ αὐτῶν πρῶτοι πρὸς ἀλλήλους ἔσονται, κἂν οἱ ἐξ ἀρχῆς τοὺς γενομένους πολλαπλασιάσαντες ποιῶσί τινας, κἀκεῖνοι πρῶτοι πρὸς ἀλλήλους ἔσονται [καὶ ἀεὶ περὶ τοὺς ἄκρους τοῦτο συμβαίνει].

Ἔστωσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ Α, Β, καὶ ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Γ ποιείτω, τὸν δὲ Γ πολλαπλασιάσας τὸν Δ ποιείτω, ὁ δὲ Β ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε ποιείτω, τὸν δὲ Ε πολλαπλασιάσας τὸν Ζ ποιείτω· λέγω, ὅτι οἵ τε Γ, Ε καὶ οἱ Δ, Ζ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Ἐπεὶ γὰρ οἱ Α, Β πρῶτοι πρὸς ἀλλήλους εἰσίν, καὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Γ πεποίηκεν, οἱ Γ, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐπεὶ οὖν οἱ Γ, Β πρῶτοι πρὸς ἀλλήλους εἰσίν, καὶ ὁ Β ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηκεν, οἱ Γ, Ε ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν.

πάλιν, ἐπεὶ οἱ Α, Β πρῶτοι πρὸς ἀλλήλους εἰσίν, καὶ ὁ Β ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηκεν, οἱ Α, Ε ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐπεὶ οὖν δύο ἀριθμοὶ οἱ Α, Γ πρὸς δύο ἀριθμοὺς τοὺς Β, Ε ἀμφότεροι πρὸς ἑκάτερον πρῶτοί εἰσιν, καὶ ὁ ἐκ τῶν Α, Γ ἄρα γενόμενος πρὸς τὸν ἐκ τῶν Β, Ε πρῶτός ἐστιν. καί ἐστιν ὁ μὲν ἐκ τῶν Α, Γ ὁ Δ, ὁ δὲ ἐκ τῶν Β, Ε ὁ Ζ. οἱ Δ, Ζ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

κη΄. Ἐὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, καὶ συναμφότερος πρὸς ἑκάτερον αὐτῶν πρῶτος ἔσται· καὶ ἐὰν συναμφότερος πρὸς ἕνα τινὰ αὐτῶν πρῶτος ᾖ, καὶ οἱ ἐξ ἀρχῆς ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ἔσονται.

Συγκείσθωσαν γὰρ δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ ΑΒ, ΒΓ· λέγω, ὅτι καὶ συναμφότερος ὁ ΑΓ πρὸς ἑκάτερον τῶν ΑΒ, ΒΓ πρῶτός ἐστιν.

Εἰ γὰρ μή εἰσιν οἱ ΓΑ, ΑΒ πρῶτοι πρὸς ἀλλήλους, μετρήσει τις τοὺς ΓΑ, ΑΒ ἀριθμός. μετρείτω, καὶ ἔστω ὁ Δ. ἐπεὶ οὖν ὁ Δ τοὺς ΓΑ, ΑΒ μετρεῖ, καὶ λοιπὸν ἄρα τὸν ΒΓ μετρήσει. μετρεῖ δὲ καὶ τὸν ΒΑ· ὁ Δ ἄρα τοὺς ΑΒ, ΒΓ μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς ΓΑ, ΑΒ ἀριθμοὺς ἀριθμός τις μετρήσει· οἱ ΓΑ, ΑΒ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. διὰ τὰ αὐτὰ δὴ καὶ οἱ ΑΓ, ΓΒ πρῶτοι πρὸς ἀλλήλους εἰσίν. ὁ ΓΑ ἄρα πρὸς ἑκάτερον τῶν ΑΒ, ΒΓ πρῶτός ἐστιν.

Ἔστωσαν δὴ πάλιν οἱ ΓΑ, ΑΒ πρῶτοι πρὸς ἀλλήλους· λέγω, ὅτι καὶ οἱ ΑΒ, ΒΓ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ γὰρ μή εἰσιν οἱ ΑΒ, ΒΓ πρῶτοι πρὸς ἀλλήλους, μετρήσει τις τοὺς ΑΒ, ΒΓ ἀριθμός. μετρείτω, καὶ ἔστω ὁ Δ. καὶ ἐπεὶ ὁ Δ ἑκάτερον τῶν ΑΒ, ΒΓ μετρεῖ, καὶ ὅλον ἄρα τὸν ΓΑ μετρήσει. μετρεῖ δὲ καὶ τὸν ΑΒ· ὁ Δ ἄρα τοὺς ΓΑ, ΑΒ μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς ΑΒ, ΒΓ ἀριθμοὺς ἀριθμός τις μετρήσει. οἱ ΑΒ, ΒΓ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

κθ΄. Ἅπας πρῶτος ἀριθμὸς πρὸς ἅπαντα ἀριθμόν, ὃν μὴ μετρεῖ, πρῶτός ἐστιν.

Ἔστω πρῶτος ἀριθμὸς ὁ Α καὶ τὸν Β μὴ μετρείτω· λέγω, ὅτι οἱ Β, Α πρῶτοι πρὸς ἀλλήλους εἰσίν. εἰ γὰρ μή εἰσιν οἱ Β, Α πρῶτοι πρὸς ἀλλήλους, μετρήσει τις αὐτοὺς ἀριθμός. μετρείτω ὁ Γ. ἐπεὶ ὁ Γ τὸν Β μετρεῖ, ὁ δὲ Α τὸν Β οὐ μετρεῖ, ὁ Γ ἄρα τῷ Α οὐκ ἐστιν ὁ αὐτός. καὶ ἐπεὶ ὁ Γ τοὺς Β, Α μετρεῖ, καὶ τὸν Α ἄρα μετρεῖ πρῶτον ὄντα μὴ ὢν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Β, Α μετρήσει τις ἀριθμός. οἱ Α, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

λ΄. Ἐὰν δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, τὸν δὲ γενόμενον ἐξ αὐτῶν μετρῇ τις πρῶτος ἀριθμός, καὶ ἕνα τῶν ἐξ ἀρχῆς μετρήσει.

Δύο γὰρ ἀριθμοὶ οἱ Α, Β πολλαπλασιάσαντες ἀλλήλους τὸν Γ ποιείτωσαν, τὸν δὲ Γ μετρείτω τις πρῶτος ἀριθμὸς ὁ Δ· λέγω, ὅτι ὁ Δ ἕνα τῶν Α, Β μετρεῖ.

Τὸν γὰρ Α μὴ μετρείτω· καί ἐστι πρῶτος ὁ Δ· οἱ Α, Δ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ὁσάκις ὁ Δ τὸν Γ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε. ἐπεὶ οὖν ὁ Δ τὸν Γ μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας, ὁ Δ ἄρα τὸν Ε πολλαπλασιάσας τὸν Γ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν· ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν Δ, Ε τῷ ἐκ τῶν Α, Β. ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Α, οὕτως ὁ Β πρὸς τὸν Ε. οἱ δὲ Δ, Α πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον· ὁ Δ ἄρα τὸν Β μετρεῖ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἐὰν τὸν Β μὴ μετρῇ, τὸν Α μετρήσει. ὁ Δ ἄρα ἕνα τῶν Α, Β μετρεῖ· ὅπερ ἔδει δεῖξαι.

λα΄. Ἅπας σύνθετος ἀριθμὸς ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται.

Ἔστω σύνθετος ἀριθμὸς ὁ Α· λέγω, ὅτι ὁ Α ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται.

Ἐπεὶ γὰρ σύνθετός ἐστιν ὁ Α, μετρήσει τις αὐτὸν ἀριθμός. μετρείτω, καὶ ἔστω ὁ Β. καὶ εἰ μὲν πρῶτός ἐστιν ὁ Β, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν. εἰ δὲ σύνθετος, μετρήσει τις αὐτὸν ἀριθμός. μετρείτω, καὶ ἔστω ὁ Γ. καὶ ἐπεὶ ὁ Γ τὸν Β μετρεῖ, ὁ δὲ Β τὸν Α μετρεῖ, καὶ ὁ Γ ἄρα τὸν Α μετρεῖ. καὶ εἰ μὲν πρῶτός ἐστιν ὁ Γ, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν. εἰ δὲ σύνθετος, μετρήσει τις αὐτὸν ἀριθμός. τοιαύτης δὴ γινομένης ἐπισκέψεως ληφθήσεταί τις πρῶτος ἀριθμός, ὃς μετρήσει. εἰ γὰρ οὐ ληφθήσεται, μετρήσουσι τὸν Α ἀριθμὸν ἄπειροι ἀριθμοί, ὧν ἕτερος ἑτέρου ἐλάσσων ἐστίν· ὅπερ ἐστὶν ἀδύνατον ἐν ἀριθμοῖς. ληφθήσεταί τις ἄρα πρῶτος ἀριθμός, ὃς μετρήσει τὸν πρὸ ἑαυτοῦ, ὃς καὶ τὸν Α μετρήσει.

Ἅπας ἄρα σύνθετος ἀριθμὸς ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται· ὅπερ ἔδει δεῖξαι.

λβ΄. Ἅπας ἀριθμὸς ἤτοι πρῶτός ἐστιν ἢ ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται.

Ἔστω ἀριθμὸς ὁ Α· λέγω, ὅτι ὁ Α ἤτοι πρῶτός ἐστιν ἢ ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται.

Εἰ μὲν οὖν πρῶτός ἐστιν ὁ Α, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν. εἰ δὲ σύνθετος, μετρήσει τις αὐτὸν πρῶτος ἀριθμός.

Ἅπας ἄρα ἀριθμὸς ἤτοι πρῶτός ἐστιν ἢ ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται· ὅπερ ἔδει δεῖξαι.

λγ΄. Ἀριθμῶν δοθέντων ὁποσωνοῦν εὑρεῖν τοὺς ἐλαχίστους τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Ἔστωσαν οἱ δοθέντες ὁποσοιοῦν ἀριθμοὶ οἱ Α, Β, Γ· δεῖ δὴ εὑρεῖν τοὺς ἐλαχίστους τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β, Γ.

Οἱ Α, Β, Γ γὰρ ἤτοι πρῶτοι πρὸς ἀλλήλους εἰσὶν ἢ οὔ. εἰ μὲν οὖν οἱ Α, Β, Γ πρῶτοι πρὸς ἀλλήλους εἰσίν, ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Εἰ δὲ οὔ, εἰλήφθω τῶν Α, Β, Γ τὸ μέγιστον κοινὸν μέτρον ὁ Δ, καὶ ὁσάκις ὁ Δ ἕκαστον τῶν Α, Β, Γ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν ἑκάστῳ τῶν Ε, Ζ, Η. καὶ ἕκαστος ἄρα τῶν Ε, Ζ, Η ἕκαστον τῶν Α, Β, Γ μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας. οἱ Ε, Ζ, Η ἄρα τοὺς Α, Β, Γ ἰσάκις μετροῦσιν· οἱ Ε, Ζ, Η ἄρα τοῖς Α, Β, Γ ἐν τῷ αὐτῷ λόγῳ εἰσίν. λέγω δή, ὅτι καὶ ἐλάχιστοι. εἰ γὰρ μή εἰσιν οἱ Ε, Ζ, Η ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β, Γ, ἔσονται [τινες] τῶν Ε, Ζ, Η ἐλάσσονες ἀριθμοὶ ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς Α, Β, Γ. ἔστωσαν οἱ Θ, Κ, Λ· ἰσάκις ἄρα ὁ Θ τὸν Α μετρεῖ καὶ ἑκάτερος τῶν Κ, Λ ἑκάτερον τῶν Β, Γ. ὁσάκις δὲ ὁ Θ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Μ· καὶ ἑκάτερος ἄρα τῶν Κ, Λ ἑκάτερον τῶν Β, Γ μετρεῖ κατὰ τὰς ἐν τῷ Μ μονάδας. καὶ ἐπεὶ ὁ Θ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Μ μονάδας, καὶ ὁ Μ ἄρα τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Θ μονάδας. διὰ τὰ αὐτὰ δὴ ὁ Μ καὶ ἑκάτερον τῶν Β, Γ μετρεῖ κατὰ τὰς ἐν ἑκατέρῳ τῶν Κ, Λ μονάδας· ὁ Μ ἄρα τοὺς Α, Β, Γ μετρεῖ. καὶ ἐπεὶ ὁ Θ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Μ μονάδας, ὁ Θ ἄρα τὸν Μ πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Ε τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν. ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν Ε, Δ τῷ ἐκ τῶν Θ, Μ. ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Μ πρὸς τὸν Δ. μείζων δὲ ὁ Ε τοῦ Θ· μείζων ἄρα καὶ ὁ Μ τοῦ Δ. καὶ μετρεῖ τοὺς Α, Β, Γ· ὅπερ ἐστὶν ἀδύνατον· ὑπόκειται γὰρ ὁ Δ τῶν Α, Β, Γ τὸ μέγιστον κοινὸν μέτρον. οὐκ ἄρα ἔσονταί τινες τῶν Ε, Ζ, Η ἐλάσσονες ἀριθμοὶ ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς Α, Β, Γ. οἱ Ε, Ζ, Η ἄρα ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β, Γ· ὅπερ ἔδει δεῖξαι.

λδ΄. Δύο ἀριθμῶν δοθέντων εὑρεῖν, ὃν ἐλάχιστον μετροῦσιν ἀριθμόν.

Ἔστωσαν οἱ δοθέντες δύο ἀριθμοὶ οἱ Α, Β· δεῖ δὴ εὑρεῖν, ὃν ἐλάχιστον μετροῦσιν ἀριθμόν.

Οἱ Α, Β γὰρ ἤτοι πρῶτοι πρὸς ἀλλήλους εἰσὶν ἢ οὔ. ἔστωσαν πρότερον οἱ Α, Β πρῶτοι πρὸς ἀλλήλους, καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· καὶ ὁ Β ἄρα τὸν Α πολλαπλασιάσας τὸν Γ πεποίηκεν. οἱ Α, Β ἄρα τὸν Γ μετροῦσιν. λέγω δή, ὅτι καὶ ἐλάχιστον. εἰ γὰρ μή, μετρήσουσί τινα ἀριθμὸν οἱ Α, Β ἐλάσσονα ὄντα τοῦ Γ. μετρείτωσαν τὸν Δ. καὶ ὁσάκις ὁ Α τὸν Δ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε, ὁσάκις δὲ ὁ Β τὸν Δ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ζ· ὁ μὲν Α ἄρα τὸν Ε πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ δὲ Β τὸν Ζ πολλαπλασιάσας τὸν Δ πεποίηκεν· ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν Α, Ε τῷ ἐκ τῶν Β, Ζ. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Ζ πρὸς τὸν Ε. οἱ δὲ Α, Β πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα· ὁ Β ἄρα τὸν Ε μετρεῖ, ὡς ἑπόμενος ἑπόμενον. καὶ ἐπεὶ ὁ Α τοὺς Β, Ε πολλαπλασιάσας τοὺς Γ, Δ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Β πρὸς τὸν Ε, οὕτως ὁ Γ πρὸς τὸν Δ. μετρεῖ δὲ ὁ Β τὸν Ε· μετρεῖ ἄρα καὶ ὁ Γ τὸν Δ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οἱ Α, Β μετροῦσί τινα ἀριθμὸν ἐλάσσονα ὄντα τοῦ Γ. ὁ Γ ἄρα ἐλάχιστος ὢν ὑπὸ τῶν Α, Β μετρεῖται.

Μὴ ἔστωσαν δὴ οἱ Α, Β πρῶτοι πρὸς ἀλλήλους, καὶ εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β οἱ Ζ, Ε· ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν Α, Ε τῷ ἐκ τῶν Β, Ζ. καὶ ὁ Α τὸν Ε πολλαπλασιάσας τὸν Γ ποιείτω· καὶ ὁ Β ἄρα τὸν Ζ πολλαπλασιάσας τὸν Γ πεποίηκεν· οἱ Α, Β ἄρα τὸν Γ μετροῦσιν. λέγω δή, ὅτι καὶ ἐλάχιστον. εἰ γὰρ μή, μετρήσουσί τινα ἀριθμὸν οἱ Α, Β ἐλάσσονα ὄντα τοῦ Γ. μετρείτωσαν τὸν Δ. καὶ ὁσάκις μὲν ὁ Α τὸν Δ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Η, ὁσάκις δὲ ὁ Β τὸν Δ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Θ. ὁ μὲν Α ἄρα τὸν Η πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ δὲ Β τὸν Θ πολλαπλασιάσας τὸν Δ

πεποίηκεν. ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν Α, Η τῷ ἐκ τῶν Β, Θ· ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Θ πρὸς τὸν Η. ὡς δὲ ὁ Α πρὸς τὸν Β, οὕτως ὁ Ζ πρὸς τὸν Ε· καὶ ὡς ἄρα ὁ Ζ πρὸς τὸν Ε, οὕτως ὁ Θ πρὸς τὸν Η. οἱ δὲ Ζ, Ε ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα· ὁ Ε ἄρα τὸν Η μετρεῖ. καὶ ἐπεὶ ὁ Α τοὺς Ε, Η πολλαπλασιάσας τοὺς Γ, Δ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Η, οὕτως ὁ Γ πρὸς τὸν Δ. ὁ δὲ Ε τὸν Η μετρεῖ· καὶ ὁ Γ ἄρα τὸν Δ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οἱ Α, Β μετρήσουσί τινα ἀριθμὸν ἐλάσσονα ὄντα τοῦ Γ. ὁ Γ ἄρα ἐλάχιστος ὢν ὑπὸ τῶν Α, Β μετρεῖται· ὅπερ ἔδει δεῖξαι

λε΄. Ἐὰν δύο ἀριθμοὶ ἀριθμόν τινα μετρῶσιν, καὶ ὁ ἐλάχιστος ὑπ᾽ αὐτῶν μετρούμενος τὸν αὐτὸν μετρήσει.

Δύο γὰρ ἀριθμοὶ οἱ Α, Β ἀριθμόν τινα τὸν ΓΔ μετρείτωσαν, ἐλάχιστον δὲ τὸν Ε· λέγω, ὅτι καὶ ὁ Ε τὸν ΓΔ μετρεῖ.

Εἰ γὰρ οὐ μετρεῖ ὁ Ε τὸν ΓΔ, ὁ Ε τὸν ΔΖ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ΓΖ. καὶ ἐπεὶ οἱ Α, Β τὸν Ε μετροῦσιν, ὁ δὲ Ε τὸν ΔΖ μετρεῖ, καὶ οἱ Α, Β ἄρα τὸν ΔΖ μετρήσουσιν. μετροῦσι δὲ καὶ ὅλον τὸν ΓΔ· καὶ λοιπὸν ἄρα τὸν ΓΖ μετρήσουσιν ἐλάσσονα ὄντα τοῦ Ε· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οὐ μετρεῖ ὁ Ε τὸν ΓΔ· μετρεῖ ἄρα· ὅπερ ἔδει δεῖξαι.

λς΄. Τριῶν ἀριθμῶν δοθέντων εὑρεῖν, ὃν ἐλάχιστον μετροῦσιν ἀριθμόν.

Ἔστωσαν οἱ δοθέντες τρεῖς ἀριθμοὶ οἱ Α, Β, Γ· δεῖ δὴ εὑρεῖν, ὃν ἐλάχιστον μετροῦσιν ἀριθμόν.

Εἰλήφθω γὰρ ὑπὸ δύο τῶν Α, Β ἐλάχιστος μετρούμενος ὁ Δ. ὁ δὴ Γ τὸν Δ ἤτοι μετρεῖ ἢ οὐ μετρεῖ. μετρείτω πρότερον. μετροῦσι δὲ καὶ οἱ Α, Β τὸν Δ· οἱ Α, Β, Γ ἄρα τὸν Δ μετροῦσιν. λέγω δή, ὅτι καὶ ἐλάχιστον. εἰ γὰρ μή, μετρήσουσιν [τινα] ἀριθμὸν οἱ Α, Β, Γ ἐλάσσονα ὄντα τοῦ Δ. μετρείτωσαν

τὸν Ε. ἐπεὶ οἱ Α, Β, Γ τὸν Ε μετροῦσιν, καὶ οἱ Α, Β ἄρα τὸν Ε μετροῦσιν. καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν Α, Β μετρούμενος [τὸν Ε] μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Α, Β μετρούμενός ἐστιν ὁ Δ· ὁ Δ ἄρα τὸν Ε μετρήσει ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οἱ Α, Β, Γ μετρήσουσί τινα ἀριθμὸν ἐλάσσονα ὄντα τοῦ Δ· οἱ Α, Β, Γ ἄρα ἐλάχιστον τὸν Δ μετροῦσιν.

Μὴ μετρείτω δὴ πάλιν ὁ Γ τὸν Δ, καὶ εἰλήφθω ὑπὸ τῶν Γ, Δ ἐλάχιστος μετρούμενος ἀριθμὸς ὁ Ε. ἐπεὶ οἱ Α, Β τὸν Δ μετροῦσιν, ὁ δὲ Δ τὸν Ε μετρεῖ, καὶ οἱ Α, Β ἄρα τὸν Ε μετροῦσιν. μετρεῖ δὲ καὶ ὁ Γ [τὸν Ε· καὶ] οἱ Α, Β, Γ ἄρα τὸν Ε μετροῦσιν. λέγω δή, ὅτι καὶ ἐλάχιστον. εἰ γὰρ μή, μετρήσουσί τινα οἱ Α, Β, Γ ἐλάσσονα ὄντα τοῦ Ε. μετρείτωσαν τὸν Ζ. ἐπεὶ οἱ Α, Β, Γ τὸν Ζ μετροῦσιν, καὶ οἱ Α, Β ἄρα τὸν Ζ μετροῦσιν· καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν Α, Β μετρούμενος τὸν Ζ μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Α, Β μετρούμενός ἐστιν ὁ Δ· ὁ Δ ἄρα τὸν Ζ μετρεῖ. μετρεῖ δὲ καὶ ὁ Γ τὸν Ζ· οἱ Δ, Γ ἄρα τὸν Ζ μετροῦσιν· ὥστε καὶ ὁ ἐλάχιστος ὑπὸ τῶν Δ, Γ μετρούμενος τὸν Ζ μετρήσει. ὁ δὲ ἐλάχιστος ὑπὸ τῶν Γ, Δ μετρούμενός ἐστιν ὁ Ε· ὁ Ε ἄρα τὸν Ζ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οἱ Α, Β, Γ μετρήσουσί τινα ἀριθμὸν ἐλάσσονα ὄντα τοῦ Ε. ὁ Ε ἄρα ἐλάχιστος ὢν ὑπὸ τῶν Α, Β, Γ μετρεῖται· ὅπερ ἔδει δεῖξαι.

λζ΄. Ἐὰν ἀριθμὸς ὑπό τινος ἀριθμοῦ μετρῆται, ὁ μετρούμενος ὁμώνυμον μέρος ἔξει τῷ μετροῦντι.

Ἀριθμὸς γὰρ ὁ Α ὑπό τινος ἀριθμοῦ τοῦ Β μετρείσθω· λέγω, ὅτι ὁ Α ὁμώνυμον μέρος ἔχει τῷ Β.

Ὁσάκις γὰρ ὁ Β τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Γ. ἐπεὶ ὁ Β τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδας, μετρεῖ δὲ καὶ ἡ Δ μονὰς τὸν Γ ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας, ἰσάκις ἄρα ἡ Δ μονὰς τὸν Γ ἀριθμὸν μετρεῖ καὶ ὁ Β τὸν Α. ἐναλλὰξ ἄρα ἰσάκις ἡ Δ μονὰς τὸν Β ἀριθμὸν μετρεῖ καὶ ὁ Γ τὸν Α· ὃ ἄρα μέρος ἐστὶν ἡ Δ μονὰς τοῦ Β ἀριθμοῦ, τὸ αὐτὸ μέρος

ἐστὶ καὶ ὁ Γ τοῦ Α. ἡ δὲ Δ μονὰς τοῦ Β ἀριθμοῦ μέρος ἐστὶν ὁμώνυμον αὐτῷ· καὶ ὁ Γ ἄρα τοῦ Α μέρος ἐστὶν ὁμώνυμον τῷ Β. ὥστε ὁ Α μέρος ἔχει τὸν Γ ὁμώνυμον ὄντα τῷ Β· ὅπερ ἔδει δεῖξαι.

λη΄. Ἐὰν ἀριθμὸς μέρος ἔχῃ ὁτιοῦν, ὑπὸ ὁμωνύμου ἀριθμοῦ μετρηθήσεται τῷ μέρει.

Ἀριθμὸς γὰρ ὁ Α μέρος ἐχέτω ὁτιοῦν τὸν Β, καὶ τῷ Β μέρει ὁμώνυμος ἔστω [ἀριθμὸς] ὁ Γ· λέγω, ὅτι ὁ Γ τὸν Α μετρεῖ.

Ἐπεὶ γὰρ ὁ Β τοῦ Α μέρος ἐστὶν ὁμώνυμον τῷ Γ, ἔστι δὲ καὶ ἡ Δ μονὰς τοῦ Γ μέρος ὁμώνυμον αὐτῷ, ὃ ἄρα μέρος ἐστὶν ἡ Δ μονὰς τοῦ Γ ἀριθμοῦ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Β τοῦ Α· ἰσάκις ἄρα ἡ Δ μονὰς τὸν Γ ἀριθμὸν μετρεῖ καὶ ὁ Β τὸν Α. ἐναλλὰξ ἄρα ἰσάκις ἡ Δ μονὰς τὸν Β ἀριθμὸν μετρεῖ καὶ ὁ Γ τὸν Α. ὁ Γ ἄρα τὸν Α μετρεῖ· ὅπερ ἔδει δεῖξαι.

λθ΄. Ἀριθμὸν εὑρεῖν, ὃς ἐλάχιστος ὢν ἕξει τὰ δοθέντα μέρη.

Ἔστω τὰ δοθέντα μέρη τὰ Α, Β, Γ· δεῖ δὴ ἀριθμὸν εὑρεῖν, ὃς ἐλάχιστος ὢν ἕξει τὰ Α, Β, Γ μέρη.

Ἔστωσαν γὰρ τοῖς Α, Β, Γ μέρεσιν ὁμώνυμοι ἀριθμοὶ οἱ Δ, Ε, Ζ, καὶ εἰλήφθω ὑπὸ τῶν Δ, Ε, Ζ ἐλάχιστος μετρούμενος ἀριθμὸς ὁ Η.

Ὁ Η ἄρα ὁμώνυμα μέρη ἔχει τοῖς Δ, Ε, Ζ. τοῖς δὲ Δ, Ε, Ζ ὁμώνυμα μέρη ἐστὶ τὰ Α, Β, Γ· ὁ Η ἄρα ἔχει τὰ Α, Β, Γ μέρη. λέγω δή, ὅτι καὶ ἐλάχιστος ὤν. εἰ γὰρ μή, ἔσται τις τοῦ Η ἐλάσσων ἀριθμός, ὃς ἕξει τὰ Α, Β, Γ μέρη. ἔστω ὁ Θ. ἐπεὶ ὁ Θ ἔχει τὰ Α, Β, Γ μέρη, ὁ Θ ἄρα ὑπὸ ὁμωνύμων ἀριθμῶν μετρηθήσεται τοῖς Α, Β, Γ μέρεσιν. τοῖς δὲ Α, Β, Γ μέρεσιν ὁμώνυμοι ἀριθμοί εἰσιν οἱ Δ, Ε, Ζ· ὁ Θ ἄρα ὑπὸ τῶν Δ, Ε, Ζ μετρεῖται. καί ἐστιν ἐλάσσων τοῦ Η· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἔσται τις τοῦ Η ἐλάσσων ἀριθμός, ὃς ἕξει τὰ Α, Β, Γ μέρη· ὅπερ ἔδει δεῖξαι.

BOOK VIII.

Προτάσεις κζ΄.

α΄. Ἐὰν ὦσιν ὁσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, οἱ δὲ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους ὦσιν, ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Ἔστωσαν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, οἱ δὲ ἄκροι αὐτῶν οἱ Α, Δ πρῶτοι πρὸς ἀλλήλους ἔστωσαν· λέγω, ὅτι οἱ Α, Β, Γ, Δ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Εἰ γὰρ μή, ἔστωσαν ἐλάττονες τῶν Α, Β, Γ, Δ οἱ Ε, Ζ, Η, Θ ἐν τῷ αὐτῷ λόγῳ ὄντες αὐτοῖς. καὶ ἐπεὶ οἱ Α, Β, Γ, Δ ἐν τῷ αὐτῷ λόγῳ εἰσὶ τοῖς Ε, Ζ, Η, Θ, καί ἐστιν ἴσον τὸ πλῆθος [τῶν Α, Β, Γ, Δ] τῷ πλήθει [τῶν Ε, Ζ, Η, Θ], δι’ ἴσου ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Δ, ὁ Ε πρὸς τὸν Θ. οἱ δὲ Α, Δ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον. μετρεῖ ἄρα ὁ Α τὸν Ε ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οἱ Ε, Ζ, Η, Θ ἐλάσσονες ὄντες τῶν Α, Β, Γ, Δ ἐν τῷ αὐτῷ λόγῳ εἰσὶν αὐτοῖς. οἱ Α, Β, Γ, Δ ἄρα ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς· ὅπερ ἔδει δεῖξαι.

β΄. Ἀριθμοὺς εὑρεῖν ἑξῆς ἀνάλογον ἐλαχίστους, ὅσους ἂν ἐπιτάξῃ τις, ἐν τῷ δοθέντι λόγῳ.

Ἔστω ὁ δοθεὶς λόγος ἐν ἐλαχίστοις ἀριθμοῖς ὁ τοῦ Α πρὸς τὸν Β· δεῖ δὴ ἀριθμοὺς εὑρεῖν ἑξῆς ἀνάλογον ἐλαχίστους, ὅσους ἄν τις ἐπιτάξῃ, ἐν τῷ τοῦ Α πρὸς τὸν Β λόγῳ.

Ἐπιτετάχθωσαν δὴ τέσσαρες, καὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Γ ποιείτω, τὸν δὲ Β πολλαπλασιάσας τὸν Δ ποιείτω, καὶ ἔτι ὁ Β ἑαυτὸν

πολλαπλασιάσας τὸν Ε ποιείτω, καὶ ἔτι ὁ Α τοὺς Γ, Δ, Ε πολλαπλασιάσας τοὺς Ζ, Η, Θ ποιείτω, ὁ δὲ Β τὸν Ε πολλαπλασιάσας τὸν Κ ποιείτω.

Καὶ ἐπεὶ ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Γ πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Δ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, [οὕτως] ὁ Γ πρὸς τὸν Δ. πάλιν, ἐπεὶ ὁ μὲν Α τὸν Β πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ δὲ Β ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηκεν, ἑκάτερος ἄρα τῶν Α, Β τὸν Β πολλαπλασιάσας ἑκάτερον τῶν Δ, Ε πεποίηκεν. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε. ἀλλ’ ὡς ὁ Α πρὸς τὸν Β, ὁ Γ πρὸς τὸν Δ· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Δ, ὁ Δ πρὸς τὸν Ε. καὶ ἐπεὶ ὁ Α τοὺς Γ, Δ πολλαπλασιάσας τοὺς Ζ, Η πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, [οὕτως] ὁ Ζ πρὸς τὸν Η. ὡς δὲ ὁ Γ πρὸς τὸν Δ, οὕτως ἦν ὁ Α πρὸς τὸν Β· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, ὁ Ζ πρὸς τὸν Η. πάλιν, ἐπεὶ ὁ Α τοὺς Δ, Ε πολλαπλασιάσας τοὺς Η, Θ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ε, ὁ Η πρὸς τὸν Θ. ἀλλ’ ὡς ὁ Δ πρὸς τὸν Ε, ὁ Α πρὸς τὸν Β. καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, οὕτως ὁ Η πρὸς τὸν Θ. καὶ ἐπεὶ οἱ Α, Β τὸν Ε πολλαπλασιάσαντες τοὺς Θ, Κ πεποιήκασιν, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Θ πρὸς τὸν Κ. ἀλλ’ ὡς ὁ Α πρὸς τὸν Β, οὕτως ὅ τε Ζ πρὸς τὸν Η καὶ ὁ Η πρὸς τὸν Θ. καὶ ὡς ἄρα ὁ Ζ πρὸς τὸν Η, οὕτως ὅ τε Η πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν Κ· οἱ Γ, Δ, Ε ἄρα καὶ οἱ Ζ, Η, Θ, Κ ἀνάλογόν εἰσιν ἐν τῷ τοῦ Α πρὸς τὸν Β λόγῳ. λέγω δή, ὅτι καὶ ἐλάχιστοι. ἐπεὶ γὰρ οἱ Α, Β ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, οἱ δὲ ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων πρῶτοι πρὸς ἀλλήλους εἰσίν, οἱ Α, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἑκάτερος μὲν τῶν Α, Β ἑαυτὸν πολλαπλασιάσας ἑκάτερον τῶν Γ, Ε πεποίηκεν, ἑκάτερον δὲ τῶν Γ, Ε πολλαπλασιάσας ἑκάτερον τῶν Ζ, Κ πεποίηκεν· οἱ Γ, Ε ἄρα καὶ οἱ Ζ, Κ πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐὰν δὲ ὦσιν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, οἱ δὲ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους ὦσιν, ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς. οἱ Γ, Δ, Ε ἄρα καὶ οἱ Ζ, Η, Θ, Κ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β· ὅπερ ἔδει δεῖξαι.

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν τρεῖς ἀριθμοὶ ἑξῆς ἀνάλογον ἐλάχιστοι ὦσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, οἱ ἄκροι αὐτῶν τετράγωνοί εἰσιν, ἐὰν δὲ τέσσαρες, κύβοι.

γ΄. Ἐὰν ὦσιν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, οἱ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν.

Ἔστωσαν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς οἱ Α, Β, Γ, Δ· λέγω, ὅτι οἱ ἄκροι αὐτῶν οἱ Α, Δ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰλήφθωσαν γὰρ δύο μὲν ἀριθμοὶ ἐλάχιστοι ἐν τῷ τῶν Α, Β, Γ, Δ λόγῳ οἱ Ε, Ζ, τρεῖς δὲ οἱ Η, Θ, Κ, καὶ ἑξῆς ἑνὶ πλείους, ἕως τὸ λαμβανόμενον πλῆθος ἴσον γένηται τῷ πλήθει τῶν Α, Β, Γ, Δ. εἰλήφθωσαν καὶ ἔστωσαν οἱ Λ, Μ, Ν, Ξ.

Καὶ ἐπεὶ οἱ Ε, Ζ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ ἑκάτερος τῶν Ε, Ζ ἑαυτὸν μὲν πολλαπλασιάσας ἑκάτερον τῶν Η, Κ πεποίηκεν, ἑκάτερον δὲ τῶν Η, Κ πολλαπλασιάσας ἑκάτερον τῶν Λ, Ξ πεποίηκεν, καὶ οἱ Η, Κ ἄρα καὶ οἱ Λ, Ξ πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ οἱ Α, Β, Γ, Δ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, εἰσὶ δὲ καὶ οἱ Λ, Μ, Ν, Ξ ἐλάχιστοι ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς Α, Β, Γ, Δ, καί ἐστιν ἴσον τὸ πλῆθος τῶν Α, Β, Γ, Δ τῷ πλήθει τῶν Λ, Μ, Ν, Ξ, ἕκαστος ἄρα τῶν Α, Β, Γ, Δ ἑκάστῳ τῶν Λ, Μ, Ν, Ξ ἴσος ἐστίν· ἴσος ἄρα ἐστὶν ὁ μὲν Α τῷ Λ, ὁ δὲ Δ τῷ Ξ. καί εἰσιν οἱ Λ, Ξ πρῶτοι πρὸς ἀλλήλους. καὶ οἱ Α, Δ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

δ΄. Λόγων δοθέντων ὁποσωνοῦν ἐν ἐλαχίστοις ἀριθμοῖς ἀριθμοὺς εὑρεῖν ἑξῆς ἀνάλογον ἐλαχίστους ἐν τοῖς δοθεῖσι λόγοις.

Ἔστωσαν οἱ δοθέντες λόγοι ἐν ἐλαχίστοις ἀριθμοῖς ὅ τε τοῦ Α πρὸς τὸν Β καὶ ὁ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι ὁ τοῦ Ε πρὸς τὸν Ζ· δεῖ δὴ ἀριθμοὺς εὑρεῖν ἑξῆς ἀνάλογον ἐλαχίστους ἔν τε τῷ τοῦ Α πρὸς τὸν Β λόγῳ καὶ ἐν τῷ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι ἐν τῷ τοῦ Ε πρὸς τὸν Ζ.

Εἰλήφθω γὰρ ὁ ὑπὸ τῶν Β, Γ ἐλάχιστος μετρούμενος ἀριθμὸς ὁ Η. καὶ ὁσάκις μὲν ὁ Β τὸν Η μετρεῖ, τοσαυτάκις καὶ ὁ Α τὸν Θ μετρείτω, ὁσάκις δὲ ὁ Γ τὸν Η μετρεῖ, τοσαυτάκις καὶ ὁ Δ τὸν Κ μετρείτω. ὁ δὲ Ε τὸν Κ ἤτοι μετρεῖ ἢ οὐ μετρεῖ. μετρείτω πρότερον. καὶ ὁσάκις ὁ Ε τὸν Κ μετρεῖ, τοσαυτάκις καὶ ὁ Ζ τὸν Λ μετρείτω. καὶ ἐπεὶ ἰσάκις ὁ Α τὸν Θ μετρεῖ καὶ ὁ Β τὸν Η, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Θ πρὸς τὸν Η. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Η πρὸς τὸν Κ, καὶ ἔτι ὡς ὁ Ε πρὸς τὸν Ζ, οὕτως ὁ Κ πρὸς τὸν Λ· οἱ Θ, Η, Κ, Λ ἄρα ἑξῆς ἀνάλογόν εἰσιν ἔν τε τῷ τοῦ Α πρὸς τὸν Β καὶ ἐν τῷ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι ἐν τῷ τοῦ Ε πρὸς τὸν Ζ λόγῳ. λέγω δή, ὅτι καὶ ἐλάχιστοι. εἰ γὰρ μή εἰσιν οἱ Θ, Η, Κ, Λ ἑξῆς ἀνάλογον ἐλάχιστοι ἔν τε τοῖς τοῦ Α πρὸς τὸν Β καὶ τοῦ Γ πρὸς τὸν Δ καὶ ἐν τῷ τοῦ Ε πρὸς τὸν Ζ λόγοις, ἔστωσαν οἱ Ν, Ξ, Μ, Ο. καὶ ἐπεί ἐστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Ν πρὸς τὸν Ξ, οἱ δὲ Α, Β ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον, ὁ Β ἄρα τὸν Ξ μετρεῖ. διὰ τὰ αὐτὰ δὴ καὶ ὁ Γ τὸν Ξ μετρεῖ· οἱ Β, Γ ἄρα τὸν Ξ μετροῦσιν· καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν Β, Γ μετρούμενος τὸν Ξ μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Β, Γ μετρεῖται ὁ Η· ὁ Η ἄρα τὸν Ξ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἔσονταί τινες τῶν Θ, Η, Κ, Λ ἐλάσσονες ἀριθμοὶ ἑξῆς ἔν τε τῷ τοῦ Α πρὸς τὸν Β καὶ τῷ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τῷ τοῦ Ε πρὸς τὸν Ζ λόγῳ.

Μὴ μετρείτω δὴ ὁ Ε τὸν Κ. καὶ εἰλήφθω ὑπὸ τῶν Ε, Κ ἐλάχιστος μετρούμενος ἀριθμὸς ὁ Μ. καὶ ὁσάκις μὲν ὁ Κ τὸν Μ μετρεῖ, τοσαυτάκις καὶ ἑκάτερος τῶν Θ, Η ἑκάτερον τῶν Ν, Ξ μετρείτω, ὁσάκις δὲ ὁ Ε τὸν Μ μετρεῖ,

τοσαυτάκις καὶ ὁ Ζ τὸν Ο μετρείτω. ἐπεὶ ἰσάκις ὁ Θ τὸν Ν μετρεῖ καὶ ὁ Η τὸν Ξ, ἔστιν ἄρα ὡς ὁ Θ πρὸς τὸν Η, οὕτως ὁ Ν πρὸς τὸν Ξ. ὡς δὲ ὁ Θ πρὸς τὸν Η, οὕτως ὁ Α πρὸς τὸν Β· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, οὕτως ὁ Ν πρὸς τὸν Ξ. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ξ πρὸς τὸν Μ. πάλιν, ἐπεὶ ἰσάκις ὁ Ε τὸν Μ μετρεῖ καὶ ὁ Ζ τὸν Ο, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Ζ, οὕτως ὁ Μ πρὸς τὸν Ο· οἱ Ν, Ξ, Μ, Ο ἄρα ἑξῆς ἀνάλογόν εἰσιν ἐν τοῖς τοῦ τε Α πρὸς τὸν Β καὶ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τοῦ Ε πρὸς τὸν Ζ λόγοις. λέγω δή, ὅτι καὶ ἐλάχιστοι ἐν τοῖς ΑΒ, ΓΔ, ΕΖ λόγοις. εἰ γὰρ μή, ἔσονταί τινες τῶν Ν, Ξ, Μ, Ο ἐλάσσονες ἀριθμοὶ ἑξῆς ἀνάλογον ἐν τοῖς ΑΒ, ΓΔ, ΕΖ λόγοις. ἔστωσαν οἱ Π, Ρ, Σ, Τ. καὶ ἐπεί ἐστιν ὡς ὁ Π πρὸς τὸν Ρ, οὕτως ὁ Α πρὸς τὸν Β, οἱ δὲ Α, Β ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς ἰσάκις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον, ὁ Β ἄρα τὸν Ρ μετρεῖ. διὰ τὰ αὐτὰ δὴ καὶ ὁ Γ τὸν Ρ μετρεῖ· οἱ Β, Γ ἄρα τὸν Ρ μετροῦσιν. καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν Β, Γ μετρούμενος τὸν Ρ μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Β, Γ μετρούμενός ἐστιν ὁ Η· ὁ Η ἄρα τὸν Ρ μετρεῖ. καί ἐστιν ὡς ὁ Η πρὸς τὸν Ρ, οὕτως ὁ Κ πρὸς τὸν Σ· καὶ ὁ Κ ἄρα τὸν Σ μετρεῖ. μετρεῖ δὲ καὶ ὁ Ε τὸν Σ· οἱ Ε, Κ ἄρα τὸν Σ μετροῦσιν. καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν Ε, Κ μετρούμενος τὸν Σ μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Ε, Κ μετρούμενός ἐστιν ὁ Μ· ὁ Μ ἄρα τὸν Σ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἔσονταί τινες τῶν Ν, Ξ, Μ, Ο ἐλάσσονες ἀριθμοὶ ἑξῆς ἀνάλογον ἔν τε τοῖς τοῦ Α πρὸς τὸν Β καὶ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τοῦ Ε πρὸς τὸν Ζ λόγοις· οἱ Ν, Ξ, Μ, Ο ἄρα ἑξῆς ἀνάλογον ἐλάχιστοί εἰσιν ἐν τοῖς ΑΒ, ΓΔ, ΕΖ λόγοις· ὅπερ ἔδει δεῖξαι.

ε΄. Οἱ ἐπίπεδοι ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχουσι τὸν συγκείμενον ἐκ τῶν πλευρῶν.

Ἔστωσαν ἐπίπεδοι ἀριθμοὶ οἱ Α, Β, καὶ τοῦ μὲν Α πλευραὶ ἔστωσαν οἱ Γ, Δ ἀριθμοί, τοῦ δὲ Β οἱ Ε, Ζ· λέγω, ὅτι ὁ Α πρὸς τὸν Β λόγον ἔχει τὸν

συγκείμενον ἐκ τῶν πλευρῶν.

Λόγων γὰρ δοθέντων τοῦ τε ὃν ἔχει ὁ Γ πρὸς τὸν Ε καὶ ὁ Δ πρὸς τὸν Ζ εἰλήφθωσαν ἀριθμοὶ ἑξῆς ἐλάχιστοι ἐν τοῖς ΓΕ, ΔΖ λόγοις, οἱ Η, Θ, Κ, ὥστε εἶναι ὡς μὲν τὸν Γ πρὸς τὸν Ε, οὕτως τὸν Η πρὸς τὸν Θ, ὡς δὲ τὸν Δ πρὸς τὸν Ζ, οὕτως τὸν Θ πρὸς τὸν Κ. καὶ ὁ Δ τὸν Ε πολλαπλασιάσας τὸν Λ ποιείτω.

Καὶ ἐπεὶ ὁ Δ τὸν μὲν Γ πολλαπλασιάσας τὸν Α πεποίηκεν, τὸν δὲ Ε πολλαπλασιάσας τὸν Λ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Λ. ὡς δὲ ὁ Γ πρὸς τὸν Ε, οὕτως ὁ Η πρὸς τὸν Θ· καὶ ὡς ἄρα ὁ Η πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Λ. πάλιν, ἐπεὶ ὁ Ε τὸν Δ πολλαπλασιάσας τὸν Λ πεποίηκεν, ἀλλὰ μὴν καὶ τὸν Ζ πολλαπλασιάσας τὸν Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Λ πρὸς τὸν Β. ἀλλ’ ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Θ πρὸς τὸν Κ· καὶ ὡς ἄρα ὁ Θ πρὸς τὸν Κ, οὕτως ὁ Λ πρὸς τὸν Β. ἐδείχθη δὲ καὶ ὡς ὁ Η πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Λ· δι’ ἴσου ἄρα ἐστὶν ὡς ὁ Η πρὸς τὸν Κ, [οὕτως] ὁ Α πρὸς τὸν Β, ὁ δὲ Η πρὸς τὸν Κ λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν· καὶ ὁ Α ἄρα πρὸς τὸν Β λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν· ὅπερ ἔδει δεῖξαι

ς΄. Ἐὰν ὦσιν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, ὁ δὲ πρῶτος τὸν δεύτερον μὴ μετρῇ, οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει.

Ἔστωσαν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, Ε, ὁ δὲ Α τὸν Β μὴ μετρείτω· λέγω, ὅτι οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει.

Ὅτι μὲν οὖν οἱ Α, Β, Γ, Δ, Ε ἑξῆς ἀλλήλους οὐ μετροῦσιν, φανερόν· οὐδὲ γὰρ ὁ Α τὸν Β μετρεῖ. λέγω δή, ὅτι οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει. εἰ γὰρ δυνατόν, μετρείτω ὁ Α τὸν Γ. καὶ ὅσοι εἰσὶν οἱ Α, Β, Γ, τοσοῦτοι εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β, Γ οἱ Ζ, Η, Θ. καὶ ἐπεὶ οἱ Ζ, Η, Θ ἐν τῷ αὐτῷ λόγῳ εἰσὶ τοῖς Α, Β, Γ, καί ἐστιν ἴσον τὸ πλῆθος τῶν Α, Β, Γ τῷ πλήθει τῶν Ζ, Η, Θ, δι’ ἴσου ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Γ, οὕτως ὁ Ζ πρὸς τὸν Θ. καὶ ἐπεί ἐστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ

Ζ πρὸς τὸν Η, οὐ μετρεῖ δὲ ὁ Α τὸν Β, οὐ μετρεῖ ἄρα οὐδὲ ὁ Ζ τὸν Η· οὐκ ἄρα μονάς ἐστιν ὁ Ζ· ἡ γὰρ μονὰς πάντα ἀριθμὸν μετρεῖ. καί εἰσιν οἱ Ζ, Θ πρῶτοι πρὸς ἀλλήλους [οὐδὲ ὁ Ζ ἄρα τὸν Θ μετρεῖ]. καί ἐστιν ὡς ὁ Ζ πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Γ· οὐδὲ ὁ Α ἄρα τὸν Γ μετρεῖ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει· ὅπερ ἔδει δεῖξαι.

ζ΄. Ἐὰν ὦσιν ὁποσοιοῦν ἀριθμοὶ [ἑξῆς] ἀνάλογον, ὁ δὲ πρῶτος τὸν ἔσχατον μετρῇ, καὶ τὸν δεύτερον μετρήσει.

Ἔστωσαν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, ὁ δὲ Α τὸν Δ μετρείτω· λέγω, ὅτι καὶ ὁ Α τὸν Β μετρεῖ. εἰ γὰρ οὐ μετρεῖ ὁ Α τὸν Β, οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει· μετρεῖ δὲ ὁ Α τὸν Δ. μετρεῖ ἄρα καὶ ὁ Α τὸν Β· ὅπερ ἔδει δεῖξαι.

η΄. Ἐὰν δύο ἀριθμῶν μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς τὸν αὐτὸν λόγον ἔχοντας [αὐτοῖς] μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

Δύο γὰρ ἀριθμῶν τῶν Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπιπτέτωσαν ἀριθμοὶ οἱ Γ, Δ, καὶ πεποιήσθω ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Ε πρὸς τὸν Ζ· λέγω, ὅτι ὅσοι εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς Ε, Ζ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

Ὅσοι γάρ εἰσι τῷ πλήθει οἱ Α, Β, Γ, Δ, τοσοῦτοι εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Γ, Δ, Β οἱ Η, Θ, Κ, Λ· οἱ ἄρα ἄκροι αὐτῶν οἱ Η, Λ πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ οἱ Α, Γ, Δ, Β τοῖς Η, Θ, Κ, Λ ἐν τῷ αὐτῷ λόγῳ εἰσίν, καί ἐστιν ἴσον τὸ πλῆθος τῶν Α, Γ, Δ, Β τῷ πλήθει τῶν Η, Θ, Κ, Λ, δι’ ἴσου ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Η πρὸς τὸν Λ. ὡς δὲ ὁ Α πρὸς τὸν Β, οὕτως ὁ Ε πρὸς τὸν Ζ· καὶ ὡς ἄρα ὁ Η

πρὸς τὸν Λ, οὕτως ὁ Ε πρὸς τὸν Ζ. οἱ δὲ Η, Λ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον. ἰσάκις ἄρα ὁ Η τὸν Ε μετρεῖ καὶ ὁ Λ τὸν Ζ. ὁσάκις δὴ ὁ Η τὸν Ε μετρεῖ, τοσαυτάκις καὶ ἑκάτερος τῶν Θ, Κ ἑκάτερον τῶν Μ, Ν μετρείτω· οἱ Η, Θ, Κ, Λ ἄρα τοὺς Ε, Μ, Ν, Ζ ἰσάκις μετροῦσιν. οἱ Η, Θ, Κ, Λ ἄρα τοῖς Ε, Μ, Ν, Ζ ἐν τῷ αὐτῷ λόγῳ εἰσίν. ἀλλὰ οἱ Η, Θ, Κ, Λ τοῖς Α, Γ, Δ, Β ἐν τῷ αὐτῷ λόγῳ εἰσίν· καὶ οἱ Α, Γ, Δ, Β ἄρα τοῖς Ε, Μ, Ν, Ζ ἐν τῷ αὐτῷ λόγῳ εἰσίν. οἱ δὲ Α, Γ, Δ, Β ἑξῆς ἀνάλογόν εἰσιν· καὶ οἱ Ε, Μ, Ν, Ζ ἄρα ἑξῆς ἀνάλογόν εἰσιν. ὅσοι ἄρα εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς Ε, Ζ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί· ὅπερ ἔδει δεῖξαι.

θ΄. Ἐὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, καὶ εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ ἑκατέρου αὐτῶν καὶ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

Ἔστωσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ Α, Β καὶ εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπιπτέτωσαν οἱ Γ, Δ, καὶ ἐκκείσθω ἡ Ε μονάς· λέγω, ὅτι ὅσοι εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ ἑκατέρου τῶν Α, Β καὶ τῆς μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

Εἰλήφθωσαν γὰρ δύο μὲν ἀριθμοὶ ἐλάχιστοι ἐν τῷ τῶν Α, Γ, Δ, Β λόγῳ ὄντες οἱ Ζ, Η, τρεῖς δὲ οἱ Θ, Κ, Λ, καὶ ἀεὶ ἑξῆς ἑνὶ πλείους, ἕως ἂν ἴσον γένηται τὸ πλῆθος αὐτῶν τῷ πλήθει τῶν Α, Γ, Δ, Β. εἰλήφθωσαν, καὶ ἔστωσαν οἱ Μ, Ν, Ξ, Ο. φανερὸν δή, ὅτι ὁ μὲν Ζ ἑαυτὸν πολλαπλασιάσας τὸν Θ πεποίηκεν, τὸν δὲ Θ πολλαπλασιάσας τὸν Μ πεποίηκεν, καὶ ὁ Η ἑαυτὸν

μὲν πολλαπλασιάσας τὸν Λ πεποίηκεν, τὸν δὲ Λ πολλαπλασιάσας τὸν Ο πεποίηκεν. καὶ ἐπεὶ οἱ Μ, Ν, Ξ, Ο ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Ζ, Η, εἰσὶ δὲ καὶ οἱ Α, Γ, Δ, Β ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Ζ, Η, καί ἐστιν ἴσον τὸ πλῆθος τῶν Μ, Ν, Ξ, Ο τῷ πλήθει τῶν Α, Γ, Δ, Β, ἕκαστος ἄρα τῶν Μ, Ν, Ξ, Ο ἑκάστῳ τῶν Α, Γ, Δ, Β ἴσος ἐστίν· ἴσος ἄρα ἐστὶν ὁ μὲν Μ τῷ Α, ὁ δὲ Ο τῷ Β. καὶ ἐπεὶ ὁ Ζ ἑαυτὸν πολλαπλασιάσας τὸν Θ πεποίηκεν, ὁ Ζ ἄρα τὸν Θ μετρεῖ κατὰ τὰς ἐν τῷ Ζ μονάδας. μετρεῖ δὲ καὶ ἡ Ε μονὰς τὸν Ζ κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκις ἄρα ἡ Ε μονὰς τὸν Ζ ἀριθμὸν μετρεῖ καὶ ὁ Ζ τὸν Θ. ἔστιν ἄρα ὡς ἡ Ε μονὰς πρὸς τὸν Ζ ἀριθμόν, οὕτως ὁ Ζ πρὸς τὸν Θ. πάλιν, ἐπεὶ ὁ Ζ τὸν Θ πολλαπλασιάσας τὸν Μ πεποίηκεν, ὁ Θ ἄρα τὸν Μ μετρεῖ κατὰ τὰς ἐν τῷ Ζ μονάδας. μετρεῖ δὲ καὶ ἡ Ε μονὰς τὸν Ζ ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκις ἄρα ἡ Ε μονὰς τὸν Ζ ἀριθμὸν μετρεῖ καὶ ὁ Θ τὸν Μ. ἔστιν ἄρα ὡς ἡ Ε μονὰς πρὸς τὸν Ζ ἀριθμόν, οὕτως ὁ Θ πρὸς τὸν Μ. ἐδείχθη δὲ καὶ ὡς ἡ Ε μονὰς πρὸς τὸν Ζ ἀριθμόν, οὕτως ὁ Ζ πρὸς τὸν Θ· καὶ ὡς ἄρα ἡ Ε μονὰς πρὸς τὸν Ζ ἀριθμόν, οὕτως ὁ Ζ πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν Μ. ἴσος δὲ ὁ Μ τῷ Α· ἔστιν ἄρα ὡς ἡ Ε μονὰς πρὸς τὸν Ζ ἀριθμόν, οὕτως ὁ Ζ πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν Α. διὰ τὰ αὐτὰ δὴ καὶ ὡς ἡ Ε μονὰς πρὸς τὸν Η ἀριθμόν, οὕτως ὁ Η πρὸς τὸν Λ καὶ ὁ Λ πρὸς τὸν Β. ὅσοι ἄρα εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ ἑκατέρου τῶν Α, Β καὶ μονάδος τῆς Ε μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί· ὅπερ ἔδει δεῖξαι.

ι΄. Ἐὰν δύο ἀριθμῶν ἑκατέρου καὶ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι ἑκατέρου αὐτῶν καὶ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

Δύο γὰρ ἀριθμῶν τῶν Α, Β καὶ μονάδος τῆς Γ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπιπτέτωσαν ἀριθμοὶ οἵ τε Δ, Ε καὶ οἱ Ζ, Η· λέγω, ὅτι ὅσοι ἑκατέρου τῶν Α, Β καὶ μονάδος τῆς Γ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

Ὁ Δ γὰρ τὸν Ζ πολλαπλασιάσας τὸν Θ ποιείτω, ἑκάτερος δὲ τῶν Δ, Ζ τὸν Θ πολλαπλασιάσας ἑκάτερον τῶν Κ, Λ ποιείτω.

Καὶ ἐπεί ἐστιν ὡς ἡ Γ μονὰς πρὸς τὸν Δ ἀριθμόν, οὕτως ὁ Δ πρὸς τὸν Ε, ἰσάκις ἄρα ἡ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ καὶ ὁ Δ τὸν Ε. ἡ δὲ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· καὶ ὁ Δ ἄρα ἀριθμὸς τὸν Ε μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· ὁ Δ ἄρα ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηκεν. πάλιν, ἐπεί ἐστιν ὡς ἡ Γ [μονὰς] πρὸς τὸν Δ ἀριθμὸν, οὕτως ὁ Ε πρὸς τὸν Α, ἰσάκις ἄρα ἡ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ καὶ ὁ Ε τὸν Α. ἡ δὲ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· καὶ ὁ Ε ἄρα τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· ὁ Δ ἄρα τὸν Ε πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ μὲν Ζ ἑαυτὸν πολλαπλασιάσας τὸν Η πεποίηκεν, τὸν δὲ Η πολλαπλασιάσας τὸν Β πεποίηκεν. καὶ ἐπεὶ ὁ Δ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε πεποίηκεν, τὸν δὲ Ζ πολλαπλασιάσας τὸν Θ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Ε πρὸς τὸν Θ. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Θ πρὸς τὸν Η. καὶ ὡς ἄρα ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Θ πρὸς τὸν Η. πάλιν, ἐπεὶ ὁ Δ ἑκάτερον τῶν Ε, Θ πολλαπλασιάσας ἑκάτερον τῶν Α, Κ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Κ. ἀλλ’ ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Δ πρὸς τὸν Ζ· καὶ ὡς ἄρα ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Κ. πάλιν, ἐπεὶ ἑκάτερος τῶν Δ, Ζ τὸν Θ πολλαπλασιάσας ἑκάτερον τῶν Κ, Λ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Κ πρὸς τὸν Λ. ἀλλ’ ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Κ· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Κ, οὕτως ὁ Κ πρὸς τὸν Λ. ἔτι ἐπεὶ ὁ Ζ ἑκάτερον τῶν Θ, Η πολλαπλασιάσας ἑκάτερον τῶν Λ, Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Θ πρὸς τὸν

Η, οὕτως ὁ Λ πρὸς τὸν Β. ὡς δὲ ὁ Θ πρὸς τὸν Η, οὕτως ὁ Δ πρὸς τὸν Ζ· καὶ ὡς ἄρα ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Λ πρὸς τὸν Β. ἐδείχθη δὲ καὶ ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὅ τε Α πρὸς τὸν Κ καὶ ὁ Κ πρὸς τὸν Λ· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Κ, οὕτως ὁ Κ πρὸς τὸν Λ καὶ ὁ Λ πρὸς τὸν Β. οἱ Α, Κ, Λ, Β ἄρα κατὰ τὸ συνεχὲς ἑξῆς εἰσιν ἀνάλογον. ὅσοι ἄρα ἑκατέρου τῶν Α, Β καὶ τῆς Γ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἐμπεσοῦνται· ὅπερ ἔδει δεῖξαι.

ια΄. Δύο τετραγώνων ἀριθμῶν εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ τετράγωνος πρὸς τὸν τετράγωνον διπλασίονα λόγον ἔχει ἥπερ ἡ πλευρὰ πρὸς τὴν πλευράν.

Ἔστωσαν τετράγωνοι ἀριθμοὶ οἱ Α, Β, καὶ τοῦ μὲν Α πλευρὰ ἔστω ὁ Γ, τοῦ δὲ Β ὁ Δ· λέγω, ὅτι τῶν Α, Β εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Δ.

Ὁ Γ γὰρ τὸν Δ πολλαπλασιάσας τὸν Ε ποιείτω. καὶ ἐπεὶ τετράγωνός ἐστιν ὁ Α, πλευρὰ δὲ αὐτοῦ ἐστιν ὁ Γ, ὁ Γ ἄρα ἑαυτὸν πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Δ ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν. ἐπεὶ οὖν ὁ Γ ἑκάτερον τῶν Γ, Δ πολλαπλασιάσας ἑκάτερον τῶν Α, Ε πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Ε. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Β. καὶ ὡς ἄρα ὁ Α πρὸς τὸν Ε, οὕτως ὁ Ε πρὸς τὸν Β. τῶν Α, Β ἄρα εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός.

Λέγω δή, ὅτι καὶ ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Δ. ἐπεὶ γὰρ τρεῖς ἀριθμοὶ ἀνάλογόν εἰσιν οἱ Α, Ε, Β, ὁ Α ἄρα πρὸς τὸν Β διπλασίονα λόγον ἔχει ἥπερ ὁ Α πρὸς τὸν Ε. ὡς δὲ ὁ Α πρὸς τὸν Ε, οὕτως ὁ Γ πρὸς τὸν Δ. ὁ Α ἄρα πρὸς τὸν Β διπλασίονα λόγον ἔχει ἥπερ ἡ Γ πλευρὰ πρὸς τὴν Δ· ὅπερ ἔδει δεῖξαι.

ιβ΄. Δύο κύβων ἀριθμῶν δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί, καὶ ὁ κύβος πρὸς τὸν κύβον τριπλασίονα λόγον ἔχει ἥπερ ἡ πλευρὰ πρὸς τὴν πλευράν.

Ἔστωσαν κύβοι ἀριθμοὶ οἱ Α, Β καὶ τοῦ μὲν Α πλευρὰ ἔστω ὁ Γ, τοῦ δὲ Β ὁ Δ· λέγω, ὅτι τῶν Α, Β δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί, καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Δ.

Ὁ γὰρ Γ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε ποιείτω, τὸν δὲ Δ πολλαπλασιάσας τὸν Ζ ποιείτω, ὁ δὲ Δ ἑαυτὸν πολλαπλασιάσας τὸν Η ποιείτω, ἑκάτερος δὲ τῶν Γ, Δ τὸν Ζ πολλαπλασιάσας ἑκάτερον τῶν Θ, Κ ποιείτω.

Καὶ ἐπεὶ κύβος ἐστὶν ὁ Α, πλευρὰ δὲ αὐτοῦ ὁ Γ, καὶ ὁ Γ ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηκεν, ὁ Γ ἄρα ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε πεποίηκεν, τὸν δὲ Ε πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Δ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Η πεποίηκεν, τὸν δὲ Η πολλαπλασιάσας τὸν Β πεποίηκεν. καὶ ἐπεὶ ὁ Γ ἑκάτερον τῶν Γ, Δ πολλαπλασιάσας ἑκάτερον τῶν Ε, Ζ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Ζ. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ζ πρὸς τὸν Η. πάλιν, ἐπεὶ ὁ Γ ἑκάτερον τῶν Ε, Ζ πολλαπλασιάσας ἑκάτερον τῶν Α, Θ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Θ. ὡς δὲ ὁ Ε πρὸς τὸν Ζ, οὕτως ὁ Γ πρὸς τὸν Δ· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Θ. πάλιν, ἐπεὶ ἑκάτερος τῶν Γ, Δ τὸν Ζ πολλαπλασιάσας ἑκάτερον τῶν Θ, Κ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Θ πρὸς τὸν Κ. πάλιν, ἐπεὶ ὁ Δ ἑκάτερον τῶν Ζ, Η πολλαπλασιάσας ἑκάτερον τῶν Κ, Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ζ πρὸς τὸν Η, οὕτως ὁ Κ πρὸς τὸν Β. ὡς δὲ ὁ Ζ πρὸς τὸν Η, οὕτως ὁ Γ πρὸς τὸν Δ· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Δ, οὕτως ὅ τε Α πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν Κ καὶ ὁ Κ πρὸς τὸν Β. τῶν Α, Β ἄρα δύο μέσοι ἀνάλογόν εἰσιν οἱ Θ, Κ.

Λέγω δή, ὅτι καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Δ. ἐπεὶ γὰρ τέσσαρες ἀριθμοὶ ἀνάλογόν εἰσιν οἱ Α, Θ, Κ, Β, ὁ Α ἄρα

πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ὁ Α πρὸς τὸν Θ. ὡς δὲ ὁ Α πρὸς τὸν Θ, οὕτως ὁ Γ πρὸς τὸν Δ· καὶ ὁ Α [ἄρα] πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Δ. ὅπερ ἔδει δεῖξαι.

ιγ΄. Ἐὰν ὦσιν ὁσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, καὶ πολλαπλασιάσας ἕκαστος ἑαυτὸν ποιῇ τινα, οἱ γενόμενοι ἐξ αὐτῶν ἀνάλογον ἔσονται· καὶ ἐὰν οἱ ἐξ ἀρχῆς τοὺς γενομένους πολλαπλασιάσαντες ποιῶσί τινας, καὶ αὐτοὶ ἀνάλογον ἔσονται [καὶ ἀεὶ περὶ τοὺς ἄκρους τοῦτο συμβαίνει].

Ἔστωσαν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, οἱ Α, Β, Γ, ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς τὸν Γ, καὶ οἱ Α, Β, Γ ἑαυτοὺς μὲν πολλαπλασιάσαντες τοὺς Δ, Ε, Ζ ποιείτωσαν, τοὺς δὲ Δ, Ε, Ζ πολλαπλασιάσαντες τοὺς Η, Θ, Κ ποιείτωσαν· λέγω, ὅτι οἵ τε Δ, Ε, Ζ καὶ οἱ Η, Θ, Κ ἑξῆς ἀνάλογόν εἰσιν.

Ὁ μὲν γὰρ Α τὸν Β πολλαπλασιάσας τὸν Λ ποιείτω, ἑκάτερος δὲ τῶν Α, Β τὸν Λ πολλαπλασιάσας ἑκάτερον τῶν Μ, Ν ποιείτω. καὶ πάλιν ὁ μὲν Β τὸν Γ πολλαπλασιάσας τὸν Ξ ποιείτω, ἑκάτερος δὲ τῶν Β, Γ τὸν Ξ πολλαπλασιάσας ἑκάτερον τῶν Ο, Π ποιείτω.

Ὁμοίως δὴ τοῖς ἐπάνω δείξομεν, ὅτι οἱ Δ, Λ, Ε καὶ οἱ Η, Μ, Ν, Θ ἑξῆς εἰσιν ἀνάλογον ἐν τῷ τοῦ Α πρὸς τὸν Β λόγῳ, καὶ ἔτι οἱ Ε, Ξ, Ζ καὶ οἱ Θ, Ο, Π, Κ ἑξῆς εἰσιν ἀνάλογον ἐν τῷ τοῦ Β πρὸς τὸν Γ λόγῳ. καί ἐστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς τὸν Γ· καὶ οἱ Δ, Λ, Ε ἄρα τοῖς Ε, Ξ, Ζ ἐν τῷ αὐτῷ λόγῳ εἰσὶ καὶ ἔτι οἱ Η, Μ, Ν, Θ τοῖς Θ, Ο, Π, Κ. καί ἐστιν ἴσον τὸ μὲν τῶν Δ, Λ, Ε πλῆθος τῷ τῶν Ε, Ξ, Ζ πλήθει, τὸ δὲ τῶν Η, Μ, Ν, Θ τῷ τῶν Θ, Ο, Π, Κ· δι’ ἴσου ἄρα ἐστὶν ὡς μὲν ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Ε πρὸς τὸν Ζ, ὡς δὲ ὁ Η πρὸς τὸν Θ, οὕτως ὁ Θ πρὸς τὸν Κ· ὅπερ ἔδει δεῖξαι.

ιδ΄. Ἐὰν τετράγωνος τετράγωνον μετρῇ, καὶ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· καὶ ἐὰν ἡ πλευρὰ τὴν πλευρὰν μετρῇ, καὶ ὁ τετράγωνος τὸν τετράγωνον μετρήσει.

Ἔστωσαν τετράγωνοι ἀριθμοὶ οἱ Α, Β, πλευραὶ δὲ αὐτῶν ἔστωσαν οἱ Γ, Δ, ὁ δὲ Α τὸν Β μετρείτω· λέγω, ὅτι καὶ ὁ Γ τὸν Δ μετρεῖ.

Ὁ Γ γὰρ τὸν Δ πολλαπλασιάσας τὸν Ε ποιείτω· οἱ Α, Ε, Β ἄρα ἑξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ οἱ Α, Ε, Β ἑξῆς ἀνάλογόν εἰσιν, καὶ μετρεῖ ὁ Α τὸν Β, μετρεῖ ἄρα καὶ ὁ Α τὸν Ε. καί ἐστιν ὡς ὁ Α πρὸς τὸν Ε, οὕτως ὁ Γ πρὸς τὸν Δ· μετρεῖ ἄρα καὶ ὁ Γ τὸν Δ.

Πάλιν δὴ ὁ Γ τὸν Δ μετρείτω· λέγω, ὅτι καὶ ὁ Α τὸν Β μετρεῖ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι οἱ Α, Ε, Β ἑξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Ε, μετρεῖ δὲ ὁ Γ τὸν Δ, μετρεῖ ἄρα καὶ ὁ Α τὸν Ε. καί εἰσιν οἱ Α, Ε, Β ἑξῆς ἀνάλογον· μετρεῖ ἄρα καὶ ὁ Α τὸν Β.

Ἐὰν ἄρα τετράγωνος τετράγωνον μετρῇ, καὶ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· καὶ ἐὰν ἡ πλευρὰ τὴν πλευρὰν μετρῇ, καὶ ὁ τετράγωνος τὸν τετράγωνον μετρήσει· ὅπερ ἔδει δεῖξαι.

ιε΄. Ἐὰν κύβος ἀριθμὸς κύβον ἀριθμὸν μετρῇ, καὶ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· καὶ ἐὰν ἡ πλευρὰ τὴν πλευρὰν μετρῇ, καὶ ὁ κύβος τὸν κύβον μετρήσει.

Κύβος γὰρ ἀριθμὸς ὁ Α κύβον τὸν Β μετρείτω, καὶ τοῦ μὲν Α πλευρὰ ἔστω ὁ Γ, τοῦ δὲ Β ὁ Δ· λέγω, ὅτι ὁ Γ τὸν Δ μετρεῖ.

Ὁ Γ γὰρ ἑαυτὸν πολλαπλασιάσας τὸν Ε ποιείτω, ὁ δὲ Δ ἑαυτὸν πολλαπλασιάσας τὸν Η ποιείτω, καὶ ἔτι ὁ Γ τὸν Δ πολλαπλασιάσας τὸν Ζ [ποιείτω], ἑκάτερος δὲ τῶν Γ, Δ τὸν Ζ πολλαπλασιάσας ἑκάτερον τῶν Θ, Κ ποιείτω. φανερὸν δή, ὅτι οἱ Ε, Ζ,

Η καὶ οἱ Α, Θ, Κ, Β ἑξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ οἱ Α, Θ, Κ, Β ἑξῆς ἀνάλογόν εἰσιν, καὶ μετρεῖ ὁ Α τὸν Β, μετρεῖ ἄρα καὶ τὸν Θ. καί ἐστιν ὡς ὁ Α πρὸς τὸν Θ, οὕτως ὁ Γ πρὸς τὸν Δ· μετρεῖ ἄρα καὶ ὁ Γ τὸν Δ.

Ἀλλὰ δὴ μετρείτω ὁ Γ τὸν Δ· λέγω, ὅτι καὶ ὁ Α τὸν Β μετρήσει.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δὴ δείξομεν, ὅτι οἱ Α, Θ, Κ, Β ἑξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ ὁ Γ τὸν Δ μετρεῖ, καί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Θ, καὶ ὁ Α ἄρα τὸν Θ μετρεῖ· ὥστε καὶ τὸν Β μετρεῖ ὁ Α· ὅπερ ἔδει δεῖξαι.

ις΄. Ἐὰν τετράγωνος ἀριθμὸς τετράγωνον ἀριθμὸν μὴ μετρῇ, οὐδὲ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· κἂν ἡ πλευρὰ τὴν πλευρὰν μὴ μετρῇ, οὐδὲ ὁ τετράγωνος τὸν τετράγωνον μετρήσει.

Ἔστωσαν τετράγωνοι ἀριθμοὶ οἱ Α, Β, πλευραὶ δὲ αὐτῶν ἔστωσαν οἱ Γ, Δ, καὶ μὴ μετρείτω ὁ Α τὸν Β· λέγω, ὅτι οὐδὲ ὁ Γ τὸν Δ μετρεῖ.

Εἰ γὰρ μετρεῖ ὁ Γ τὸν Δ, μετρήσει καὶ ὁ Α τὸν Β. οὐ μετρεῖ δὲ ὁ Α τὸν Β· οὐδὲ ἄρα ὁ Γ τὸν Δ μετρήσει.

Μὴ μετρείτω [δὴ] πάλιν ὁ Γ τὸν Δ· λέγω, ὅτι οὐδὲ ὁ Α τὸν Β μετρήσει.

Εἰ γὰρ μετρεῖ ὁ Α τὸν Β, μετρήσει καὶ ὁ Γ τὸν Δ. οὐ μετρεῖ δὲ ὁ Γ τὸν Δ· οὐδ’ ἄρα ὁ Α τὸν Β μετρήσει· ὅπερ ἔδει δεῖξαι.

ιζ΄. Ἐὰν κύβος ἀριθμὸς κύβον ἀριθμὸν μὴ μετρῇ, οὐδὲ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· κἂν ἡ πλευρὰ τὴν πλευρὰν μὴ μετρῇ, οὐδὲ ὁ κύβος τὸν κύβον μετρήσει.

Κύβος γὰρ ἀριθμὸς ὁ Α κύβον ἀριθμὸν τὸν Β μὴ μετρείτω, καὶ τοῦ μὲν Α πλευρὰ ἔστω ὁ Γ, τοῦ δὲ Β ὁ Δ· λέγω, ὅτι ὁ Γ τὸν Δ οὐ μετρήσει.

Εἰ γὰρ μετρεῖ ὁ Γ τὸν Δ, καὶ ὁ Α τὸν Β μετρήσει. οὐ μετρεῖ δὲ ὁ Α τὸν Β· οὐδ’ ἄρα ὁ Γ τὸν Δ μετρεῖ.

Ἀλλὰ δὴ μὴ μετρείτω ὁ Γ τὸν Δ· λέγω, ὅτι οὐδὲ ὁ Α τὸν Β μετρήσει.

Εἰ γὰρ ὁ Α τὸν Β μετρεῖ, καὶ ὁ Γ τὸν Δ μετρήσει. οὐ μετρεῖ δὲ ὁ Γ τὸν Δ· οὐδ’ ἄρα ὁ Α τὸν Β μετρήσει· ὅπερ ἔδει δεῖξαι.

ιη΄. Δύο ὁμοίων ἐπιπέδων ἀριθμῶν εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός· καὶ ὁ ἐπίπεδος πρὸς τὸν ἐπίπεδον διπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν.

Ἔστωσαν δύο ὅμοιοι ἐπίπεδοι ἀριθμοὶ οἱ Α, Β, καὶ τοῦ μὲν Α πλευραὶ ἔστωσαν οἱ Γ, Δ ἀριθμοί, τοῦ δὲ Β οἱ Ε, Ζ. καὶ ἐπεὶ ὅμοιοι ἐπίπεδοί εἰσιν οἱ ἀνάλογον ἔχοντες τὰς πλευράς, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Ζ. λέγω οὖν, ὅτι τῶν Α, Β εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Ε ἢ ὁ Δ πρὸς τὸν Ζ, τουτέστιν ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον [πλευράν].

Καὶ ἐπεί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Ζ, ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Γ πρὸς τὸν Ε, ὁ Δ πρὸς τὸν Ζ. καὶ ἐπεὶ ἐπίπεδός ἐστιν ὁ Α, πλευραὶ δὲ αὐτοῦ οἱ Γ, Δ, ὁ Δ ἄρα τὸν Γ πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Ε τὸν Ζ πολλαπλασιάσας τὸν Β πεποίηκεν. ὁ Δ δὴ τὸν Ε πολλαπλασιάσας τὸν Η ποιείτω. καὶ ἐπεὶ ὁ Δ τὸν μὲν Γ πολλαπλασιάσας τὸν Α πεποίηκεν, τὸν δὲ Ε πολλαπλασιάσας τὸν Η πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Η. ἀλλ’ ὡς ὁ Γ πρὸς τὸν Ε, [οὕτως] ὁ Δ πρὸς τὸν Ζ· καὶ ὡς ἄρα ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Η. πάλιν, ἐπεὶ ὁ Ε τὸν μὲν Δ πολλαπλασιάσας τὸν Η πεποίηκεν, τὸν δὲ Ζ πολλαπλασιάσας τὸν Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Η πρὸς τὸν Β. ἐδείχθη δὲ καὶ ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Η· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Η, οὕτως ὁ Η πρὸς τὸν Β. οἱ Α, Η, Β ἄρα ἑξῆς ἀνάλογόν εἰσιν. τῶν Α, Β ἄρα εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός.

Λέγω δή, ὅτι καὶ ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἥπερ ὁ Γ πρὸς τὸν Ε ἢ ὁ Δ πρὸς τὸν Ζ. ἐπεὶ γὰρ οἱ Α, Η, Β ἑξῆς ἀνάλογόν εἰσιν, ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἥπερ πρὸς τὸν Η. καί ἐστιν ὡς ὁ Α πρὸς τὸν Η, οὕτως ὅ τε Γ πρὸς τὸν Ε καὶ ὁ Δ πρὸς τὸν Ζ. καὶ ὁ Α ἄρα πρὸς τὸν Β διπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Ε ἢ ὁ Δ πρὸς τὸν Ζ· ὅπερ ἔδει δεῖξαι.

ιθ΄. Δύο ὁμοίων στερεῶν ἀριθμῶν δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί· καὶ ὁ στερεὸς πρὸς τὸν ὅμοιον στερεὸν τριπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν.

Ἔστωσαν δύο ὅμοιοι στερεοὶ οἱ Α, Β, καὶ τοῦ μὲν Α πλευραὶ ἔστωσαν οἱ Γ, Δ, Ε, τοῦ δὲ Β οἱ Ζ, Η, Θ. καὶ ἐπεὶ ὅμοιοι στερεοί εἰσιν οἱ ἀνάλογον ἔχοντες τὰς πλευράς, ἔστιν ἄρα ὡς μὲν ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ζ πρὸς τὸν Η, ὡς δὲ ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Η πρὸς τὸν Θ. λέγω, ὅτι τῶν Α, Β δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί, καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ.

Ὁ Γ γὰρ τὸν Δ πολλαπλασιάσας τὸν Κ ποιείτω, ὁ δὲ Ζ τὸν Η πολλαπλασιάσας τὸν Λ ποιείτω. καὶ ἐπεὶ οἱ Γ, Δ τοῖς Ζ, Η ἐν τῷ αὐτῷ λόγῳ εἰσίν, καὶ ἐκ μὲν τῶν Γ, Δ ἐστιν ὁ Κ, ἐκ δὲ τῶν Ζ, Η ὁ Λ, οἱ Κ, Λ [ἄρα] ὅμοιοι ἐπίπεδοί εἰσιν ἀριθμοί· τῶν Κ, Λ ἄρα εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός. ἔστω ὁ Μ. ὁ Μ ἄρα ἐστὶν ὁ ἐκ τῶν Δ, Ζ, ὡς ἐν τῷ πρὸ τούτου θεωρήματι ἐδείχθη. καὶ ἐπεὶ ὁ Δ τὸν μὲν Γ πολλαπλασιάσας τὸν Κ πεποίηκεν, τὸν δὲ Ζ πολλαπλασιάσας τὸν Μ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Ζ, οὕτως ὁ Κ πρὸς τὸν Μ. ἀλλ’ ὡς ὁ Κ πρὸς τὸν Μ, ὁ Μ πρὸς τὸν Λ. οἱ Κ, Μ, Λ ἄρα ἑξῆς εἰσιν ἀνάλογον ἐν τῷ τοῦ Γ πρὸς τὸν Ζ λόγῳ. καὶ ἐπεί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ζ πρὸς τὸν Η, ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Γ πρὸς τὸν Ζ, οὕτως ὁ Δ πρὸς τὸν Η. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Δ πρὸς τὸν Η, οὕτως ὁ Ε πρὸς τὸν Θ. οἱ Κ, Μ, Λ ἄρα ἑξῆς εἰσιν ἀνάλογον ἔν τε τῷ τοῦ Γ πρὸς τὸν Ζ λόγῳ καὶ τῷ τοῦ Δ πρὸς τὸν Η καὶ ἔτι τῷ τοῦ Ε πρὸς τὸν Θ. ἑκάτερος δὴ τῶν Ε, Θ τὸν Μ πολλαπλασιάσας ἑκάτερον τῶν Ν, Ξ ποιείτω. καὶ ἐπεὶ στερεός ἐστιν ὁ Α, πλευραὶ δὲ αὐτοῦ εἰσιν οἱ Γ, Δ, Ε, ὁ Ε ἄρα τὸν ἐκ τῶν Γ, Δ πολλαπλασιάσας τὸν Α πεποίηκεν. ὁ δὲ ἐκ τῶν Γ, Δ ἐστιν ὁ Κ· ὁ Ε ἄρα τὸν Κ πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Θ τὸν Λ πολλαπλασιάσας τὸν Β πεποίηκεν. καὶ ἐπεὶ ὁ Ε τὸν Κ πολλαπλασιάσας τὸν Α πεποίηκεν, ἀλλὰ μὴν καὶ τὸν Μ πολλαπλασιάσας τὸν Ν πεποίηκεν, ἔστιν ἄρα

ὡς ὁ Κ πρὸς τὸν Μ, οὕτως ὁ Α πρὸς τὸν Ν. ὡς δὲ ὁ Κ πρὸς τὸν Μ, οὕτως ὅ τε Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Ν. πάλιν, ἐπεὶ ἑκάτερος τῶν Ε, Θ τὸν Μ πολλαπλασιάσας ἑκάτερον τῶν Ν, Ξ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Ν πρὸς τὸν Ξ. ἀλλ’ ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὅ τε Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ, οὕτως ὅ τε Α πρὸς τὸν Ν καὶ ὁ Ν πρὸς τὸν Ξ. πάλιν, ἐπεὶ ὁ Θ τὸν Μ πολλαπλασιάσας τὸν Ξ πεποίηκεν, ἀλλὰ μὴν καὶ τὸν Λ πολλαπλασιάσας τὸν Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Μ πρὸς τὸν Λ, οὕτως ὁ Ξ πρὸς τὸν Β. ἀλλ’ ὡς ὁ Μ πρὸς τὸν Λ, οὕτως ὅ τε Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ. καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ, οὕτως οὐ μόνον ὁ Ξ πρὸς τὸν Β, ἀλλὰ καὶ ὁ Α πρὸς τὸν Ν καὶ ὁ Ν πρὸς τὸν Ξ. οἱ Α, Ν, Ξ, Β ἄρα ἑξῆς εἰσιν ἀνάλογον ἐν τοῖς εἰρημένοις τῶν πλευρῶν λόγοις.

Λέγω, ὅτι καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἥπερ ὁ Γ ἀριθμὸς πρὸς τὸν Ζ ἢ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ. ἐπεὶ γὰρ τέσσαρες ἀριθμοὶ ἑξῆς ἀνάλογόν εἰσιν οἱ Α, Ν, Ξ, Β, ὁ Α ἄρα πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ὁ Α πρὸς τὸν Ν. ἀλλ’ ὡς ὁ Α πρὸς τὸν Ν, οὕτως ἐδείχθη ὅ τε Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ. καὶ ὁ Α ἄρα πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἥπερ ὁ Γ ἀριθμὸς πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ· ὅπερ ἔδει δεῖξαι.

κ΄. Ἐὰν δύο ἀριθμῶν εἷς μέσος ἀνάλογον ἐμπίπτῃ ἀριθμός, ὅμοιοι ἐπίπεδοι ἔσονται οἱ ἀριθμοί.

Δύο γὰρ ἀριθμῶν τῶν Α, Β εἷς μέσος ἀνάλογον ἐμπιπτέτω ἀριθμὸς ὁ Γ· λέγω, ὅτι οἱ Α, Β ὅμοιοι ἐπίπεδοί εἰσιν ἀριθμοί.

Εἰλήφθωσαν [γὰρ] ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Γ οἱ Δ, Ε· ἰσάκις ἄρα ὁ Δ τὸν Α μετρεῖ καὶ ὁ Ε τὸν Γ. ὁσάκις δὴ ὁ Δ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ζ· ὁ Ζ ἄρα τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν. ὥστε ὁ Α ἐπίπεδός ἐστιν, πλευραὶ δὲ αὐτοῦ οἱ Δ, Ζ. πάλιν, ἐπεὶ οἱ Δ, Ε ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Γ, Β, ἰσάκις ἄρα ὁ Δ τὸν Γ μετρεῖ καὶ ὁ Ε τὸν Β. ὁσάκις δὴ ὁ Ε τὸν Β μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Η. ὁ Ε ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν τῷ Η μονάδας· ὁ Η ἄρα τὸν Ε πολλαπλασιάσας τὸν Β πεποίηκεν. ὁ Β ἄρα ἐπίπεδός ἐστι, πλευραὶ δὲ αὐτοῦ εἰσιν οἱ Ε, Η. οἱ Α, Β ἄρα ἐπίπεδοί εἰσιν ἀριθμοί. λέγω δή, ὅτι καὶ ὅμοιοι. ἐπεὶ γὰρ ὁ Ζ τὸν μὲν Δ πολλαπλασιάσας τὸν Α πεποίηκεν, τὸν δὲ Ε πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Γ, τουτέστιν ὁ Γ πρὸς τὸν Β. πάλιν, ἐπεὶ ὁ Ε ἑκάτερον τῶν Ζ, Η πολλαπλασιάσας τοὺς Γ, Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ζ πρὸς τὸν Η, οὕτως ὁ Γ πρὸς τὸν Β. ὡς δὲ ὁ Γ πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε· καὶ ὡς ἄρα ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Ζ πρὸς τὸν Η. καὶ ἐναλλὰξ ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Ε πρὸς τὸν Η. οἱ Α, Β ἄρα ὅμοιοι ἐπίπεδοί εἰσιν ἀριθμοί· αἱ γὰρ πλευραὶ αὐτῶν ἀνάλογόν εἰσιν· ὅπερ ἔδει δεῖξαι.

κα΄. Ἐὰν δύο ἀριθμῶν δύο μέσοι ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅμοιοι στερεοί εἰσιν οἱ ἀριθμοί.

Δύο γὰρ ἀριθμῶν τῶν Α, Β δύο μέσοι ἀνάλογον ἐμπιπτέτωσαν ἀριθμοὶ οἱ Γ, Δ· λέγω, ὅτι οἱ Α, Β ὅμοιοι στερεοί εἰσιν.

Εἰλήφθωσαν γὰρ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Γ, Δ τρεῖς οἱ Ε, Ζ, Η· οἱ ἄρα ἄκροι αὐτῶν οἱ Ε, Η πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ τῶν Ε, Η εἷς μέσος ἀνάλογον ἐμπέπτωκεν ἀριθμὸς ὁ Ζ, οἱ Ε, Η ἄρα ἀριθμοὶ ὅμοιοι ἐπίπεδοί εἰσιν. ἔστωσαν οὖν τοῦ μὲν Ε πλευραὶ οἱ Θ, Κ, τοῦ δὲ Η οἱ Λ, Μ. φανερὸν ἄρα ἐστὶν ἐκ τοῦ πρὸ τούτου, ὅτι οἱ Ε, Ζ, Η ἑξῆς εἰσιν ἀνάλογον ἔν τε τῷ τοῦ Θ πρὸς τὸν Λ λόγῳ καὶ τῷ τοῦ Κ πρὸς τὸν Μ.

καὶ ἐπεὶ οἱ Ε, Ζ, Η ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Γ, Δ, καί ἐστιν ἴσον τὸ πλῆθος τῶν Ε, Ζ, Η τῷ πλήθει τῶν Α, Γ, Δ, δι' ἴσου ἄρα ἐστὶν ὡς ὁ Ε πρὸς τὸν Η, οὕτως ὁ Α πρὸς τὸν Δ. οἱ δὲ Ε, Η πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον· ἰσάκις ἄρα ὁ Ε τὸν Α μετρεῖ καὶ ὁ Η τὸν Δ. ὁσάκις δὴ ὁ Ε τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ν. ὁ Ν ἄρα τὸν Ε πολλαπλασιάσας τὸν Α πεποίηκεν. ὁ δὲ Ε ἐστιν ὁ ἐκ τῶν Θ, Κ· ὁ Ν ἄρα τὸν ἐκ τῶν Θ, Κ πολλαπλασιάσας τὸν Α πεποίηκεν. στερεὸς ἄρα ἐστὶν ὁ Α, πλευραὶ δὲ αὐτοῦ εἰσιν οἱ Θ, Κ, Ν. πάλιν, ἐπεὶ οἱ Ε, Ζ, Η ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Γ, Δ, Β, ἰσάκις ἄρα ὁ Ε τὸν Γ μετρεῖ καὶ ὁ Η τὸν Β. ὁσάκις δὴ ὁ Ε τὸν Γ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ξ. ὁ Η ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν τῷ Ξ μονάδας· ὁ Ξ ἄρα τὸν Η πολλαπλασιάσας τὸν Β πεποίηκεν. ὁ δὲ Η ἐστιν ὁ ἐκ τῶν Λ, Μ· ὁ Ξ ἄρα τὸν ἐκ τῶν Λ, Μ πολλαπλασιάσας τὸν Β πεποίηκεν. στερεὸς ἄρα ἐστὶν ὁ Β, πλευραὶ δὲ αὐτοῦ εἰσιν οἱ Λ, Μ, Ξ· οἱ Α, Β ἄρα στερεοί εἰσιν.

Λέγω [δή], ὅτι καὶ ὅμοιοι. ἐπεὶ γὰρ οἱ Ν, Ξ τὸν Ε πολλαπλασιάσαντες τοὺς Α, Γ πεποιήκασιν, ἔστιν ἄρα ὡς ὁ Ν πρὸς τὸν Ξ, ὁ Α πρὸς τὸν Γ, τουτέστιν ὁ Ε πρὸς τὸν Ζ. ἀλλ' ὡς ὁ Ε πρὸς τὸν Ζ, ὁ Θ πρὸς τὸν Λ καὶ ὁ Κ πρὸς τὸν Μ· καὶ ὡς ἄρα ὁ Θ πρὸς τὸν Λ, οὕτως ὁ Κ πρὸς τὸν Μ καὶ ὁ Ν πρὸς τὸν Ξ. καί εἰσιν οἱ μὲν Θ, Κ, Ν πλευραὶ τοῦ Α, οἱ δὲ Ξ, Λ, Μ πλευραὶ τοῦ Β. οἱ Α, Β ἄρα ἀριθμοὶ ὅμοιοι στερεοί εἰσιν· ὅπερ ἔδει δεῖξαι.

κβ΄. Ἐὰν τρεῖς ἀριθμοὶ ἑξῆς ἀνάλογον ὦσιν, ὁ δὲ πρῶτος τετράγωνος ᾖ, καὶ ὁ τρίτος τετράγωνος ἔσται.

Ἔστωσαν τρεῖς ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, ὁ δὲ πρῶτος ὁ Α τετράγωνος ἔστω· λέγω, ὅτι καὶ ὁ τρίτος ὁ Γ τετράγωνός ἐστιν.

Ἐπεὶ γὰρ τῶν Α, Γ εἷς μέσος ἀνάλογόν ἐστιν ἀριθμὸς ὁ Β, οἱ Α, Γ ἄρα ὅμοιοι ἐπίπεδοί εἰσιν. τετράγωνος δὲ ὁ Α· τετράγωνος ἄρα καὶ ὁ Γ· ὅπερ ἔδει δεῖξαι.

κγ΄. Ἐὰν τέσσαρες ἀριθμοὶ ἑξῆς ἀνάλογον ὦσιν, ὁ δὲ πρῶτος κύβος ᾖ, καὶ ὁ τέταρτος κύβος ἔσται.

Ἔστωσαν τέσσαρες ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, ὁ δὲ Α κύβος ἔστω· λέγω, ὅτι καὶ ὁ Δ κύβος ἐστίν.

Ἐπεὶ γὰρ τῶν Α, Δ δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοὶ οἱ Β, Γ, οἱ Α, Δ ἄρα ὅμοιοί εἰσι στερεοὶ ἀριθμοί. κύβος δὲ ὁ Α· κύβος ἄρα καὶ ὁ Δ· ὅπερ ἔδει δεῖξαι.

κδ΄. Ἐὰν δύο ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχωσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, ὁ δὲ πρῶτος τετράγωνος ᾖ, καὶ ὁ δεύτερος τετράγωνος ἔσται.

Δύο γὰρ ἀριθμοὶ οἱ Α, Β πρὸς ἀλλήλους λόγον ἐχέτωσαν, ὃν τετράγωνος ἀριθμὸς ὁ Γ πρὸς τετράγωνον ἀριθμὸν τὸν Δ, ὁ δὲ Α τετράγωνος ἔστω· λέγω, ὅτι καὶ ὁ Β τετράγωνός ἐστιν.

Ἐπεὶ γὰρ οἱ Γ, Δ τετράγωνοί εἰσιν, οἱ Γ, Δ ἄρα ὅμοιοι ἐπίπεδοί εἰσιν. τῶν Γ, Δ ἄρα εἷς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. καί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ, ὁ Α πρὸς τὸν Β· καὶ τῶν Α, Β ἄρα εἷς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. καί ἐστιν ὁ Α τετράγωνος· καὶ ὁ Β ἄρα τετράγωνός ἐστιν· ὅπερ ἔδει δεῖξαι.

κε΄. Ἐὰν δύο ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχωσιν, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν, ὁ δὲ πρῶτος κύβος ᾖ, καὶ ὁ δεύτερος κύβος ἔσται.

Δύο γὰρ ἀριθμοὶ οἱ Α, Β πρὸς ἀλλήλους λόγον ἐχέτωσαν, ὃν κύβος ἀριθμὸς ὁ Γ πρὸς κύβον ἀριθμὸν τὸν Δ, κύβος δὲ ἔστω ὁ Α· λέγω [δή], ὅτι

καὶ ὁ Β κύβος ἐστίν.

Ἐπεὶ γὰρ οἱ Γ, Δ κύβοι εἰσίν, οἱ Γ, Δ ὅμοιοι στερεοί εἰσιν· τῶν Γ, Δ ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. ὅσοι δὲ εἰς τοὺς Γ, Δ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν, τοσοῦτοι καὶ εἰς τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς· ὥστε καὶ τῶν Α, Β δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. ἐμπιπτέτωσαν οἱ Ε, Ζ. ἐπεὶ οὖν τέσσαρες ἀριθμοὶ οἱ Α, Ε, Ζ, Β ἑξῆς ἀνάλογόν εἰσιν, καί ἐστι κύβος ὁ Α, κύβος ἄρα καὶ ὁ Β· ὅπερ ἔδει δεῖξαι.

κς΄. Οἱ ὅμοιοι ἐπίπεδοι ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν.

Ἔστωσαν ὅμοιοι ἐπίπεδοι ἀριθμοὶ οἱ Α, Β· λέγω, ὅτι ὁ Α πρὸς τὸν Β λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν.

Ἐπεὶ γὰρ οἱ Α, Β ὅμοιοι ἐπίπεδοί εἰσιν, τῶν Α, Β ἄρα εἷς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. ἐμπιπτέτω καὶ ἔστω ὁ Γ, καὶ εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Γ, Β οἱ Δ, Ε, Ζ· οἱ ἄρα ἄκροι αὐτῶν οἱ Δ, Ζ τετράγωνοί εἰσιν. καὶ ἐπεί ἐστιν ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Β, καί εἰσιν οἱ Δ, Ζ τετράγωνοι, ὁ Α ἄρα πρὸς τὸν Β λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ὅπερ ἔδει δεῖξαι.

κζ΄. Οἱ ὅμοιοι στερεοὶ ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν.

Ἔστωσαν ὅμοιοι στερεοὶ ἀριθμοὶ οἱ Α, Β· λέγω, ὅτι ὁ Α πρὸς τὸν Β λόγον ἔχει, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν.

Ἐπεὶ γὰρ οἱ Α, Β ὅμοιοι στερεοί εἰσιν, τῶν Α, Β ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. ἐμπιπτέτωσαν οἱ Γ, Δ, καὶ εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Γ, Δ, Β ἴσοι αὐτοῖς τὸ πλῆθος οἱ Ε, Ζ, Η, Θ· οἱ ἄρα ἄκροι αὐτῶν οἱ Ε, Θ κύβοι εἰσίν. καί ἐστιν ὡς ὁ Ε πρὸς

τὸν Θ, οὕτως ὁ Α πρὸς τὸν Β· καὶ ὁ Α ἄρα πρὸς τὸν Β λόγον ἔχει, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν· ὅπερ ἔδει δεῖξαι.

BOOK IX.

Προτάσεις λς΄.

α΄. Ἐὰν δύο ὅμοιοι ἐπίπεδοι ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος τετράγωνος ἔσται.

Ἔστωσαν δύο ὅμοιοι ἐπίπεδοι ἀριθμοὶ οἱ Α, Β, καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι ὁ Γ τετράγωνός ἐστιν.

Ὁ γὰρ Α ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω. ὁ Δ ἄρα τετράγωνός ἐστιν. ἐπεὶ οὖν ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Δ πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Γ. καὶ ἐπεὶ οἱ Α, Β ὅμοιοι ἐπίπεδοί εἰσιν ἀριθμοί, τῶν Α, Β ἄρα εἷς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. ἐὰν δὲ δύο ἀριθμῶν μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι εἰς αὐτοὺς ἐμπίπτουσι, τοσοῦτοι καὶ εἰς τοὺς τὸν αὐτὸν λόγον ἔχοντας· ὥστε καὶ τῶν Δ, Γ εἷς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. καί ἐστι τετράγωνος ὁ Δ· τετράγωνος ἄρα καὶ ὁ Γ· ὅπερ ἔδει δεῖξαι.

β΄. Ἐὰν δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσι τετράγωνον, ὅμοιοι ἐπίπεδοί εἰσιν ἀριθμοί.

Ἔστωσαν δύο ἀριθμοὶ οἱ Α, Β, καὶ ὁ Α τὸν Β πολλαπλασιάσας τετράγωνον τὸν Γ ποιείτω· λέγω, ὅτι οἱ Α, Β ὅμοιοι ἐπίπεδοί εἰσιν ἀριθμοί.

Ὁ γὰρ Α ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω· ὁ Δ ἄρα τετράγωνός ἐστιν. καὶ ἐπεὶ ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Δ πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, ὁ Δ πρὸς τὸν Γ. καὶ ἐπεὶ ὁ Δ τετράγωνός ἐστιν, ἀλλὰ καὶ ὁ Γ, οἱ Δ, Γ ἄρα ὅμοιοι ἐπίπεδοί εἰσιν. τῶν Δ, Γ ἄρα εἷς μέσος ἀνάλογον ἐμπίπτει. καί ἐστιν ὡς ὁ Δ πρὸς τὸν Γ, οὕτως ὁ Α πρὸς τὸν Β· καὶ τῶν Α, Β ἄρα εἷς μέσος ἀνάλογον

ἐμπίπτει. ἐὰν δὲ δύο ἀριθμῶν εἷς μέσος ἀνάλογον ἐμπίπτῃ, ὅμοιοι ἐπίπεδοί εἰσιν [οἱ] ἀριθμοί· οἱ ἄρα Α, Β ὅμοιοί εἰσιν ἐπίπεδοι· ὅπερ ἔδει δεῖξαι.

γ΄. Ἐὰν κύβος ἀριθμὸς ἑαυτὸν πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος κύβος ἔσται.

Κύβος γὰρ ἀριθμὸς ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β ποιείτω· λέγω, ὅτι ὁ Β κύβος ἐστίν.

Εἰλήφθω γὰρ τοῦ Α πλευρὰ ὁ Γ, καὶ ὁ Γ ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω. φανερὸν δή ἐστιν, ὅτι ὁ Γ τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν. καὶ ἐπεὶ ὁ Γ ἑαυτὸν πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ Γ ἄρα τὸν Δ μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας. ἀλλὰ μὴν καὶ ἡ μονὰς τὸν Γ μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν Γ, ὁ Γ πρὸς τὸν Δ. πάλιν, ἐπεὶ ὁ Γ τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν, ὁ Δ ἄρα τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδας. μετρεῖ δὲ καὶ ἡ μονὰς τὸν Γ κατὰ τὰς ἐν αὐτῷ μονάδας· ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν Γ, ὁ Δ πρὸς τὸν Α. ἀλλ’ ὡς ἡ μονὰς πρὸς τὸν Γ, ὁ Γ πρὸς τὸν Δ· καὶ ὡς ἄρα ἡ μονὰς πρὸς τὸν Γ, οὕτως ὁ Γ πρὸς τὸν Δ καὶ ὁ Δ πρὸς τὸν Α. τῆς ἄρα μονάδος καὶ τοῦ Α ἀριθμοῦ δύο μέσοι ἀνάλογον κατὰ τὸ συνεχὲς ἐμπεπτώκασιν ἀριθμοὶ οἱ Γ, Δ. πάλιν, ἐπεὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν, ὁ Α ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας. μετρεῖ δὲ καὶ ἡ μονὰς τὸν Α κατὰ τὰς ἐν αὐτῷ μονάδας· ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν Α, ὁ Α πρὸς τὸν Β. τῆς δὲ μονάδος καὶ τοῦ Α δύο μέσοι ἀνάλογον ἐμπεπτώκασιν ἀριθμοί· καὶ τῶν Α, Β ἄρα δύο μέσοι ἀνάλογον ἐμπεσοῦνται ἀριθμοί. ἐὰν δὲ δύο ἀριθμῶν δύο μέσοι ἀνάλογον ἐμπίπτωσιν, ὁ δὲ πρῶτος κύβος ᾖ, καὶ ὁ δεύτερος κύβος ἔσται. καί ἐστιν ὁ Α κύβος· καὶ ὁ Β ἄρα κύβος ἐστίν· ὅπερ ἔδει δεῖξαι.

δ΄. Ἐὰν κύβος ἀριθμὸς κύβον ἀριθμὸν πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος κύβος ἔσται.

Κύβος γὰρ ἀριθμὸς ὁ Α κύβον ἀριθμὸν τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι ὁ Γ κύβος ἐστίν.

Ὁ γὰρ Α ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω· ὁ Δ ἄρα κύβος ἐστίν. καὶ ἐπεὶ ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Δ πεποίηκεν, τὸν δὲ Β ολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Γ. καὶ ἐπεὶ οἱ Α, Β κύβοι εἰσίν, ὅμοιοι στερεοί εἰσιν οἱ Α, Β. τῶν Α, Β ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί· ὥστε καὶ τῶν Δ, Γ δύο μέσοι ἀνάλογον ἐμπεσοῦνται ἀριθμοί. καί ἐστι κύβος ὁ Δ· κύβος ἄρα καὶ ὁ Γ· ὅπερ ἔδει δεῖξαι.

ε΄. Ἐὰν κύβος ἀριθμὸς ἀριθμόν τινα πολλαπλασιάσας κύβον ποιῇ, καὶ ὁ πολλαπλασιασθεὶς κύβος ἔσται.

Κύβος γὰρ ἀριθμὸς ὁ Α ἀριθμόν τινα τὸν Β πολλαπλασιάσας κύβον τὸν Γ ποιείτω· λέγω, ὅτι ὁ Β κύβος ἐστίν.

Ὁ γὰρ Α ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω· κύβος ἄρα ἐστίν ὁ Δ. καὶ ἐπεὶ ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Δ πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, ὁ Δ πρὸς τὸν Γ. καὶ ἐπεὶ οἱ Δ, Γ κύβοι εἰσίν, ὅμοιοι στερεοί εἰσιν. τῶν Δ, Γ ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. καί ἐστιν ὡς ὁ Δ πρὸς τὸν Γ, οὕτως ὁ Α πρὸς τὸν Β· καὶ τῶν Α, Β ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. καί ἐστι κύβος ὁ Α· κύβος ἄρα ἐστὶ καὶ ὁ Β· ὅπερ ἔδει δεῖξαι.

ς΄. Ἐὰν ἀριθμὸς ἑαυτὸν πολλαπλασιάσας κύβον ποιῇ, καὶ αὐτὸς κύβος ἔσται.

Ἀριθμὸς γὰρ ὁ Α ἑαυτὸν πολλαπλασιάσας κύβον τὸν Β ποιείτω· λέγω, ὅτι καὶ ὁ Α κύβος ἐστίν. ὁ γὰρ Α τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω. ἐπεὶ οὖν ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Β πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ Γ ἄρα κύβος ἐστίν. καὶ ἐπεὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν, ὁ Α ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας. μετρεῖ δὲ

καὶ ἡ μονὰς τὸν Α κατὰ τὰς ἐν αὐτῷ μονάδας. ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν Α, οὕτως ὁ Α πρὸς τὸν Β. καὶ ἐπεὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ Β ἄρα τὸν Γ μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας. μετρεῖ δὲ καὶ ἡ μονὰς τὸν Α κατὰ τὰς ἐν αὐτῷ μονάδας. ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν Α, οὕτως ὁ Β πρὸς τὸν Γ. ἀλλ᾽ ὡς ἡ μονὰς πρὸς τὸν Α, οὕτως ὁ Α πρὸς τὸν Β· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, ὁ Β πρὸς τὸν Γ. καὶ ἐπεὶ οἱ Β, Γ κύβοι εἰσίν, ὅμοιοι στερεοί εἰσιν. τῶν Β, Γ ἄρα δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί. καί ἐστιν ὡς ὁ Β πρὸς τὸν Γ, ὁ Α πρὸς τὸν Β. καὶ τῶν Α, Β ἄρα δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί. καί ἐστι κύβος ὁ Β· κύβος ἄρα ἐστὶ καὶ ὁ Α· ὅπερ ἔδει δεῖξαι.

ζ΄. Ἐὰν σύνθετος ἀριθμὸς ἀριθμόν τινα πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος στερεὸς ἔσται.

Σύνθετος γὰρ ἀριθμὸς ὁ Α ἀριθμόν τινα τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι ὁ Γ στερεός ἐστιν.

Ἐπεὶ γὰρ ὁ Α σύνθετός ἐστιν, ὑπὸ ἀριθμοῦ τινος μετρηθήσεται. μετρείσθω ὑπὸ τοῦ Δ, καὶ ὁσάκις ὁ Δ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε. ἐπεὶ οὖν ὁ Δ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας, ὁ Ε ἄρα τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν. καὶ ἐπεὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ δὲ Α ἐστιν ὁ ἐκ τῶν Δ, Ε, ὁ ἄρα ἐκ τῶν Δ, Ε τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν. ὁ Γ ἄρα στερεός ἐστιν, πλευραὶ δὲ αὐτοῦ εἰσιν οἱ Δ, Ε, Β· ὅπερ ἔδει δεῖξαι.

η΄. Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ὦσιν, ὁ μὲν τρίτος ἀπὸ τῆς μονάδος τετράγωνος ἔσται καὶ οἱ ἕνα διαλείποντες, ὁ δὲ τέταρτος κύβος καὶ οἱ δύο διαλείποντες πάντες, ὁ δὲ ἕβδομος κύβος ἅμα καὶ τετράγωνος καὶ οἱ πέντε διαλείποντες.

Ἔστωσαν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, Ε, Ζ· λέγω, ὅτι ὁ μὲν τρίτος ἀπὸ τῆς μονάδος ὁ Β τετράγωνός ἐστι καὶ οἱ ἕνα διαλείποντες πάντες, ὁ δὲ τέταρτος ὁ Γ κύβος καὶ οἱ δύο διαλείποντες πάντες, ὁ δὲ ἕβδομος ὁ Ζ κύβος ἅμα καὶ τετράγωνος καὶ οἱ πέντε διαλείποντες πάντες.

Ἐπεὶ γάρ ἐστιν ὡς ἡ μονὰς πρὸς τὸν Α, οὕτως ὁ Α πρὸς τὸν Β, ἰσάκις ἄρα ἡ μονὰς τὸν Α ἀριθμὸν μετρεῖ καὶ ὁ Α τὸν Β. ἡ δὲ μονὰς τὸν Α ἀριθμὸν μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· καὶ ὁ Α ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας. ὁ Α ἄρα ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν· τετράγωνος ἄρα ἐστὶν ὁ Β. καὶ ἐπεὶ οἱ Β, Γ, Δ ἑξῆς ἀνάλογόν εἰσιν, ὁ δὲ Β τετράγωνός ἐστιν, καὶ ὁ Δ ἄρα τετράγωνός ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Ζ τετράγωνός ἐστιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ οἱ ἕνα διαλείποντες πάντες τετράγωνοί εἰσιν. λέγω δή, ὅτι καὶ ὁ τέταρτος ἀπὸ τῆς μονάδος ὁ Γ κύβος ἐστὶ καὶ οἱ δύο διαλείποντες πάντες. ἐπεὶ γάρ ἐστιν ὡς ἡ μονὰς πρὸς τὸν Α, οὕτως ὁ Β πρὸς τὸν Γ, ἰσάκις ἄρα ἡ μονὰς τὸν Α ἀριθμὸν μετρεῖ καὶ ὁ Β τὸν Γ. ἡ δὲ μονὰς τὸν Α ἀριθμὸν μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας· καὶ ὁ Β ἄρα τὸν Γ μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας· ὁ Α ἄρα τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν. ἐπεὶ οὖν ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Β πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηκεν, κύβος ἄρα ἐστὶν ὁ Γ. καὶ ἐπεὶ οἱ Γ, Δ, Ε, Ζ ἑξῆς ἀνάλογόν εἰσιν, ὁ δὲ Γ κύβος ἐστίν, καὶ ὁ Ζ ἄρα κύβος ἐστίν. ἐδείχθη δὲ καὶ τετράγωνος· ὁ ἄρα ἕβδομος ἀπὸ τῆς μονάδος κύβος τέ ἐστι καὶ τετράγωνος. ὁμοίως δὴ δείξομεν, ὅτι καὶ οἱ πέντε διαλείποντες πάντες κύβοι τέ εἰσι καὶ τετράγωνοι· ὅπερ ἔδει δεῖξαι.

θ΄. Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἑξῆς κατὰ τὸ συνεχὲς ἀριθμοὶ ἀνάλογον ὦσιν, ὁ δὲ μετὰ τὴν μονάδα τετράγωνος ᾖ, καὶ οἱ λοιποὶ πάντες τετράγωνοι ἔσονται. καὶ ἐὰν ὁ μετὰ τὴν μονάδα κύβος ᾖ, καὶ οἱ λοιποὶ πάντες κύβοι ἔσονται.

Ἔστωσαν ἀπὸ μονάδος ἑξῆς ἀνάλογον ὁσοιδηποτοῦν ἀριθμοὶ οἱ Α, Β, Γ, Δ, Ε, Ζ, ὁ δὲ μετὰ τὴν μονάδα ὁ Α τετράγωνος ἔστω· λέγω, ὅτι καὶ οἱ λοιποὶ πάντες τετράγωνοι ἔσονται.

Ὅτι μὲν οὖν ὁ τρίτος ἀπὸ τῆς μονάδος ὁ Β τετράγωνός ἐστι καὶ οἱ ἕνα διαλείποντες πάντες, δέδεικται· λέγω [δή], ὅτι καὶ οἱ λοιποὶ πάντες τετράγωνοί εἰσιν. ἐπεὶ γὰρ οἱ Α, Β, Γ ἑξῆς ἀνάλογόν εἰσιν, καί ἐστιν ὁ Α τετράγωνος, καὶ ὁ Γ [ἄρα] τετράγωνός ἐστιν. πάλιν, ἐπεὶ [καὶ] οἱ Β, Γ, Δ ἑξῆς ἀνάλογόν εἰσιν, καί ἐστιν ὁ Β τετράγωνος, καὶ ὁ Δ [ἄρα] τετράγωνός ἐστιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ οἱ λοιποὶ πάντες τετράγωνοί εἰσιν.

Ἀλλὰ δὴ ἔστω ὁ Α κύβος· λέγω, ὅτι καὶ οἱ λοιποὶ πάντες κύβοι εἰσίν.

Ὅτι μὲν οὖν ὁ τέταρτος ἀπὸ τῆς μονάδος ὁ Γ κύβος ἐστὶ καὶ οἱ δύο διαλείποντες πάντες, δέδεικται· λέγω [δή], ὅτι καὶ οἱ λοιποὶ πάντες κύβοι εἰσίν. ἐπεὶ γάρ ἐστιν ὡς ἡ μονὰς πρὸς τὸν Α, οὕτως ὁ Α πρὸς τὸν Β, ἰσάκις ἄρα ἡ μονὰς τὸν Α μετρεῖ καὶ ὁ Α τὸν Β. ἡ δὲ μονὰς τὸν Α μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· καὶ ὁ Α ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· ὁ Α ἄρα ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν. καί ἐστιν ὁ Α κύβος. ἐὰν δὲ κύβος ἀριθμὸς ἑαυτὸν πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος κύβος ἐστίν· καὶ ὁ Β ἄρα κύβος ἐστίν. καὶ ἐπεὶ τέσσαρες ἀριθμοὶ οἱ Α, Β, Γ, Δ ἑξῆς ἀνάλογόν εἰσιν, καί ἐστιν ὁ Α κύβος, καὶ ὁ Δ ἄρα κύβος ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Ε κύβος ἐστίν, καὶ ὁμοίως οἱ λοιποὶ πάντες κύβοι εἰσίν· ὅπερ ἔδει δεῖξαι.

ι΄. Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ [ἑξῆς] ἀνάλογον ὦσιν, ὁ δὲ μετὰ τὴν μονάδα μὴ ᾖ τετράγωνος, οὐδ᾽ ἄλλος οὐδεὶς τετράγωνος ἔσται χωρὶς τοῦ τρίτου ἀπὸ τῆς μονάδος καὶ τῶν ἕνα διαλειπόντων πάντων. καὶ ἐὰν ὁ μετὰ τὴν μονάδα κύβος μὴ ᾖ, οὐδὲ ἄλλος οὐδεὶς κύβος ἔσται χωρὶς τοῦ τετάρτου ἀπὸ τῆς μονάδος καὶ τῶν δύο διαλειπόντων πάντων.

Ἔστωσαν ἀπὸ μονάδος ἑξῆς ἀνάλογον ὁσοιδηποτοῦν ἀριθμοὶ οἱ Α, Β, Γ, Δ, Ε, Ζ, ὁ δὲ μετὰ τὴν μονάδα ὁ Α μὴ ἔστω τετράγωνος· λέγω, ὅτι οὐδὲ ἄλλος οὐδεὶς τετράγωνος ἔσται χωρὶς τοῦ τρίτου ἀπὸ τῆς μονάδος [καὶ τῶν ἕνα διαλειπόντων].

Εἰ γὰρ δυνατόν, ἔστω ὁ Γ τετράγωνος. ἔστι δὲ καὶ ὁ Β τετράγωνος· οἱ Β, Γ ἄρα πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καί ἐστιν ὡς ὁ Β πρὸς τὸν Γ, ὁ Α πρὸς τὸν Β· οἱ Α, Β ἄρα πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ὥστε οἱ Α, Β ὅμοιοι ἐπίπεδοί εἰσιν. καί ἐστι τετράγωνος ὁ Β· τετράγωνος ἄρα ἐστὶ καὶ ὁ Α· ὅπερ οὐχ ὑπέκειτο. οὐκ ἄρα ὁ Γ τετράγωνός ἐστιν. ὁμοίως δὴ δείξομεν, ὅτι οὐδ’ ἄλλος οὐδεὶς τετράγωνός ἐστι χωρὶς τοῦ τρίτου ἀπὸ τῆς μονάδος καὶ τῶν ἕνα διαλειπόντων.

Ἀλλὰ δὴ μὴ ἔστω ὁ Α κύβος. λέγω, ὅτι οὐδ’ ἄλλος οὐδεὶς κύβος ἔσται χωρὶς τοῦ τετάρτου ἀπὸ τῆς μονάδος καὶ τῶν δύο διαλειπόντων.

Εἰ γὰρ δυνατόν, ἔστω ὁ Δ κύβος. ἔστι δὲ καὶ ὁ Γ κύβος· τέταρτος γάρ ἐστιν ἀπὸ τῆς μονάδος. καί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ, ὁ Β πρὸς τὸν Γ· καὶ ὁ Β ἄρα πρὸς τὸν Γ λόγον ἔχει, ὃν κύβος πρὸς κύβον. καί ἐστιν ὁ Γ κύβος· καὶ ὁ Β ἄρα κύβος ἐστίν. καὶ ἐπεί ἐστιν ὡς ἡ μονὰς πρὸς τὸν Α, ὁ Α πρὸς τὸν Β, ἡ δὲ μονὰς τὸν Α μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας, καὶ ὁ Α ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· ὁ Α ἄρα ἑαυτὸν πολλαπλασιάσας κύβον τὸν Β πεποίηκεν. ἐὰν δὲ ἀριθμὸς ἑαυτὸν πολλαπλασιάσας κύβον ποιῇ, καὶ αὐτὸς κύβος ἔσται. κύβος ἄρα καὶ ὁ Α· ὅπερ οὐχ ὑπόκειται. οὐκ ἄρα ὁ Δ κύβος ἐστίν. ὁμοίως δὴ δείξομεν, ὅτι οὐδ’ ἄλλος οὐδεὶς κύβος ἐστὶ χωρὶς τοῦ τετάρτου ἀπὸ τῆς μονάδος καὶ τῶν δύο διαλειπόντων· ὅπερ ἔδει δεῖξαι.

ια΄. Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ὦσιν, ὁ ἐλάττων τὸν μείζονα μετρεῖ κατά τινα τῶν ὑπαρχόντων ἐν τοῖς ἀνάλογον ἀριθμοῖς.

Ἔστωσαν ἀπὸ μονάδος τῆς Α ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Β, Γ, Δ, Ε· λέγω, ὅτι τῶν Β, Γ, Δ, Ε ὁ ἐλάχιστος ὁ Β τὸν Ε μετρεῖ κατά τινα τῶν Γ, Δ.

Ἐπεὶ γάρ ἐστιν ὡς ἡ Α μονὰς πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε, ἰσάκις ἄρα ἡ Α μονὰς τὸν Β ἀριθμὸν μετρεῖ καὶ ὁ Δ τὸν Ε· ἐναλλὰξ ἄρα ἰσάκις ἡ Α μονὰς τὸν Δ μετρεῖ καὶ ὁ Β τὸν Ε. ἡ δὲ Α μονὰς τὸν Δ μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· καὶ ὁ Β ἄρα τὸν Ε μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· ὥστε ὁ ἐλάσσων ὁ Β τὸν μείζονα τὸν Ε μετρεῖ κατά τινα ἀριθμὸν τῶν ὑπαρχόντων ἐν τοῖς ἀνάλογον ἀριθμοῖς.

Πόρισμα Καὶ φανερόν, ὅτι ἣν ἔχει τάξιν ὁ μετρῶν ἀπὸ μονάδος, τὴν αὐτὴν ἔχει καὶ ὁ καθ’ ὃν μετρεῖ ἀπὸ τοῦ μετρουμένου ἐπὶ τὸ πρὸ αὐτοῦ. ¯ὅπερ ἔδει δεῖξαι.

ιβ΄. Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ὦσιν, ὑφ’ ὅσων ἂν ὁ ἔσχατος πρώτων ἀριθμῶν μετρῆται, ὑπὸ τῶν αὐτῶν καὶ ὁ παρὰ τὴν μονάδα μετρηθήσεται.

Ἔστωσαν ἀπὸ μονάδος ὁποσοιδηποτοῦν ἀριθμοὶ ἀνάλογον οἱ Α, Β, Γ, Δ· λέγω, ὅτι ὑφ’ ὅσων ἂν ὁ Δ πρώτων ἀριθμῶν μετρῆται, ὑπὸ τῶν αὐτῶν καὶ ὁ Α μετρηθήσεται.

Μετρείσθω γὰρ ὁ Δ ὑπό τινος πρώτου ἀριθμοῦ τοῦ Ε· λέγω, ὅτι ὁ Ε τὸν Α μετρεῖ. μὴ γάρ· καί ἐστιν ὁ Ε πρῶτος, ἅπας δὲ πρῶτος ἀριθμὸς πρὸς ἅπαντα, ὃν μὴ μετρεῖ, πρῶτός ἐστιν· οἱ Ε, Α ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ ὁ Ε τὸν Δ μετρεῖ, μετρείτω αὐτὸν κατὰ τὸν Ζ· ὁ Ε ἄρα τὸν Ζ πολλαπλασιάσας τὸν Δ πεποίηκεν. πάλιν, ἐπεὶ ὁ Α τὸν Δ μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδας, ὁ Α ἄρα τὸν Γ πολλαπλασιάσας τὸν Δ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Ε τὸν Ζ πολλαπλασιάσας τὸν Δ πεποίηκεν· ὁ ἄρα ἐκ τῶν Α, Γ ἴσος ἐστὶ τῷ ἐκ τῶν Ε, Ζ. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Ε, ὁ Ζ πρὸς τὸν Γ. οἱ δὲ Α, Ε πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν

λόγον ἔχοντας ἰσάκις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον· μετρεῖ ἄρα ὁ Ε τὸν Γ. μετρείτω αὐτὸν κατὰ τὸν Η· ὁ Ε ἄρα τὸν Η πολλαπλασιάσας τὸν Γ πεποίηκεν. ἀλλὰ μὴν διὰ τὸ πρὸ τούτου καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν. ὁ ἄρα ἐκ τῶν Α, Β ἴσος ἐστὶ τῷ ἐκ τῶν Ε, Η. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Ε, ὁ Η πρὸς τὸν Β. οἱ δὲ Α, Ε πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς ἰσάκις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον· μετρεῖ ἄρα ὁ Ε τὸν Β. μετρείτω αὐτὸν κατὰ τὸν Θ· ὁ Ε ἄρα τὸν Θ πολλαπλασιάσας τὸν Β πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν· ὁ ἄρα ἐκ τῶν Ε, Θ ἴσος ἐστὶ τῷ ἀπὸ τοῦ Α. ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Α, ὁ Α πρὸς τὸν Θ. οἱ δὲ Α, Ε πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον· μετρεῖ ἄρα ὁ Ε τὸν Α ὡς ἡγούμενος ἡγούμενον. ἀλλὰ μὴν καὶ οὐ μετρεῖ· ὅπερ ἀδύνατον. οὐκ ἄρα οἱ Ε, Α πρῶτοι πρὸς ἀλλήλους εἰσίν. σύνθετοι ἄρα. οἱ δὲ σύνθετοι ὑπὸ [πρώτου] ἀριθμοῦ τινος μετροῦνται. καὶ ἐπεὶ ὁ Ε πρῶτος ὑπόκειται, ὁ δὲ πρῶτος ὑπὸ ἑτέρου ἀριθμοῦ οὐ μετρεῖται ἢ ὑφ' ἑαυτοῦ, ὁ Ε ἄρα τοὺς Α, Ε μετρεῖ· ὥστε ὁ Ε τὸν Α μετρεῖ. μετρεῖ δὲ καὶ τὸν Δ· ὁ Ε ἄρα τοὺς Α, Δ μετρεῖ. ὁμοίως δὴ δείξομεν, ὅτι ὑφ' ὅσων ἂν ὁ Δ πρώτων ἀριθμῶν μετρῆται, ὑπὸ τῶν αὐτῶν καὶ ὁ Α μετρηθήσεται· ὅπερ ἔδει δεῖξαι.

ιγ΄. Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ὦσιν, ὁ δὲ μετὰ τὴν μονάδα πρῶτος ᾖ, ὁ μέγιστος ὑπ' οὐδενὸς [ἄλλου] μετρηθήσεται παρὲξ τῶν ὑπαρχόντων ἐν τοῖς ἀνάλογον ἀριθμοῖς.

Ἔστωσαν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, ὁ δὲ μετὰ τὴν μονάδα ὁ Α πρῶτος ἔστω· λέγω, ὅτι ὁ μέγιστος αὐτῶν ὁ Δ ὑπ' οὐδενὸς ἄλλου μετρηθήσεται παρὲξ τῶν Α, Β, Γ.

Εἰ γὰρ δυνατόν, μετρείσθω ὑπὸ τοῦ Ε, καὶ ὁ Ε μηδενὶ τῶν Α, Β, Γ ἔστω ὁ αὐτός. φανερὸν δή, ὅτι ὁ Ε πρῶτος οὐκ ἐστιν. εἰ γὰρ ὁ Ε πρῶτός ἐστι καὶ μετρεῖ τὸν Δ, καὶ τὸν Α μετρήσει πρῶτον ὄντα μὴ ὢν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ὁ Ε πρῶτός ἐστιν. σύνθετος ἄρα. πᾶς δὲ σύνθετος ἀριθμὸς ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται· ὁ Ε ἄρα ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται. λέγω δή, ὅτι ὑπ᾽ οὐδενὸς ἄλλου πρώτου μετρηθήσεται πλὴν τοῦ Α. εἰ γὰρ ὑφ᾽ ἑτέρου μετρεῖται ὁ Ε, ὁ δὲ Ε τὸν Δ μετρεῖ, κἀκεῖνος ἄρα τὸν Δ μετρήσει· ὥστε καὶ τὸν Α μετρήσει πρῶτον ὄντα μὴ ὢν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον. ὁ Α ἄρα τὸν Ε μετρεῖ. καὶ ἐπεὶ ὁ Ε τὸν Δ μετρεῖ, μετρείτω αὐτὸν κατὰ τὸν Ζ. λέγω, ὅτι ὁ Ζ οὐδενὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός. εἰ γὰρ ὁ Ζ ἑνὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτὸς καὶ μετρεῖ τὸν Δ κατὰ τὸν Ε, καὶ εἷς ἄρα τῶν Α, Β, Γ τὸν Δ μετρεῖ κατὰ τὸν Ε. ἀλλὰ εἷς τῶν Α, Β, Γ τὸν Δ μετρεῖ κατά τινα τῶν Α, Β, Γ· καὶ ὁ Ε ἄρα ἑνὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός· ὅπερ οὐχ ὑπόκειται. οὐκ ἄρα ὁ Ζ ἑνὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός. ὁμοίως δὴ δείξομεν, ὅτι μετρεῖται ὁ Ζ ὑπὸ τοῦ Α, δεικνύντες πάλιν, ὅτι ὁ Ζ οὐκ ἐστι πρῶτος. εἰ γάρ, καὶ μετρεῖ τὸν Δ, καὶ τὸν Α μετρήσει πρῶτον ὄντα μὴ ὢν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα πρῶτός ἐστιν ὁ Ζ· σύνθετος ἄρα. ἅπας δὲ σύνθετος ἀριθμὸς ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται· ὁ Ζ ἄρα ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται. λέγω δή, ὅτι ὑφ᾽ ἑτέρου πρώτου οὐ μετρηθήσεται πλὴν τοῦ Α. εἰ γὰρ ἕτερός τις πρῶτος τὸν Ζ μετρεῖ, ὁ δὲ Ζ τὸν Δ μετρεῖ, κἀκεῖνος ἄρα τὸν Δ μετρήσει· ὥστε καὶ τὸν Α μετρήσει πρῶτον ὄντα μὴ ὢν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον. ὁ Α ἄρα τὸν Ζ μετρεῖ. καὶ ἐπεὶ ὁ Ε τὸν Δ μετρεῖ κατὰ τὸν Ζ, ὁ Ε ἄρα τὸν Ζ πολλαπλασιάσας τὸν Δ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Α τὸν Γ πολλαπλασιάσας τὸν Δ πεποίηκεν· ὁ ἄρα ἐκ τῶν Α, Γ ἴσος ἐστὶ τῷ ἐκ τῶν Ε, Ζ. ἀνάλογον ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Ε, οὕτως ὁ Ζ πρὸς τὸν Γ. ὁ δὲ Α τὸν Ε μετρεῖ· καὶ ὁ Ζ ἄρα τὸν Γ μετρεῖ. μετρείτω αὐτὸν κατὰ τὸν Η. ὁμοίως δὴ δείξομεν, ὅτι ὁ Η οὐδενὶ τῶν Α, Β ἐστιν ὁ αὐτός, καὶ ὅτι μετρεῖται ὑπὸ τοῦ Α. καὶ ἐπεὶ ὁ Ζ τὸν Γ μετρεῖ κατὰ τὸν Η, ὁ Ζ ἄρα τὸν Η

πολλαπλασιάσας τὸν Γ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν· ὁ ἄρα ἐκ τῶν Α, Β ἴσος ἐστὶ τῷ ἐκ τῶν Ζ, Η. ἀνάλογον ἄρα ὡς ὁ Α πρὸς τὸν Ζ, ὁ Η πρὸς τὸν Β. μετρεῖ δὲ ὁ Α τὸν Ζ· μετρεῖ ἄρα καὶ ὁ Η τὸν Β. μετρείτω αὐτὸν κατὰ τὸν Θ. ὁμοίως δὴ δείξομεν, ὅτι ὁ Θ τῷ Α οὐκ ἔστιν ὁ αὐτός. καὶ ἐπεὶ ὁ Η τὸν Β μετρεῖ κατὰ τὸν Θ, ὁ Η ἄρα τὸν Θ πολλαπλασιάσας τὸν Β πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν· ὁ ἄρα ὑπὸ Θ, Η ἴσος ἐστὶ τῷ ἀπὸ τοῦ Α τετραγώνῳ. ἔστιν ἄρα ὡς ὁ Θ πρὸς τὸν Α, ὁ Α πρὸς τὸν Η. μετρεῖ δὲ ὁ Α τὸν Η· μετρεῖ ἄρα καὶ ὁ Θ τὸν Α πρῶτον ὄντα μὴ ὢν αὐτῷ ὁ αὐτός· ὅπερ ἄτοπον. οὐκ ἄρα ὁ μέγιστος ὁ Δ ὑπὸ ἑτέρου ἀριθμοῦ μετρηθήσεται παρὲξ τῶν Α, Β, Γ· ὅπερ ἔδει δεῖξαι.

ιδ΄. Ἐὰν ἐλάχιστος ἀριθμὸς ὑπὸ πρώτων ἀριθμῶν μετρῆται, ὑπ᾽ οὐδενὸς ἄλλου πρώτου ἀριθμοῦ μετρηθήσεται παρὲξ τῶν ἐξ ἀρχῆς μετρούντων.

Ἐλάχιστος γὰρ ἀριθμὸς ὁ Α ὑπὸ πρώτων ἀριθμῶν τῶν Β, Γ, Δ μετρείσθω· λέγω, ὅτι ὁ Α ὑπ᾽ οὐδενὸς ἄλλου πρώτου ἀριθμοῦ μετρηθήσεται παρὲξ τῶν Β, Γ, Δ.

Εἰ γὰρ δυνατόν, μετρείσθω ὑπὸ πρώτου τοῦ Ε, καὶ ὁ Ε μηδενὶ τῶν Β, Γ, Δ ἔστω ὁ αὐτός. καὶ ἐπεὶ ὁ Ε τὸν Α μετρεῖ, μετρείτω αὐτὸν κατὰ τὸν Ζ· ὁ Ε ἄρα τὸν Ζ πολλαπλασιάσας τὸν Α πεποίηκεν. καὶ μετρεῖται ὁ Α ὑπὸ πρώτων ἀριθμῶν τῶν Β, Γ, Δ. ἐὰν δὲ δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, τὸν δὲ γενόμενον ἐξ αὐτῶν μετρῇ τις πρῶτος ἀριθμός, καὶ ἕνα τῶν ἐξ ἀρχῆς μετρήσει· οἱ Β, Γ, Δ ἄρα ἕνα τῶν Ε, Ζ μετρήσουσιν. τὸν μὲν οὖν Ε οὐ μετρήσουσιν· ὁ γὰρ Ε πρῶτός ἐστι καὶ οὐδενὶ τῶν Β, Γ, Δ ὁ αὐτός. τὸν Ζ ἄρα μετροῦσιν ἐλάσσονα ὄντα τοῦ Α· ὅπερ ἀδύνατον. ὁ γὰρ Α ὑπόκειται ἐλάχιστος ὑπὸ τῶν Β, Γ, Δ μετρούμενος. οὐκ ἄρα τὸν Α μετρήσει πρῶτος ἀριθμὸς παρὲξ τῶν Β, Γ, Δ· ὅπερ ἔδει δεῖξαι.

ιε΄. Ἐὰν τρεῖς ἀριθμοὶ ἑξῆς ἀνάλογον ὦσιν ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, δύο ὁποιοιοῦν συντεθέντες πρὸς τὸν λοιπὸν πρῶτοί εἰσιν.

Ἔστωσαν τρεῖς ἀριθμοὶ ἑξῆς ἀνάλογον ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς οἱ Α, Β, Γ· λέγω, ὅτι τῶν Α, Β, Γ δύο ὁποιοιοῦν συντεθέντες πρὸς τὸν λοιπὸν πρῶτοί εἰσιν, οἱ μὲν Α, Β πρὸς τὸν Γ, οἱ δὲ Β, Γ πρὸς τὸν Α καὶ ἔτι οἱ Α, Γ πρὸς τὸν Β.

Εἰλήφθωσαν γὰρ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β, Γ δύο οἱ ΔΕ, ΕΖ. φανερὸν δή, ὅτι ὁ μὲν ΔΕ ἑαυτὸν πολλαπλασιάσας τὸν Α πεποίηκεν, τὸν δὲ ΕΖ πολλαπλασιάσας τὸν Β πεποίηκεν, καὶ ἔτι ὁ ΕΖ ἑαυτὸν πολλαπλασιάσας τὸν Γ πεποίηκεν. καὶ ἐπεὶ οἱ ΔΕ, ΕΖ ἐλάχιστοί εἰσιν, πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐὰν δὲ δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, καὶ συναμφότερος πρὸς ἑκάτερον πρῶτός ἐστιν· καὶ ὁ ΔΖ ἄρα πρὸς ἑκάτερον τῶν ΔΕ, ΕΖ πρῶτός ἐστιν. ἀλλὰ μὴν καὶ ὁ ΔΕ πρὸς τὸν ΕΖ πρῶτός ἐστιν· οἱ ΔΖ, ΔΕ ἄρα πρὸς τὸν ΕΖ πρῶτοί εἰσιν. ἐὰν δὲ δύο ἀριθμοὶ πρός τινα ἀριθμὸν πρῶτοι ὦσιν, καὶ ὁ ἐξ αὐτῶν γενόμενος πρὸς τὸν λοιπὸν πρῶτός ἐστιν· ὥστε ὁ ἐκ τῶν ΖΔ, ΔΕ πρὸς τὸν ΕΖ πρῶτός ἐστιν· ὥστε καὶ ὁ ἐκ τῶν ΖΔ, ΔΕ πρὸς τὸν ἀπὸ τοῦ ΕΖ πρῶτός ἐστιν. [ἐὰν γὰρ δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, ὁ ἐκ τοῦ ἑνὸς αὐτῶν γενόμενος πρὸς τὸν λοιπὸν πρῶτός ἐστιν]. ἀλλ’ ὁ ἐκ τῶν ΖΔ, ΔΕ ὁ ἀπὸ τοῦ ΔΕ ἐστι μετὰ τοῦ ἐκ τῶν ΔΕ, ΕΖ· ὁ ἄρα ἀπὸ τοῦ ΔΕ μετὰ τοῦ ἐκ τῶν ΔΕ, ΕΖ πρὸς τὸν ἀπὸ τοῦ ΕΖ πρῶτός ἐστιν. καί ἐστιν ὁ μὲν ἀπὸ τοῦ ΔΕ ὁ Α, ὁ δὲ ἐκ τῶν ΔΕ, ΕΖ ὁ Β, ὁ δὲ ἀπὸ τοῦ ΕΖ ὁ Γ· οἱ Α, Β ἄρα συντεθέντες πρὸς τὸν Γ πρῶτοί εἰσιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ οἱ Β, Γ πρὸς τὸν Α πρῶτοί εἰσιν. λέγω δή, ὅτι καὶ οἱ Α, Γ πρὸς τὸν Β πρῶτοί εἰσιν. ἐπεὶ γὰρ ὁ ΔΖ πρὸς ἑκάτερον τῶν ΔΕ, ΕΖ πρῶτός ἐστιν, καὶ ὁ ἀπὸ τοῦ ΔΖ πρὸς τὸν ἐκ τῶν ΔΕ, ΕΖ πρῶτός ἐστιν. ἀλλὰ τῷ ἀπὸ τοῦ ΔΖ ἴσοι εἰσὶν οἱ ἀπὸ τῶν ΔΕ, ΕΖ μετὰ τοῦ δὶς ἐκ τῶν ΔΕ, ΕΖ· καὶ οἱ ἀπὸ τῶν ΔΕ, ΕΖ ἄρα μετὰ τοῦ δὶς ὑπὸ τῶν ΔΕ, ΕΖ πρὸς τὸν ὑπὸ τῶν ΔΕ, ΕΖ πρῶτοί [εἰσι]. διελόντι οἱ ἀπὸ τῶν ΔΕ, ΕΖ μετὰ τοῦ ἅπαξ ὑπὸ ΔΕ, ΕΖ πρὸς τὸν ὑπὸ ΔΕ, ΕΖ πρῶτοί

εἰσιν. ἔτι διελόντι οἱ ἀπὸ τῶν ΔΕ, ΕΖ ἄρα πρὸς τὸν ὑπὸ ΔΕ, ΕΖ πρῶτοί εἰσιν. καί ἐστιν ὁ μὲν ἀπὸ τοῦ ΔΕ ὁ Α, ὁ δὲ ὑπὸ τῶν ΔΕ, ΕΖ ὁ Β, ὁ δὲ ἀπὸ τοῦ ΕΖ ὁ Γ. οἱ Α, Γ ἄρα συντεθέντες πρὸς τὸν Β πρῶτοί εἰσιν· ὅπερ ἔδει δεῖξαι.

ις΄. Ἐὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, οὐκ ἔσται ὡς ὁ πρῶτος πρὸς τὸν δεύτερον, οὕτως ὁ δεύτερος πρὸς ἄλλον τινά.

Δύο γὰρ ἀριθμοὶ οἱ Α, Β πρῶτοι πρὸς ἀλλήλους ἔστωσαν· λέγω, ὅτι οὐκ ἔστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς ἄλλον τινά.

Εἰ γὰρ δυνατόν, ἔστω ὡς ὁ Α πρὸς τὸν Β, ὁ Β πρὸς τὸν Γ. οἱ δὲ Α, Β πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον· μετρεῖ ἄρα ὁ Α τὸν Β ὡς ἡγούμενος ἡγούμενον. μετρεῖ δὲ καὶ ἑαυτόν· ὁ Α ἄρα τοὺς Α, Β μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἄτοπον. οὐκ ἄρα ἔσται ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς τὸν Γ· ὅπερ ἔδει δεῖξαι.

ιζ΄. Ἐὰν ὦσιν ὁσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, οἱ δὲ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους ὦσιν, οὐκ ἔσται ὡς ὁ πρῶτος πρὸς τὸν δεύτερον, οὕτως ὁ ἔσχατος πρὸς ἄλλον τινά.

Ἔστωσαν ὁσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, οἱ δὲ ἄκροι αὐτῶν οἱ Α, Δ πρῶτοι πρὸς ἀλλήλους ἔστωσαν· λέγω, ὅτι οὐκ ἔστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς ἄλλον τινά.

Εἰ γὰρ δυνατόν, ἔστω ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε· ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Δ, ὁ Β πρὸς τὸν Ε. οἱ δὲ Α, Δ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον. μετρεῖ ἄρα ὁ Α τὸν Β. καί ἐστιν ὡς ὁ Α πρὸς τὸν Β, ὁ Β πρὸς τὸν Γ. καὶ ὁ Β ἄρα τὸν Γ μετρεῖ· ὥστε καὶ ὁ Α τὸν Γ μετρεῖ. καὶ ἐπεί ἐστιν ὡς ὁ Β πρὸς τὸν

Γ, ὁ Γ πρὸς τὸν Δ, μετρεῖ δὲ ὁ Β τὸν Γ, μετρεῖ ἄρα καὶ ὁ Γ τὸν Δ. ἀλλ᾽ ὁ Α τὸν Γ ἐμέτρει· ὥστε ὁ Α καὶ τὸν Δ μετρεῖ. μετρεῖ δὲ καὶ ἑαυτόν. ὁ Α ἄρα τοὺς Α, Δ μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἔσται ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς ἄλλον τινά· ὅπερ ἔδει δεῖξαι.

ιη΄. Δύο ἀριθμῶν δοθέντων ἐπισκέψασθαι, εἰ δυνατόν ἐστιν αὐτοῖς τρίτον ἀνάλογον προσευρεῖν.

Ἔστωσαν οἱ δοθέντες δύο ἀριθμοὶ οἱ Α, Β, καὶ δέον ἔστω ἐπισκέψασθαι, εἰ δυνατόν ἐστιν αὐτοῖς τρίτον ἀνάλογον προσευρεῖν.

Οἱ δὴ Α, Β ἤτοι πρῶτοι πρὸς ἀλλήλους εἰσὶν ἢ οὔ. καὶ εἰ πρῶτοι πρὸς ἀλλήλους εἰσίν, δέδεικται, ὅτι ἀδύνατόν ἐστιν αὐτοῖς τρίτον ἀνάλογον προσευρεῖν.

Ἀλλὰ δὴ μὴ ἔστωσαν οἱ Α, Β πρῶτοι πρὸς ἀλλήλους, καὶ ὁ Β ἑαυτὸν πολλαπλασιάσας τὸν Γ ποιείτω· ὁ Α δὴ τὸν Γ ἤτοι μετρεῖ ἢ οὐ μετρεῖ. μετρείτω πρότερον κατὰ τὸν Δ· ὁ Α ἄρα τὸν Δ πολλαπλασιάσας τὸν Γ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Β ἑαυτὸν πολλαπλασιάσας τὸν Γ πεποίηκεν· ὁ ἄρα ἐκ τῶν Α, Δ ἴσος ἐστὶ τῷ ἀπὸ τοῦ Β. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, ὁ Β πρὸς τὸν Δ· τοῖς Α, Β ἄρα τρίτος ἀριθμὸς ἀνάλογον προσηύρηται ὁ Δ.

Ἀλλὰ δὴ μὴ μετρείτω ὁ Α τὸν Γ· λέγω, ὅτι τοῖς Α, Β ἀδύνατόν ἐστι τρίτον ἀνάλογον προσευρεῖν ἀριθμόν. εἰ γὰρ δυνατόν, προσηυρήσθω ὁ Δ. ὁ ἄρα ἐκ τῶν Α, Δ ἴσος ἐστὶ τῷ ἀπὸ τοῦ Β. ὁ δὲ ἀπὸ τοῦ Β ἐστιν ὁ Γ· ὁ ἄρα ἐκ τῶν Α, Δ ἴσος ἐστὶ τῷ Γ. ὥστε ὁ Α τὸν Δ πολλαπλασιάσας τὸν Γ πεποίηκεν· ὁ Α ἄρα τὸν Γ μετρεῖ κατὰ τὸν Δ. ἀλλὰ μὴν ὑπόκειται καὶ μὴ μετρῶν· ὅπερ ἄτοπον. οὐκ ἄρα δυνατόν ἐστι τοῖς Α, Β τρίτον ἀνάλογον προσευρεῖν ἀριθμόν, ὅταν ὁ Α τὸν Γ μὴ μετρῇ· ὅπερ ἔδει δεῖξαι.

ιθ΄. Τριῶν ἀριθμῶν δοθέντων ἐπισκέψασθαι, πότε δυνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσευρεῖν.

Ἔστωσαν οἱ δοθέντες τρεῖς ἀριθμοὶ οἱ Α, Β, Γ, καὶ δέον ἔστω ἐπισκέψασθαι, πότε δυνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσευρεῖν.

Ἤτοι οὖν οὐκ εἰσιν ἑξῆς ἀνάλογον, καὶ οἱ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν, ἢ ἑξῆς εἰσιν ἀνάλογον, καὶ οἱ ἄκροι αὐτῶν οὐκ εἰσι πρῶτοι πρὸς ἀλλήλους, ἢ οὔτε ἑξῆς εἰσιν ἀνάλογον, οὔτε οἱ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν, ἢ καὶ ἑξῆς εἰσιν ἀνάλογον, καὶ οἱ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ μὲν οὖν οἱ Α, Β, Γ ἑξῆς εἰσιν ἀνάλογον, καὶ οἱ ἄκροι αὐτῶν οἱ Α, Γ πρῶτοι πρὸς ἀλλήλους εἰσίν, δέδεικται, ὅτι ἀδύνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσευρεῖν ἀριθμόν. μὴ ἔστωσαν δὴ οἱ Α, Β, Γ ἑξῆς ἀνάλογον τῶν ἄκρων πάλιν ὄντων πρώτων πρὸς ἀλλήλους. λέγω, ὅτι καὶ οὕτως ἀδύνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσευρεῖν. εἰ γὰρ δυνατόν, προσευρήσθω ὁ Δ, ὥστε εἶναι ὡς τὸν Α πρὸς τὸν Β, τὸν Γ πρὸς τὸν Δ, καὶ γεγονέτω ὡς ὁ Β πρὸς τὸν Γ, ὁ Δ πρὸς τὸν Ε. καὶ ἐπεί ἐστιν ὡς μὲν ὁ Α πρὸς τὸν Β, ὁ Γ πρὸς τὸν Δ, ὡς δὲ ὁ Β πρὸς τὸν Γ, ὁ Δ πρὸς τὸν Ε, δι’ ἴσου ἄρα ὡς ὁ Α πρὸς τὸν Γ, ὁ Γ πρὸς τὸν Ε. οἱ δὲ Α, Γ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον. μετρεῖ ἄρα ὁ Α τὸν Γ ὡς ἡγούμενος ἡγούμενον. μετρεῖ δὲ καὶ ἑαυτόν· ὁ Α ἄρα τοὺς Α, Γ μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοῖς Α, Β, Γ δυνατόν ἐστι τέταρτον ἀνάλογον προσευρεῖν.

Ἀλλὰ δὴ πάλιν ἔστωσαν οἱ Α, Β, Γ ἑξῆς ἀνάλογον, οἱ δὲ Α, Γ μὴ ἔστωσαν πρῶτοι πρὸς ἀλλήλους. λέγω, ὅτι δυνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσευρεῖν. ὁ γὰρ Β τὸν Γ πολλαπλασιάσας τὸν Δ ποιείτω· ὁ Α ἄρα τὸν Δ ἤτοι μετρεῖ ἢ οὐ μετρεῖ. μετρείτω αὐτὸν πρότερον κατὰ τὸν Ε· ὁ Α ἄρα τὸν Ε πολλαπλασιάσας τὸν Δ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Β τὸν Γ πολλαπλασιάσας τὸν Δ πεποίηκεν· ὁ ἄρα ἐκ τῶν Α, Ε ἴσος ἐστὶ τῷ ἐκ τῶν Β, Γ. ἀνάλογον ἄρα

[ἐστὶν] ὡς ὁ Α πρὸς τὸν Β, ὁ Γ πρὸς τὸν Ε· τοῖς Α, Β, Γ ἄρα τέταρτος ἀνάλογον προσηύρηται ὁ Ε.

Ἀλλὰ δὴ μὴ μετρείτω ὁ Α τὸν Δ· λέγω, ὅτι ἀδύνατόν ἐστι τοῖς Α, Β, Γ τέταρτον ἀνάλογον προσευρεῖν ἀριθμόν. εἰ γὰρ δυνατόν, προσευρήσθω ὁ Ε· ὁ ἄρα ἐκ τῶν Α, Ε ἴσος ἐστὶ τῷ ἐκ τῶν Β, Γ. ἀλλὰ ὁ ἐκ τῶν Β, Γ ἐστιν ὁ Δ· καὶ ὁ ἐκ τῶν Α, Ε ἄρα ἴσος ἐστὶ τῷ Δ. ὁ Α ἄρα τὸν Ε πολλαπλασιάσας τὸν Δ πεποίηκεν· ὁ Α ἄρα τὸν Δ μετρεῖ κατὰ τὸν Ε· ὥστε μετρεῖ ὁ Α τὸν Δ. ἀλλὰ καὶ οὐ μετρεῖ· ὅπερ ἄτοπον. οὐκ ἄρα δυνατόν ἐστι τοῖς Α, Β, Γ τέταρτον ἀνάλογον προσευρεῖν ἀριθμόν, ὅταν ὁ Α τὸν Δ μὴ μετρῇ. ἀλλὰ δὴ οἱ Α, Β, Γ μήτε ἑξῆς ἔστωσαν ἀνάλογον μήτε οἱ ἄκροι πρῶτοι πρὸς ἀλλήλους. καὶ ὁ Β τὸν Γ πολλαπλασιάσας τὸν Δ ποιείτω. ὁμοίως δὴ δειχθήσεται, ὅτι εἰ μὲν μετρεῖ ὁ Α τὸν Δ, δυνατόν ἐστιν αὐτοῖς ἀνάλογον προσευρεῖν, εἰ δὲ οὐ μετρεῖ, ἀδύνατον· ὅπερ ἔδει δεῖξαι.

κ΄. Οἱ πρῶτοι ἀριθμοὶ πλείους εἰσὶ παντὸς τοῦ προτεθέντος πλήθους πρώτων ἀριθμῶν.

Ἔστωσαν οἱ προτεθέντες πρῶτοι ἀριθμοὶ οἱ Α, Β, Γ· λέγω, ὅτι τῶν Α, Β, Γ πλείους εἰσὶ πρῶτοι ἀριθμοί.

Εἰλήφθω γὰρ ὁ ὑπὸ τῶν Α, Β, Γ ἐλάχιστος μετρούμενος καὶ ἔστω ὁ ΔΕ, καὶ προσκείσθω τῷ ΔΕ μονὰς ἡ ΔΖ. ὁ δὴ ΕΖ ἤτοι πρῶτός ἐστιν ἢ οὔ. ἔστω πρότερον πρῶτος· εὑρημένοι ἄρα εἰσὶ πρῶτοι ἀριθμοὶ οἱ Α, Β, Γ, ΕΖ πλείους τῶν Α, Β, Γ.

Ἀλλὰ δὴ μὴ ἔστω ὁ ΕΖ πρῶτος· ὑπὸ πρώτου ἄρα τινὸς ἀριθμοῦ μετρεῖται. μετρείσθω ὑπὸ πρώτου τοῦ Η· λέγω, ὅτι ὁ Η οὐδενὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός. εἰ γὰρ δυνατόν, ἔστω. οἱ δὲ Α, Β, Γ τὸν ΔΕ μετροῦσιν· καὶ ὁ Η ἄρα τὸν ΔΕ μετρήσει. μετρεῖ δὲ καὶ τὸν ΕΖ· καὶ λοιπὴν τὴν ΔΖ μονάδα μετρήσει ὁ Η ἀριθμὸς ὤν· ὅπερ ἄτοπον. οὐκ ἄρα ὁ Η ἑνὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός.

καὶ ὑπόκειται πρῶτος. εὑρημένοι ἄρα εἰσὶ πρῶτοι ἀριθμοὶ πλείους τοῦ προτεθέντος πλήθους τῶν Α, Β, Γ οἱ Α, Β, Γ, Η· ὅπερ ἔδει δεῖξαι.

κα΄. Ἐὰν ἄρτιοι ἀριθμοὶ ὁποσοιοῦν συντεθῶσιν, ὁ ὅλος ἄρτιός ἐστιν.

Συγκείσθωσαν γὰρ ἄρτιοι ἀριθμοὶ ὁποσοιοῦν οἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ· λέγω, ὅτι ὅλος ὁ ΑΕ ἄρτιός ἐστιν.

Ἐπεὶ γὰρ ἕκαστος τῶν ΑΒ, ΒΓ, ΓΔ, ΔΕ ἄρτιός ἐστιν, ἔχει μέρος ἥμισυ· ὥστε καὶ ὅλος ὁ ΑΕ ἔχει μέρος ἥμισυ. ἄρτιος δὲ ἀριθμός ἐστιν ὁ δίχα διαιρούμενος· ἄρτιος ἄρα ἐστὶν ὁ ΑΕ· ὅπερ ἔδει δεῖξαι.

κβ΄. Ἐὰν περισσοὶ ἀριθμοὶ ὁποσοιοῦν συντεθῶσιν, τὸ δὲ πλῆθος αὐτῶν ἄρτιον ᾖ, ὁ ὅλος ἄρτιος ἔσται.

Συγκείσθωσαν γὰρ περισσοὶ ἀριθμοὶ ὁσοιδηποτοῦν ἄρτιοι τὸ πλῆθος οἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ· λέγω, ὅτι ὅλος ὁ ΑΕ ἄρτιός ἐστιν.

Ἐπεὶ γὰρ ἕκαστος τῶν ΑΒ, ΒΓ, ΓΔ, ΔΕ περιττός ἐστιν, ἀφαιρεθείσης μονάδος ἀφ’ ἑκάστου ἕκαστος τῶν λοιπῶν ἄρτιος ἔσται· ὥστε καὶ ὁ συγκείμενος ἐξ αὐτῶν ἄρτιος ἔσται. ἔστι δὲ καὶ τὸ πλῆθος τῶν μονάδων ἄρτιον. καὶ ὅλος ἄρα ὁ ΑΕ ἄρτιός ἐστιν· ὅπερ ἔδει δεῖξαι.

κγ΄. Ἐὰν περισσοὶ ἀριθμοὶ ὁποσοιοῦν συντεθῶσιν, τὸ δὲ πλῆθος αὐτῶν περισσὸν ᾖ, καὶ ὁ ὅλος περισσὸς ἔσται.

Συγκείσθωσαν γὰρ ὁποσοιοῦν περισσοὶ ἀριθμοί, ὧν τὸ πλῆθος περισσὸν ἔστω, οἱ ΑΒ, ΒΓ, ΓΔ· λέγω, ὅτι καὶ ὅλος ὁ ΑΔ περισσός ἐστιν.

Ἀφῃρήσθω ἀπὸ τοῦ ΓΔ μονὰς ἡ ΔΕ· λοιπὸς ἄρα ὁ ΓΕ ἄρτιός ἐστιν. ἔστι δὲ καὶ ὁ ΓΑ ἄρτιος· καὶ ὅλος ἄρα ὁ ΑΕ ἄρτιός ἐστιν. καί ἐστι μονὰς ἡ ΔΕ. περισσὸς ἄρα ἐστὶν ὁ ΑΔ· ὅπερ ἔδει δεῖξαι.

κδ΄. Ἐὰν ἀπὸ ἀρτίου ἀριθμοῦ ἄρτιος ἀφαιρεθῇ, ὁ λοιπὸς ἄρτιος ἔσται.

Ἀπὸ γὰρ ἀρτίου τοῦ ΑΒ ἄρτιος ἀφῃρήσθω ὁ ΒΓ· λέγω, ὅτι ὁ λοιπὸς ὁ ΓΑ ἄρτιός ἐστιν.

Ἐπεὶ γὰρ ὁ ΑΒ ἄρτιός ἐστιν, ἔχει μέρος ἥμισυ. διὰ τὰ αὐτὰ δὴ καὶ ὁ ΒΓ ἔχει μέρος ἥμισυ· ὥστε καὶ λοιπὸς [ὁ ΓΑ ἔχει μέρος ἥμισυ] ἄρτιος [ἄρα] ἐστὶν ὁ ΑΓ· ὅπερ ἔδει δεῖξαι.

κε΄. Ἐὰν ἀπὸ ἀρτίου ἀριθμοῦ περισσὸς ἀφαιρεθῇ, ὁ λοιπὸς περισσὸς ἔσται.

Ἀπὸ γὰρ ἀρτίου τοῦ ΑΒ περισσὸς ἀφῃρήσθω ὁ ΒΓ· λέγω, ὅτι ὁ λοιπὸς ὁ ΓΑ περισσός ἐστιν.

Ἀφῃρήσθω γὰρ ἀπὸ τοῦ ΒΓ μονὰς ἡ ΓΔ· ὁ ΔΒ ἄρα ἄρτιός ἐστιν. ἔστι δὲ καὶ ὁ ΑΒ ἄρτιος· καὶ λοιπὸς ἄρα ὁ ΑΔ ἄρτιός ἐστιν. καί ἐστι μονὰς ἡ ΓΔ· ὁ ΓΑ ἄρα περισσός ἐστιν· ὅπερ ἔδει δεῖξαι.

κς΄. Ἐὰν ἀπὸ περισσοῦ ἀριθμοῦ περισσὸς ἀφαιρεθῇ, ὁ λοιπὸς ἄρτιος ἔσται.

Ἀπὸ γὰρ περισσοῦ τοῦ ΑΒ περισσὸς ἀφῃρήσθω ὁ ΒΓ· λέγω, ὅτι ὁ λοιπὸς ὁ ΓΑ ἄρτιός ἐστιν.

Ἐπεὶ γὰρ ὁ ΑΒ περισσός ἐστιν, ἀφῃρήσθω μονὰς ἡ ΒΔ· λοιπὸς ἄρα ὁ ΑΔ ἄρτιός ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ὁ ΓΔ ἄρτιός ἐστιν· ὥστε καὶ λοιπὸς ὁ ΓΑ ἄρτιός ἐστιν· ὅπερ ἔδει δεῖξαι.

κζ΄. Ἐὰν ἀπὸ περισσοῦ ἀριθμοῦ ἄρτιος ἀφαιρεθῇ, ὁ λοιπὸς περισσὸς ἔσται.

Ἀπὸ γὰρ περισσοῦ τοῦ ΑΒ ἄρτιος ἀφῃρήσθω ὁ ΒΓ· λέγω, ὅτι ὁ λοιπὸς ὁ ΓΑ περισσός ἐστιν.

Ἀφῃρήσθω [γὰρ] μονὰς ἡ ΑΔ· ὁ ΔΒ ἄρα ἄρτιός ἐστιν. ἔστι δὲ καὶ ὁ ΒΓ ἄρτιος· καὶ λοιπὸς ἄρα ὁ ΓΔ ἄρτιός ἐστιν. περισσὸς ἄρα ὁ ΓΑ· ὅπερ ἔδει δεῖξαι.

κη΄. Ἐὰν περισσὸς ἀριθμὸς ἄρτιον πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος ἄρτιος ἔσται.

Περισσὸς γὰρ ἀριθμὸς ὁ Α ἄρτιον τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι ὁ Γ ἄρτιός ἐστιν.

ἐπεὶ γὰρ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ Γ ἄρα σύγκειται ἐκ τοσούτων ἴσων τῷ Β, ὅσαι εἰσὶν ἐν τῷ Α μονάδες. καί ἐστιν ὁ Β ἄρτιος· ὁ Γ ἄρα σύγκειται ἐξ ἀρτίων. ἐὰν δὲ ἄρτιοι ἀριθμοὶ ὁποσοιοῦν συντεθῶσιν, ὁ ὅλος ἄρτιός ἐστιν. ἄρτιος ἄρα ἐστὶν ὁ Γ· ὅπερ ἔδει δεῖξαι.

κθ΄. Ἐὰν περισσὸς ἀριθμὸς περισσὸν ἀριθμὸν πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος περισσὸς ἔσται.

Περισσὸς γὰρ ἀριθμὸς ὁ Α περισσὸν τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι ὁ Γ περισσός ἐστιν.

Ἐπεὶ γὰρ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ Γ ἄρα σύγκειται ἐκ τοσούτων ἴσων τῷ Β, ὅσαι εἰσὶν ἐν τῷ Α μονάδες. καί ἐστιν ἑκάτερος τῶν Α, Β περισσός· ὁ Γ ἄρα σύγκειται ἐκ περισσῶν ἀριθμῶν, ὧν τὸ πλῆθος περισσόν ἐστιν. ὥστε ὁ Γ περισσός ἐστιν· ὅπερ ἔδει δεῖξαι.

λ΄. Ἐὰν περισσὸς ἀριθμὸς ἄρτιον ἀριθμὸν μετρῇ, καὶ τὸν ἥμισυν αὐτοῦ μετρήσει.

Περισσὸς γὰρ ἀριθμὸς ὁ Α ἄρτιον τὸν Β μετρείτω· λέγω, ὅτι καὶ τὸν ἥμισυν αὐτοῦ μετρήσει.

Ἐπεὶ γὰρ ὁ Α τὸν Β μετρεῖ, μετρείτω αὐτὸν κατὰ τὸν Γ· λέγω, ὅτι ὁ Γ οὐκ ἔστι περισσός. εἰ γὰρ δυνατόν, ἔστω. καὶ ἐπεὶ ὁ Α τὸν Β μετρεῖ κατὰ τὸν Γ, ὁ Α ἄρα τὸν Γ πολλαπλασιάσας τὸν Β πεποίηκεν. ὁ Β ἄρα σύγκειται ἐκ περισσῶν ἀριθμῶν, ὧν τὸ πλῆθος περισσόν ἐστιν. ὁ Β ἄρα περισσός ἐστιν· ὅπερ ἄτοπον· ὑπόκειται γὰρ ἄρτιος. οὐκ ἄρα ὁ Γ περισσός ἐστιν· ἄρτιος ἄρα

ἐστὶν ὁ Γ. ὥστε ὁ Α τὸν Β μετρεῖ ἀρτιάκις. διὰ δὴ τοῦτο καὶ τὸν ἥμισυν αὐτοῦ μετρήσει· ὅπερ ἔδει δεῖξαι.

λα΄. Ἐὰν περισσὸς ἀριθμὸς πρός τινα ἀριθμὸν πρῶτος ᾖ, καὶ πρὸς τὸν διπλασίονα αὐτοῦ πρῶτος ἔσται.

Περισσὸς γὰρ ἀριθμὸς ὁ Α πρός τινα ἀριθμὸν τὸν Β πρῶτος ἔστω, τοῦ δὲ Β διπλασίων ἔστω ὁ Γ· λέγω, ὅτι ὁ Α [καὶ] πρὸς τὸν Γ πρῶτός ἐστιν.

Εἰ γὰρ μή εἰσιν [οἱ Α, Γ] πρῶτοι, μετρήσει τις αὐτοὺς ἀριθμός. μετρείτω, καὶ ἔστω ὁ Δ. καί ἐστιν ὁ Α περισσός· περισσὸς ἄρα καὶ ὁ Δ. καὶ ἐπεὶ ὁ Δ περισσὸς ὢν τὸν Γ μετρεῖ, καί ἐστιν ὁ Γ ἄρτιος, καὶ τὸν ἥμισυν ἄρα τοῦ Γ μετρήσει [ὁ Δ]. τοῦ δὲ Γ ἥμισύ ἐστιν ὁ Β· ὁ Δ ἄρα τὸν Β μετρεῖ. μετρεῖ δὲ καὶ τὸν Α. ὁ Δ ἄρα τοὺς Α, Β μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ὁ Α πρὸς τὸν Γ πρῶτος οὔκ ἐστιν. οἱ Α, Γ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

λβ΄. Τῶν ἀπὸ δυάδος διπλασιαζομένων ἀριθμῶν ἕκαστος ἀρτιάκις ἄρτιός ἐστι μόνον.

Ἀπὸ γὰρ δυάδος τῆς Α δεδιπλασιάσθωσαν ὁσοιδηποτοῦν ἀριθμοὶ οἱ Β, Γ, Δ· λέγω, ὅτι οἱ Β, Γ, Δ ἀρτιάκις ἄρτιοί εἰσι μόνον.

Ὅτι μὲν οὖν ἕκαστος [τῶν Β, Γ, Δ] ἀρτιάκις ἄρτιός ἐστιν, φανερόν· ἀπὸ γὰρ δυάδος ἐστὶ διπλασιασθείς. λέγω, ὅτι καὶ μόνον. ἐκκείσθω γὰρ μονάς. ἐπεὶ οὖν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἐξῆς ἀνάλογόν εἰσιν, ὁ δὲ μετὰ τὴν μονάδα ὁ Α πρῶτός ἐστιν, ὁ μέγιστος τῶν Α, Β, Γ, Δ ὁ Δ ὑπ᾽ οὐδενὸς ἄλλου μετρηθήσεται παρὲξ τῶν Α, Β, Γ. καί ἐστιν ἕκαστος τῶν Α, Β, Γ ἄρτιος· ὁ Δ ἄρα ἀρτιάκις ἄρτιός ἐστι μόνον. ὁμοίως δὴ δείξομεν, ὅτι [καὶ] ἑκάτερος τῶν Β, Γ ἀρτιάκις ἄρτιός ἐστι μόνον· ὅπερ ἔδει δεῖξαι.

λγ΄. Ἐὰν ἀριθμὸς τὸν ἥμισυν ἔχῃ περισσόν, ἀρτιάκις περισσός ἐστι μόνον.

Ἀριθμὸς γὰρ ὁ Α τὸν ἥμισυν ἐχέτω περισσόν· λέγω, ὅτι ὁ Α ἀρτιάκις περισσός ἐστι μόνον.

Ὅτι μὲν οὖν ἀρτιάκις περισσός ἐστιν, φανερόν· ὁ γὰρ ἥμισυς αὐτοῦ περισσὸς ὢν μετρεῖ αὐτὸν ἀρτιάκις. λέγω δή, ὅτι καὶ μόνον. εἰ γὰρ ἔσται ὁ Α καὶ ἀρτιάκις ἄρτιος, μετρηθήσεται ὑπὸ ἀρτίου κατὰ ἄρτιον ἀριθμόν· ὥστε καὶ ὁ ἥμισυς αὐτοῦ μετρηθήσεται ὑπὸ ἀρτίου ἀριθμοῦ περισσὸς ὤν· ὅπερ ἐστὶν ἄτοπον. ὁ Α ἄρα ἀρτιάκις περισσός ἐστι μόνον· ὅπερ ἔδει δεῖξαι.

λδ΄. Ἐὰν ἀριθμὸς μήτε τῶν ἀπὸ δυάδος διπλασιαζομένων ᾖ μήτε τὸν ἥμισυν ἔχῃ περισσόν, ἀρτιάκις τε ἄρτιός ἐστι καὶ ἀρτιάκις περισσός.

Ἀριθμὸς γὰρ ὁ Α μήτε τῶν ἀπὸ δυάδος διπλασιαζομένων ἔστω μήτε τὸν ἥμισυν ἐχέτω περισσόν· λέγω, ὅτι ὁ Α ἀρτιάκις τέ ἐστιν ἄρτιος καὶ ἀρτιάκις περισσός.

Ὅτι μὲν οὖν ὁ Α ἀρτιάκις ἐστὶν ἄρτιος, φανερόν· τὸν γὰρ ἥμισυν οὐκ ἔχει περισσόν. λέγω δή, ὅτι καὶ ἀρτιάκις περισσός ἐστιν. ἐὰν γὰρ τὸν Α τέμνωμεν δίχα καὶ τὸν ἥμισυν αὐτοῦ δίχα καὶ τοῦτο ἀεὶ ποιῶμεν, καταντήσομεν εἴς τινα ἀριθμὸν περισσόν, ὃς μετρήσει τὸν Α κατὰ ἄρτιον ἀριθμόν. εἰ γὰρ οὔ, καταντήσομεν εἰς δυάδα, καὶ ἔσται ὁ Α τῶν ἀπὸ δυάδος διπλασιαζομένων· ὅπερ οὐχ ὑπόκειται. ὥστε ὁ Α ἀρτιάκις περισσός ἐστιν. ἐδείχθη δὲ καὶ ἀρτιάκις ἄρτιος. ὁ Α ἄρα ἀρτιάκις τε ἄρτιός ἐστι καὶ ἀρτιάκις περισσός· ὅπερ ἔδει δεῖξαι.

λε΄. Ἐὰν ὦσιν ὁσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, ἀφαιρεθῶσι δὲ ἀπό τε τοῦ δευτέρου καὶ τοῦ ἐσχάτου ἴσοι τῷ πρώτῳ, ἔσται ὡς ἡ τοῦ δευτέρου ὑπεροχὴ πρὸς τὸν πρῶτον, οὕτως ἡ τοῦ ἐσχάτου ὑπεροχὴ πρὸς τοὺς πρὸ ἑαυτοῦ πάντας.

Ἔστωσαν ὁποσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, ΒΓ, Δ, ΕΖ ἀρχόμενοι ἀπὸ ἐλαχίστου τοῦ Α, καὶ ἀφῃρήσθω ἀπὸ τοῦ ΒΓ καὶ τοῦ ΕΖ τῷ Α

ἴσος ἑκάτερος τῶν ΒΗ, ΖΘ· λέγω, ὅτι ἐστὶν ὡς ὁ ΗΓ πρὸς τὸν Α, οὕτως ὁ ΕΘ πρὸς τοὺς Α, ΒΓ, Δ.

Κείσθω γὰρ τῷ μὲν ΒΓ ἴσος ὁ ΖΚ, τῷ δὲ Δ ἴσος ὁ ΖΛ. καὶ ἐπεὶ ὁ ΖΚ τῷ ΒΓ ἴσος ἐστίν, ὧν ὁ ΖΘ τῷ ΒΗ ἴσος ἐστίν, λοιπὸς ἄρα ὁ ΘΚ λοιπῷ τῷ ΗΓ ἐστιν ἴσος. καὶ ἐπεί ἐστιν ὡς ὁ ΕΖ πρὸς τὸν Δ, οὕτως ὁ Δ πρὸς τὸν ΒΓ καὶ ὁ ΒΓ πρὸς τὸν Α, ἴσος δὲ ὁ μὲν Δ τῷ ΖΛ, ὁ δὲ ΒΓ τῷ ΖΚ, ὁ δὲ Α τῷ ΖΘ, ἔστιν ἄρα ὡς ὁ ΕΖ πρὸς τὸν ΖΛ, οὕτως ὁ ΛΖ πρὸς τὸν ΖΚ καὶ ὁ ΖΚ πρὸς τὸν ΖΘ. διελόντι, ὡς ὁ ΕΛ πρὸς τὸν ΛΖ, οὕτως ὁ ΛΚ πρὸς τὸν ΖΚ καὶ ὁ ΚΘ πρὸς τὸν ΖΘ. ἔστιν ἄρα καὶ ὡς εἷς τῶν ἡγουμένων πρὸς ἕνα τῶν ἑπομένων, οὕτως ἅπαντες οἱ ἡγούμενοι πρὸς ἅπαντας τοὺς ἑπομένους· ἔστιν ἄρα ὡς ὁ ΚΘ πρὸς τὸν ΖΘ, οὕτως οἱ ΕΛ, ΛΚ, ΚΘ πρὸς τοὺς ΛΖ, ΖΚ, ΘΖ. ἴσος δὲ ὁ μὲν ΚΘ τῷ ΓΗ, ὁ δὲ ΖΘ τῷ Α, οἱ δὲ ΛΖ, ΖΚ, ΘΖ τοῖς Δ, ΒΓ, Α· ἔστιν ἄρα ὡς ὁ ΓΗ πρὸς τὸν Α, οὕτως ὁ ΕΘ πρὸς τοὺς Δ, ΒΓ, Α. ἔστιν ἄρα ὡς ἡ τοῦ δευτέρου ὑπεροχὴ πρὸς τὸν πρῶτον, οὕτως ἡ τοῦ ἐσχάτου ὑπεροχὴ πρὸς τοὺς πρὸ ἑαυτοῦ πάντας· ὅπερ ἔδει δεῖξαι.

λς΄. Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἐκτεθῶσιν ἐν τῇ διπλασίονι ἀναλογίᾳ, ἕως οὗ ὁ σύμπας συντεθεὶς πρῶτος γένηται, καὶ ὁ σύμπας ἐπὶ τὸν ἔσχατον πολλαπλασιασθεὶς ποιῇ τινα, ὁ γενόμενος τέλειος ἔσται.

Ἀπὸ γὰρ μονάδος ἐκκείσθωσαν ὁσοιδηποτοῦν ἀριθμοὶ ἐν τῇ διπλασίονι ἀναλογίᾳ, ἕως οὗ ὁ σύμπας συντεθεὶς πρῶτος γένηται, οἱ Α, Β, Γ, Δ, καὶ τῷ σύμπαντι ἴσος ἔστω ὁ Ε, καὶ ὁ Ε τὸν Δ πολλαπλασιάσας τὸν ΖΗ ποιείτω. λέγω, ὅτι ὁ ΖΗ τέλειός ἐστιν.

Ὅσοι γάρ εἰσιν οἱ Α, Β, Γ, Δ τῷ πλήθει, τοσοῦτοι ἀπὸ τοῦ Ε εἰλήφθωσαν ἐν τῇ διπλασίονι ἀναλογίᾳ οἱ Ε, ΘΚ, Λ, Μ· δι’ ἴσου ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Μ. ὁ ἄρα ἐκ τῶν Ε, Δ ἴσος ἐστὶ τῷ ἐκ τῶν Α, Μ. καί ἐστιν ὁ ἐκ τῶν Ε, Δ ὁ ΖΗ· καὶ ὁ ἐκ τῶν Α, Μ ἄρα ἐστὶν ὁ ΖΗ. ὁ Α ἄρα

τὸν Μ πολλαπλασιάσας τὸν ΖΗ πεποίηκεν· ὁ Μ ἄρα τὸν ΖΗ μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας. καί ἐστι δυὰς ὁ Α· διπλάσιος ἄρα ἐστὶν ὁ ΖΗ τοῦ Μ.

εἰσὶ δὲ καὶ οἱ Μ, Λ, ΘΚ, Ε ἑξῆς διπλάσιοι ἀλλήλων· οἱ Ε, ΘΚ, Λ, Μ, ΖΗ ἄρα ἑξῆς ἀνάλογόν εἰσιν ἐν τῇ διπλασίονι ἀναλογίᾳ. ἀφῃρήσθω δὴ ἀπὸ τοῦ δευτέρου τοῦ ΘΚ καὶ τοῦ ἐσχάτου τοῦ ΖΗ τῷ πρώτῳ τῷ Ε ἴσος ἑκάτερος τῶν ΘΝ, ΖΞ· ἔστιν ἄρα ὡς ἡ τοῦ δευτέρου ἀριθμοῦ ὑπεροχὴ πρὸς τὸν πρῶτον, οὕτως ἡ τοῦ ἐσχάτου ὑπεροχὴ πρὸς τοὺς πρὸ ἑαυτοῦ πάντας. ἔστιν ἄρα ὡς ὁ ΝΚ πρὸς τὸν Ε, οὕτως ὁ ΞΗ πρὸς τοὺς Μ, Λ, ΚΘ, Ε. καί ἐστιν ὁ ΝΚ ἴσος τῷ Ε· καὶ ὁ ΞΗ ἄρα ἴσος ἐστὶ τοῖς Μ, Λ, ΘΚ, Ε. ἔστι δὲ καὶ ὁ ΖΞ τῷ Ε ἴσος, ὁ δὲ Ε τοῖς Α, Β, Γ, Δ καὶ τῇ μονάδι. ὅλος ἄρα ὁ ΖΗ ἴσος ἐστὶ τοῖς τε Ε, ΘΚ, Λ, Μ καὶ τοῖς Α, Β, Γ, Δ καὶ τῇ μονάδι· καὶ μετρεῖται ὑπ' αὐτῶν. λέγω, ὅτι καὶ ὁ ΖΗ ὑπ' οὐδενὸς ἄλλου μετρηθήσεται παρὲξ τῶν Α, Β, Γ, Δ, Ε, ΘΚ, Λ, Μ καὶ τῆς μονάδος. εἰ γὰρ δυνατόν, μετρείτω τις τὸν ΖΗ ὁ Ο, καὶ ὁ Ο μηδενὶ τῶν Α, Β, Γ, Δ, Ε, ΘΚ, Λ, Μ ἔστω ὁ αὐτός. καὶ ὁσάκις ὁ Ο τὸν ΖΗ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Π· ὁ Π ἄρα τὸν Ο πολλαπλασιάσας τὸν ΖΗ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Ε τὸν Δ πολλαπλασιάσας τὸν ΖΗ πεποίηκεν· ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Π, ὁ Ο πρὸς τὸν Δ. καὶ ἐπεὶ ἀπὸ μονάδος ἑξῆς ἀνάλογόν εἰσιν οἱ Α, Β, Γ, Δ, ὁ Δ ἄρα ὑπ' οὐδενὸς ἄλλου ἀριθμοῦ μετρηθήσεται παρὲξ τῶν Α, Β, Γ. καὶ ὑπόκειται ὁ Ο οὐδενὶ τῶν Α, Β, Γ ὁ αὐτός· οὐκ ἄρα μετρήσει ὁ Ο τὸν Δ. ἀλλ' ὡς ὁ Ο πρὸς τὸν Δ, ὁ Ε πρὸς τὸν Π· οὐδὲ ὁ Ε ἄρα τὸν Π μετρεῖ. καί ἐστιν ὁ Ε πρῶτος· πᾶς δὲ πρῶτος ἀριθμὸς πρὸς ἅπαντα, ὃν μὴ μετρεῖ, πρῶτος [ἐστιν]. οἱ Ε, Π ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον· καί ἐστιν ὡς ὁ Ε πρὸς τὸν Π, ὁ Ο πρὸς τὸν Δ· ἰσάκις ἄρα ὁ Ε τὸν Ο μετρεῖ καὶ ὁ Π τὸν Δ· ἰσάκις ἄρα ὁ Ε τὸν Ο μετρεῖ καὶ ὁ Π τὸν Δ. ὁ δὲ Δ ὑπ' οὐδενὸς ἄλλου μετρεῖται παρὲξ τῶν Α, Β, Γ· ὁ Π ἄρα ἑνὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός. ἔστω τῷ Β ὁ αὐτός. καὶ ὅσοι εἰσὶν οἱ Β, Γ, Δ τῷ πλήθει

τοσοῦτοι εἰλήφθωσαν ἀπὸ τοῦ Ε οἱ Ε, ΘΚ, Λ. καί εἰσιν οἱ Ε, ΘΚ, Λ τοῖς Β, Γ, Δ ἐν τῷ αὐτῷ λόγῳ· δι’ ἴσου ἄρα ἐστὶν ὡς ὁ Β πρὸς τὸν Δ, ὁ Ε πρὸς τὸν Λ. ὁ ἄρα ἐκ τῶν Β, Λ ἴσος ἐστὶ τῷ ἐκ τῶν Δ, Ε· ἀλλ’ ὁ ἐκ τῶν Δ, Ε ἴσος ἐστὶ τῷ ἐκ τῶν Π, Ο· καὶ ὁ ἐκ τῶν Π, Ο ἄρα ἴσος ἐστὶ τῷ ἐκ τῶν Β, Λ. ἔστιν ἄρα ὡς ὁ Π πρὸς τὸν Β, ὁ Λ πρὸς τὸν Ο. καί ἐστιν ὁ Π τῷ Β ὁ αὐτός· καὶ ὁ Λ ἄρα τῷ Ο ἐστιν ὁ αὐτός· ὅπερ ἀδύνατον· ὁ γὰρ Ο ὑπόκειται μηδενὶ τῶν ἐκκειμένων ὁ αὐτός. οὐκ ἄρα τὸν ΖΗ μετρήσει τις ἀριθμὸς παρὲξ τῶν Α, Β, Γ, Δ, Ε, ΘΚ, Λ, Μ καὶ τῆς μονάδος. καὶ ἐδείχθη ὁ ΖΗ τοῖς Α, Β, Γ, Δ, Ε, ΘΚ, Λ, Μ καὶ τῇ μονάδι ἴσος. τέλειος δὲ ἀριθμός ἐστιν ὁ τοῖς ἑαυτοῦ μέρεσιν ἴσος ὤν· τέλειος ἄρα ἐστὶν ὁ ΖΗ· ὅπερ ἔδει δεῖξαι.

BOOK X.

Ὅροι δ΄.

α΄. Σύμμετρα μεγέθη λέγεται τὰ τῷ αὐτῷ μέτρῳ μετρούμενα, ἀσύμμετρα δέ, ὧν μηδὲν ἐνδέχεται κοινὸν μέτρον γενέσθαι.

β΄. Εὐθεῖαι δυνάμει σύμμετροί εἰσιν, ὅταν τὰ ἀπ᾽ αὐτῶν τετράγωνα τῷ αὐτῷ χωρίῳ μετρῆται, ἀσύμμετροι δέ, ὅταν τοῖς ἀπ᾽ αὐτῶν τετραγώνοις μηδὲν ἐνδέχηται χωρίον κοινὸν μέτρον γενέσθαι.

γ΄. Τούτων ὑποκειμένων δείκνυται, ὅτι τῇ προτεθείσῃ εὐθείᾳ ὑπάρχουσιν εὐθεῖαι πλήθει ἄπειροι σύμμετροί τε καὶ ἀσύμμετροι αἱ μὲν μήκει μόνον, αἱ δὲ καὶ δυνάμει. καλείσθω οὖν ἡ μὲν προτεθεῖσα εὐθεῖα ῥητή, καὶ αἱ ταύτῃ σύμμετροι εἴτε μήκει καὶ δυνάμει εἴτε δυνάμει μόνον ῥηταί, αἱ δὲ ταύτῃ ἀσύμμετροι ἄλογοι καλείσθωσαν.

δ΄. Καὶ τὸ μὲν ἀπὸ τῆς προτεθείσης εὐθείας τετράγωνον ῥητόν, καὶ τὰ τούτῳ σύμμετρα ῥητά, τὰ δὲ τούτῳ ἀσύμμετρα ἄλογα καλείσθω, καὶ αἱ δυνάμεναι αὐτὰ ἄλογοι, εἰ μὲν τετράγωνα εἴη, αὐταὶ αἱ πλευραί, εἰ δὲ ἕτερά τινα εὐθύγραμμα, αἱ ἴσα αὐτοῖς τετράγωνα ἀναγράφουσαι.

Προτάσεις ριε΄.(μέρος 1)

α΄. Δύο μεγεθῶν ἀνίσων ἐκκειμένων, ἐὰν ἀπὸ τοῦ μείζονος ἀφαιρεθῇ μεῖζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μεῖζον ἢ τὸ ἥμισυ, καὶ τοῦτο ἀεὶ γίγνηται, λειφθήσεταί τι μέγεθος, ὃ ἔσται ἔλασσον τοῦ ἐκκειμένου ἐλάσσονος μεγέθους.

Ἔστω δύο μεγέθη ἄνισα τὰ ΑΒ, Γ, ὧν μεῖζον τὸ ΑΒ· λέγω, ὅτι, ἐὰν ἀπὸ τοῦ ΑΒ ἀφαιρεθῇ μεῖζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μεῖζον ἢ τὸ ἥμισυ, καὶ τοῦτο ἀεὶ γίγνηται, λειφθήσεταί τι μέγεθος, ὃ ἔσται ἔλασσον τοῦ Γ μεγέθους.

Τὸ Γ γὰρ πολλαπλασιαζόμενον ἔσται ποτὲ τοῦ ΑΒ μεῖζον. πεπολλαπλασιάσθω, καὶ ἔστω τὸ ΔΕ τοῦ μὲν Γ πολλαπλάσιον, τοῦ δὲ ΑΒ μεῖζον, καὶ διῃρήσθω τὸ ΔΕ εἰς τὰ τῷ Γ ἴσα τὰ ΔΖ, ΖΗ, ΗΕ, καὶ ἀφῃρήσθω ἀπὸ μὲν τοῦ ΑΒ μεῖζον ἢ τὸ ἥμισυ τὸ ΒΘ, ἀπὸ δὲ τοῦ ΑΘ μεῖζον ἢ τὸ ἥμισυ τὸ ΘΚ, καὶ τοῦτο ἀεὶ γιγνέσθω, ἕως ἂν αἱ ἐν τῷ ΑΒ διαιρέσεις ἰσοπληθεῖς γένωνται ταῖς ἐν τῷ ΔΕ διαιρέσεσιν.

Ἔστωσαν οὖν αἱ ΑΚ, ΚΘ, ΘΒ διαιρέσεις ἰσοπληθεῖς οὖσαι ταῖς ΔΖ, ΖΗ, ΗΕ· καὶ ἐπεὶ μεῖζόν ἐστι τὸ ΔΕ τοῦ ΑΒ, καὶ ἀφῄρηται ἀπὸ μὲν τοῦ ΔΕ ἔλασσον τοῦ ἡμίσεος τὸ ΕΗ, ἀπὸ δὲ τοῦ ΑΒ μεῖζον ἢ τὸ ἥμισυ τὸ ΒΘ, λοιπὸν ἄρα τὸ ΗΔ λοιποῦ τοῦ ΘΑ μεῖζόν ἐστιν. καὶ ἐπεὶ μεῖζόν ἐστι τὸ ΗΔ τοῦ ΘΑ, καὶ ἀφῄρηται τοῦ μὲν ΗΔ ἥμισυ τὸ ΗΖ, τοῦ δὲ ΘΑ μεῖζον ἢ τὸ ἥμισυ τὸ ΘΚ, λοιπὸν ἄρα τὸ ΔΖ λοιποῦ τοῦ ΑΚ μεῖζόν ἐστιν. ἴσον δὲ τὸ ΔΖ τῷ Γ· καὶ τὸ Γ ἄρα τοῦ ΑΚ μεῖζόν ἐστιν. ἔλασσον ἄρα τὸ ΑΚ τοῦ Γ.

Καταλείπεται ἄρα ἀπὸ τοῦ ΑΒ μεγέθους τὸ ΑΚ μέγεθος ἔλασσον ὂν τοῦ ἐκκειμένου ἐλάσσονος μεγέθους τοῦ Γ· ὅπερ ἔδει δεῖξαι.-ὁμοίως δὲ δειχθήσεται, κἂν ἡμίση ᾖ τὰ ἀφαιρούμενα.

β΄. Ἐὰν δύο μεγεθῶν [ἐκκειμένων] ἀνίσων ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος τὸ καταλειπόμενον μηδέποτε καταμετρῇ τὸ πρὸ

ἑαυτοῦ, ἀσύμμετρα ἔσται τὰ μεγέθη.

Δύο γὰρ μεγεθῶν ὄντων ἀνίσων τῶν ΑΒ, ΓΔ καὶ ἐλάσσονος τοῦ ΑΒ ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος τὸ περιλειπόμενον μηδέποτε καταμετρείτω τὸ πρὸ ἑαυτοῦ· λέγω, ὅτι ἀσύμμετρά ἐστι τὰ ΑΒ, ΓΔ μεγέθη.

Εἰ γάρ ἐστι σύμμετρα, μετρήσει τι αὐτὰ μέγεθος. μετρείτω, εἰ δυνατόν, καὶ ἔστω τὸ Ε· καὶ τὸ μὲν ΑΒ τὸ ΖΔ καταμετροῦν λειπέτω ἑαυτοῦ ἔλασσον τὸ ΓΖ, τὸ δὲ ΓΖ τὸ ΒΗ καταμετροῦν λειπέτω ἑαυτοῦ ἔλασσον τὸ ΑΗ, καὶ τοῦτο ἀεὶ γινέσθω, ἕως οὗ λειφθῇ τι μέγεθος, ὅ ἐστιν ἔλασσον τοῦ Ε. γεγονέτω,

καὶ λελείφθω τὸ ΑΗ ἔλασσον τοῦ Ε. ἐπεὶ οὖν τὸ Ε τὸ ΑΒ μετρεῖ, ἀλλὰ τὸ ΑΒ τὸ ΔΖ μετρεῖ, καὶ τὸ Ε ἄρα τὸ ΖΔ μετρήσει. μετρεῖ δὲ καὶ ὅλον τὸ ΓΔ· καὶ λοιπὸν ἄρα τὸ ΓΖ μετρήσει. ἀλλὰ τὸ ΓΖ τὸ ΒΗ μετρεῖ· καὶ τὸ Ε ἄρα τὸ ΒΗ μετρεῖ. μετρεῖ δὲ καὶ ὅλον τὸ ΑΒ· καὶ λοιπὸν ἄρα τὸ ΑΗ μετρήσει, τὸ μεῖζον τὸ ἔλασσον. ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὰ ΑΒ, ΓΔ μεγέθη μετρήσει τι μέγεθος· ἀσύμμετρα ἄρα ἐστὶ τὰ ΑΒ, ΓΔ μεγέθη.

Ἐὰν ἄρα δύο μεγεθῶν ἀνίσων, καὶ τὰ ἑξῆς.

γ΄. Δύο μεγεθῶν συμμέτρων δοθέντων τὸ μέγιστον αὐτῶν κοινὸν μέτρον εὑρεῖν.

Ἔστω τὰ δοθέντα δύο μεγέθη σύμμετρα τὰ ΑΒ, ΓΔ, ὧν ἔλασσον τὸ ΑΒ· δεῖ δὴ τῶν ΑΒ, ΓΔ τὸ μέγιστον κοινὸν μέτρον εὑρεῖν.

Τὸ ΑΒ γὰρ μέγεθος ἤτοι μετρεῖ τὸ ΓΔ ἢ οὔ. εἰ μὲν οὖν μετρεῖ, μετρεῖ δὲ καὶ ἑαυτό, τὸ ΑΒ ἄρα τῶν ΑΒ, ΓΔ κοινὸν μέτρον ἐστίν· καὶ φανερόν, ὅτι καὶ μέγιστον. μεῖζον γὰρ τοῦ ΑΒ μεγέθους τὸ ΑΒ οὐ μετρήσει.

Μὴ μετρείτω δὴ τὸ ΑΒ τὸ ΓΔ. καὶ ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος, τὸ περιλειπόμενον μετρήσει ποτὲ τὸ πρὸ ἑαυτοῦ διὰ τὸ μὴ εἶναι ἀσύμμετρα τὰ ΑΒ, ΓΔ· καὶ τὸ μὲν ΑΒ τὸ ΕΔ καταμετροῦν λειπέτω

ἑαυτοῦ ἔλασσον τὸ ΕΓ, τὸ δὲ ΕΓ τὸ ΖΒ καταμετροῦν λειπέτω ἑαυτοῦ ἔλασσον τὸ ΑΖ, τὸ δὲ ΑΖ τὸ ΓΕ μετρείτω.

Ἐπεὶ οὖν τὸ ΑΖ τὸ ΓΕ μετρεῖ, ἀλλὰ τὸ ΓΕ τὸ ΖΒ μετρεῖ, καὶ τὸ ΑΖ ἄρα τὸ ΖΒ μετρήσει. μετρεῖ δὲ καὶ ἑαυτό· καὶ ὅλον ἄρα τὸ ΑΒ μετρήσει τὸ ΑΖ. ἀλλὰ τὸ ΑΒ τὸ ΔΕ μετρεῖ· καὶ τὸ ΑΖ ἄρα τὸ ΕΔ μετρήσει. μετρεῖ δὲ καὶ τὸ ΓΕ· καὶ ὅλον ἄρα τὸ ΓΔ μετρεῖ· τὸ ΑΖ ἄρα τῶν ΑΒ, ΓΔ κοινὸν μέτρον ἐστίν. λέγω δή, ὅτι καὶ μέγιστον. εἰ γὰρ μή, ἔσται τι μέγεθος μεῖζον τοῦ ΑΖ, ὃ μετρήσει τὰ ΑΒ, ΓΔ. ἔστω τὸ Η. ἐπεὶ οὖν τὸ Η τὸ ΑΒ μετρεῖ, ἀλλὰ τὸ ΑΒ τὸ ΕΔ μετρεῖ, καὶ τὸ Η ἄρα τὸ ΕΔ μετρήσει. μετρεῖ δὲ καὶ ὅλον τὸ ΓΔ· καὶ λοιπὸν ἄρα τὸ ΓΕ μετρήσει τὸ Η. ἀλλὰ τὸ ΓΕ τὸ ΖΒ μετρεῖ· καὶ τὸ Η ἄρα τὸ ΖΒ μετρήσει. μετρεῖ δὲ καὶ ὅλον τὸ ΑΒ, καὶ λοιπὸν τὸ ΑΖ μετρήσει, τὸ μεῖζον τὸ ἔλασσον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα μεῖζόν τι μέγεθος τοῦ ΑΖ τὰ ΑΒ, ΓΔ μετρήσει· τὸ ΑΖ ἄρα τῶν ΑΒ, ΓΔ τὸ μέγιστον κοινὸν μέτρον ἐστίν.

Δύο ἄρα μεγεθῶν συμμέτρων δοθέντων τῶν ΑΒ, ΓΔ τὸ μέγιστον κοινὸν μέτρον ηὕρηται· ὅπερ ἔδει δεῖξαι.

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι, ἐὰν μέγεθος δύο μεγέθη μετρῇ, καὶ τὸ μέγιστον αὐτῶν κοινὸν μέτρον μετρήσει.

δ΄. Τριῶν μεγεθῶν συμμέτρων δοθέντων τὸ μέγιστον αὐτῶν κοινὸν μέτρον εὑρεῖν.

Ἔστω τὰ δοθέντα τρία μεγέθη σύμμετρα τὰ Α, Β, Γ· δεῖ δὴ τῶν Α, Β, Γ τὸ μέγιστον κοινὸν μέτρον εὑρεῖν.

Εἰλήφθω γὰρ δύο τῶν Α, Β τὸ μέγιστον κοινὸν μέτρον, καὶ ἔστω τὸ Δ· τὸ δὴ Δ τὸ Γ ἤτοι μετρεῖ ἢ οὔ [μετρεῖ]. μετρείτω πρότερον. ἐπεὶ οὖν τὸ Δ τὸ Γ μετρεῖ, μετρεῖ δὲ καὶ τὰ Α, Β, τὸ Δ ἄρα τὰ Α, Β, Γ μετρεῖ· τὸ Δ ἄρα τῶν Α, Β,

Γ κοινὸν μέτρον ἐστίν. καὶ φανερόν, ὅτι καὶ μέγιστον· μεῖζον γὰρ τοῦ Δ μεγέθους τὰ Α, Β οὐ μετρεῖ.

Μὴ μετρείτω δὴ τὸ Δ τὸ Γ. λέγω πρῶτον, ὅτι σύμμετρά ἐστι τὰ Γ, Δ. ἐπεὶ γὰρ σύμμετρά ἐστι τὰ Α, Β, Γ, μετρήσει τι αὐτὰ μέγεθος, ὃ δηλαδὴ καὶ τὰ Α, Β μετρήσει· ὥστε καὶ τὸ τῶν Α, Β μέγιστον κοινὸν μέτρον τὸ Δ μετρήσει. μετρεῖ δὲ καὶ τὸ Γ· ὥστε τὸ εἰρημένον μέγεθος μετρήσει τὰ Γ, Δ· σύμμετρα ἄρα ἐστὶ τὰ Γ, Δ. εἰλήφθω οὖν αὐτῶν τὸ μέγιστον κοινὸν μέτρον, καὶ ἔστω τὸ Ε. ἐπεὶ οὖν τὸ Ε τὸ Δ μετρεῖ, ἀλλὰ τὸ Δ τὰ Α, Β μετρεῖ, καὶ τὸ Ε ἄρα τὰ Α, Β μετρήσει. μετρεῖ δὲ καὶ τὸ Γ. τὸ Ε ἄρα τὰ Α, Β, Γ μετρεῖ· τὸ Ε ἄρα τῶν Α, Β, Γ κοινόν ἐστι μέτρον. λέγω δή, ὅτι καὶ μέγιστον. εἰ γὰρ δυνατόν, ἔστω τι τοῦ Ε μεῖζον μέγεθος τὸ Ζ, καὶ μετρείτω τὰ Α, Β, Γ. καὶ ἐπεὶ τὸ Ζ τὰ Α, Β, Γ μετρεῖ, καὶ τὰ Α, Β ἄρα μετρήσει καὶ τὸ τῶν Α, Β μέγιστον κοινὸν μέτρον μετρήσει. τὸ δὲ τῶν Α, Β μέγιστον κοινὸν μέτρον ἐστὶ τὸ Δ· τὸ Ζ ἄρα τὸ Δ μετρεῖ. μετρεῖ δὲ καὶ τὸ Γ· τὸ Ζ ἄρα τὰ Γ, Δ μετρεῖ· καὶ τὸ τῶν Γ, Δ ἄρα μέγιστον κοινὸν μέτρον μετρήσει τὸ Ζ. ἔστι δὲ τὸ Ε· τὸ Ζ ἄρα τὸ Ε μετρήσει, τὸ μεῖζον τὸ ἔλασσον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα μεῖζόν τι τοῦ Ε μεγέθους [μέγεθος] τὰ Α, Β, Γ μετρεῖ· τὸ Ε ἄρα τῶν Α, Β, Γ τὸ μέγιστον κοινὸν μέτρον ἐστίν, ἐὰν μὴ μετρῇ τὸ Δ τὸ Γ, ἐὰν δὲ μετρῇ, αὐτὸ τὸ Δ.

Τριῶν ἄρα μεγεθῶν συμμέτρων δοθέντων τὸ μέγιστον κοινὸν μέτρον ηὕρηται [ὅπερ ἔδει δεῖξαι].

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι, ἐὰν μέγεθος τρία μεγέθη μετρῇ, καὶ τὸ μέγιστον αὐτῶν κοινὸν μέτρον μετρήσει.

Ὁμοίως δὴ καὶ ἐπὶ πλειόνων τὸ μέγιστον κοινὸν μέτρον ληφθήσεται, καὶ τὸ πόρισμα προχωρήσει. ὅπερ ἔδει δεῖξαι.

ε΄. Τὰ σύμμετρα μεγέθη πρὸς ἄλληλα λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν.

Ἔστω σύμμετρα μεγέθη τὰ Α, Β· λέγω, ὅτι τὸ Α πρὸς τὸ Β λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν.

Ἐπεὶ γὰρ σύμμετρά ἐστι τὰ Α, Β, μετρήσει τι αὐτὰ μέγεθος. μετρείτω, καὶ ἔστω τὸ Γ. καὶ ὁσάκις τὸ Γ τὸ Α μετρεῖ τοσαῦται μονάδες ἔστωσαν ἐν τῷ Δ, ὁσάκις δὲ τὸ Γ τὸ Β μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε.

Ἐπεὶ οὖν τὸ Γ τὸ Α μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας, μετρεῖ δὲ καὶ ἡ μονὰς τὸν Δ κατὰ τὰς ἐν αὐτῷ μονάδας, ἰσάκις ἄρα ἡ μονὰς τὸν Δ μετρεῖ ἀριθμὸν καὶ τὸ Γ μέγεθος τὸ Α· ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Α, οὕτως ἡ μονὰς πρὸς τὸν Δ· ἀνάπαλιν ἄρα, ὡς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Δ πρὸς τὴν μονάδα. πάλιν ἐπεὶ τὸ Γ τὸ Β μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας, μετρεῖ δὲ καὶ ἡ μονὰς τὸν Ε κατὰ τὰς ἐν αὐτῷ μονάδας, ἰσάκις ἄρα ἡ μονὰς τὸν Ε μετρεῖ καὶ τὸ Γ τὸ Β· ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Β, οὕτως ἡ μονὰς πρὸς τὸν Ε. ἐδείχθη δὲ καὶ ὡς τὸ Α πρὸς τὸ Γ, ὁ Δ πρὸς τὴν μονάδα· δι’ ἴσου ἄρα ἐστὶν ὡς τὸ Α πρὸς τὸ Β, οὕτως ὁ Δ ἀριθμὸς πρὸς τὸν Ε.

Τὰ ἄρα σύμμετρα μεγέθη τὰ Α, Β πρὸς ἄλληλα λόγον ἔχει, ὃν ἀριθμὸς ὁ Δ πρὸς ἀριθμὸν τὸν Ε· ὅπερ ἔδει δεῖξαι.

ς΄. Ἐὰν δύο μεγέθη πρὸς ἄλληλα λόγον ἔχῃ, ὃν ἀριθμὸς πρὸς ἀριθμόν, σύμμετρα ἔσται τὰ μεγέθη.

Δύο γὰρ μεγέθη τὰ Α, Β πρὸς ἄλληλα λόγον ἐχέτω, ὃν ἀριθμὸς ὁ Δ πρὸς ἀριθμὸν τὸν Ε· λέγω, ὅτι σύμμετρά ἐστι τὰ Α, Β μεγέθη.

Ὅσαι γάρ εἰσιν ἐν τῷ Δ μονάδες, εἰς τοσαῦτα ἴσα διῃρήσθω τὸ Α, καὶ ἑνὶ αὐτῶν ἴσον ἔστω τὸ Γ· ὅσαι δέ εἰσιν ἐν τῷ Ε μονάδες, ἐκ τοσούτων μεγεθῶν ἴσων τῷ Γ συγκείσθω τὸ Ζ.

Ἐπεὶ οὖν, ὅσαι εἰσὶν ἐν τῷ Δ μονάδες, τοσαῦτά εἰσι καὶ ἐν τῷ Α μεγέθη ἴσα τῷ Γ, ὃ ἄρα μέρος ἐστὶν ἡ μονὰς τοῦ Δ, τὸ αὐτὸ μέρος ἐστὶ καὶ τὸ Γ τοῦ Α· ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Α, οὕτως ἡ μονὰς πρὸς τὸν Δ. μετρεῖ δὲ ἡ μονὰς τὸν Δ ἀριθμόν· μετρεῖ ἄρα καὶ τὸ Γ τὸ Α. καὶ ἐπεί ἐστιν ὡς τὸ Γ πρὸς

τὸ Α, οὕτως ἡ μονὰς πρὸς τὸν Δ [ἀριθμόν], ἀνάπαλιν ἄρα ὡς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Δ ἀριθμὸς πρὸς τὴν μονάδα. πάλιν ἐπεί, ὅσαι εἰσὶν ἐν τῷ Ε μονάδες, τοσαῦτά εἰσι καὶ ἐν τῷ Ζ ἴσα τῷ Γ, ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Ζ, οὕτως ἡ μονὰς πρὸς τὸν Ε [ἀριθμόν]. ἐδείχθη δὲ καὶ ὡς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Δ πρὸς τὴν μονάδα· δι' ἴσου ἄρα ἐστὶν ὡς τὸ Α πρὸς τὸ Ζ, οὕτως ὁ Δ πρὸς τὸν Ε. ἀλλ' ὡς ὁ Δ πρὸς τὸν Ε, οὕτως ἐστὶ τὸ Α πρὸς τὸ Β· καὶ ὡς ἄρα τὸ Α πρὸς τὸ Β, οὕτως καὶ πρὸς τὸ Ζ. τὸ Α ἄρα πρὸς ἑκάτερον τῶν Β, Ζ τὸν αὐτὸν ἔχει λόγον· ἴσον ἄρα ἐστὶ τὸ Β τῷ Ζ. μετρεῖ δὲ τὸ Γ τὸ Ζ· μετρεῖ ἄρα καὶ τὸ Β. ἀλλὰ μὴν καὶ τὸ Α· τὸ Γ ἄρα τὰ Α, Β μετρεῖ. σύμμετρον ἄρα ἐστὶ τὸ Α τῷ Β.

Ἐὰν ἄρα δύο μεγέθη πρὸς ἄλληλα, καὶ τὰ ἑξῆς.

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι, ἐὰν ὦσι δύο ἀριθμοί, ὡς οἱ Δ, Ε, καὶ εὐθεῖα, ὡς ἡ Α, δύνατόν ἐστι ποιῆσαι ὡς ὁ Δ ἀριθμὸς πρὸς τὸν Ε ἀριθμόν, οὕτως τὴν εὐθεῖαν πρὸς εὐθεῖαν. ἐὰν δὲ καὶ τῶν Α, Ζ μέση ἀνάλογον ληφθῇ, ὡς ἡ Β, ἔσται ὡς ἡ Α πρὸς τὴν Ζ, οὕτως τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς Β, τουτέστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον. ἀλλ' ὡς ἡ Α πρὸς τὴν Ζ, οὕτως ἐστὶν ὁ Δ ἀριθμὸς πρὸς τὸν Ε ἀριθμόν· γέγονεν ἄρα καὶ ὡς ὁ Δ ἀριθμὸς πρὸς τὸν Ε ἀριθμόν, οὕτως τὸ ἀπὸ τῆς Α εὐθείας πρὸς τὸ ἀπὸ τῆς Β εὐθείας· ὅπερ ἔδει δεῖξαι.

ζ΄. Τὰ ἀσύμμετρα μεγέθη πρὸς ἄλληλα λόγον οὐκ ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν.

Ἔστω ἀσύμμετρα μεγέθη τὰ Α, Β· λέγω, ὅτι τὸ Α πρὸς τὸ Β λόγον οὐκ ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. εἰ γὰρ ἔχει τὸ Α πρὸς τὸ Β λόγον, ὃν ἀριθμὸς πρὸς ἀριθμόν, σύμμετρον ἔσται τὸ Α τῷ Β. οὐκ ἔστι δέ· οὐκ ἄρα τὸ Α πρὸς τὸ Β λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν.

Τὰ ἄρα ἀσύμμετρα μεγέθη πρὸς ἄλληλα λόγον οὐκ ἔχει, καὶ τὰ ἑξῆς.

η΄. Ἐὰν δύο μεγέθη πρὸς ἄλληλα λόγον μὴ ἔχῃ, ὃν ἀριθμὸς πρὸς ἀριθμόν, ἀσύμμετρα ἔσται τὰ μεγέθη.

Δύο γὰρ μεγέθη τὰ Α, Β πρὸς ἄλληλα λόγον μὴ ἐχέτω, ὃν ἀριθμὸς πρὸς ἀριθμόν· λέγω, ὅτι ἀσύμμετρά ἐστι τὰ Α, Β μεγέθη.

Εἰ γὰρ ἔσται σύμμετρα, τὸ Α πρὸς τὸ Β λόγον ἕξει, ὃν ἀριθμὸς πρὸς ἀριθμόν. οὐκ ἔχει δέ. ἀσύμμετρα ἄρα ἐστὶ τὰ Α, Β μεγέθη.

Ἐὰν ἄρα δύο μεγέθη πρὸς ἄλληλα, καὶ τὰ ἑξῆς.

θ΄. Τὰ ἀπὸ τῶν μήκει συμμέτρων εὐθειῶν τετράγωνα πρὸς ἄλληλα λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ τὰ τετράγωνα τὰ πρὸς ἄλληλα λόγον ἔχοντα, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ τὰς πλευρὰς ἕξει μήκει συμμέτρους. τὰ δὲ ἀπὸ τῶν μήκει ἀσυμμέτρων εὐθειῶν τετράγωνα πρὸς ἄλληλα λόγον οὐκ ἔχει, ὅνπερ τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ τὰ τετράγωνα τὰ πρὸς ἄλληλα λόγον μὴ ἔχοντα, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὰς πλευρὰς ἕξει μήκει συμμέτρους.

Ἔστωσαν γὰρ αἱ Α, Β μήκει σύμμετροι· λέγω, ὅτι τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β τετράγωνον λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν.

Ἐπεὶ γὰρ σύμμετρός ἐστιν ἡ Α τῇ Β μήκει, ἡ Α ἄρα πρὸς τὴν Β λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. ἐχέτω, ὃν ὁ Γ πρὸς τὸν Δ. ἐπεὶ οὖν ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ὁ Γ πρὸς τὸν Δ, ἀλλὰ τοῦ μὲν τῆς Α πρὸς τὴν Β λόγου διπλασίων ἐστὶν ὁ τοῦ ἀπὸ τῆς Α τετραγώνου πρὸς τὸ ἀπὸ τῆς Β τετράγωνον· τὰ γὰρ ὅμοια σχήματα ἐν διπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν· τοῦ δὲ τοῦ Γ [ἀριθμοῦ] πρὸς τὸν Δ [ἀριθμὸν] λόγου διπλασίων ἐστὶν ὁ τοῦ ἀπὸ τοῦ Γ τετραγώνου πρὸς τὸν ἀπὸ τοῦ Δ τετράγωνον· δύο γὰρ τετραγώνων ἀριθμῶν εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ τετράγωνος πρὸς τὸν

τετράγωνον [ἀριθμὸν] διπλασίονα λόγον ἔχει, ἥπερ ἡ πλευρὰ πρὸς τὴν πλευράν· ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β τετράγωνον, οὕτως ὁ ἀπὸ τοῦ Γ τετράγωνος [ἀριθμὸς] πρὸς τὸν ἀπὸ τοῦ Δ [ἀριθμοῦ] τετράγωνον [ἀριθμόν].

Ἀλλὰ δὴ ἔστω ὡς τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β, οὕτως ὁ ἀπὸ τοῦ Γ τετράγωνος πρὸς τὸν ἀπὸ τοῦ Δ [τετράγωνον]· λέγω, ὅτι σύμμετρός ἐστιν ἡ Α τῇ Β μήκει.

Ἐπεὶ γάρ ἐστιν ὡς τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον], οὕτως ὁ ἀπὸ τοῦ Γ τετράγωνος πρὸς τὸν ἀπὸ τοῦ Δ [τετράγωνον], ἀλλ' ὁ μὲν τοῦ ἀπὸ τῆς Α τετραγώνου πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον] λόγος διπλασίων ἐστὶ τοῦ τῆς Α πρὸς τὴν Β λόγου, ὁ δὲ τοῦ ἀπὸ τοῦ Γ [ἀριθμοῦ] τετραγώνου [ἀριθμοῦ] πρὸς τὸν ἀπὸ τοῦ Δ [ἀριθμοῦ] τετράγωνον [ἀριθμὸν] λόγος διπλασίων ἐστὶ τοῦ τοῦ Γ [ἀριθμοῦ] πρὸς τὸν Δ [ἀριθμὸν] λόγου, ἔστιν ἄρα καὶ ὡς ἡ Α πρὸς τὴν Β, οὕτως ὁ Γ [ἀριθμὸς] πρὸς τὸν Δ [ἀριθμόν]. ἡ Α ἄρα πρὸς τὴν Β, λόγον ἔχει, ὃν ἀριθμὸς ὁ Γ πρὸς ἀριθμὸν τὸν Δ· σύμμετρος ἄρα ἐστὶν ἡ Α τῇ Β μήκει.

Ἀλλὰ δὴ ἀσύμμετρος ἔστω ἡ Α τῇ Β μήκει· λέγω, ὅτι τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον] λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν.

Εἰ γὰρ ἔχει τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον] λόγον, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, σύμμετρος ἔσται ἡ Α τῇ Β. οὐκ ἔστι δέ· οὐκ ἄρα τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον] λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν.

Πάλιν δὴ τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον] λόγον μὴ ἐχέτω, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· λέγω, ὅτι ἀσύμμετρός ἐστιν ἡ Α τῇ Β μήκει.

Εἰ γάρ ἐστι σύμμετρος ἡ Α τῇ Β, ἕξει τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς Β λόγον, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. οὐκ ἔχει δέ· οὐκ

ἄρα σύμμετρός ἐστιν ἡ Α τῇ Β μήκει.

Τὰ ἄρα ἀπὸ τῶν μήκει συμμέτρων, καὶ τὰ ἐξῆς.

Πόρισμα

Καὶ φανερὸν ἐκ τῶν δεδειγμένων ἔσται, ὅτι αἱ μήκει σύμμετροι πάντως καὶ δυνάμει, αἱ δὲ δυνάμει οὐ πάντως καὶ μήκει [εἴπερ τὰ ἀπὸ τῶν μήκει συμμέτρων εὐθειῶν τετράγωνα λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, τὰ δὲ λόγον ἔχοντα, ὃν ἀριθμὸς πρὸς ἀριθμόν, σύμμετρά ἐστιν. ὥστε αἱ μήκει σύμμετροι εὐθεῖαι οὐ μόνον [εἰσὶ] μήκει σύμμετροι, ἀλλὰ καὶ δυνάμει. πάλιν ἐπεί, ὅσα τετράγωνα πρὸς ἄλληλα λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, μήκει ἐδείχθη σύμμετρα καὶ δυνάμει ὄντα σύμμετρα τῷ τὰ τετράγωνα λόγον ἔχειν, ὃν ἀριθμὸς πρὸς ἀριθμόν, ὅσα ἄρα τετράγωνα λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, ἀλλὰ ἁπλῶς, ὃν ἀριθμὸς πρὸς ἀριθμόν, σύμμετρα μὲν ἔσται αὐτὰ τὰ τετράγωνα δυνάμει, οὐκέτι δὲ καὶ μήκει· ὥστε τὰ μὲν μήκει σύμμετρα πάντως καὶ δυνάμει, τὰ δὲ δυνάμει οὐ πάντως καὶ μήκει, εἰ μὴ καὶ λόγον ἔχοιεν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. λέγω δή, ὅτι [καὶ] αἱ μήκει ἀσύμμετροι οὐ πάντως καὶ δυνάμει, ἐπειδήπερ αἱ δυνάμει σύμμετροι δύνανται λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ διὰ τοῦτο δυνάμει οὖσαι σύμμετροι μήκει εἰσὶν ἀσύμμετροι. ὥστε οὐχ αἱ τῷ μήκει ἀσύμμετροι πάντως καὶ δυνάμει, ἀλλὰ δύνανται μήκει οὖσαι ἀσύμμετροι δυνάμει εἶναι καὶ ἀσύμμετροι καὶ σύμμετροι. αἱ δὲ δυνάμει ἀσύμμετροι πάντως καὶ μήκει ἀσύμμετροι· εἰ γὰρ [εἰσι] μήκει σύμμετροι, ἔσονται καὶ δυνάμει σύμμετροι. ὑπόκεινται δὲ καὶ ἀσύμμετροι· ὅπερ ἄτοπον. αἱ ἄρα δυνάμει ἀσύμμετροι πάντως καὶ μήκει].

Λῆμμα

Δέδεικται ἐν τοῖς ἀριθμητικοῖς, ὅτι οἱ ὅμοιοι ἐπίπεδοι ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ ὅτι, ἐὰν δύο ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχωσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, ὅμοιοί εἰσιν ἐπίπεδοι. καὶ δῆλον ἐκ τούτων, ὅτι οἱ μὴ ὅμοιοι ἐπίπεδοι ἀριθμοί, τουτέστιν οἱ μὴ ἀνάλογον ἔχοντες τὰς πλευράς, πρὸς ἀλλήλους λόγον οὐκ ἔχουσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. εἰ γὰρ ἔξουσιν, ὅμοιοι ἐπίπεδοι ἔσονται· ὅπερ οὐχ ὑπόκειται. οἱ ἄρα μὴ ὅμοιοι ἐπίπεδοι πρὸς ἀλλήλους λόγον οὐκ ἔχουσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν.

ι΄. Τῇ προτεθείσῃ εὐθείᾳ προσευρεῖν δύο εὐθείας ἀσυμμέτρους, τὴν μὲν μήκει μόνον, τὴν δὲ καὶ δυνάμει.

Ἔστω ἡ προτεθεῖσα εὐθεῖα ἡ Α· δεῖ δὴ τῇ Α προσευρεῖν δύο εὐθείας ἀσυμμέτρους, τὴν μὲν μήκει μόνον, τὴν δὲ καὶ δυνάμει.

Ἐκκείσθωσαν γὰρ δύο ἀριθμοὶ οἱ Β, Γ πρὸς ἀλλήλους λόγον μὴ ἔχοντες, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, τουτέστι μὴ ὅμοιοι ἐπίπεδοι, καὶ γεγονέτω ὡς ὁ Β πρὸς τὸν Γ, οὕτως τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Δ τετράγωνον· ἐμάθομεν γάρ· σύμμετρον ἄρα τὸ ἀπὸ τῆς Α τῷ ἀπὸ τῆς Δ. καὶ ἐπεὶ ὁ Β πρὸς τὸν Γ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ’ ἄρα τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς Δ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Α τῇ Δ μήκει. εἰλήφθω τῶν Α, Δ μέση ἀνάλογον ἡ Ε· ἔστιν ἄρα ὡς ἡ Α πρὸς τὴν Δ, οὕτως τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Ε. ἀσύμμετρος δέ ἐστιν ἡ Α τῇ Δ μήκει· ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς Α τετράγωνον τῷ ἀπὸ τῆς Ε τετραγώνῳ· ἀσύμμετρος ἄρα ἐστὶν ἡ Α τῇ Ε δυνάμει. Τῇ ἄρα προτεθείσῃ εὐθείᾳ τῇ Α προσεύρηνται δύο εὐθεῖαι ἀσύμμετροι αἱ Δ, Ε, μήκει μὲν μόνον ἡ Δ, δυνάμει δὲ καὶ μήκει δηλαδὴ ἡ Ε [ὅπερ ἔδει δεῖξαι].

ια΄. Ἐὰν τέσσαρα μεγέθη ἀνάλογον ᾖ, τὸ δὲ πρῶτον τῷ δευτέρῳ σύμμετρον ᾖ, καὶ τὸ τρίτον τῷ τετάρτῳ σύμμετρον ἔσται· κἂν τὸ πρῶτον τῷ δευτέρῳ ἀσύμμετρον ᾖ, καὶ τὸ τρίτον τῷ τετάρτῳ ἀσύμμετρον ἔσται.

Ἔστωσαν τέσσαρα μεγέθη ἀνάλογον τὰ Α, Β, Γ, Δ, ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, τὸ Α δὲ τῷ Β σύμμετρον ἔστω· λέγω, ὅτι καὶ τὸ Γ τῷ Δ σύμμετρον ἔσται.

Ἐπεὶ γὰρ σύμμετρόν ἐστι τὸ Α τῷ Β, τὸ Α ἄρα πρὸς τὸ Β λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. καί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ· καὶ τὸ Γ ἄρα πρὸς τὸ Δ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· σύμμετρον ἄρα ἐστὶ τὸ Γ τῷ Δ.

Ἀλλὰ δὴ τὸ Α τῷ Β ἀσύμμετρον ἔστω· λέγω, ὅτι καὶ τὸ Γ τῷ Δ ἀσύμμετρον ἔσται. ἐπεὶ γὰρ ἀσύμμετρόν ἐστι τὸ Α τῷ Β, τὸ Α ἄρα πρὸς τὸ Β λόγον οὐκ ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. καί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ· οὐδὲ τὸ Γ ἄρα πρὸς τὸ Δ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· ἀσύμμετρον ἄρα ἐστὶ τὸ Γ τῷ Δ.

Ἐὰν ἄρα τέσσαρα μεγέθη, καὶ τὰ ἑξῆς.

ιβ΄. Τὰ τῷ αὐτῷ μεγέθει σύμμετρα καὶ ἀλλήλοις ἐστὶ σύμμετρα.

Ἑκάτερον γὰρ τῶν Α, Β τῷ Γ ἔστω σύμμετρον. λέγω, ὅτι καὶ τὸ Α τῷ Β ἐστι σύμμετρον.

Ἐπεὶ γὰρ σύμμετρόν ἐστι τὸ Α τῷ Γ, τὸ Α ἄρα πρὸς τὸ Γ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. ἐχέτω, ὃν ὁ Δ πρὸς τὸν Ε. πάλιν, ἐπεὶ σύμμετρόν ἐστι τὸ Γ τῷ Β, τὸ Γ ἄρα πρὸς τὸ Β λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. ἐχέτω, ὃν ὁ Ζ πρὸς τὸν Η. καὶ λόγων δοθέντων ὁποσωνοῦν τοῦ τε, ὃν ἔχει ὁ Δ πρὸς τὸν Ε, καὶ ὁ Ζ πρὸς τὸν Η εἰλήφθωσαν ἀριθμοὶ ἑξῆς ἐν τοῖς δοθεῖσι λόγοις οἱ Θ, Κ, Λ· ὥστε εἶναι ὡς μὲν τὸν Δ πρὸς τὸν Ε, οὕτως τὸν Θ πρὸς τὸν Κ, ὡς δὲ τὸν Ζ πρὸς τὸν Η, οὕτως τὸν Κ πρὸς τὸν Λ.

Ἐπεὶ οὖν ἐστιν ὡς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Δ πρὸς τὸν Ε, ἀλλ᾽ ὡς ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Θ πρὸς τὸν Κ, ἔστιν ἄρα καὶ ὡς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Θ πρὸς τὸν Κ. πάλιν, ἐπεί ἐστιν ὡς τὸ Γ πρὸς τὸ Β, οὕτως ὁ Ζ πρὸς τὸν Η, ἀλλ᾽ ὡς ὁ Ζ πρὸς τὸν Η, [οὕτως] ὁ Κ πρὸς τὸν Λ, καὶ ὡς ἄρα τὸ Γ πρὸς τὸ Β, οὕτως ὁ Κ πρὸς τὸν Λ. ἔστι δὲ καὶ ὡς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Θ πρὸς τὸν Κ· δι᾽ ἴσου ἄρα ἐστὶν ὡς τὸ Α πρὸς τὸ Β, οὕτως ὁ Θ πρὸς τὸν Λ. τὸ Α ἄρα πρὸς τὸ Β λόγον ἔχει, ὃν ἀριθμὸς ὁ Θ πρὸς ἀριθμὸν τὸν Λ· σύμμετρον ἄρα ἐστὶ τὸ Α τῷ Β.

Τὰ ἄρα τῷ αὐτῷ μεγέθει σύμμετρα καὶ ἀλλήλοις ἐστὶ σύμμετρα· ὅπερ ἔδει δεῖξαι.

ιγ΄. Ἐὰν ᾖ δύο μεγέθη σύμμετρα, τὸ δὲ ἕτερον αὐτῶν μεγέθει τινὶ ἀσύμμετρον ᾖ, καὶ τὸ λοιπὸν τῷ αὐτῷ ἀσύμμετρον ἔσται.

Ἔστω δύο μεγέθη σύμμετρα τὰ Α, Β, τὸ δὲ ἕτερον αὐτῶν τὸ Α ἄλλῳ τινὶ τῷ Γ ἀσύμμετρον ἔστω· λέγω, ὅτι καὶ τὸ λοιπὸν τὸ Β τῷ Γ ἀσύμμετρόν ἐστιν.

Εἰ γάρ ἐστι σύμμετρον τὸ Β τῷ Γ, ἀλλὰ καὶ τὸ Α τῷ Β σύμμετρόν ἐστιν, καὶ τὸ Α ἄρα τῷ Γ σύμμετρόν ἐστιν. ἀλλὰ καὶ ἀσύμμετρον· ὅπερ ἀδύνατον. οὐκ ἄρα σύμμετρόν ἐστι τὸ Β τῷ Γ· ἀσύμμετρον ἄρα.

Ἐὰν ἄρα ᾖ δύο μεγέθη σύμμετρα, καὶ τὰ ἑξῆς.

Λῆμμα

Δύο δοθεισῶν εὐθειῶν ἀνίσων εὑρεῖν, τίνι μεῖζον δύναται ἡ μείζων τῆς ἐλάσσονος.

Ἔστωσαν αἱ δοθεῖσαι δύο ἄνισοι εὐθεῖαι αἱ ΑΒ, Γ, ὧν μείζων ἔστω ἡ ΑΒ· δεῖ δὴ εὑρεῖν, τίνι μεῖζον δύναται ἡ ΑΒ τῆς Γ.

Γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ εἰς αὐτὸ ἐνηρμόσθω τῇ Γ ἴση ἡ ΑΔ, καὶ ἐπεζεύχθω ἡ ΔΒ. φανερὸν δή, ὅτι ὀρθή ἐστιν ἡ ὑπὸ ΑΔΒ γωνία, καὶ ὅτι ἡ ΑΒ τῆς ΑΔ, τουτέστι τῆς Γ, μεῖζον δύναται τῇ ΔΒ.

Ὁμοίως δὲ καὶ δύο δοθεισῶν εὐθειῶν ἡ δυναμένη αὐτὰς εὑρίσκεται οὕτως.

Ἔστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι αἱ ΑΔ, ΔΒ, καὶ δέον ἔστω εὑρεῖν τὴν δυναμένην αὐτάς. κείσθωσαν γάρ, ὥστε ὀρθὴν γωνίαν περιέχειν τὴν ὑπὸ ΑΔ, ΔΒ, καὶ ἐπεζεύχθω ἡ ΑΒ· φανερὸν πάλιν, ὅτι ἡ τὰς ΑΔ, ΔΒ δυναμένη ἐστὶν ἡ ΑΒ· ὅπερ ἔδει δεῖξαι.

ιδ΄. Ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, δύνηται δὲ ἡ πρώτη τῆς δευτέρας μεῖζον τῷ ἀπὸ συμμέτρου ἑαυτῇ [μήκει], καὶ ἡ τρίτη τῆς τετάρτης μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῇ [μήκει]. καὶ ἐὰν ἡ πρώτη τῆς δευτέρας μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ [μήκει], καὶ ἡ τρίτη τῆς τετάρτης μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ [μήκει].

Ἔστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἱ Α, Β, Γ, Δ, ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ, καὶ ἡ Α μὲν τῆς Β μεῖζον δυνάσθω τῷ ἀπὸ τῆς Ε, ἡ δὲ Γ τῆς Δ μεῖζον δυνάσθω τῷ ἀπὸ τῆς Ζ· λέγω, ὅτι, εἴτε σύμμετρός ἐστιν ἡ Α τῇ Ε, σύμμετρός ἐστι καὶ ἡ Γ τῇ Ζ, εἴτε ἀσύμμετρός ἐστιν ἡ Α τῇ Ε, ἀσύμμετρός ἐστι καὶ ἡ Γ τῇ Ζ.

Ἐπεὶ γάρ ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ, ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς Β, οὕτως τὸ ἀπὸ τῆς Γ πρὸς τὸ ἀπὸ τῆς Δ. ἀλλὰ τῷ μὲν ἀπὸ τῆς Α ἴσα ἐστὶ τὰ ἀπὸ τῶν Ε, Β, τῷ δὲ ἀπὸ τῆς Γ ἴσα ἐστὶ τὰ ἀπὸ τῶν Δ, Ζ. ἔστιν ἄρα ὡς τὰ ἀπὸ τῶν Ε, Β πρὸς τὸ ἀπὸ τῆς Β, οὕτως τὰ ἀπὸ τῶν Δ, Ζ πρὸς τὸ ἀπὸ τῆς Δ· διελόντι ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς Β, οὕτως τὸ ἀπὸ τῆς Ζ πρὸς τὸ ἀπὸ τῆς Δ· ἔστιν ἄρα καὶ ὡς ἡ Ε πρὸς τὴν Β, οὕτως ἡ Ζ πρὸς τὴν Δ· ἀνάπαλιν ἄρα ἐστὶν ὡς ἡ Β πρὸς τὴν Ε, οὕτως ἡ Δ πρὸς τὴν Ζ. ἔστι δὲ καὶ ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ· δι’ ἴσου ἄρα ἐστὶν ὡς ἡ Α πρὸς τὴν Ε, οὕτως ἡ Γ πρὸς τὴν Ζ. εἴτε οὖν σύμμετρός ἐστιν ἡ Α τῇ Ε, σύμμετρός ἐστι καὶ ἡ Γ τῇ Ζ, εἴτε ἀσύμμετρός ἐστιν ἡ Α τῇ Ε, ἀσύμμετρός ἐστι καὶ ἡ Γ τῇ Ζ.

Ἐὰν ἄρα, καὶ τὰ ἑξῆς.

ιε΄. Ἐὰν δύο μεγέθη σύμμετρα συντεθῇ, καὶ τὸ ὅλον ἑκατέρῳ αὐτῶν σύμμετρον ἔσται· κἂν τὸ ὅλον ἑνὶ αὐτῶν σύμμετρον ᾖ, καὶ τὰ ἐξ ἀρχῆς μεγέθη σύμμετρα ἔσται.

Συγκείσθω γὰρ δύο μεγέθη σύμμετρα τὰ ΑΒ, ΒΓ· λέγω, ὅτι καὶ ὅλον τὸ ΑΓ ἑκατέρῳ τῶν ΑΒ, ΒΓ ἐστι σύμμετρον.

Ἐπεὶ γὰρ σύμμετρά ἐστι τὰ ΑΒ, ΒΓ, μετρήσει τι αὐτὰ μέγεθος. μετρείτω, καὶ ἔστω τὸ Δ. ἐπεὶ οὖν τὸ Δ τὰ ΑΒ, ΒΓ μετρεῖ, καὶ ὅλον τὸ ΑΓ μετρήσει. μετρεῖ δὲ καὶ τὰ ΑΒ, ΒΓ. τὸ Δ ἄρα τὰ ΑΒ, ΒΓ, ΑΓ μετρεῖ· σύμμετρον ἄρα ἐστὶ τὸ ΑΓ ἑκατέρῳ τῶν ΑΒ, ΒΓ.

Ἀλλὰ δὴ τὸ ΑΓ ἔστω σύμμετρον τῷ ΑΒ· λέγω δή, ὅτι καὶ τὰ ΑΒ, ΒΓ σύμμετρά ἐστιν.

Ἐπεὶ γὰρ σύμμετρά ἐστι τὰ ΑΓ, ΑΒ, μετρήσει τι αὐτὰ μέγεθος. μετρείτω, καὶ ἔστω τὸ Δ. ἐπεὶ οὖν τὸ Δ τὰ ΓΑ, ΑΒ μετρεῖ, καὶ λοιπὸν ἄρα τὸ ΒΓ μετρήσει. μετρεῖ δὲ καὶ τὸ ΑΒ· τὸ Δ ἄρα τὰ ΑΒ, ΒΓ μετρήσει· σύμμετρα ἄρα ἐστὶ τὰ ΑΒ, ΒΓ.

Ἐὰν ἄρα δύο μεγέθη, καὶ τὰ ἑξῆς.

ις΄. Ἐὰν δύο μεγέθη ἀσύμμετρα συντεθῇ, καὶ τὸ ὅλον ἑκατέρῳ αὐτῶν ἀσύμμετρον ἔσται· κἂν τὸ ὅλον ἑνὶ αὐτῶν ἀσύμμετρον ᾖ, καὶ τὰ ἐξ ἀρχῆς μεγέθη ἀσύμμετρα ἔσται.

Συγκείσθω γὰρ δύο μεγέθη ἀσύμμετρα τὰ ΑΒ, ΒΓ· λέγω, ὅτι καὶ ὅλον τὸ ΑΓ ἑκατέρῳ τῶν ΑΒ, ΒΓ ἀσύμμετρόν ἐστιν.

Εἰ γὰρ μή ἐστιν ἀσύμμετρα τὰ ΓΑ, ΑΒ, μετρήσει τι [αὐτὰ] μέγεθος. μετρείτω, εἰ δυνατόν, καὶ ἔστω τὸ Δ. ἐπεὶ οὖν τὸ Δ τὰ ΓΑ, ΑΒ μετρεῖ, καὶ λοιπὸν ἄρα τὸ ΒΓ μετρήσει. μετρεῖ δὲ καὶ τὸ ΑΒ· τὸ Δ ἄρα τὰ ΑΒ, ΒΓ μετρεῖ. σύμμετρα ἄρα ἐστὶ τὰ ΑΒ, ΒΓ· ὑπέκειντο δὲ καὶ ἀσύμμετρα· ὅπερ

ἐστὶν ἀδύνατον. οὐκ ἄρα τὰ ΓΑ, ΑΒ μετρήσει τι μέγεθος· ἀσύμμετρα ἄρα ἐστὶ τὰ ΓΑ, ΑΒ. ὁμοίως δὴ δείξομεν, ὅτι καὶ τὰ ΑΓ, ΓΒ ἀσύμμετρά ἐστιν. τὸ ΑΓ ἄρα ἑκατέρῳ τῶν ΑΒ, ΒΓ ἀσύμμετρόν ἐστιν.

Ἀλλὰ δὴ τὸ ΑΓ ἑνὶ τῶν ΑΒ, ΒΓ ἀσύμμετρον ἔστω. ἔστω δὴ πρότερον τῷ ΑΒ· λέγω, ὅτι καὶ τὰ ΑΒ, ΒΓ ἀσύμμετρά ἐστιν. εἰ γὰρ ἔσται σύμμετρα, μετρήσει τι αὐτὰ μέγεθος. μετρείτω, καὶ ἔστω τὸ Δ. ἐπεὶ οὖν τὸ Δ τὰ ΑΒ, ΒΓ μετρεῖ, καὶ ὅλον ἄρα τὸ ΑΓ μετρήσει. μετρεῖ δὲ καὶ τὸ ΑΒ· τὸ Δ ἄρα τὰ ΓΑ, ΑΒ μετρεῖ. σύμμετρα ἄρα ἐστὶ τὰ ΓΑ, ΑΒ· ὑπέκειτο δὲ καὶ ἀσύμμετρα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὰ ΑΒ, ΒΓ μετρήσει τι μέγεθος· ἀσύμμετρα ἄρα ἐστὶ τὰ ΑΒ, ΒΓ.

Ἐὰν ἄρα δύο μεγέθη, καὶ τὰ ἑξῆς.

Λῆμμα

Ἐὰν παρά τινα εὐθεῖαν παραβληθῇ παραλληλόγραμμον ἐλλεῖπον εἴδει τετραγώνῳ, τὸ παραβληθὲν ἴσον ἐστὶ τῷ ὑπὸ τῶν ἐκ τῆς παραβολῆς γενομένων τμημάτων τῆς εὐθείας.

Παρὰ γὰρ εὐθεῖαν τὴν ΑΒ παραβεβλήσθω παραλληλόγραμμον τὸ ΑΔ ἐλλεῖπον εἴδει τετραγώνῳ τῷ ΔΒ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΔ τῷ ὑπὸ τῶν ΑΓ, ΓΒ.

Καί ἐστιν αὐτόθεν φανερόν· ἐπεὶ γὰρ τετράγωνόν ἐστι τὸ ΔΒ, ἴση ἐστὶν ἡ ΔΓ τῇ ΓΒ, καί ἐστι τὸ ΑΔ τὸ ὑπὸ τῶν ΑΓ, ΓΔ, τουτέστι τὸ ὑπὸ τῶν ΑΓ, ΓΒ.

Ἐὰν ἄρα παρά τινα εὐθεῖαν, καὶ τὰ ἑξῆς.

ιζ΄. Ἐὰν ὦσι δύο εὐθεῖαι ἄνισοι, τῷ δὲ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ καὶ εἰς σύμμετρα αὐτὴν διαιρῇ μήκει, ἡ μείζων τῆς ἐλάσσονος μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῇ [μήκει]. καὶ ἐὰν ἡ μείζων τῆς ἐλάσσονος μεῖζον δύνηται τῷ ἀπὸ συμμέτρου ἑαυτῇ [μήκει], τῷ δὲ τετάρτῳ τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον

παρὰ τὴν μείζονα παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς σύμμετρα αὐτὴν διαιρεῖ μήκει.

Ἔστωσαν δύο εὐθεῖαι ἄνισοι αἱ Α, ΒΓ, ὧν μείζων ἡ ΒΓ, τῷ δὲ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος τῆς Α, τουτέστι τῷ ἀπὸ τῆς ἡμισείας τῆς Α, ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΒΔ, ΔΓ, σύμμετρος δὲ ἔστω ἡ ΒΔ τῇ ΔΓ μήκει· λέγω, ὅτι ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ.

Τετμήσθω γὰρ ἡ ΒΓ δίχα κατὰ τὸ Ε σημεῖον, καὶ κείσθω τῇ ΔΕ ἴση ἡ ΕΖ. λοιπὴ ἄρα ἡ ΔΓ ἴση ἐστὶ τῇ ΒΖ. καὶ ἐπεὶ εὐθεῖα ἡ ΒΓ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Ε, εἰς δὲ ἄνισα κατὰ τὸ Δ, τὸ ἄρα ὑπὸ ΒΔ, ΔΓ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΕΔ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΓ τετραγώνῳ· καὶ τὰ τετραπλάσια· τὸ ἄρα τετράκις ὑπὸ τῶν ΒΔ, ΔΓ μετὰ τοῦ τετραπλασίου τοῦ ἀπὸ τῆς ΔΕ ἴσον ἐστὶ τῷ τετράκις ἀπὸ τῆς ΕΓ τετραγώνῳ. ἀλλὰ τῷ μέν τετραπλασίῳ τοῦ ὑπὸ τῶν ΒΔ, ΔΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς Α τετράγωνον, τῷ δὲ τετραπλασίῳ τοῦ ἀπὸ τῆς ΔΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΔΖ τετράγωνον· διπλασίων γάρ ἐστιν ἡ ΔΖ τῆς ΔΕ. τῷ δὲ τετραπλασίῳ τοῦ ἀπὸ τῆς ΕΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΒΓ τετράγωνον· διπλασίων γάρ ἐστι πάλιν ἡ ΒΓ τῆς ΓΕ. τὰ ἄρα ἀπὸ τῶν Α, ΔΖ τετράγωνα ἴσα ἐστὶ τῷ ἀπὸ τῆς ΒΓ τετραγώνῳ· ὥστε τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς Α μεῖζόν ἐστι τῷ ἀπὸ τῆς ΔΖ· ἡ ΒΓ ἄρα τῆς Α μεῖζον δύναται τῇ ΔΖ. δεικτέον, ὅτι καὶ σύμμετρός ἐστιν ἡ ΒΓ τῇ ΔΖ. ἐπεὶ γὰρ σύμμετρός ἐστιν ἡ ΒΔ τῇ ΔΓ μήκει, σύμμετρος ἄρα ἐστὶ καὶ ἡ ΒΓ τῇ ΓΔ μήκει. ἀλλὰ ἡ ΓΔ ταῖς ΓΔ, ΒΖ ἐστι σύμμετρος μήκει· ἴση γάρ ἐστιν ἡ ΓΔ τῇ ΒΖ. καὶ ἡ ΒΓ ἄρα σύμμετρός ἐστι ταῖς ΒΖ, ΓΔ μήκει· ὥστε καὶ λοιπῇ τῇ ΖΔ σύμμετρός ἐστιν ἡ ΒΓ μήκει· ἡ ΒΓ ἄρα τῆς Α μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ.

Ἀλλὰ δὴ ἡ ΒΓ τῆς Α μεῖζον δυνάσθω τῷ ἀπὸ συμμέτρου ἑαυτῇ, τῷ δὲ τετάρτῳ τοῦ ἀπὸ τῆς Α ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω ἐλλεῖπον εἴδει

τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΒΔ, ΔΓ. δεικτέον, ὅτι σύμμετρός ἐστιν ἡ ΒΔ τῇ ΔΓ μήκει.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ τῆς ΖΔ. δύναται δὲ ἡ ΒΓ τῆς Α μεῖζον τῷ ἀπὸ συμμέτρου ἑαυτῇ. σύμμετρος ἄρα ἐστὶν ἡ ΒΓ τῇ ΖΔ μήκει· ὥστε καὶ λοιπῇ συναμφοτέρῳ τῇ ΒΖ, ΔΓ σύμμετρός ἐστιν ἡ ΒΓ μήκει. ἀλλὰ συναμφότερος ἡ ΒΖ, ΔΓ σύμμετρός ἐστι τῇ ΔΓ [μήκει]. ὥστε καὶ ἡ ΒΓ τῇ ΓΔ σύμμετρός ἐστι μήκει· καὶ διελόντι ἄρα ἡ ΒΔ τῇ ΔΓ ἐστι σύμμετρος μήκει.

Ἐὰν ἄρα ὦσι δύο εὐθεῖαι ἄνισοι, καὶ τὰ ἑξῆς.

ιη΄. Ἐὰν ὦσι δύο εὐθεῖαι ἄνισοι, τῷ δὲ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, καὶ εἰς ἀσύμμετρα αὐτὴν διαιρῇ [μήκει], ἡ μείζων τῆς ἐλάσσονος μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ ἐὰν ἡ μείζων τῆς ἐλάσσονος μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, τῷ δὲ τετάρτῳ τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς ἀσύμμετρα αὐτὴν διαιρεῖ [μήκει].

Ἔστωσαν δύο εὐθεῖαι ἄνισοι αἱ Α, ΒΓ, ὧν μείζων ἡ ΒΓ, τῷ δὲ τετάρτῳ [μέρει] τοῦ ἀπὸ τῆς ἐλάσσονος τῆς Α ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΒΔΓ, ἀσύμμετρος δὲ ἔστω ἡ ΒΔ τῇ ΔΓ μήκει· λέγω, ὅτι ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων τῷ πρότερον ὁμοίως δείξομεν, ὅτι ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ τῆς ΖΔ. δεικτέον [οὖν], ὅτι ἀσύμμετρός ἐστιν ἡ ΒΓ τῇ ΔΖ μήκει. ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ ΒΔ τῇ ΔΓ μήκει, ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΒΓ τῇ ΓΔ μήκει. ἀλλὰ ἡ ΔΓ σύμμετρός ἐστι συναμφοτέραις ταῖς ΒΖ, ΔΓ· καὶ ἡ ΒΓ ἄρα ἀσύμμετρός ἐστι συναμφοτέραις ταῖς ΒΖ, ΔΓ. ὥστε καὶ λοιπῇ τῇ ΖΔ ἀσύμμετρός ἐστιν ἡ ΒΓ μήκει. καὶ ἡ ΒΓ

τῆς Α μεῖζον δύναται τῷ ἀπὸ τῆς ΖΔ· ἡ ΒΓ ἄρα τῆς Α μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ.

Δυνάσθω δὴ πάλιν ἡ ΒΓ τῆς Α μεῖζον τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, τῷ δὲ τετάρτῳ τοῦ ἀπὸ τῆς Α ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΒΔ, ΔΓ. δεικτέον, ὅτι ἀσύμμετρός ἐστιν ἡ ΒΔ τῇ ΔΓ μήκει.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ τῆς ΖΔ. ἀλλὰ ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΓ τῇ ΖΔ μήκει· ὥστε καὶ λοιπῇ συναμφοτέρῳ τῇ ΒΖ, ΔΓ ἀσύμμετρός ἐστιν ἡ ΒΓ. ἀλλὰ συναμφότερος ἡ ΒΖ, ΔΓ τῇ ΔΓ σύμμετρός ἐστι μήκει· καὶ ἡ ΒΓ ἄρα τῇ ΔΓ ἀσύμμετρός ἐστι μήκει· ὥστε καὶ διελόντι ἡ ΒΔ τῇ ΔΓ ἀσύμμετρός ἐστι μήκει.

Ἐὰν ἄρα ὦσι δύο εὐθεῖαι, καὶ τὰ ἑξῆς.

Λῆμμα

Ἐπεὶ δέδεικται, ὅτι αἱ μήκει σύμμετροι πάντως καὶ δυνάμει [εἰσὶ σύμμετροι], αἱ δὲ δυνάμει οὐ πάντως καὶ μήκει, ἀλλὰ δὴ δύνανται μήκει καὶ σύμμετροι εἶναι καὶ ἀσύμμετροι, φανερόν, ὅτι, ἐὰν τῇ ἐκκειμένῃ ῥητῇ σύμμετρός τις ᾖ μήκει, λέγεται ῥητὴ καὶ σύμμετρος αὐτῇ οὐ μόνον μήκει, ἀλλὰ καὶ δυνάμει, ἐπεὶ αἱ μήκει σύμμετροι πάντως καὶ δυνάμει. ἐὰν δὲ τῇ ἐκκειμένῃ ῥητῇ σύμμετρός τις ᾖ δυνάμει, εἰ μὲν καὶ μήκει, λέγεται καὶ οὕτως ῥητὴ καὶ σύμμετρος αὐτῇ μήκει καὶ δυνάμει· εἰ δὲ τῇ ἐκκειμένῃ πάλιν ῥητῇ σύμμετρός τις οὖσα δυνάμει μήκει αὐτῇ ᾗ ἀσύμμετρος, λέγεται καὶ οὕτως ῥητὴ δυνάμει μόνον σύμμετρος.

ιθ΄. Τὸ ὑπὸ ῥητῶν μήκει συμμέτρων κατά τινα τῶν προειρημένων τρόπων εὐθειῶν περιεχόμενον ὀρθογώνιον ῥητόν ἐστιν.

Ὑπὸ γὰρ ῥητῶν μήκει συμμέτρων εὐθειῶν τῶν ΑΒ, ΒΓ ὀρθογώνιον περιεχέσθω τὸ ΑΓ· λέγω, ὅτι ῥητόν ἐστι τὸ ΑΓ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔ· ῥητὸν ἄρα ἐστὶ τὸ ΑΔ. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει, ἴση δέ ἐστιν ἡ ΑΒ τῇ ΒΔ, σύμμετρος ἄρα ἐστὶν ἡ ΒΔ τῇ ΒΓ μήκει. καί ἐστιν ὡς ἡ ΒΔ πρὸς τὴν ΒΓ, οὕτως τὸ ΔΑ πρὸς τὸ ΑΓ. σύμμετρον ἄρα ἐστὶ τὸ ΔΑ τῷ ΑΓ. ῥητὸν δὲ τὸ ΔΑ· ῥητὸν ἄρα ἐστὶ καὶ τὸ ΑΓ.

Τὸ ἄρα ὑπὸ ῥητῶν μήκει συμμέτρων, καὶ τὰ ἑξῆς.

κ΄. Ἐὰν ῥητὸν παρὰ ῥητὴν παραβληθῇ, πλάτος ποιεῖ ῥητὴν καὶ σύμμετρον τῇ, παρ᾽ ἣν παράκειται, μήκει.

Ῥητὸν γὰρ τὸ ΑΓ παρὰ ῥητὴν κατά τινα πάλιν τῶν προειρημένων τρόπων τὴν ΑΒ παραβεβλήσθω πλάτος ποιοῦν τὴν ΒΓ· λέγω, ὅτι ῥητή ἐστιν ἡ ΒΓ καὶ σύμμετρος τῇ ΒΑ μήκει.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔ· ῥητὸν ἄρα ἐστὶ τὸ ΑΔ. ῥητὸν δὲ καὶ τὸ ΑΓ· σύμμετρον ἄρα ἐστὶ τὸ ΔΑ τῷ ΑΓ. καί ἐστιν ὡς τὸ ΔΑ πρὸς τὸ ΑΓ, οὕτως ἡ ΔΒ πρὸς τὴν ΒΓ. σύμμετρος ἄρα ἐστὶ καὶ ἡ ΔΒ τῇ ΒΓ· ἴση δὲ ἡ ΔΒ τῇ ΒΑ· σύμμετρος ἄρα καὶ ἡ ΑΒ τῇ ΒΓ. ῥητὴ δέ ἐστιν ἡ ΑΒ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΒΓ καὶ σύμμετρος τῇ ΑΒ μήκει.

Ἐὰν ἄρα ῥητὸν παρὰ ῥητὴν παραβληθῇ, καὶ τὰ ἑξῆς.

κα΄. Τὸ ὑπὸ ῥητῶν δυνάμει μόνον συμμέτρων εὐθειῶν περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν, καλείσθω δὲ μέση.

Ὑπὸ γὰρ ῥητῶν δυνάμει μόνον συμμέτρων εὐθειῶν τῶν ΑΒ, ΒΓ ὀρθογώνιον περιεχέσθω τὸ ΑΓ· λέγω, ὅτι ἄλογόν ἐστι τὸ ΑΓ, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν, καλείσθω δὲ μέση.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔ· ῥητὸν ἄρα ἐστὶ τὸ ΑΔ. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει· δυνάμει γὰρ μόνον ὑπόκεινται σύμμετροι· ἴση δὲ ἡ ΑΒ τῇ ΒΔ, ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΔΒ τῇ ΒΓ μήκει. καί ἐστιν ὡς ἡ ΔΒ πρὸς τὴν ΒΓ, οὕτως τὸ ΑΔ πρὸς τὸ ΑΓ· ἀσύμμετρον ἄρα [ἐστὶ] τὸ ΔΑ τῷ ΑΓ. ῥητὸν δὲ τὸ ΔΑ· ἄλογον ἄρα ἐστὶ τὸ ΑΓ· ὥστε καὶ ἡ δυναμένη τὸ ΑΓ [τουτέστιν ἡ ἴσον αὐτῷ τετράγωνον δυναμένη] ἄλογός ἐστιν, καλείσθω δὲ μέση· ὅπερ ἔδει δεῖξαι.

Λῆμμα: Ἐὰν ὦσι δύο εὐθεῖαι, ἔστιν ὡς ἡ πρώτη πρὸς τὴν δευτέραν, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ὑπὸ τῶν δύο εὐθειῶν.

Ἔστωσαν δύο εὐθεῖαι αἱ ΖΕ, ΕΗ. λέγω, ὅτι ἐστὶν ὡς ἡ ΖΕ πρὸς τὴν ΕΗ, οὕτως τὸ ἀπὸ τῆς ΖΕ πρὸς τὸ ὑπὸ τῶν ΖΕ, ΕΗ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΖΕ τετράγωνον τὸ ΔΖ, καὶ συμπεπληρώσθω τὸ ΗΔ. ἐπεὶ οὖν ἐστιν ὡς ἡ ΖΕ πρὸς τὴν ΕΗ, οὕτως τὸ ΖΔ πρὸς τὸ ΔΗ, καί ἐστι τὸ μὲν ΖΔ τὸ ἀπὸ τῆς ΖΕ, τὸ δὲ ΔΗ τὸ ὑπὸ τῶν ΔΕ, ΕΗ, τουτέστι τὸ ὑπὸ τῶν ΖΕ, ΕΗ, ἔστιν ἄρα ὡς ἡ ΖΕ τὴν ΕΗ, οὕτως τὸ ἀπὸ τῆς ΖΕ πρὸς τὸ ὑπὸ τῶν ΖΕ, ΕΗ. ὁμοίως δὲ καὶ ὡς τὸ ὑπὸ τῶν ΗΕ, ΕΖ πρὸς τὸ ἀπὸ τῆς ΕΖ, τουτέστιν ὡς τὸ ΗΔ πρὸς τὸ ΖΔ, οὕτως ἡ ΗΕ πρὸς τὴν ΕΖ· ὅπερ ἔδει δεῖξαι.

κβ΄. Τὸ ἀπὸ μέσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ῥητὴν καὶ ἀσύμμετρον τῇ, παρ᾽ ἣν παράκειται, μήκει.

Ἔστω μέση μὲν ἡ Α, ῥητὴ δὲ ἡ ΓΒ, καὶ τῷ ἀπὸ τῆς Α ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω χωρίον ὀρθογώνιον τὸ ΒΔ πλάτος ποιοῦν τὴν ΓΔ· λέγω, ὅτι ῥητή ἐστιν ἡ ΓΔ καὶ ἀσύμμετρος τῇ ΓΒ μήκει.

Ἐπεὶ γὰρ μέση ἐστὶν ἡ Α, δύναται χωρίον περιεχόμενον ὑπὸ ῥητῶν δυνάμει μόνον συμμέτρων. δυνάσθω τὸ ΗΖ. δύναται δὲ καὶ τὸ ΒΔ· ἴσον ἄρα ἐστὶ τὸ ΒΔ τῷ ΗΖ. ἔστι δὲ αὐτῷ καὶ ἰσογώνιον· τῶν δὲ ἴσων τε καὶ ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας

γωνίας· ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΓ πρὸς τὴν ΕΗ, οὕτως ἡ ΕΖ πρὸς τὴν ΓΔ. ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς ΒΓ πρὸς τὸ ἀπὸ τῆς ΕΗ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΓΔ. σύμμετρον δέ ἐστι τὸ ἀπὸ τῆς ΓΒ τῷ ἀπὸ τῆς ΕΗ· ῥητὴ γάρ ἐστιν ἑκατέρα αὐτῶν· σύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς ΓΔ. ῥητὸν δέ ἐστι τὸ ἀπὸ τῆς ΕΖ· ῥητὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΓΔ· ῥητὴ ἄρα ἐστὶν ἡ ΓΔ. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΕΖ τῇ ΕΗ μήκει· δυνάμει γὰρ μόνον εἰσὶ σύμμετροι· ὡς δὲ ἡ ΕΖ πρὸς τὴν ΕΗ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ὑπὸ τῶν ΖΕ, ΕΗ, ἀσύμμετρον ἄρα [ἐστὶ] τὸ ἀπὸ τῆς ΕΖ τῷ ὑπὸ τῶν ΖΕ, ΕΗ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΕΖ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΓΔ· ῥηταὶ γάρ εἰσι δυνάμει· τῷ δὲ ὑπὸ τῶν ΖΕ, ΕΗ σύμμετρόν ἐστι τὸ ὑπὸ τῶν ΑΓ, ΓΒ· ἴσα γάρ ἐστι τῷ ἀπὸ τῆς Α· ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΓΔ τῷ ὑπὸ τῶν ΔΓ, ΓΒ. ὡς δὲ τὸ ἀπὸ τῆς ΓΔ πρὸς τὸ ὑπὸ τῶν ΔΓ, ΓΒ, οὕτως ἐστὶν ἡ ΔΓ πρὸς τὴν ΓΒ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΔΓ τῇ ΓΒ μήκει. ῥητὴ ἄρα ἐστὶν ἡ ΓΔ καὶ ἀσύμμετρος τῇ ΓΒ μήκει· ὅπερ ἔδει δεῖξαι.

κγ΄. Ἡ τῇ μέσῃ σύμμετρος μέση ἐστίν.

Ἔστω μέση ἡ Α, καὶ τῇ Α σύμμετρος ἔστω ἡ Β· λέγω, ὅτι καὶ ἡ Β μέση ἐστίν.

Ἐκκείσθω γὰρ ῥητὴ ἡ ΓΔ, καὶ τῷ μὲν ἀπὸ τῆς Α ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω χωρίον ὀρθογώνιον τὸ ΓΕ πλάτος ποιοῦν τὴν ΕΔ· ῥητὴ ἄρα ἐστὶν ἡ ΕΔ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. τῷ δὲ ἀπὸ τῆς Β ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω χωρίον ὀρθογώνιον τὸ ΓΖ πλάτος ποιοῦν τὴν ΔΖ. ἐπεὶ οὖν σύμμετρός ἐστιν ἡ Α τῇ Β, σύμμετρόν ἐστι καὶ τὸ ἀπὸ τῆς Α τῷ ἀπὸ τῆς Β. ἀλλὰ τῷ μὲν ἀπὸ τῆς Α ἴσον ἐστὶ τὸ ΕΓ, τῷ δὲ ἀπὸ τῆς Β ἴσον ἐστὶ τὸ ΓΖ· σύμμετρον ἄρα ἐστὶ τὸ ΕΓ τῷ ΓΖ. καί ἐστιν ὡς τὸ ΕΓ πρὸς τὸ ΓΖ, οὕτως ἡ ΕΔ πρὸς τὴν ΔΖ· σύμμετρος ἄρα ἐστὶν ἡ ΕΔ τῇ ΔΖ μήκει. ῥητὴ δέ ἐστιν ἡ ΕΔ καὶ ἀσύμμετρος τῇ ΔΓ μήκει· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΔΖ καὶ ἀσύμμετρος τῇ ΔΓ μήκει· αἱ ΓΔ, ΔΖ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ἡ δὲ τὸ ὑπὸ

ῥητῶν δυνάμει μόνον συμμέτρων δυναμένη μέση ἐστίν. ἡ ἄρα τὸ ὑπὸ τῶν ΓΔ, ΔΖ δυναμένη μέση ἐστίν· καὶ δύναται τὸ ὑπὸ τῶν ΓΔ, ΔΖ ἡ Β· μέση ἄρα ἐστὶν ἡ Β.

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι τὸ τῷ μέσῳ χωρίῳ σύμμετρον μέσον ἐστίν. [δύνανται γὰρ αὐτὰ εὐθεῖαι, αἵ εἰσι δυνάμει σύμμετροι, ὧν ἡ ἑτέρα μέση· ὥστε καὶ ἡ λοιπὴ μέση ἐστίν.]

Ὡσαύτως δὲ τοῖς ἐπὶ τῶν ῥητῶν εἰρημένοις καὶ ἐπὶ τῶν μέσων ἐξακολουθεῖ, τὴν τῇ μέσῃ μήκει σύμμετρον λέγεσθαι μέσην καὶ σύμμετρον αὐτῇ μὴ μόνον μήκει, ἀλλὰ καὶ δυνάμει, ἐπειδήπερ καθόλου αἱ μήκει σύμμετροι πάντως καὶ δυνάμει. ἐὰν δὲ τῇ μέσῃ σύμμετρός τις ᾖ δυνάμει, εἰ μὲν καὶ μήκει, λέγονται καὶ οὕτως μέσαι καὶ σύμμετροι μήκει καὶ δυνάμει, εἰ δὲ δυνάμει μόνον, λέγονται μέσαι δυνάμει μόνον σύμμετροι.

κδ΄. Τὸ ὑπὸ μέσων μήκει συμμέτρων εὐθειῶν κατά τινα τῶν εἰρημένων τρόπων περιεχόμενον ὀρθογώνιον μέσον ἐστίν.

Ὑπὸ γὰρ μέσων μήκει συμμέτρων εὐθειῶν τῶν ΑΒ, ΒΓ περιεχέσθω ὀρθογώνιον τὸ ΑΓ· λέγω, ὅτι τὸ ΑΓ μέσον ἐστίν.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔ· μέσον ἄρα ἐστὶ τὸ ΑΔ. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει, ἴση δὲ ἡ ΑΒ τῇ ΒΔ, σύμμετρος ἄρα ἐστὶ καὶ ἡ ΔΒ τῇ ΒΓ μήκει· ὥστε καὶ τὸ ΔΑ τῷ ΑΓ σύμμετρόν ἐστιν. μέσον δὲ τὸ ΔΑ· μέσον ἄρα καὶ τὸ ΑΓ· ὅπερ ἔδει δεῖξαι.

κε΄. Τὸ ὑπὸ μέσων δυνάμει μόνον συμμέτρων εὐθειῶν περιεχόμενον ὀρθογώνιον ἤτοι ῥητὸν ἢ μέσον ἐστίν.

Ὑπὸ γὰρ μέσων δυνάμει μόνον συμμέτρων εὐθειῶν τῶν ΑΒ, ΒΓ ὀρθογώνιον περιεχέσθω τὸ ΑΓ· λέγω, ὅτι τὸ ΑΓ ἤτοι ῥητὸν ἢ μέσον ἐστίν.

Ἀναγεγράφθω γὰρ ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα τὰ ΑΔ, ΒΕ· μέσον ἄρα ἐστὶν ἑκάτερον τῶν ΑΔ, ΒΕ. καὶ ἐκκείσθω ῥητὴ ἡ ΖΗ, καὶ τῷ μὲν ΑΔ ἴσον παρὰ τὴν ΖΗ παραβεβλήσθω ὀρθογώνιον παραλληλόγραμμον τὸ ΗΘ πλάτος ποιοῦν τὴν ΖΘ, τῷ δὲ ΑΓ ἴσον παρὰ τὴν ΘΜ παραβεβλήσθω ὀρθογώνιον παραλληλόγραμμον τὸ ΜΚ πλάτος ποιοῦν τὴν ΘΚ, καὶ ἔτι τῷ ΒΕ ἴσον ὁμοίως παρὰ τὴν ΚΝ παραβεβλήσθω τὸ ΝΛ πλάτος ποιοῦν τὴν ΚΛ· ἐπ’ εὐθείας ἄρα εἰσὶν αἱ ΖΘ, ΘΚ, ΚΛ. ἐπεὶ οὖν μέσον ἐστὶν ἑκάτερον τῶν ΑΔ, ΒΕ, καί ἐστιν ἴσον τὸ μὲν ΑΔ τῷ ΗΘ, τὸ δὲ ΒΕ τῷ ΝΛ, μέσον ἄρα καὶ ἑκάτερον τῶν ΗΘ, ΝΛ. καὶ παρὰ ῥητὴν τὴν ΖΗ παράκειται· ῥητὴ ἄρα ἐστὶν ἑκατέρα τῶν ΖΘ, ΚΛ καὶ ἀσύμμετρος τῇ ΖΗ μήκει. καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ΑΔ τῷ ΒΕ, σύμμετρον ἄρα ἐστὶ καὶ τὸ ΗΘ τῷ ΝΛ. καί ἐστιν ὡς τὸ ΗΘ πρὸς τὸ ΝΛ, οὕτως ἡ ΖΘ πρὸς τὴν ΚΛ· σύμμετρος ἄρα ἐστὶν ἡ ΖΘ τῇ ΚΛ μήκει. αἱ ΖΘ, ΚΛ ἄρα ῥηταί εἰσι μήκει σύμμετροι· ῥητὸν ἄρα ἐστὶ τὸ ὑπὸ τῶν ΖΘ, ΚΛ. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΔΒ τῇ ΒΑ, ἡ δὲ ΞΒ τῇ ΒΓ, ἔστιν ἄρα ὡς ἡ ΔΒ πρὸς τὴν ΒΓ, οὕτως ἡ ΑΒ πρὸς τὴν ΒΞ. ἀλλ’ ὡς μὲν ἡ ΔΒ πρὸς τὴν ΒΓ, οὕτως τὸ ΔΑ πρὸς τὸ ΑΓ· ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΞ, οὕτως τὸ ΑΓ πρὸς τὸ ΓΞ· ἔστιν ἄρα ὡς τὸ ΔΑ πρὸς τὸ ΑΓ, οὕτως τὸ ΑΓ πρὸς τὸ ΓΞ. ἴσον δέ ἐστι τὸ μὲν ΑΔ τῷ ΗΘ, τὸ δὲ ΑΓ τῷ ΜΚ, τὸ δὲ ΓΞ τῷ ΝΛ· ἔστιν ἄρα ὡς τὸ ΗΘ πρὸς τὸ ΜΚ, οὕτως τὸ ΜΚ πρὸς τὸ ΝΛ· ἔστιν ἄρα καὶ ὡς ἡ ΖΘ πρὸς τὴν ΘΚ, οὕτως ἡ ΘΚ πρὸς τὴν ΚΛ· τὸ ἄρα ὑπὸ τῶν ΖΘ, ΚΛ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΘΚ. ῥητὸν δὲ τὸ ὑπὸ τῶν ΖΘ, ΚΛ· ῥητὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΘΚ· ῥητὴ ἄρα ἐστὶν ἡ ΘΚ. καὶ εἰ μὲν σύμμετρός ἐστι τῇ ΖΗ μήκει, ῥητόν ἐστι τὸ ΘΝ· εἰ δὲ ἀσύμμετρός ἐστι τῇ ΖΗ μήκει, αἱ ΚΘ, ΘΜ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· μέσον ἄρα τὸ ΘΝ. τὸ ΘΝ ἄρα ἤτοι ῥητὸν ἢ μέσον ἐστίν. ἴσον δὲ τὸ ΘΝ τῷ ΑΓ· τὸ ΑΓ ἄρα ἤτοι ῥητὸν ἢ μέσον ἐστίν.

Τὸ ἄρα ὑπὸ μέσων δυνάμει μόνον συμμέτρων, καὶ τὰ ἑξῆς.

κς΄. Μέσον μέσου οὐχ ὑπερέχει ῥητῷ.

Εἰ γὰρ δυνατόν, μέσον τὸ ΑΒ μέσου τοῦ ΑΓ ὑπερεχέτω ῥητῷ τῷ ΔΒ, καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ τῷ ΑΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω παραλληλόγραμμον ὀρθογώνιον τὸ ΖΘ πλάτος ποιοῦν τὴν ΕΘ, τῷ δὲ ΑΓ ἴσον ἀφῃρήσθω τὸ ΖΗ· λοιπὸν ἄρα τὸ ΒΔ λοιπῷ τῷ ΚΘ ἐστιν ἴσον. ῥητὸν δέ ἐστι τὸ ΔΒ· ῥητὸν ἄρα ἐστὶ καὶ τὸ ΚΘ. ἐπεὶ οὖν μέσον ἐστὶν ἑκάτερον τῶν ΑΒ, ΑΓ, καί ἐστι τὸ μὲν ΑΒ τῷ ΖΘ ἴσον, τὸ δὲ ΑΓ τῷ ΖΗ, μέσον ἄρα καὶ ἑκάτερον τῶν ΖΘ, ΖΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται· ῥητὴ ἄρα ἐστὶν ἑκατέρα τῶν ΘΕ, ΕΗ καὶ ἀσύμμετρος τῇ ΕΖ μήκει. καὶ ἐπεὶ ῥητόν ἐστι τὸ ΔΒ καί ἐστιν ἴσον τῷ ΚΘ, ῥητὸν ἄρα ἐστὶ καὶ τὸ ΚΘ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ ΗΘ καὶ σύμμετρος τῇ ΕΖ μήκει. ἀλλὰ καὶ ἡ ΕΗ ῥητή ἐστι καὶ ἀσύμμετρος τῇ ΕΖ μήκει· ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΗ τῇ ΗΘ μήκει. καί ἐστιν ὡς ἡ ΕΗ πρὸς τὴν ΗΘ, οὕτως τὸ ἀπὸ τῆς ΕΗ πρὸς τὸ ὑπὸ τῶν ΕΗ, ΗΘ· ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΕΗ τῷ ὑπὸ τῶν ΕΗ, ΗΘ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΕΗ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΕΗ, ΗΘ τετράγωνα· ῥητὰ γὰρ ἀμφότερα· τῷ δὲ ὑπὸ τῶν ΕΗ, ΗΘ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΕΗ, ΗΘ· διπλάσιον γάρ ἐστιν αὐτοῦ· ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΕΗ, ΗΘ τῷ δὶς ὑπὸ τῶν ΕΗ, ΗΘ· καὶ συναμφότερα ἄρα τά τε ἀπὸ τῶν ΕΗ, ΗΘ καὶ τὸ δὶς ὑπὸ τῶν ΕΗ, ΗΘ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΕΘ, ἀσύμμετρόν ἐστι τοῖς ἀπὸ τῶν ΕΗ, ΗΘ. ῥητὰ δὲ τὰ ἀπὸ τῶν ΕΗ, ΗΘ· ἄλογον ἄρα τὸ ἀπὸ τῆς ΕΘ. ἄλογος ἄρα ἐστὶν ἡ ΕΘ. ἀλλὰ καὶ ῥητή· ὅπερ ἐστὶν ἀδύνατον.

Μέσον ἄρα μέσου οὐχ ὑπερέχει ῥητῷ· ὅπερ ἔδει δεῖξαι.

κζ΄. Μέσας εὑρεῖν δυνάμει μόνον συμμέτρους ῥητὸν περιεχούσας.

Ἐκκείσθωσαν δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ Α, Β, καὶ εἰλήφθω τῶν Α, Β μέση ἀνάλογον ἡ Γ, καὶ γεγονέτω ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ.

Καὶ ἐπεὶ αἱ Α, Β ῥηταί εἰσι δυνάμει μόνον σύμμετροι, τὸ ἄρα ὑπὸ τῶν Α, Β, τουτέστι τὸ ἀπὸ τῆς Γ, μέσον ἐστίν. μέση ἄρα ἡ Γ. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, [οὕτως] ἡ Γ πρὸς τὴν Δ, αἱ δὲ Α, Β δυνάμει μόνον [εἰσὶ] σύμμετροι, καὶ αἱ Γ, Δ ἄρα δυνάμει μόνον εἰσὶ σύμμετροι. καί ἐστι μέση ἡ Γ· μέση ἄρα καὶ ἡ Δ. αἱ Γ, Δ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. λέγω, ὅτι καὶ ῥητὸν περιέχουσιν. ἐπεὶ γάρ ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ, ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ Α πρὸς τὴν Γ, ἡ Β πρὸς τὴν Δ. ἀλλ᾽ ὡς ἡ Α πρὸς τὴν Γ, ἡ Γ πρὸς τὴν Β· καὶ ὡς ἄρα ἡ Γ πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Δ· τὸ ἄρα ὑπὸ τῶν Γ, Δ ἴσον ἐστὶ τῷ ἀπὸ τῆς Β. ῥητὸν δὲ τὸ ἀπὸ τῆς Β· ῥητὸν ἄρα [ἐστὶ] καὶ τὸ ὑπὸ τῶν Γ, Δ.

Εὕρηνται ἄρα μέσαι δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι· ὅπερ ἔδει δεῖξαι.

κη΄. Μέσας εὑρεῖν δυνάμει μόνον συμμέτρους μέσον περιεχούσας.

Ἐκκείσθωσαν [τρεῖς] ῥηταὶ δυνάμει μόνον σύμμετροι αἱ Α, Β, Γ, καὶ εἰλήφθω τῶν Α, Β μέση ἀνάλογον ἡ Δ, καὶ γεγονέτω ὡς ἡ Β πρὸς τὴν Γ, ἡ Δ πρὸς τὴν Ε.

Ἐπεὶ αἱ Α, Β ῥηταί εἰσι δυνάμει μόνον σύμμετροι, τὸ ἄρα ὑπὸ τῶν Α, Β, τουτέστι τὸ ἀπὸ τῆς Δ, μέσον ἐστίν. μέση ἄρα ἡ Δ. καὶ ἐπεὶ αἱ Β, Γ δυνάμει μόνον εἰσὶ σύμμετροι, καί ἐστιν ὡς ἡ Β πρὸς τὴν Γ, ἡ Δ πρὸς τὴν Ε, καὶ αἱ Δ, Ε ἄρα δυνάμει μόνον εἰσὶ σύμμετροι. μέση δὲ ἡ Δ· μέση ἄρα καὶ ἡ Ε· αἱ Δ, Ε ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. λέγω δή, ὅτι καὶ μέσον περιέχουσιν. ἐπεὶ γάρ ἐστιν ὡς ἡ Β πρὸς τὴν Γ, ἡ Δ πρὸς τὴν Ε, ἐναλλὰξ ἄρα ὡς ἡ Β πρὸς τὴν Δ, ἡ Γ πρὸς τὴν Ε. ὡς δὲ ἡ Β πρὸς τὴν Δ, ἡ Δ πρὸς τὴν Α· καὶ ὡς ἄρα ἡ Δ πρὸς τὴν Α, ἡ Γ πρὸς τὴν Ε· τὸ ἄρα ὑπὸ τῶν Α, Γ ἴσον ἐστὶ τῷ ὑπὸ τῶν Δ, Ε. μέσον δὲ τὸ ὑπὸ τῶν Α, Γ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν Δ, Ε.

Εὕρηνται ἄρα μέσαι δυνάμει μόνον σύμμετροι μέσον περιέχουσαι· ὅπερ ἔδει δεῖξαι.

Λῆμμα

Εὑρεῖν δύο τετραγώνους ἀριθμούς, ὥστε καὶ τὸν συγκείμενον ἐξ αὐτῶν εἶναι τετράγωνον.

Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΒ, ΒΓ, ἔστωσαν δὲ ἤτοι ἄρτιοι ἢ περιττοί. καὶ ἐπεί, ἐάν τε ἀπὸ ἀρτίου ἄρτιος ἀφαιρεθῇ, ἐάν τε ἀπὸ περισσοῦ περισσός, ὁ λοιπὸς ἄρτιός ἐστιν, ὁ λοιπὸς ἄρα ὁ ΑΓ ἄρτιός ἐστιν. τετμήσθω ὁ ΑΓ δίχα κατὰ τὸ Δ. ἔστωσαν δὲ καὶ οἱ ΑΒ, ΒΓ ἤτοι ὅμοιοι ἐπίπεδοι ἢ τετράγωνοι, οἳ καὶ αὐτοὶ ὅμοιοί εἰσιν ἐπίπεδοι· ὁ ἄρα ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ [τοῦ] ΓΔ τετραγώνου ἴσος ἐστὶ τῷ ἀπὸ τοῦ ΒΔ τετραγώνῳ. καί ἐστι τετράγωνος ὁ ἐκ τῶν ΑΒ, ΒΓ, ἐπειδήπερ ἐδείχθη, ὅτι, ἐὰν δύο ὅμοιοι ἐπίπεδοι πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος τετράγωνός ἐστιν. εὕρηνται ἄρα δύο τετράγωνοι ἀριθμοὶ ὅ τε ἐκ τῶν ΑΒ, ΒΓ καὶ ὁ ἀπὸ τοῦ ΓΔ, οἳ συντεθέντες ποιοῦσι τὸν ἀπὸ τοῦ ΒΔ τετράγωνον.

Καὶ φανερόν, ὅτι εὕρηνται πάλιν δύο τετράγωνοι ὅ τε ἀπὸ τοῦ ΒΔ καὶ ὁ ἀπὸ τοῦ ΓΔ, ὥστε τὴν ὑπεροχὴν αὐτῶν τὸν ὑπὸ ΑΒ, ΒΓ εἶναι τετράγωνον, ὅταν οἱ ΑΒ, ΒΓ ὅμοιοι ὦσιν ἐπίπεδοι. ὅταν δὲ μὴ ὦσιν ὅμοιοι ἐπίπεδοι, εὕρηνται δύο τετράγωνοι ὅ τε ἀπὸ τοῦ ΒΔ καὶ ὁ ἀπὸ τοῦ ΔΓ, ὧν ἡ ὑπεροχὴ ὁ ὑπὸ τῶν ΑΒ, ΒΓ οὐκ ἔστι τετράγωνος· ὅπερ ἔδει δεῖξαι.

Λῆμμα

Εὑρεῖν δύο τετραγώνους ἀριθμούς, ὥστε τὸν ἐξ αὐτῶν συγκείμενον μὴ εἶναι τετράγωνον.

Ἔστω γὰρ ὁ ἐκ τῶν ΑΒ, ΒΓ, ὡς ἔφαμεν, τετράγωνος, καὶ ἄρτιος ὁ ΓΑ, καὶ τετμήσθω ὁ ΓΑ δίχα τῷ Δ. φανερὸν δή, ὅτι ὁ ἐκ τῶν ΑΒ, ΒΓ τετράγωνος μετὰ τοῦ ἀπὸ [τοῦ] ΓΔ τετραγώνου ἴσος ἐστὶ τῷ ἀπὸ [τοῦ] ΒΔ τετραγώνῳ. ἀφῃρήσθω μονὰς ἡ ΔΕ· ὁ ἄρα ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ [τοῦ] ΓΕ

ἐλάσσων ἐστὶ τοῦ ἀπὸ [τοῦ] ΒΔ τετραγώνου. λέγω οὖν, ὅτι ὁ ἐκ τῶν ΑΒ, ΒΓ τετράγωνος μετὰ τοῦ ἀπὸ [τοῦ] ΓΕ οὐκ ἔσται τετράγωνος.

Εἰ γὰρ ἔσται τετράγωνος, ἤτοι ἴσος ἐστὶ τῷ ἀπὸ [τοῦ] ΒΕ ἢ ἐλάσσων τοῦ ἀπὸ [τοῦ] ΒΕ, οὐκέτι δὲ καὶ μείζων, ἵνα μὴ τμηθῇ ἡ μονάς. ἔστω, εἰ δυνατόν, πρότερον ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος τῷ ἀπὸ ΒΕ, καὶ ἔστω τῆς ΔΕ μονάδος διπλασίων ὁ ΗΑ. ἐπεὶ οὖν ὅλος ὁ ΑΓ ὅλου τοῦ ΓΔ ἐστι διπλασίων, ὧν ὁ ΑΗ τοῦ ΔΕ ἐστι διπλασίων, καὶ λοιπὸς ἄρα ὁ ΗΓ λοιποῦ τοῦ ΕΓ ἐστι διπλασίων· δίχα ἄρα τέτμηται ὁ ΗΓ τῷ Ε. ὁ ἄρα ἐκ τῶν ΗΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ἐστὶ τῷ ἀπὸ ΒΕ τετραγώνῳ. ἀλλὰ καὶ ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ὑπόκειται τῷ ἀπὸ [τοῦ] ΒΕ τετραγώνῳ· ὁ ἄρα ἐκ τῶν ΗΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ἐστὶ τῷ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ. καὶ κοινοῦ ἀφαιρεθέντος τοῦ ἀπὸ ΓΕ συνάγεται ὁ ΑΒ ἴσος τῷ ΗΒ· ὅπερ ἄτοπον. οὐκ ἄρα ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ [τοῦ] ΓΕ ἴσος ἐστὶ τῷ ἀπὸ ΒΕ. λέγω δή, ὅτι οὐδὲ ἐλάσσων τοῦ ἀπὸ ΒΕ. εἰ γὰρ δυνατόν, ἔστω τῷ ἀπὸ ΒΖ ἴσος, καὶ τοῦ ΔΖ διπλασίων ὁ ΘΑ. καὶ συναχθήσεται πάλιν διπλασίων ὁ ΘΓ τοῦ ΓΖ· ὥστε καὶ τὸν ΓΘ δίχα τετμῆσθαι κατὰ τὸ Ζ, καὶ διὰ τοῦτο τὸν ἐκ τῶν ΘΒ, ΒΓ μετὰ τοῦ ἀπὸ ΖΓ ἴσον γίνεσθαι τῷ ἀπὸ ΒΖ. ὑπόκειται δὲ καὶ ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος τῷ ἀπὸ ΒΖ. ὥστε καὶ ὁ ἐκ τῶν ΘΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΖ ἴσος ἔσται τῷ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ· ὅπερ ἄτοπον. οὐκ ἄρα ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ἐστὶ [τῷ] ἐλάσσονι τοῦ ἀπὸ ΒΕ. ἐδείχθη δέ, ὅτι οὐδὲ [αὐτῷ] τῷ ἀπὸ ΒΕ. οὐκ ἄρα ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ τετράγωνός ἐστιν. [δυνατοῦ δὲ ὄντος καὶ κατὰ πλείονας τρόπους τοὺς εἰρημένους ἀριθμοὺς ἐπιδεικνύειν, ἀρκείσθωσαν ἡμῖν οἱ εἰρημένοι, ἵνα μὴ μακροτέρας οὔσης τῆς πραγματείας ἐπὶ πλέον αὐτὴν μηκύνωμεν.] ὅπερ ἔδει δεῖξαι.

κθ΄. Εὑρεῖν δύο ῥητὰς δυνάμει μόνον συμμέτρους, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει.

Ἐκκείσθω γάρ τις ῥητὴ ἡ ΑΒ καὶ δύο τετράγωνοι ἀριθμοὶ οἱ ΓΔ, ΔΕ, ὥστε τὴν ὑπεροχὴν αὐτῶν τὸν ΓΕ μὴ εἶναι τετράγωνον, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΖΒ, καὶ πεποιήσθω ὡς ὁ ΔΓ πρὸς τὸν ΓΕ, οὕτως τὸ ἀπὸ τῆς ΒΑ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΑΖ τετράγωνον καὶ ἐπεζεύχθω ἡ ΖΒ.

Ἐπεὶ [οὖν] ἐστιν ὡς τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ, οὕτως ὁ ΔΓ πρὸς τὸν ΓΕ, τὸ ἀπὸ τῆς ΒΑ ἄρα πρὸς τὸ ἀπὸ τῆς ΑΖ λόγον ἔχει, ὃν ἀριθμὸς ὁ ΔΓ πρὸς ἀριθμὸν τὸν ΓΕ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΑ τῷ ἀπὸ τῆς ΑΖ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΑΒ· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΑΖ· ῥητὴ ἄρα καὶ ἡ ΑΖ. καὶ ἐπεὶ ὁ ΔΓ πρὸς τὸν ΓΕ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΒΑ ἄρα πρὸς τὸ ἀπὸ τῆς ΑΖ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΒ τῇ ΑΖ μήκει· αἱ ΒΑ, ΑΖ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ ἐπεί [ἐστιν] ὡς ὁ ΔΓ πρὸς τὸν ΓΕ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ, ἀναστρέψαντι ἄρα ὡς ὁ ΓΔ πρὸς τὸν ΔΕ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΖ. ὁ δὲ ΓΔ πρὸς τὸν ΔΕ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ τὸ ἀπὸ τῆς ΑΒ ἄρα πρὸς τὸ ἀπὸ τῆς ΒΖ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· σύμμετρος ἄρα ἐστὶν ἡ ΑΒ τῇ ΒΖ μήκει. καί ἐστι τὸ ἀπὸ τῆς ΑΒ ἴσον τοῖς ἀπὸ τῶν ΑΖ, ΖΒ· ἡ ΑΒ ἄρα τῆς ΑΖ μεῖζον δύναται τῇ ΒΖ συμμέτρῳ ἑαυτῇ.

Εὕρηνται ἄρα δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ ΒΑ, ΑΖ, ὥστε τὴν μείζονα τὴν ΑΒ τῆς ἐλάσσονος τῆς ΑΖ μεῖζον δύνασθαι τῷ ἀπὸ τῆς ΒΖ συμμέτρου ἑαυτῇ μήκει· ὅπερ ἔδει δεῖξαι.

λ΄. Εὑρεῖν δύο ῥητὰς δυνάμει μόνον συμμέτρους, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει.

Ἐκκείσθω ῥητὴ ἡ ΑΒ καὶ δύο τετράγωνοι ἀριθμοὶ οἱ ΓΕ, ΕΔ, ὥστε τὸν συγκείμενον ἐξ αὐτῶν τὸν ΓΔ μὴ εἶναι τετράγωνον, καὶ γεγράφθω ἐπὶ τῆς

ΑΒ ἡμικύκλιον τὸ ΑΖΒ, καὶ πεποιήσθω ὡς ὁ ΔΓ πρὸς τὸν ΓΕ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ, καὶ ἐπεζεύχθω ἡ ΖΒ.

Ὁμοίως δὴ δείξομεν τῷ πρὸ τούτου, ὅτι αἱ ΒΑ, ΑΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ ἐπεί ἐστιν ὡς ὁ ΔΓ πρὸς τὸν ΓΕ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ, ἀναστρέψαντι ἄρα ὡς ὁ ΓΔ πρὸς τὸν ΔΕ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΖ. ὁ δὲ ΓΔ πρὸς τὸν ΔΕ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ’ ἄρα τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΖ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΒ τῇ ΒΖ μήκει. καὶ δύναται ἡ ΑΒ τῆς ΑΖ μεῖζον τῷ ἀπὸ τῆς ΖΒ ἀσυμμέτρου ἑαυτῇ.

Αἱ ΑΒ, ΑΖ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΒ τῆς ΑΖ μεῖζον δύναται τῷ ἀπὸ τῆς ΖΒ συμμέτρου ἑαυτῇ μήκει· ὅπερ ἔδει δεῖξαι.

λα΄. Εὑρεῖν δύο μέσας δυνάμει μόνον συμμέτρους ῥητὸν περιεχούσας, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει.

Ἐκκείσθωσαν δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ Α, Β, ὥστε τὴν Α μείζονα οὖσαν τῆς ἐλάσσονος τῆς Β μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. καὶ τῷ ὑπὸ τῶν Α, Β ἴσον ἔστω τὸ ἀπὸ τῆς Γ. μέσον δὲ τὸ ὑπὸ τῶν Α, Β· μέσον ἄρα καὶ τὸ ἀπὸ τῆς Γ· μέση ἄρα καὶ ἡ Γ. τῷ δὲ ἀπὸ τῆς Β ἴσον ἔστω τὸ ὑπὸ τῶν Γ, Δ. ῥητὸν δὲ τὸ ἀπὸ τῆς Β· ῥητὸν ἄρα καὶ τὸ ὑπὸ τῶν Γ, Δ. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως τὸ ὑπὸ τῶν Α, Β πρὸς τὸ ἀπὸ τῆς Β, ἀλλὰ τῷ μὲν ὑπὸ τῶν Α, Β ἴσον ἐστὶ τὸ ἀπὸ τῆς Γ, τῷ δὲ ἀπὸ τῆς Β ἴσον τὸ ὑπὸ τῶν Γ, Δ, ὡς ἄρα ἡ Α πρὸς τὴν Β, οὕτως τὸ ἀπὸ τῆς Γ πρὸς τὸ ὑπὸ τῶν Γ, Δ.

ὡς δὲ τὸ ἀπὸ τῆς Γ πρὸς τὸ ὑπὸ τῶν Γ, Δ, οὕτως ἡ Γ πρὸς τὴν Δ· καὶ ὡς ἄρα ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ. σύμμετρος δὲ ἡ Α τῇ Β δυνάμει μόνον· σύμμετρος ἄρα καὶ ἡ Γ τῇ Δ δυνάμει μόνον. καί ἐστι μέση ἡ Γ· μέση ἄρα καὶ ἡ Δ. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, ἡ Γ πρὸς τὴν Δ, ἡ δὲ Α τῆς Β

μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ Γ ἄρα τῆς Δ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ.

Εὕρηνται ἄρα δύο μέσαι δυνάμει μόνον σύμμετροι αἱ Γ, Δ ῥητὸν περιέχουσαι, καὶ ἡ Γ τῆς Δ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει.

Ὁμοίως δὴ δειχθήσεται καὶ τῷ ἀπὸ ἀσυμμέτρου, ὅταν ἡ Α τῆς Β μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ.

λβ΄. Εὑρεῖν δύο μέσας δυνάμει μόνον συμμέτρους μέσον περιεχούσας, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ.

Ἐκκείσθωσαν τρεῖς ῥηταὶ δυνάμει μόνον σύμμετροι αἱ Α, Β, Γ, ὥστε τὴν Α τῆς Γ μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ τῷ μὲν ὑπὸ τῶν Α, Β ἴσον ἔστω τὸ ἀπὸ τῆς Δ. μέσον ἄρα τὸ ἀπὸ τῆς Δ· καὶ ἡ Δ ἄρα μέση ἐστίν. τῷ δὲ ὑπὸ τῶν Β, Γ ἴσον ἔστω τὸ ὑπὸ τῶν Δ, Ε. καὶ ἐπεί ἐστιν ὡς τὸ ὑπὸ τῶν Α, Β πρὸς τὸ ὑπὸ τῶν Β, Γ, οὕτως ἡ Α πρὸς τὴν Γ, ἀλλὰ τῷ μὲν ὑπὸ τῶν Α, Β ἴσον ἐστὶ τὸ ἀπὸ τῆς Δ, τῷ δὲ ὑπὸ τῶν Β, Γ ἴσον τὸ ὑπὸ τῶν Δ, Ε, ἔστιν ἄρα ὡς ἡ Α πρὸς τὴν Γ, οὕτως τὸ ἀπὸ τῆς Δ πρὸς τὸ ὑπὸ τῶν Δ, Ε.

ὡς δὲ τὸ ἀπὸ τῆς Δ πρὸς τὸ ὑπὸ τῶν Δ, Ε, οὕτως ἡ Δ πρὸς τὴν Ε· καὶ ὡς ἄρα ἡ Α πρὸς τὴν Γ, οὕτως ἡ Δ πρὸς τὴν Ε· σύμμετρος δὲ ἡ Α τῇ Γ δυνάμει [μόνον]. σύμμετρος ἄρα καὶ ἡ Δ τῇ Ε δυνάμει μόνον. μέση δὲ ἡ Δ· μέση ἄρα καὶ ἡ Ε. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Γ, ἡ Δ πρὸς τὴν Ε, ἡ δὲ Α τῆς Γ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ Δ ἄρα τῆς Ε μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῇ. λέγω δή, ὅτι καὶ μέσον ἐστὶ τὸ ὑπὸ τῶν Δ, Ε. ἐπεὶ γὰρ ἴσον ἐστὶ τὸ ὑπὸ τῶν Β, Γ τῷ ὑπὸ τῶν Δ, Ε, μέσον δὲ τὸ ὑπὸ τῶν Β, Γ [αἱ γὰρ Β, Γ ῥηταί εἰσι δυνάμει μόνον σύμμετροι], μέσον ἄρα καὶ τὸ ὑπὸ τῶν Δ, Ε.

Εὕρηνται ἄρα δύο μέσαι δυνάμει μόνον σύμμετροι αἱ Δ, Ε μέσον περιέχουσαι, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ.

Ὁμοίως δὴ πάλιν δειχθήσεται καὶ τῷ ἀπὸ ἀσυμμέτρου, ὅταν ἡ Α τῆς Γ μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ.

Λῆμμα

Ἔστω τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν ἔχον τὴν Α, καὶ ἤχθω κάθετος ἡ ΑΔ· λέγω, ὅτι τὸ μὲν ὑπὸ τῶν ΓΒΔ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΒΑ, τὸ δὲ ὑπὸ τῶν ΒΓΔ ἴσον τῷ ἀπὸ τῆς ΓΑ, καὶ τὸ ὑπὸ τῶν ΒΔ, ΔΓ ἴσον τῷ ἀπὸ τῆς ΑΔ, καὶ ἔτι τὸ ὑπὸ τῶν ΒΓ, ΑΔ ἴσον [ἐστὶ] τῷ ὑπὸ τῶν ΒΑ, ΑΓ.

Καὶ πρῶτον, ὅτι τὸ ὑπὸ τῶν ΓΒΔ ἴσον [ἐστὶ] τῷ ἀπὸ τῆς ΒΑ.

Ἐπεὶ γὰρ ἐν ὀρθογωνίῳ τριγώνῳ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἦκται ἡ ΑΔ, τὰ ΑΒΔ, ΑΔΓ ἄρα τρίγωνα ὅμοιά ἐστι τῷ τε ὅλῳ τῷ ΑΒΓ καὶ ἀλλήλοις. καὶ ἐπεὶ ὅμοιόν ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΑΒΔ τριγώνῳ, ἔστιν ἄρα ὡς ἡ ΓΒ πρὸς τὴν ΒΑ, οὕτως ἡ ΒΑ πρὸς τὴν ΒΔ· τὸ ἄρα ὑπὸ τῶν ΓΒΔ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ.

Διὰ τὰ αὐτὰ δὴ καὶ τὸ ὑπὸ τῶν ΒΓΔ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ.

Καὶ ἐπεί, ἐὰν ἐν ὀρθογωνίῳ τριγώνῳ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἀχθῇ, ἡ ἀχθεῖσα τῶν τῆς βάσεως τμημάτων μέση ἀνάλογόν ἐστιν, ἔστιν ἄρα ὡς ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΑΔ πρὸς τὴν ΔΓ· τὸ ἄρα ὑπὸ τῶν ΒΔ, ΔΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΑ.

Λέγω, ὅτι καὶ τὸ ὑπὸ τῶν ΒΓ, ΑΔ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΒΑ, ΑΓ. ἐπεὶ γάρ, ὡς ἔφαμεν, ὅμοιόν ἐστι τὸ ΑΒΓ τῷ ΑΒΔ, ἔστιν ἄρα ὡς ἡ ΒΓ πρὸς τὴν ΓΑ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΔ. [ἐὰν δὲ τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, τὸ ὑπὸ τῶν ἄκρων ἴσον ἐστὶ τῷ ὑπὸ τῶν μέσων.] τὸ ἄρα ὑπὸ τῶν ΒΓ, ΑΔ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΒΑ, ΑΓ· ὅπερ ἔδει δεῖξαι.

λγ΄. Εὑρεῖν δύο εὐθείας δυνάμει ἀσυμμέτρους ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δ᾽ ὑπ᾽ αὐτῶν μέσον.

Ἐκκείσθωσαν δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ, ὥστε τὴν μείζονα τὴν ΑΒ τῆς ἐλάσσονος τῆς ΒΓ μεῖζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ τετμήσθω ἡ ΒΓ δίχα κατὰ τὸ Δ, καὶ τῷ ἀφ' ὁποτέρας τῶν ΒΔ, ΔΓ ἴσον παρὰ τὴν ΑΒ παραβεβλήσθω παραλληλόγραμμον ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΑΕΒ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΖΒ, καὶ ἤχθω τῇ ΑΒ πρὸς ὀρθὰς ἡ ΕΖ, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΖΒ.

Καὶ ἐπεὶ [δύο] εὐθεῖαι ἄνισοί εἰσιν αἱ ΑΒ, ΒΓ, καὶ ἡ ΑΒ τῆς ΒΓ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, τῷ δὲ τετάρτῳ τοῦ ἀπὸ τῆς ΒΓ, τουτέστι τῷ ἀπὸ τῆς ἡμισείας αὐτῆς, ἴσον παρὰ τὴν ΑΒ παραβέβληται παραλληλόγραμμον ἐλλεῖπον εἴδει τετραγώνῳ καὶ ποιεῖ τὸ ὑπὸ τῶν ΑΕΒ, ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΕ τῇ ΕΒ. καί ἐστιν ὡς ἡ ΑΕ πρὸς ΕΒ, οὕτως τὸ ὑπὸ τῶν ΒΑ, ΑΕ πρὸς τὸ ὑπὸ τῶν ΑΒ, ΒΕ, ἴσον δὲ τὸ μὲν ὑπὸ τῶν ΒΑ, ΑΕ τῷ ἀπὸ τῆς ΑΖ, τὸ δὲ ὑπὸ τῶν ΑΒ, ΒΕ τῷ ἀπὸ τῆς ΒΖ· ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΖ τῷ ἀπὸ τῆς ΖΒ· αἱ ΑΖ, ΖΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι. καὶ ἐπεὶ ἡ ΑΒ ῥητή ἐστιν, ῥητὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΑΒ· ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΖ, ΖΒ ῥητόν ἐστιν. καὶ ἐπεὶ πάλιν τὸ ὑπὸ τῶν ΑΕ, ΕΒ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΖ, ὑπόκειται δὲ τὸ ὑπὸ τῶν ΑΕ, ΕΒ καὶ τῷ ἀπὸ τῆς ΒΔ ἴσον, ἴση ἄρα ἐστὶν ἡ ΖΕ τῇ ΒΔ· διπλῆ ἄρα ἡ ΒΓ τῆς ΖΕ· ὥστε καὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τῷ ὑπὸ τῶν ΑΒ, ΕΖ. μέσον δὲ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΑΒ, ΕΖ. ἴσον δὲ τὸ ὑπὸ τῶν ΑΒ, ΕΖ τῷ ὑπὸ τῶν ΑΖ, ΖΒ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΑΖ, ΖΒ. ἐδείχθη δὲ καὶ ῥητὸν τὸ συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων.

Εὕρηνται ἄρα δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ ΑΖ, ΖΒ ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων ῥητόν, τὸ δὲ ὑπ' αὐτῶν μέσον· ὅπερ ἔδει δεῖξαι.

λδ΄. Εὑρεῖν δύο εὐθείας δυνάμει ἀσυμμέτρους ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δ' ὑπ' αὐτῶν ῥητόν.

Ἐκκείσθωσαν δύο μέσαι δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ ῥητὸν περιέχουσαι τὸ ὑπ᾽ αὐτῶν, ὥστε τὴν ΑΒ τῆς ΒΓ μεῖζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ τὸ ΑΔΒ ἡμικύκλιον, καὶ τετμήσθω ἡ ΒΓ δίχα κατὰ τὸ Ε, καὶ παραβεβλήσθω παρὰ τὴν ΑΒ τῷ ἀπὸ τῆς ΒΕ ἴσον παραλληλόγραμμον ἐλλεῖπον εἴδει τετραγώνῳ τὸ ὑπὸ τῶν ΑΖΒ· ἀσύμμετρος ἄρα [ἐστὶν] ἡ ΑΖ τῇ ΖΒ μήκει. καὶ ἤχθω ἀπὸ τοῦ Ζ τῇ ΑΒ πρὸς ὀρθὰς ἡ ΖΔ, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΒ.

Ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΖ τῇ ΖΒ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΒΑ, ΑΖ τῷ ὑπὸ τῶν ΑΒ, ΒΖ. ἴσον δὲ τὸ μὲν ὑπὸ τῶν ΒΑ, ΑΖ τῷ ἀπὸ τῆς ΑΔ, τὸ δὲ ὑπὸ τῶν ΑΒ, ΒΖ τῷ ἀπὸ τῆς ΔΒ· ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΑΔ τῷ ἀπὸ τῆς ΔΒ. καὶ ἐπεὶ μέσον ἐστὶ τὸ ἀπὸ τῆς ΑΒ, μέσον ἄρα καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΒΓ τῆς ΔΖ, διπλάσιον ἄρα καὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ τοῦ ὑπὸ τῶν ΑΒ, ΖΔ. ῥητὸν δὲ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· ῥητὸν ἄρα καὶ τὸ ὑπὸ τῶν ΑΒ, ΖΔ. τὸ δὲ ὑπὸ τῶν ΑΒ, ΖΔ ἴσον τῷ ὑπὸ τῶν ΑΔ, ΔΒ· ὥστε καὶ τὸ ὑπὸ τῶν ΑΔ, ΔΒ ῥητόν ἐστιν.

Εὕρηνται ἄρα δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ ΑΔ, ΔΒ ποιοῦσαι τὸ [μὲν] συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον, τὸ δ᾽ ὑπ᾽ αὐτῶν ῥητόν· ὅπερ ἔδει δεῖξαι.

λε΄. Εὑρεῖν δύο εὐθείας δυνάμει ἀσυμμέτρους ποιούσας τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ᾽ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῳ ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνῳ.

Ἐκκείσθωσαν δύο μέσαι δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ μέσον περιέχουσαι, ὥστε τὴν ΑΒ τῆς ΒΓ μεῖζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ τὰ λοιπὰ γεγονέτω τοῖς ἐπάνω ὁμοίως.

Καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΖ τῇ ΖΒ μήκει, ἀσύμμετρός ἐστι καὶ ἡ ΑΔ τῇ ΔΒ δυνάμει.

καὶ ἐπεὶ μέσον ἐστὶ τὸ ἀπὸ τῆς ΑΒ, μέσον ἄρα καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΖ, ΖΒ ἴσον ἐστὶ τῷ ἀφ' ἑκατέρας τῶν ΒΕ, ΔΖ, ἴση ἄρα ἐστὶν ἡ ΒΕ τῇ ΔΖ· διπλῆ ἄρα ἡ ΒΓ τῆς ΖΔ· ὥστε καὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ διπλάσιόν ἐστι τοῦ ὑπὸ τῶν ΑΒ, ΖΔ. μέσον δὲ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΑΒ, ΖΔ. καί ἐστιν ἴσον τῷ ὑπὸ τῶν ΑΔ, ΔΒ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει, σύμμετρος δὲ ἡ ΓΒ τῇ ΒΕ, ἀσύμμετρος ἄρα καὶ ἡ ΑΒ τῇ ΒΕ μήκει· ὥστε καὶ τὸ ἀπὸ τῆς ΑΒ τῷ ὑπὸ τῶν ΑΒ, ΒΕ ἀσύμμετρόν ἐστιν. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΒ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ, τῷ δὲ ὑπὸ τῶν ΑΒ, ΒΕ ἴσον ἐστὶ τὸ ὑπὸ τῶν ΑΒ, ΖΔ, τουτέστι τὸ ὑπὸ τῶν ΑΔ, ΔΒ· ἀσύμμετρον ἄρα ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ τῷ ὑπὸ τῶν ΑΔ, ΔΒ.

Εὕρηνται ἄρα δύο εὐθεῖαι αἱ ΑΔ, ΔΒ δυνάμει ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ' αὐτῶν μέσον καὶ τὸ ὑπ' αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῳ ἐκ τῶν ἀπ' αὐτῶν τετραγώνων· ὅπερ ἔδει δεῖξαι.

λς΄. Ἐὰν δύο ῥηταὶ δυνάμει μόνον σύμμετροι συντεθῶσιν, ἡ ὅλη ἄλογός ἐστιν, καλείσθω δὲ ἐκ δύο ὀνομάτων.

Συγκείσθωσαν γὰρ δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ· λέγω, ὅτι ὅλη ἡ ΑΓ ἄλογός ἐστιν.

Ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει· δυνάμει γὰρ μόνον εἰσὶ σύμμετροι· ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ὑπὸ τῶν ΑΒΓ πρὸς τὸ ἀπὸ τῆς ΒΓ, ἀσύμμετρον ἄρα ἐστὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ τῷ ἀπὸ τῆς ΒΓ. ἀλλὰ τῷ μὲν ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, τῷ δὲ ἀπὸ τῆς ΒΓ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ· αἱ γὰρ ΑΒ, ΒΓ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀσύμμετρον ἄρα ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ τοῖς ἀπὸ τῶν ΑΒ, ΒΓ. καὶ συνθέντι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ μετὰ τῶν ἀπὸ τῶν ΑΒ, ΒΓ, τουτέστι τὸ ἀπὸ τῆς ΑΓ, ἀσύμμετρόν ἐστι τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ. ῥητὸν δὲ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ· ἄλογον ἄρα [ἐστὶ] τὸ ἀπὸ

τῆς ΑΓ· ὥστε καὶ ἡ ΑΓ ἄλογός ἐστιν, καλείσθω δὲ ἐκ δύο ὀνομάτων· ὅπερ ἔδει δεῖξαι.

λζ΄. Ἐὰν δύο μέσαι δυνάμει μόνον σύμμετροι συντεθῶσι ῥητὸν περιέχουσαι, ἡ ὅλη ἄλογός ἐστιν, καλείσθω δὲ ἐκ δύο μέσων πρώτη.

Συγκείσθωσαν γὰρ δύο μέσαι δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ ῥητὸν περιέχουσαι· λέγω, ὅτι ὅλη ἡ ΑΓ ἄλογός ἐστιν.

Ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει, καὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ ἄρα ἀσύμμετρά ἐστι τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· καὶ συνθέντι τὰ ἀπὸ τῶν ΑΒ, ΒΓ μετὰ τοῦ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΑΓ, ἀσύμμετρόν ἐστι τῷ ὑπὸ τῶν ΑΒ, ΒΓ. ῥητὸν δὲ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· ὑπόκεινται γὰρ αἱ ΑΒ, ΒΓ ῥητὸν περιέχουσαι· ἄλογον ἄρα τὸ ἀπὸ τῆς ΑΓ· ἄλογος ἄρα ἡ ΑΓ, καλείσθω δὲ ἐκ δύο μέσων πρώτη· ὅπερ ἔδει δεῖξαι.

λη΄. Ἐὰν δύο μέσαι δυνάμει μόνον σύμμετροι συντεθῶσι μέσον περιέχουσαι, ἡ ὅλη ἄλογός ἐστιν, καλείσθω δὲ ἐκ δύο μέσων δευτέρα.

Συγκείσθωσαν γὰρ δύο μέσαι δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ μέσον περιέχουσαι· λέγω, ὅτι ἄλογός ἐστιν ἡ ΑΓ.

Ἐκκείσθω γὰρ ῥητὴ ἡ ΔΕ, καὶ τῷ ἀπὸ τῆς ΑΓ ἴσον παρὰ τὴν ΔΕ παραβεβλήσθω τὸ ΔΖ πλάτος ποιοῦν τὴν ΔΗ. καὶ ἐπεὶ τὸ ἀπὸ τῆς ΑΓ ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν ΑΒ, ΒΓ καὶ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, παραβεβλήσθω δὴ τοῖς ἀπὸ τῶν ΑΒ, ΒΓ παρὰ τὴν ΔΕ ἴσον τὸ ΕΘ· λοιπὸν ἄρα τὸ ΘΖ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. καὶ ἐπεὶ μέση ἐστὶν ἑκατέρα τῶν ΑΒ, ΒΓ, μέσα ἄρα ἐστὶ καὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ.

μέσον δὲ ὑπόκειται καὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. καί ἐστι τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον τὸ ΕΘ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ἴσον τὸ ΖΘ· μέσον ἄρα ἑκάτερον τῶν ΕΘ, ΘΖ. καὶ παρὰ ῥητὴν τὴν ΔΕ παράκειται· ῥητὴ ἄρα ἐστὶν ἑκατέρα τῶν ΔΘ, ΘΗ καὶ ἀσύμμετρος τῇ ΔΕ μήκει. ἐπεὶ οὖν ἀσύμμετρός

ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει, καί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ὑπὸ τῶν ΑΒ, ΒΓ, ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τῷ ὑπὸ τῶν ΑΒ, ΒΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΒ σύμμετρόν ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τετραγώνων, τῷ δὲ ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. ἀσύμμετρον ἄρα ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τὸ ΕΘ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τὸ ΘΖ. ἀσύμμετρον ἄρα ἐστὶ τὸ ΕΘ τῷ ΘΖ· ὥστε καὶ ἡ ΔΘ τῇ ΘΗ ἐστιν ἀσύμμετρος μήκει. αἱ ΔΘ, ΘΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ὥστε ἡ ΔΗ ἄλογός ἐστιν. ῥητὴ δὲ ἡ ΔΕ· τὸ δὲ ὑπὸ ἀλόγου καὶ ῥητῆς περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν· ἄλογον ἄρα ἐστὶ τὸ ΔΖ χωρίον, καὶ ἡ δυναμένη [αὐτὸ] ἄλογός ἐστιν. δύναται δὲ τὸ ΔΖ ἡ ΑΓ· ἄλογος ἄρα ἐστὶν ἡ ΑΓ, καλείσθω δὲ ἐκ δύο μέσων δευτέρα. ὅπερ ἔδει δεῖξαι.

λθ΄. Ἐὰν δύο εὐθεῖαι δυνάμει ἀσύμμετροι συντεθῶσι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δ᾽ ὑπ᾽ αὐτῶν μέσον, ἡ ὅλη εὐθεῖα ἄλογός ἐστιν, καλείσθω δὲ μείζων.

Συγκείσθωσαν γὰρ δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ ΑΒ, ΒΓ ποιοῦσαι τὰ προκείμενα· λέγω, ὅτι ἄλογός ἐστιν ἡ ΑΓ.

Ἐπεὶ γὰρ τὸ ὑπὸ τῶν ΑΒ, ΒΓ μέσον ἐστίν, καὶ τὸ δὶς [ἄρα] ὑπὸ τῶν ΑΒ, ΒΓ μέσον ἐστίν. τὸ δὲ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ ῥητόν· ἀσύμμετρον ἄρα ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ· ὥστε καὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ μετὰ τοῦ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΑΓ, ἀσύμμετρόν ἐστι τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ [ῥητὸν δὲ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ]· ἄλογον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΓ. ὥστε καὶ ἡ ΑΓ ἄλογός ἐστιν, καλείσθω δὲ μείζων. ὅπερ ἔδει δεῖξαι.

μ΄. Ἐὰν δύο εὐθεῖαι δυνάμει ἀσύμμετροι συντεθῶσι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον, τὸ δ᾽ ὑπ᾽ αὐτῶν ῥητόν, ἡ ὅλη εὐθεῖα ἄλογός ἐστιν, καλείσθω δὲ ῥητὸν καὶ μέσον δυναμένη.

Συγκείσθωσαν γὰρ δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ ΑΒ, ΒΓ ποιοῦσαι τὰ προκείμενα· λέγω, ὅτι ἄλογός ἐστιν ἡ ΑΓ.

Ἐπεὶ γὰρ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ μέσον ἐστίν, τὸ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ῥητόν, ἀσύμμετρον ἄρα ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· ὥστε καὶ τὸ ἀπὸ τῆς ΑΓ ἀσύμμετρόν ἐστι τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. ῥητὸν δὲ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· ἄλογον ἄρα τὸ ἀπὸ τῆς ΑΓ. ἄλογος ἄρα ἡ ΑΓ, καλείσθω δὲ ῥητὸν καὶ μέσον δυναμένη. ὅπερ ἔδει δεῖξαι.

μα΄. Ἐὰν δύο εὐθεῖαι δυνάμει ἀσύμμετροι συντεθῶσι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ᾽ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῳ ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων, ἡ ὅλη εὐθεῖα ἄλογός ἐστιν, καλείσθω δὲ δύο μέσα δυναμένη.

Συγκείσθωσαν γὰρ δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ ΑΒ, ΒΓ ποιοῦσαι τὰ προκείμενα· λέγω, ὅτι ἡ ΑΓ ἄλογός ἐστιν.

Ἐκκείσθω ῥητὴ ἡ ΔΕ, καὶ παραβεβλήσθω παρὰ τὴν ΔΕ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον τὸ ΔΖ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ἴσον τὸ ΗΘ· ὅλον ἄρα τὸ ΔΘ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ τετραγώνῳ.

* Βιβλίον ι΄ Πρότασις 41

καὶ ἐπεὶ μέσον ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ, καί ἐστιν ἴσον τῷ ΔΖ, μέσον ἄρα ἐστὶ καὶ τὸ ΔΖ. καὶ παρὰ ῥητὴν τὴν ΔΕ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ ΔΗ καὶ ἀσύμμετρος τῇ ΔΕ μήκει. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΗΚ ῥητή ἐστι καὶ ἀσύμμετρος τῇ ΗΖ, τουτέστι τῇ ΔΕ, μήκει. καὶ ἐπεὶ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, ἀσύμμετρόν ἐστι τὸ ΔΖ τῷ

ΗΘ· ὥστε καὶ ἡ ΔΗ τῇ ΗΚ ἀσύμμετρός ἐστιν. καί εἰσι ῥηταί· αἱ ΔΗ, ΗΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἄλογος ἄρα ἐστὶν ἡ ΔΚ ἡ καλουμένη ἐκ δύο ὀνομάτων. ῥητὴ δὲ ἡ ΔΕ· ἄλογον ἄρα ἐστὶ τὸ ΔΘ καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν. δύναται δὲ τὸ ΘΔ ἡ ΑΓ· ἄλογος ἄρα ἐστὶν ἡ ΑΓ, καλείσθω δὲ δύο μέσα δυναμένη. ὅπερ ἔδει δεῖξαι.

Λῆμμα

Ὅτι δὲ αἱ εἰρημέναι ἄλογοι μοναχῶς διαιροῦνται εἰς τὰς εὐθείας, ἐξ ὧν σύγκεινται ποιουσῶν τὰ προκείμενα εἴδη, δείξομεν ἤδη προεκθέμενοι λημμάτιον τοιοῦτον·

Ἐκκείσθω εὐθεῖα ἡ ΑΒ καὶ τετμήσθω ἡ ὅλη εἰς ἄνισα καθ’ ἑκάτερον τῶν Γ, Δ, ὑποκείσθω δὲ μείζων ἡ ΑΓ τῆς ΔΒ· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΓ, ΓΒ μείζονά ἐστι τῶν ἀπὸ τῶν ΑΔ, ΔΒ.

Τετμήσθω γὰρ ἡ ΑΒ δίχα κατὰ τὸ Ε. καὶ ἐπεὶ μείζων ἐστὶν ἡ ΑΓ τῆς ΔΒ, κοινὴ ἀφῃρήσθω ἡ ΔΓ· λοιπὴ ἄρα ἡ ΑΔ λοιπῆς τῆς ΓΒ μείζων ἐστίν. ἴση δὲ ἡ ΑΕ τῇ ΕΒ· ἐλάττων ἄρα ἡ ΔΕ τῆς ΕΓ· τὰ Γ, Δ ἄρα σημεῖα οὐκ ἴσον ἀπέχουσι τῆς διχοτομίας. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΓ, ΓΒ μετὰ τοῦ ἀπὸ τῆς ΕΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΒ, ἀλλὰ μὴν καὶ τὸ ὑπὸ τῶν ΑΔ, ΔΒ μετὰ τοῦ ἀπὸ ΔΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΒ, τὸ ἄρα ὑπὸ τῶν ΑΓ, ΓΒ μετὰ τοῦ ἀπὸ τῆς ΕΓ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΔ, ΔΒ μετὰ τοῦ ἀπὸ τῆς ΔΕ· ὧν τὸ ἀπὸ τῆς ΔΕ ἔλασσόν ἐστι τοῦ ἀπὸ τῆς ΕΓ· καὶ λοιπὸν ἄρα τὸ ὑπὸ τῶν ΑΓ, ΓΒ ἔλασσόν ἐστι τοῦ ὑπὸ τῶν ΑΔ, ΔΒ. ὥστε καὶ τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἔλασσόν ἐστι τοῦ δὶς ὑπὸ ΑΔ, ΔΒ. καὶ λοιπὸν ἄρα τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ μεῖζόν ἐστι τοῦ συγκειμένου ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ· ὅπερ ἔδει δεῖξαι.

μβ΄. Ἡ ἐκ δύο ὀνομάτων κατὰ ἓν μόνον σημεῖον διαιρεῖται εἰς τὰ ὀνόματα.

Ἔστω ἐκ δύο ὀνομάτων ἡ ΑΒ διῃρημένη εἰς τὰ ὀνόματα κατὰ τὸ Γ· αἱ ΑΓ, ΓΒ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. λέγω, ὅτι ἡ ΑΒ κατ’ ἄλλο

σημεῖον οὐ διαιρεῖται εἰς δύο ῥητὰς δυνάμει μόνον συμμέτρους.

Εἰ γὰρ δυνατόν, διῃρήσθω καὶ κατὰ τὸ Δ, ὥστε καὶ τὰς ΑΔ, ΔΒ ῥητὰς εἶναι δυνάμει μόνον συμμέτρους. φανερὸν δή, ὅτι ἡ ΑΓ τῇ ΔΒ οὐκ ἔστιν ἡ αὐτή. εἰ γὰρ δυνατόν, ἔστω. ἔσται δὴ καὶ ἡ ΑΔ τῇ ΓΒ ἡ αὐτή· καὶ ἔσται ὡς ἡ ΑΓ πρὸς τὴν ΓΒ, οὕτως ἡ ΒΔ πρὸς τὴν ΔΑ, καὶ ἔσται ἡ ΑΒ κατὰ τὸ αὐτὸ τῇ κατὰ τὸ Γ διαιρέσει διαιρεθεῖσα καὶ κατὰ τὸ Δ· ὅπερ οὐχ ὑπόκειται. οὐκ ἄρα ἡ ΑΓ τῇ ΔΒ ἐστιν ἡ αὐτή.

διὰ δὴ τοῦτο καὶ τὰ Γ, Δ σημεῖα οὐκ ἴσον ἀπέχουσι τῆς διχοτομίας. ᾧ ἄρα διαφέρει τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ τῶν ΑΔ, ΔΒ, τούτῳ διαφέρει καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ διὰ τὸ καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ μετὰ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ μετὰ τοῦ δὶς ὑπὸ τῶν ΑΔ, ΔΒ ἴσα εἶναι τῷ ἀπὸ τῆς ΑΒ. ἀλλὰ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ τῶν ΑΔ, ΔΒ διαφέρει ῥητῷ· ῥητὰ γὰρ ἀμφότερα· καὶ τὸ δὶς ἄρα ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ διαφέρει ῥητῷ μέσα ὄντα· ὅπερ ἄτοπον· μέσον γὰρ μέσου οὐχ ὑπερέχει ῥητῷ.

Οὐκ ἄρα ἡ ἐκ δύο ὀνομάτων κατ᾽ ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται· καθ᾽ ἓν ἄρα μόνον· ὅπερ ἔδει δεῖξαι.

μγ΄. Ἡ ἐκ δύο μέσων πρώτη καθ᾽ ἓν μόνον σημεῖον διαιρεῖται.

Ἔστω ἐκ δύο μέσων πρώτη ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, ὥστε τὰς ΑΓ, ΓΒ μέσας εἶναι δυνάμει μόνον συμμέτρους ῥητὸν περιεχούσας· λέγω, ὅτι ἡ ΑΒ κατ᾽ ἄλλο σημεῖον οὐ διαιρεῖται.

Εἰ γὰρ δυνατόν, διῃρήσθω καὶ κατὰ τὸ Δ, ὥστε καὶ τὰς ΑΔ, ΔΒ μέσας εἶναι δυνάμει μόνον συμμέτρους ῥητὸν περιεχούσας. ἐπεὶ οὖν, ᾧ διαφέρει τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, τούτῳ διαφέρει τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ τῶν ΑΔ, ΔΒ, ῥητῷ δὲ διαφέρει τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ· ῥητὰ γὰρ ἀμφότερα· ῥητῷ ἄρα διαφέρει καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ τῶν ΑΔ, ΔΒ μέσα ὄντα· ὅπερ ἄτοπον.

Οὐκ ἄρα ἡ ἐκ δύο μέσων πρώτη κατ᾽ ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται εἰς τὰ ὀνόματα· καθ᾽ ἓν ἄρα μόνον· ὅπερ ἔδει δεῖξαι

μδ΄. Ἡ ἐκ μέσων δευτέρα καθ᾽ ἓν μόνον σημεῖον διαιρεῖται.

Ἔστω ἐκ δύο μέσων δευτέρα ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, ὥστε τὰς ΑΓ, ΓΒ μέσας εἶναι δυνάμει μόνον συμμέτρους μέσον περιεχούσας· φανερὸν δή, ὅτι τὸ Γ οὐκ ἔστι κατὰ τῆς διχοτομίας, ὅτι οὐκ εἰσὶ μήκει σύμμετροι. λέγω, ὅτι ἡ ΑΒ κατ᾽ ἄλλο σημεῖον οὐ διαιρεῖται.

Εἰ γὰρ δυνατόν, διῃρήσθω καὶ κατὰ τὸ Δ, ὥστε τὴν ΑΓ τῇ ΔΒ μὴ εἶναι τὴν αὐτήν, ἀλλὰ μείζονα καθ᾽ ὑπόθεσιν τὴν ΑΓ· δῆλον δή, ὅτι καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ, ὡς ἐπάνω ἐδείξαμεν, ἐλάσσονα τῶν ἀπὸ τῶν ΑΓ, ΓΒ· καὶ τὰς ΑΔ, ΔΒ μέσας εἶναι δυνάμει μόνον συμμέτρους μέσον περιεχούσας. καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ τῷ μὲν ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΕΖ παραλληλόγραμμον ὀρθογώνιον παραβεβλήσθω τὸ ΕΚ, τοῖς δὲ ἀπὸ τῶν ΑΓ, ΓΒ ἴσον ἀφῃρήσθω τὸ ΕΗ· λοιπὸν ἄρα τὸ ΘΚ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. πάλιν δὴ τοῖς ἀπὸ τῶν ΑΔ, ΔΒ, ἅπερ ἐλάσσονα ἐδείχθη τῶν ἀπὸ τῶν ΑΓ, ΓΒ, ἴσον ἀφῃρήσθω τὸ ΕΛ· καὶ λοιπὸν ἄρα τὸ ΜΚ ἴσον τῷ δὶς ὑπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεὶ μέσα ἐστὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ, μέσον ἄρα [καὶ] τὸ ΕΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ ΕΘ καὶ ἀσύμμετρος τῇ ΕΖ μήκει.

διὰ τὰ αὐτὰ δὴ καὶ ἡ ΘΝ ῥητή ἐστι καὶ ἀσύμμετρος τῇ ΕΖ μήκει. καὶ ἐπεὶ αἱ ΑΓ, ΓΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι, ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΓ τῇ ΓΒ μήκει. ὡς δὲ ἡ ΑΓ πρὸς τὴν ΓΒ, οὕτως τὸ ἀπὸ τῆς ΑΓ πρὸς τὸ ὑπὸ τῶν ΑΓ, ΓΒ· ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΓ τῷ ὑπὸ τῶν ΑΓ, ΓΒ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΓ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΓ, ΓΒ· δυνάμει γάρ εἰσι σύμμετροι αἱ ΑΓ, ΓΒ. τῷ δὲ ὑπὸ τῶν ΑΓ, ΓΒ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ ἄρα ἀσύμμετρά ἐστι τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΑΓ, ΓΒ ἴσον ἐστὶ τὸ ΕΗ, τῷ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον τὸ ΘΚ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΕΗ τῷ ΘΚ· ὥστε καὶ ἡ ΕΘ τῇ ΘΝ

ἀσύμμετρός ἐστι μήκει. καί εἰσι ῥηταί· αἱ ΕΘ, ΘΝ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ἐὰν δὲ δύο ῥηταὶ δυνάμει μόνον σύμμετροι συντεθῶσιν, ἡ ὅλη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο ὀνομάτων· ἡ ΕΝ ἄρα ἐκ δύο ὀνομάτων ἐστὶ διῃρημένη κατὰ τὸ Θ. κατὰ τὰ αὐτὰ δὴ δειχθήσονται καὶ αἱ ΕΜ, ΜΝ ῥηταὶ δυνάμει μόνον σύμμετροι· καὶ ἔσται ἡ ΕΝ ἐκ δύο ὀνομάτων κατ’ ἄλλο καὶ ἄλλο διῃρημένη τό τε Θ καὶ τὸ Μ, καὶ οὐκ ἔστιν ἡ ΕΘ τῇ ΜΝ ἡ αὐτή, ὅτι τὰ ἀπὸ τῶν ΑΓ, ΓΒ μείζονά ἐστι τῶν ἀπὸ τῶν ΑΔ, ΔΒ. ἀλλὰ τὰ ἀπὸ τῶν ΑΔ, ΔΒ μείζονά ἐστι τοῦ δὶς ὑπὸ ΑΔ, ΔΒ· πολλῷ ἄρα καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ, τουτέστι τὸ ΕΗ, μεῖζόν ἐστι τοῦ δὶς ὑπὸ τῶν ΑΔ, ΔΒ, τουτέστι τοῦ ΜΚ· ὥστε καὶ ἡ ΕΘ τῆς ΜΝ μείζων ἐστίν. ἡ ἄρα ΕΘ τῇ ΜΝ οὐκ ἔστιν ἡ αὐτή· ὅπερ ἔδει δεῖξαι.

με΄. Ἡ μείζων κατὰ τὸ αὐτὸ μόνον σημεῖον διαιρεῖται.

Ἔστω μείζων ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, ὥστε τὰς ΑΓ, ΓΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ τετραγώνων ῥητόν, τὸ δ’ ὑπὸ τῶν ΑΓ, ΓΒ μέσον· λέγω, ὅτι ἡ ΑΒ κατ’ ἄλλο σημεῖον οὐ διαιρεῖται.

Εἰ γὰρ δυνατόν, διῃρήσθω καὶ κατὰ τὸ Δ, ὥστε καὶ τὰς ΑΔ, ΔΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ ῥητόν, τὸ δ’ ὑπ’ αὐτῶν μέσον. καὶ ἐπεί, ᾧ διαφέρει τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ τῶν ΑΔ, ΔΒ, τούτῳ διαφέρει καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, ἀλλὰ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ τῶν ΑΔ, ΔΒ ὑπερέχει ῥητῷ· ῥητὰ γὰρ ἀμφότερα· καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ ἄρα τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ μέσα ὄντα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ μείζων κατ’ ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται· κατὰ τὸ αὐτὸ ἄρα μόνον διαιρεῖται· ὅπερ ἔδει δεῖξαι.

μς΄. Ἡ ῥητὸν καὶ μέσον δυναμένη καθ’ ἓν μόνον σημεῖον διαιρεῖται.

Ἔστω ῥητὸν καὶ μέσον δυναμένη ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, ὥστε τὰς ΑΓ, ΓΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ μέσον, τὸ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ῥητόν· λέγω, ὅτι ἡ ΑΒ κατ᾽ ἄλλο σημεῖον οὐ διαιρεῖται.

Εἰ γὰρ δυνατόν, διῃρήσθω καὶ κατὰ τὸ Δ, ὥστε καὶ τὰς ΑΔ, ΔΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ μέσον, τὸ δὲ δὶς ὑπὸ τῶν ΑΔ, ΔΒ ῥητόν. ἐπεὶ οὖν, ᾧ διαφέρει τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ τοῦ δὶς ὑπὸ τῶν ΑΔ, ΔΒ, τούτῳ διαφέρει καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, τὸ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ τοῦ δὶς ὑπὸ τῶν ΑΔ, ΔΒ ὑπερέχει ῥητῷ, καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ ἄρα τῶν ἀπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ μέσα ὄντα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ῥητὸν καὶ μέσον δυναμένη κατ᾽ ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται. κατὰ ἓν ἄρα σημεῖον διαιρεῖται· ὅπερ ἔδει δεῖξαι.

μζ΄. Ἡ δύο μέσα δυναμένη καθ᾽ ἓν μόνον σημεῖον διαιρεῖται.

Ἔστω [δύο μέσα δυναμένη] ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, ὥστε τὰς ΑΓ, ΓΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τό τε συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ μέσον καὶ τὸ ὑπὸ τῶν ΑΓ, ΓΒ μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῳ ἐκ τῶν ἀπ᾽ αὐτῶν. λέγω, ὅτι ἡ ΑΒ κατ᾽ ἄλλο σημεῖον οὐ διαιρεῖται ποιοῦσα τὰ προκείμενα.

Εἰ γὰρ δυνατόν, διῃρήσθω κατὰ τὸ Δ, ὥστε πάλιν δηλονότι τὴν ΑΓ τῇ ΔΒ μὴ εἶναι τὴν αὐτήν, ἀλλὰ μείζονα καθ᾽ ὑπόθεσιν τὴν ΑΓ, καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ παραβεβλήσθω παρὰ τὴν ΕΖ τοῖς μὲν ἀπὸ τῶν ΑΓ, ΓΒ ἴσον τὸ ΕΗ, τῷ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον τὸ ΘΚ· ὅλον ἄρα τὸ ΕΚ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ τετραγώνῳ. πάλιν δὴ παραβεβλήσθω παρὰ τὴν ΕΖ τοῖς ἀπὸ τῶν ΑΔ, ΔΒ ἴσον τὸ ΕΛ· λοιπὸν ἄρα τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ λοιπῷ τῷ ΜΚ ἴσον ἐστίν. καὶ ἐπεὶ μέσον ὑπόκειται τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, μέσον ἄρα ἐστὶ καὶ τὸ ΕΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ ΘΕ καὶ

ἀσύμμετρος τῇ ΕΖ μήκει. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΘΝ ῥητή ἐστι καὶ ἀσύμμετρος τῇ ΕΖ μήκει. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, καὶ τὸ ΕΗ ἄρα τῷ ΗΝ ἀσύμμετρόν ἐστιν· ὥστε καὶ ἡ ΕΘ τῇ ΘΝ ἀσύμμετρός ἐστιν. καί εἰσι ῥηταί· αἱ ΕΘ, ΘΝ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΕΝ ἄρα ἐκ δύο ὀνομάτων ἐστὶ διῃρημένη κατὰ τὸ Θ. ὁμοίως δὴ δείξομεν, ὅτι καὶ κατὰ τὸ Μ διῄρηται. καὶ οὐκ ἔστιν ἡ ΕΘ τῇ ΜΝ ἡ αὐτή· ἡ ἄρα ἐκ δύο ὀνομάτων κατ’ ἄλλο καὶ ἄλλο σημεῖον διῄρηται· ὅπερ ἐστὶν ἄτοπον. οὐκ ἄρα ἡ δύο μέσα δυναμένη κατ’ ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται· καθ’ ἓν ἄρα μόνον [σημεῖον] διαιρεῖται.

Ὅροι δεύτεροι ς΄.

α΄. Ὑποκειμένης ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων διῃρημένης εἰς τὰ ὀνόματα, ἧς τὸ μεῖζον ὄνομα τοῦ ἐλάσσονος μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει, ἐὰν μὲν τὸ μεῖζον ὄνομα σύμμετρον ᾖ μήκει τῇ ἐκκειμένῃ ῥητῇ, καλείσθω [ἡ ὅλη] ἐκ δύο ὀνομάτων πρώτη.

β΄. Ἐὰν δὲ τὸ ἔλασσον ὄνομα σύμμετρον ᾖ μήκει τῇ ἐκκειμένῃ ῥητῇ, καλείσθω ἐκ δύο ὀνομάτων δευτέρα.

γ΄. Ἐὰν δὲ μηδέτερον τῶν ὀνομάτων σύμμετρον ᾖ μήκει τῇ ἐκκειμένῃ ῥητῇ, καλείσθω ἐκ δύο ὀνομάτων τρίτη.

δ΄. Πάλιν δὴ ἐὰν τὸ μεῖζον ὄνομα [τοῦ ἐλάσσονος] μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει, ἐὰν μὲν τὸ μεῖζον ὄνομα σύμμετρον ᾖ μήκει τῇ ἐκκειμένῃ ῥητῇ, καλείσθω ἐκ δύο ὀνομάτων τετάρτη.

ε΄. Ἐὰν δὲ τὸ ἔλασσον, πέμπτη.

ς΄. Ἐὰν δὲ μηδέτερον, ἕκτη.

Προτάσεις ριε΄.(μέρος 2)

μη΄. Εὑρεῖν τὴν ἐκ δύο ὀνομάτων πρώτην.

Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν συγκείμενον ἐξ αὐτῶν τὸν ΑΒ πρὸς μὲν τὸν ΒΓ λόγον ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, πρὸς δὲ τὸν ΓΑ λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ ἐκκείσθω τις ῥητὴ ἡ Δ, καὶ τῇ Δ σύμμετρος ἔστω μήκει ἡ ΕΖ. ῥητὴ ἄρα ἐστὶ καὶ ἡ ΕΖ. καὶ γεγονέτω ὡς ὁ ΒΑ ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ. ὁ δὲ ΑΒ πρὸς τὸν ΑΓ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· καὶ τὸ ἀπὸ τῆς ΕΖ ἄρα πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· ὥστε σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς ΖΗ. καί ἐστι ῥητὴ ἡ ΕΖ· ῥητὴ ἄρα καὶ ἡ ΖΗ. καὶ ἐπεὶ ὁ ΒΑ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΕΖ ἄρα πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΖ τῇ ΖΗ μήκει. αἱ ΕΖ, ΖΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΕΗ.

Λέγω, ὅτι καὶ πρώτη.

Ἐπεὶ γάρ ἐστιν ὡς ὁ ΒΑ ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ, μείζων δὲ ὁ ΒΑ τοῦ ΑΓ, μεῖζον ἄρα καὶ τὸ ἀπὸ τῆς ΕΖ τοῦ ἀπὸ τῆς ΖΗ. ἔστω οὖν τῷ ἀπὸ τῆς ΕΖ ἴσα τὰ ἀπὸ τῶν ΖΗ, Θ. καὶ ἐπεί ἐστιν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ, ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΑΒ πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς Θ. ὁ δὲ ΑΒ πρὸς τὸν ΒΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ τὸ ἀπὸ τῆς ΕΖ ἄρα πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. σύμμετρος ἄρα ἐστὶν ἡ ΕΖ τῇ Θ μήκει· ἡ ΕΖ ἄρα τῆς ΖΗ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καί εἰσι ῥηταὶ αἱ ΕΖ, ΖΗ, καὶ σύμμετρος ἡ ΕΖ τῇ Δ μήκει.

Ἡ ΕΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ πρώτη· ὅπερ ἔδει δεῖξαι.

μθ΄. Εὑρεῖν τὴν ἐκ δύο ὀνομάτων δευτέραν.

Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν συγκείμενον ἐξ αὐτῶν τὸν ΑΒ πρὸς μὲν τὸν ΒΓ λόγον ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, πρὸς δὲ τὸν ΑΓ λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ ἐκκείσθω ῥητὴ ἡ Δ, καὶ τῇ Δ σύμμετρος ἔστω ἡ ΕΖ μήκει· ῥητὴ ἄρα ἐστὶν ἡ ΕΖ. γεγονέτω δὴ καὶ ὡς ὁ ΓΑ ἀριθμὸς πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς ΖΗ. ῥητὴ ἄρα ἐστὶ καὶ ἡ ΖΗ. καὶ ἐπεὶ ὁ ΓΑ ἀριθμὸς πρὸς τὸν ΑΒ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΖ τῇ ΖΗ μήκει· αἱ ΕΖ, ΖΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΕΗ.

Δεικτέον δή, ὅτι καὶ δευτέρα.

Ἐπεὶ γὰρ ἀνάπαλίν ἐστιν ὡς ὁ ΒΑ ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΗΖ πρὸς τὸ ἀπὸ τῆς ΖΕ, μείζων δὲ ὁ ΒΑ τοῦ ΑΓ, μεῖζον ἄρα [καὶ] τὸ ἀπὸ τῆς ΗΖ τοῦ ἀπὸ τῆς ΖΕ. ἔστω τῷ ἀπὸ τῆς ΗΖ ἴσα τὰ ἀπὸ τῶν ΕΖ, Θ· ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΑΒ πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς Θ. ἀλλ’ ὁ ΑΒ πρὸς τὸν ΒΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ τὸ ἀπὸ τῆς ΖΗ ἄρα πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. σύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ Θ μήκει· ὥστε ἡ ΖΗ τῆς ΖΕ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καί εἰσι ῥηταὶ αἱ ΖΗ, ΖΕ δυνάμει μόνον σύμμετροι, καὶ τὸ ΕΖ ἔλασσον ὄνομα τῇ ἐκκειμένῃ ῥητῇ σύμμετρόν ἐστι τῇ Δ μήκει.

Ἡ ΕΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ δευτέρα· ὅπερ ἔδει δεῖξαι.

ν΄. Εὑρεῖν τὴν ἐκ δύο ὀνομάτων τρίτην.

Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν συγκείμενον ἐξ αὐτῶν τὸν ΑΒ πρὸς μὲν τὸν ΒΓ λόγον ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, πρὸς δὲ τὸν ΑΓ λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἐκκείσθω δέ τις καὶ ἄλλος μὴ τετράγωνος ἀριθμὸς ὁ Δ, καὶ πρὸς ἑκάτερον τῶν ΒΑ, ΑΓ λόγον μὴ ἐχέτω, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ ἐκκείσθω τις ῥητὴ εὐθεῖα ἡ Ε, καὶ γεγονέτω ὡς ὁ Δ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς Ε τῷ ἀπὸ τῆς ΖΗ. καί ἐστι ῥητὴ ἡ Ε· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΖΗ. καὶ ἐπεὶ ὁ Δ πρὸς τὸν ΑΒ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Ε τῇ ΖΗ μήκει. γεγονέτω δὴ πάλιν ὡς ὁ ΒΑ ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΗΘ. ῥητὴ δὲ ἡ ΖΗ· ῥητὴ ἄρα καὶ ἡ ΗΘ. καὶ ἐπεὶ ὁ ΒΑ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὅν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΘΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ ΗΘ μήκει. αἱ ΖΗ, ΗΘ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΖΘ ἄρα ἐκ δύο ὀνομάτων ἐστίν.

Λέγω δή, ὅτι καὶ τρίτη.

Ἐπεὶ γάρ ἐστιν ὡς ὁ Δ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ, ὡς δὲ ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, δι’ ἴσου ἄρα ἐστὶν ὡς ὁ Δ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΗΘ. ὁ δὲ Δ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδὲ τὸ ἀπὸ τῆς Ε ἄρα πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Ε τῇ ΗΘ μήκει. καὶ ἐπεί ἐστιν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, μεῖζον ἄρα τὸ ἀπὸ τῆς ΖΗ τοῦ ἀπὸ τῆς ΗΘ. ἔστω οὖν τῷ ἀπὸ τῆς ΖΗ ἴσα τὰ ἀπὸ τῶν ΗΘ, Κ· ἀναστρέψαντι ἄρα [ἐστὶν] ὡς ὁ ΑΒ πρὸς τὸν

ΒΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς Κ. ὁ δὲ ΑΒ πρὸς τὸν ΒΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ τὸ ἀπὸ τῆς ΖΗ ἄρα πρὸς τὸ ἀπὸ τῆς Κ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· σύμμετρος ἄρα [ἐστὶν] ἡ ΖΗ τῇ Κ μήκει. ἡ ΖΗ ἄρα τῆς ΗΘ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καί εἰσιν αἱ ΖΗ, ΗΘ ῥηταὶ δυνάμει μόνον σύμμετροι, καὶ οὐδετέρα αὐτῶν σύμμετρός ἐστι τῇ Ε μήκει.

Ἡ ΖΘ ἄρα ἐκ δύο ὀνομάτων ἐστὶ τρίτη· ὅπερ ἔδει δεῖξαι.

να΄. Εὑρεῖν τὴν ἐκ δύο ὀνομάτων τετάρτην.

Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν ΑΒ πρὸς τὸν ΒΓ λόγον μὴ ἔχειν μήτε μὴν πρὸς τὸν ΑΓ, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ ἐκκείσθω ῥητὴ ἡ Δ, καὶ τῇ Δ σύμμετρος ἔστω μήκει ἡ ΕΖ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΕΖ. καὶ γεγονέτω ὡς ὁ ΒΑ ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς ΖΗ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΖΗ. καὶ ἐπεὶ ὁ ΒΑ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΖ τῇ ΖΗ μήκει. αἱ ΕΖ, ΖΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ὥστε ἡ ΕΗ ἐκ δύο ὀνομάτων ἐστίν.

Λέγω δή, ὅτι καὶ τετάρτη.

Ἐπεὶ γάρ ἐστιν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ [μείζων δὲ ὁ ΒΑ τοῦ ΑΓ], μεῖζον ἄρα τὸ ἀπὸ τῆς ΕΖ τοῦ ἀπὸ τῆς ΖΗ. ἔστω οὖν τῷ ἀπὸ τῆς ΕΖ ἴσα τὰ ἀπὸ τῶν ΖΗ, Θ· ἀναστρέψαντι ἄρα ὡς ὁ ΑΒ ἀριθμὸς πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς Θ. ὁ δὲ ΑΒ πρὸς τὸν ΒΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ’ ἄρα τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΖ τῇ Θ μήκει·

ἡ ΕΖ ἄρα τῆς ΗΖ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καί εἰσιν αἱ ΕΖ, ΖΗ ῥηταὶ δυνάμει μόνον σύμμετροι, καὶ ἡ ΕΖ τῇ Δ σύμμετρός ἐστι μήκει.

Ἡ ΕΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ τετάρτη· ὅπερ ἔδει δεῖξαι.

νβ΄. Εὑρεῖν τὴν ἐκ δύο ὀνομάτων πέμπτην.

Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν ΑΒ πρὸς ἑκάτερον αὐτῶν λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ ἐκκείσθω ῥητή τις εὐθεῖα ἡ Δ, καὶ τῇ Δ σύμμετρος ἔστω [μήκει] ἡ ΕΖ· ῥητὴ ἄρα ἡ ΕΖ. καὶ γεγονέτω ὡς ὁ ΓΑ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ. ὁ δὲ ΓΑ πρὸς τὸν ΑΒ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδὲ τὸ ἀπὸ τῆς ΕΖ ἄρα πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. αἱ ΕΖ, ΖΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΕΗ.

Λέγω δή, ὅτι καὶ πέμπτη.

Ἐπεὶ γάρ ἐστιν ὡς ὁ ΓΑ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ, ἀνάπαλιν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΖΕ· μεῖζον ἄρα τὸ ἀπὸ τῆς ΗΖ τοῦ ἀπὸ τῆς ΖΕ. ἔστω οὖν τῷ ἀπὸ τῆς ΗΖ ἴσα τὰ ἀπὸ τῶν ΕΖ, Θ· ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΑΒ ἀριθμὸς πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΗΖ πρὸς τὸ ἀπὸ τῆς Θ. ὁ δὲ ΑΒ πρὸς τὸν ΒΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ’ ἄρα τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ Θ μήκει· ὥστε ἡ ΖΗ τῆς ΖΕ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καί εἰσιν αἱ ΗΖ, ΖΕ ῥηταὶ δυνάμει μόνον σύμμετροι καὶ τὸ ΕΖ ἔλαττον ὄνομα σύμμετρόν ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ Δ μήκει. Ἡ ΕΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ πέμπτη· ὅπερ ἔδει δεῖξαι.

νγ΄. Εὑρεῖν τὴν ἐκ δύο ὀνομάτων ἕκτην.

Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν ΑΒ πρὸς ἑκάτερον αὐτῶν λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἔστω δὲ καὶ ἕτερος ἀριθμὸς ὁ Δ μὴ τετράγωνος ὢν μηδὲ πρὸς ἑκάτερον τῶν ΒΑ, ΑΓ λόγον ἔχων, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ ἐκκείσθω τις ῥητὴ εὐθεῖα ἡ Ε, καὶ γεγονέτω ὡς ὁ Δ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ· σύμμετρον ἄρα τὸ ἀπὸ τῆς Ε τῷ ἀπὸ τῆς ΖΗ. καί ἐστι ῥητὴ ἡ Ε· ῥητὴ ἄρα καὶ ἡ ΖΗ. καὶ ἐπεὶ οὐκ ἔχει ὁ Δ πρὸς τὸν ΑΒ λόγον, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς Ε ἄρα πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἡ Ε τῇ ΖΗ μήκει. γεγονέτω δὴ πάλιν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ. σύμμετρον ἄρα τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΘΗ. ῥητὸν ἄρα τὸ ἀπὸ τῆς ΘΗ· ῥητὴ ἄρα ἡ ΘΗ. καὶ ἐπεὶ ὁ ΒΑ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ ΗΘ μήκει. αἱ ΖΗ, ΗΘ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΖΘ.

Δεικτέον δή, ὅτι καὶ ἕκτη.

Ἐπεὶ γάρ ἐστιν ὡς ὁ Δ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ, ἔστι δὲ καὶ ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, δι’ ἴσου ἄρα ἐστὶν ὡς ὁ Δ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΗΘ. ὁ δὲ Δ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδὲ τὸ ἀπὸ τῆς Ε ἄρα πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Ε τῇ ΗΘ μήκει. ἐδείχθη δὲ καὶ τῇ ΖΗ ἀσύμμετρος· ἑκατέρα ἄρα τῶν ΖΗ, ΗΘ ἀσύμμετρός ἐστι τῇ Ε μήκει. καὶ ἐπεί ἐστιν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, μεῖζον ἄρα τὸ ἀπὸ τῆς ΖΗ τοῦ ἀπὸ τῆς ΗΘ. ἔστω οὖν τῷ ἀπὸ [τῆς] ΖΗ ἴσα τὰ ἀπὸ τῶν ΗΘ, Κ· ἀναστρέψαντι ἄρα ὡς ὁ

ΑΒ πρὸς ΒΓ, οὕτως τὸ ἀπὸ ΖΗ πρὸς τὸ ἀπὸ τῆς Κ. ὁ δὲ ΑΒ πρὸς τὸν ΒΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ὥστε οὐδὲ τὸ ἀπὸ ΖΗ πρὸς τὸ ἀπὸ τῆς Κ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ Κ μήκει· ἡ ΖΗ ἄρα τῆς ΗΘ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καί εἰσιν αἱ ΖΗ, ΗΘ ῥηταὶ δυνάμει μόνον σύμμετροι, καὶ οὐδετέρα αὐτῶν σύμμετρός ἐστι μήκει τῇ ἐκκειμένῃ ῥητῇ τῇ Ε.

Ἡ ΖΘ ἄρα ἐκ δύο ὀνομάτων ἐστὶν ἕκτη· ὅπερ ἔδει δεῖξαι.

Λῆμμα

Ἔστω δύο τετράγωνα τὰ ΑΒ, ΒΓ καὶ κείσθωσαν ὥστε ἐπ᾽ εὐθείας εἶναι τὴν ΔΒ τῇ ΒΕ· ἐπ᾽ εὐθείας ἄρα ἐστὶ καὶ ἡ ΖΒ τῇ ΒΗ. καὶ συμπεπληρώσθω τὸ ΑΓ παραλληλόγραμμον· λέγω, ὅτι τετράγωνόν ἐστι τὸ ΑΓ, καὶ ὅτι τῶν ΑΒ, ΒΓ μέσον ἀνάλογόν ἐστι τὸ ΔΗ, καὶ ἔτι τῶν ΑΓ, ΓΒ μέσον ἀνάλογόν ἐστι τὸ ΔΓ.

Ἐπεὶ γὰρ ἴση ἐστὶν ἡ μὲν ΔΒ τῇ ΒΖ, ἡ δὲ ΒΕ τῇ ΒΗ, ὅλη ἄρα ἡ ΔΕ ὅλῃ τῇ ΖΗ ἐστιν ἴση. ἀλλ᾽ ἡ μὲν ΔΕ ἑκατέρᾳ τῶν ΑΘ, ΚΓ ἐστιν ἴση, ἡ δὲ ΖΗ ἑκατέρᾳ τῶν ΑΚ, ΘΓ ἐστιν ἴση· καὶ ἑκατέρα ἄρα τῶν ΑΘ, ΚΓ ἑκατέρᾳ τῶν ΑΚ, ΘΓ ἐστιν ἴση. ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΓ παραλληλόγραμμον· ἔστι δὲ καὶ ὀρθογώνιον· τετράγωνον ἄρα ἐστὶ τὸ ΑΓ.

Καὶ ἐπεί ἐστιν ὡς ἡ ΖΒ πρὸς τὴν ΒΗ, οὕτως ἡ ΔΒ πρὸς τὴν ΒΕ, ἀλλ᾽ ὡς μὲν ἡ ΖΒ πρὸς τὴν ΒΗ, οὕτως τὸ ΑΒ πρὸς τὸ ΔΗ, ὡς δὲ ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως τὸ ΔΗ πρὸς τὸ ΒΓ, καὶ ὡς ἄρα τὸ ΑΒ πρὸς τὸ ΔΗ, οὕτως τὸ ΔΗ πρὸς τὸ ΒΓ. τῶν ΑΒ, ΒΓ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΔΗ.

Λέγω δή, ὅτι καὶ τῶν ΑΓ, ΓΒ μέσον ἀνάλογόν [ἐστι] τὸ ΔΓ.

Ἐπεὶ γάρ ἐστιν ὡς ἡ ΑΔ πρὸς τὴν ΔΚ, οὕτως ἡ ΚΗ πρὸς τὴν ΗΓ· ἴση γάρ [ἐστιν] ἑκατέρα ἑκατέρᾳ· καὶ συνθέντι ὡς ἡ ΑΚ πρὸς ΚΔ, οὕτως ἡ ΚΓ πρὸς ΓΗ, ἀλλ᾽ ὡς μὲν ἡ ΑΚ πρὸς ΚΔ, οὕτως τὸ ΑΓ πρὸς τὸ ΓΔ, ὡς δὲ ἡ ΚΓ πρὸς

ΓΗ, οὕτως τὸ ΔΓ πρὸς ΓΒ, καὶ ὡς ἄρα τὸ ΑΓ πρὸς ΔΓ, οὕτως τὸ ΔΓ πρὸς τὸ ΒΓ. τῶν ΑΓ, ΓΒ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΔΓ· ἃ προέκειτο δεῖξαι.

νδ΄. Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων πρώτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο ὀνομάτων.

Χωρίον γὰρ τὸ ΑΓ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων πρώτης τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο ὀνομάτων.

Ἐπεὶ γὰρ ἐκ δύο ὀνομάτων ἐστὶ πρώτη ἡ ΑΔ, διῃρήσθω εἰς τὰ ὀνόματα κατὰ τὸ Ε, καὶ ἔστω τὸ μεῖζον ὄνομα τὸ ΑΕ. φανερὸν δή, ὅτι αἱ ΑΕ, ΕΔ ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΕ τῆς ΕΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ ΑΕ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΑΒ μήκει. τετμήσθω δὴ ἡ ΕΔ δίχα κατὰ τὸ Ζ σημεῖον. καὶ ἐπεὶ ἡ ΑΕ τῆς ΕΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος, τουτέστι τῷ ἀπὸ τῆς ΕΖ, ἴσον παρὰ τὴν μείζονα τὴν ΑΕ παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς σύμμετρα αὐτὴν διαιρεῖ. παραβεβλήσθω οὖν παρὰ τὴν ΑΕ τῷ ἀπὸ τῆς ΕΖ ἴσον τὸ ὑπὸ ΑΗ, ΗΕ· σύμμετρος ἄρα ἐστὶν ἡ ΑΗ τῇ ΕΗ μήκει. καὶ ἤχθωσαν ἀπὸ τῶν Η, Ε, Ζ ὁποτέρᾳ τῶν ΑΒ, ΓΔ παράλληλοι αἱ ΗΘ, ΕΚ, ΖΛ· καὶ τῷ μὲν ΑΘ παραλληλογράμμῳ ἴσον τετράγωνον συνεστάτω τὸ ΣΝ, τῷ δὲ ΗΚ ἴσον τὸ ΝΠ, καὶ κείσθω ὥστε ἐπ᾽ εὐθείας εἶναι τὴν ΜΝ τῇ ΝΞ· ἐπ᾽ εὐθείας ἄρα ἐστὶ καὶ ἡ ΡΝ τῇ ΝΟ. καὶ συμπεπληρώσθω τὸ ΣΠ παραλληλόγραμμον· τετράγωνον ἄρα ἐστὶ τὸ ΣΠ. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΗ, ΗΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΖ, ἔστιν ἄρα ὡς ἡ ΑΗ πρὸς ΕΖ, οὕτως ἡ ΖΕ πρὸς ΕΗ· καὶ ὡς ἄρα τὸ ΑΘ πρὸς ΕΛ, τὸ ΕΛ πρὸς ΚΗ· τῶν ΑΘ, ΗΚ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΕΛ. ἀλλὰ τὸ μὲν ΑΘ ἴσον ἐστὶ τῷ ΣΝ, τὸ δὲ ΗΚ ἴσον τῷ ΝΠ· τῶν ΣΝ, ΝΠ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΕΛ. ἔστι δὲ τῶν αὐτῶν τῶν ΣΝ, ΝΠ μέσον ἀνάλογον καὶ τὸ ΜΡ· ἴσον ἄρα ἐστὶ τὸ ΕΛ τῷ ΜΡ· ὥστε καὶ τῷ ΟΞ ἴσον

ἐστίν. ἔστι δὲ καὶ τὰ ΑΘ, ΗΚ τοῖς ΣΝ, ΝΠ ἴσα· ὅλον ἄρα τὸ ΑΓ ἴσον ἐστὶν ὅλῳ τῷ ΣΠ, τουτέστι τῷ ἀπὸ τῆς ΜΞ τετραγώνῳ· τὸ ΑΓ ἄρα δύναται ἡ ΜΞ.

Λέγω, ὅτι ἡ ΜΞ ἐκ δύο ὀνομάτων ἐστίν.

Ἐπεὶ γὰρ σύμμετρός ἐστιν ἡ ΑΗ τῇ ΗΕ, σύμμετρός ἐστι καὶ ἡ ΑΕ ἑκατέρᾳ τῶν ΑΗ, ΗΕ. ὑπόκειται δὲ καὶ ἡ ΑΕ τῇ ΑΒ σύμμετρος· καὶ αἱ ΑΗ, ΗΕ ἄρα τῇ ΑΒ σύμμετροί εἰσιν. καί ἐστι ῥητὴ ἡ ΑΒ· ῥητὴ ἄρα ἐστὶ καὶ ἑκατέρα τῶν ΑΗ, ΗΕ· ῥητὸν ἄρα ἐστὶν ἑκάτερον τῶν ΑΘ, ΗΚ, καί ἐστι σύμμετρον τὸ ΑΘ τῷ ΗΚ. ἀλλὰ τὸ μὲν ΑΘ τῷ ΣΝ ἴσον ἐστίν, τὸ δὲ ΗΚ τῷ ΝΠ· καὶ τὰ ΣΝ, ΝΠ ἄρα, τουτέστι τὰ ἀπὸ τῶν ΜΝ, ΝΞ, ῥητά ἐστι καὶ σύμμετρα. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΕ τῇ ΕΔ μήκει, ἀλλ’ ἡ μὲν ΑΕ τῇ ΑΗ ἐστι σύμμετρος, ἡ δὲ ΔΕ τῇ ΕΖ σύμμετρος, ἀσύμμετρος ἄρα καὶ ἡ ΑΗ τῇ ΕΖ· ὥστε καὶ τὸ ΑΘ τῷ ΕΛ ἀσύμμετρόν ἐστιν. ἀλλὰ τὸ μὲν ΑΘ τῷ ΣΝ ἐστιν ἴσον, τὸ δὲ ΕΛ τῷ ΜΡ· καὶ τὸ ΣΝ ἄρα τῷ ΜΡ ἀσύμμετρόν ἐστιν. ἀλλ’ ὡς τὸ ΣΝ πρὸς ΜΡ, ἡ ΟΝ πρὸς τὴν ΝΡ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΟΝ τῇ ΝΡ. ἴση δὲ ἡ μὲν ΟΝ τῇ ΜΝ, ἡ δὲ ΝΡ τῇ ΝΞ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΜΝ τῇ ΝΞ. καί ἐστι τὸ ἀπὸ τῆς ΜΝ σύμμετρον τῷ ἀπὸ τῆς ΝΞ, καὶ ῥητὸν ἑκάτερον· αἱ ΜΝ, ΝΞ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι.

Ἡ ΜΞ ἄρα ἐκ δύο ὀνομάτων ἐστὶ καὶ δύναται τὸ ΑΓ· ὅπερ ἔδει δεῖξαι.

νε΄. Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων δευτέρας, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο μέσων πρώτη.

Περιεχέσθω γὰρ χωρίον τὸ ΑΒΓΔ ὑπὸ ῥητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων δευτέρας τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἐκ δύο μέσων πρώτη ἐστίν.

Ἐπεὶ γὰρ ἐκ δύο ὀνομάτων δευτέρα ἐστὶν ἡ ΑΔ, διῃρήσθω εἰς τὰ ὀνόματα κατὰ τὸ Ε, ὥστε τὸ μεῖζον ὄνομα εἶναι τὸ ΑΕ· αἱ ΑΕ, ΕΔ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΕ τῆς ΕΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ τὸ ἔλαττον ὄνομα ἡ ΕΔ σύμμετρόν ἐστι τῇ ΑΒ μήκει.

τετμήσθω ἡ ΕΔ δίχα κατὰ τὸ Ζ, καὶ τῷ ἀπὸ τῆς ΕΖ ἴσον παρὰ τὴν ΑΕ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ τὸ ὑπὸ τῶν ΑΗΕ· σύμμετρος ἄρα ἡ ΑΗ τῇ ΗΕ μήκει.

καὶ διὰ τῶν Η, Ε, Ζ παράλληλοι ἤχθωσαν ταῖς ΑΒ, ΓΔ αἱ ΗΘ, ΕΚ, ΖΛ, καὶ τῷ μὲν ΑΘ παραλληλογράμμῳ ἴσον τετράγωνον συνεστάτω τὸ ΣΝ, τῷ δὲ ΗΚ ἴσον τετράγωνον τὸ ΝΠ, καὶ κείσθω ὥστε ἐπ᾽ εὐθείας εἶναι τὴν ΜΝ τῇ ΝΞ· ἐπ᾽ εὐθείας ἄρα [ἐστὶ] καὶ ἡ ΡΝ τῇ ΝΟ. καὶ συμπεπληρώσθω τὸ ΣΠ τετράγωνον· φανερὸν δὴ ἐκ τοῦ προδεδειγμένου, ὅτι τὸ ΜΡ μέσον ἀνάλογόν ἐστι τῶν ΣΝ, ΝΠ, καὶ ἴσον τῷ ΕΛ, καὶ ὅτι τὸ ΑΓ χωρίον δύναται ἡ ΜΞ. δεικτέον δή, ὅτι ἡ ΜΞ ἐκ δύο μέσων ἐστὶ πρώτη. ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΕ τῇ ΕΔ μήκει, σύμμετρος δὲ ἡ ΕΔ τῇ ΑΒ, ἀσύμμετρος ἄρα ἡ ΑΕ τῇ ΑΒ. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΑΗ τῇ ΕΗ, σύμμετρός ἐστι καὶ ἡ ΑΕ ἑκατέρᾳ τῶν ΑΗ, ΗΕ. ἀλλὰ ἡ ΑΕ ἀσύμμετρος τῇ ΑΒ μήκει· καὶ αἱ ΑΗ, ΗΕ ἄρα ἀσύμμετροί εἰσι τῇ ΑΒ. αἱ ΒΑ, ΑΗ, ΗΕ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ὥστε μέσον ἐστὶν ἑκάτερον τῶν ΑΘ, ΗΚ. ὥστε καὶ ἑκάτερον τῶν ΣΝ, ΝΠ μέσον ἐστίν. καὶ αἱ ΜΝ, ΝΞ ἄρα μέσαι εἰσίν. καὶ ἐπεὶ σύμμετρος ἡ ΑΗ τῇ ΗΕ μήκει, σύμμετρόν ἐστι καὶ τὸ ΑΘ τῷ ΗΚ, τουτέστι τὸ ΣΝ τῷ ΝΠ, τουτέστι τὸ ἀπὸ τῆς ΜΝ τῷ ἀπὸ τῆς ΝΞ [ὥστε δυνάμει εἰσὶ σύμμετροι αἱ ΜΝ, ΝΞ]. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΕ τῇ ΕΔ μήκει, ἀλλ᾽ ἡ μὲν ΑΕ σύμμετρός ἐστι τῇ ΑΗ, ἡ δὲ ΕΔ τῇ ΕΖ σύμμετρος, ἀσύμμετρος ἄρα ἡ ΑΗ τῇ ΕΖ· ὥστε καὶ τὸ ΑΘ τῷ ΕΛ ἀσύμμετρόν ἐστιν, τουτέστι τὸ ΣΝ τῷ ΜΡ, τουτέστιν ἡ ΟΝ τῇ ΝΡ, τουτέστιν ἡ ΜΝ τῇ ΝΞ ἀσύμμετρός ἐστι μήκει. ἐδείχθησαν δὲ αἱ ΜΝ, ΝΞ καὶ μέσαι οὖσαι καὶ δυνάμει σύμμετροι· αἱ ΜΝ, ΝΞ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. λέγω δή, ὅτι καὶ ῥητὸν περιέχουσιν. ἐπεὶ γὰρ ἡ ΔΕ ὑπόκειται ἑκατέρᾳ τῶν ΑΒ, ΕΖ σύμμετρος, σύμμετρος ἄρα καὶ ἡ ΕΖ τῇ ΕΚ. καὶ ῥητὴ ἑκατέρα αὐτῶν· ῥητὸν ἄρα τὸ ΕΛ, τουτέστι τὸ ΜΡ· τὸ δὲ ΜΡ ἐστι τὸ ὑπὸ τῶν ΜΝΞ. ἐὰν δὲ δύο μέσαι δυνάμει

μόνον σύμμετροι συντεθῶσι ῥητὸν περιέχουσαι, ἡ ὅλη ἄλογός ἐστιν, καλεῖται δὲ ἐκ δύο μέσων πρώτη.

Ἡ ἄρα ΜΞ ἐκ δύο μέσων ἐστὶ πρώτη· ὅπερ ἔδει δεῖξαι.

νς΄. Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων τρίτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο μέσων δευτέρα.

Χωρίον γὰρ τὸ ΑΒΓΔ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων τρίτης τῆς ΑΔ διῃρημένης εἰς τὰ ὀνόματα κατὰ τὸ Ε, ὧν μεῖζόν ἐστι τὸ ΑΕ· λέγω, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο μέσων δευτέρα.

Κατεσκευάσθω γὰρ τὰ αὐτὰ τοῖς πρότερον. καὶ ἐπεὶ ἐκ δύο ὀνομάτων ἐστὶ τρίτη ἡ ΑΔ, αἱ ΑΕ, ΕΔ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΕ τῆς ΕΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ οὐδετέρα τῶν ΑΕ, ΕΔ σύμμετρός [ἐστι] τῇ ΑΒ μήκει. ὁμοίως δὴ τοῖς προδεδειγμένοις δείξομεν, ὅτι ἡ ΜΞ ἐστιν ἡ τὸ ΑΓ χωρίον δυναμένη, καὶ αἱ ΜΝ, ΝΞ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι· ὥστε ἡ ΜΞ ἐκ δύο μέσων ἐστίν.

Δεικτέον δή, ὅτι καὶ δευτέρα.

[Καὶ] ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΔΕ τῇ ΑΒ μήκει, τουτέστι τῇ ΕΚ, σύμμετρος δὲ ἡ ΔΕ τῇ ΕΖ, ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΖ τῇ ΕΚ μήκει. καί εἰσι ῥηταί· αἱ ΖΕ, ΕΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. μέσον ἄρα [ἐστὶ] τὸ ΕΛ, τουτέστι τὸ ΜΡ· καὶ περιέχεται ὑπὸ τῶν ΜΝΞ· μέσον ἄρα ἐστὶ τὸ ὑπὸ τῶν ΜΝΞ.

Ἡ ΜΞ ἄρα ἐκ δύο μέσων ἐστὶ δευτέρα· ὅπερ ἔδει δεῖξαι.

νζ΄. Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων τετάρτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη μείζων.

Χωρίον γὰρ τὸ ΑΓ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων τετάρτης τῆς ΑΔ διῃρημένης εἰς τὰ ὀνόματα κατὰ τὸ Ε, ὧν μεῖζον

ἔστω τὸ ΑΕ· λέγω, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη μείζων.

Ἐπεὶ γὰρ ἡ ΑΔ ἐκ δύο ὀνομάτων ἐστὶ τετάρτη, αἱ ΑΕ, ΕΔ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΕ τῆς ΕΔ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ ἡ ΑΕ τῇ ΑΒ σύμμετρός [ἐστι] μήκει. τετμήσθω ἡ ΔΕ δίχα κατὰ τὸ Ζ, καὶ τῷ ἀπὸ τῆς ΕΖ ἴσον παρὰ τὴν ΑΕ παραβεβλήσθω παραλληλόγραμμον τὸ ὑπὸ ΑΗ, ΗΕ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΗ τῇ ΗΕ μήκει. ἤχθωσαν παράλληλοι τῇ ΑΒ αἱ ΗΘ, ΕΚ, ΖΛ, καὶ τὰ λοιπὰ τὰ αὐτὰ τοῖς πρὸ τούτου γεγονέτω· φανερὸν δή, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἐστὶν ἡ ΜΞ. δεικτέον δή, ὅτι ἡ ΜΞ ἄλογός ἐστιν ἡ καλουμένη μείζων. ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΗ τῇ ΕΗ μήκει, ἀσύμμετρόν ἐστι καὶ τὸ ΑΘ τῷ ΗΚ, τουτέστι τὸ ΣΝ τῷ ΝΠ· αἱ ΜΝ, ΝΞ ἄρα δυνάμει εἰσὶν ἀσύμμετροι. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΑΕ τῇ ΑΒ μήκει, ῥητόν ἐστι τὸ ΑΚ· καί ἐστιν ἴσον τοῖς ἀπὸ τῶν ΜΝ, ΝΞ· ῥητὸν ἄρα [ἐστὶ] καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ. καὶ ἐπεὶ ἀσύμμετρός [ἐστιν] ἡ ΔΕ τῇ ΑΒ μήκει, τουτέστι τῇ ΕΚ, ἀλλὰ ἡ ΔΕ σύμμετρός ἐστι τῇ ΕΖ, ἀσύμμετρος ἄρα ἡ ΕΖ τῇ ΕΚ μήκει. αἱ ΕΚ, ΕΖ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· μέσον ἄρα τὸ ΛΕ, τουτέστι τὸ ΜΡ. καὶ περιέχεται ὑπὸ τῶν ΜΝ, ΝΞ· μέσον ἄρα ἐστὶ τὸ ὑπὸ τῶν ΜΝ, ΝΞ. καὶ ῥητὸν τὸ [συγκείμενον] ἐκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ, καί εἰσιν ἀσύμμετροι αἱ ΜΝ, ΝΞ δυνάμει. ἐὰν δὲ δύο εὐθεῖαι δυνάμει ἀσύμμετροι συντεθῶσι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ’ αὐτῶν τετραγώνων ῥητόν, τὸ δ’ ὑπ’ αὐτῶν μέσον, ἡ ὅλη ἄλογός ἐστιν, καλεῖται δὲ μείζων.

Ἡ ΜΞ ἄρα ἄλογός ἐστιν ἡ καλουμένη μείζων, καὶ δύναται τὸ ΑΓ χωρίον· ὅπερ ἔδει δεῖξαι.

νη΄. Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων πέμπτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ῥητὸν καὶ μέσον δυναμένη.

Χωρίον γὰρ τὸ ΑΓ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων πέμπτης τῆς ΑΔ διῃρημένης εἰς τὰ ὀνόματα κατὰ τὸ Ε, ὥστε τὸ μεῖζον ὄνομα εἶναι τὸ ΑΕ· λέγω [δή], ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ῥητὸν καὶ μέσον δυναμένη.

Κατεσκευάσθω γὰρ τὰ αὐτὰ τοῖς πρότερον δεδειγμένοις· φανερὸν δή, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἐστὶν ἡ ΜΞ. δεικτέον δή, ὅτι ἡ ΜΞ ἐστιν ἡ ῥητὸν καὶ μέσον δυναμένη. ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ ΑΗ τῇ ΗΕ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ΑΘ τῷ ΘΕ, τουτέστι τὸ ἀπὸ τῆς ΜΝ τῷ ἀπὸ τῆς ΝΞ· αἱ ΜΝ, ΝΞ ἄρα δυνάμει εἰσὶν ἀσύμμετροι. καὶ ἐπεὶ ἡ ΑΔ ἐκ δύο ὀνομάτων ἐστὶ πέμπτη, καί [ἐστιν] ἔλασσον αὐτῆς τμῆμα τὸ ΕΔ, σύμμετρος ἄρα ἡ ΕΔ τῇ ΑΒ μήκει. ἀλλὰ ἡ ΑΕ τῇ ΕΔ ἐστιν ἀσύμμετρος· καὶ ἡ ΑΒ ἄρα τῇ ΑΕ ἐστιν ἀσύμμετρος μήκει. [αἱ ΒΑ, ΑΕ ῥηταί εἰσι δυνάμει μόνον σύμμετροι.] μέσον ἄρα ἐστὶ τὸ ΑΚ, τουτέστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΔΕ τῇ ΑΒ μήκει, τουτέστι τῇ ΕΚ, ἀλλὰ ἡ ΔΕ τῇ ΕΖ σύμμετρός ἐστιν, καὶ ἡ ΕΖ ἄρα τῇ ΕΚ σύμμετρός ἐστιν. καὶ ῥητὴ ἡ ΕΚ· ῥητὸν ἄρα καὶ τὸ ΕΛ, τουτέστι τὸ ΜΡ, τουτέστι τὸ ὑπὸ ΜΝΞ· αἱ ΜΝ, ΝΞ ἄρα δυνάμει ἀσύμμετροί εἰσι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ’ αὐτῶν τετραγώνων μέσον, τὸ δ’ ὑπ’ αὐτῶν ῥητόν.

Ἡ ΜΞ ἄρα ῥητὸν καὶ μέσον δυναμένη ἐστὶ καὶ δύναται τὸ ΑΓ χωρίον· ὅπερ ἔδει δεῖξαι.

νθ΄. Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων ἕκτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη δύο μέσα δυναμένη.

Χωρίον γὰρ τὸ ΑΒΓΔ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων ἕκτης τῆς ΑΔ διῃρημένης εἰς τὰ ὀνόματα κατὰ τὸ Ε, ὥστε τὸ μεῖζον ὄνομα εἶναι τὸ ΑΕ· λέγω, ὅτι ἡ τὸ ΑΓ δυναμένη ἡ δύο μέσα δυναμένη ἐστίν.

Κατεσκευάσθω [γὰρ] τὰ αὐτὰ τοῖς προδεδειγμένοις. φανερὸν δή, ὅτι [ἡ] τὸ ΑΓ δυναμένη ἐστὶν ἡ ΜΞ, καὶ ὅτι ἀσύμμετρός ἐστι ἡ ΜΝ τῇ ΝΞ δυνάμει. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΕΑ τῇ ΑΒ μήκει, αἱ ΕΑ, ΑΒ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· μέσον ἄρα ἐστὶ τὸ ΑΚ, τουτέστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ. πάλιν, ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΕΔ τῇ ΑΒ μήκει, ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΖΕ τῇ ΕΚ· αἱ ΖΕ, ΕΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· μέσον ἄρα ἐστὶ τὸ ΕΛ, τουτέστι τὸ ΜΡ, τουτέστι τὸ ὑπὸ τῶν ΜΝΞ. καὶ ἐπεὶ ἀσύμμετρος ἡ ΑΕ τῇ ΕΖ, καὶ τὸ ΑΚ τῷ ΕΛ ἀσύμμετρόν ἐστιν. ἀλλὰ τὸ μὲν ΑΚ ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ, τὸ δὲ ΕΛ ἐστι τὸ ὑπὸ τῶν ΜΝΞ· ἀσύμμετρον ἄρα ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝΞ τῷ ὑπὸ τῶν ΜΝΞ. καί ἐστι μέσον ἑκάτερον αὐτῶν, καὶ αἱ ΜΝ, ΝΞ δυνάμει εἰσὶν ἀσύμμετροι.

Ἡ ΜΞ ἄρα δύο μέσα δυναμένη ἐστὶ καὶ δύναται τὸ ΑΓ· ὅπερ ἔδει δεῖξαι.

Λῆμμα

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ εἰς ἄνισα, τὰ ἀπὸ τῶν ἀνίσων τετράγωνα μείζονά ἐστι τοῦ δὶς ὑπὸ τῶν ἀνίσων περιεχομένου ὀρθογωνίου.

Ἔστω εὐθεῖα ἡ ΑΒ καὶ τετμήσθω εἰς ἄνισα κατὰ τὸ Γ, καὶ ἔστω μείζων ἡ ΑΓ· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΓ, ΓΒ μείζονά ἐστι τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ.

Τετμήσθω γὰρ ἡ ΑΒ δίχα κατὰ τὸ Δ. ἐπεὶ οὖν εὐθεῖα γραμμὴ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Δ, εἰς δὲ ἄνισα κατὰ τὸ Γ, τὸ ἄρα ὑπὸ τῶν ΑΓ, ΓΒ μετὰ τοῦ ἀπὸ ΓΔ ἴσον ἐστὶ τῷ ἀπὸ ΑΔ· ὥστε τὸ ὑπὸ τῶν ΑΓ, ΓΒ ἔλαττόν ἐστι τοῦ ἀπὸ ΑΔ· τὸ ἄρα δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἔλαττον ἢ διπλάσιόν ἐστι τοῦ ἀπὸ ΑΔ. ἀλλὰ τὰ ἀπὸ τῶν ΑΓ, ΓΒ διπλάσιά [ἐστι] τῶν ἀπὸ τῶν ΑΔ, ΔΓ· τὰ ἄρα ἀπὸ τῶν ΑΓ, ΓΒ μείζονά ἐστι τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ· ὅπερ ἔδει δεῖξαι.]

ξ΄. Τὸ ἀπὸ τῆς ἐκ δύο ὀνομάτων παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων πρώτην.

Ἔστω ἐκ δύο ὀνομάτων ἡ ΑΒ διῃρημένη εἰς τὰ ὀνόματα κατὰ τὸ Γ, ὥστε τὸ μεῖζον ὄνομα εἶναι τὸ ΑΓ, καὶ ἐκκείσθω ῥητὴ ἡ ΔΕ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΔΕ παραβεβλήσθω τὸ ΔΕΖΗ πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶ πρώτη.

Παραβεβλήσθω γὰρ παρὰ τὴν ΔΕ τῷ μὲν ἀπὸ τῆς ΑΓ ἴσον τὸ ΔΘ, τῷ δὲ ἀπὸ τῆς ΒΓ ἴσον τὸ ΚΛ· λοιπὸν ἄρα τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον ἐστὶ τῷ ΜΖ. τετμήσθω ἡ ΜΗ δίχα κατὰ τὸ Ν, καὶ παράλληλος ἤχθω ἡ ΝΞ [ἑκατέρᾳ τῶν ΜΛ, ΗΖ]. ἑκάτερον ἄρα τῶν ΜΞ, ΝΖ ἴσον ἐστὶ τῷ ἅπαξ ὑπὸ τῶν ΑΓΒ. καὶ ἐπεὶ ἐκ δύο ὀνομάτων ἐστὶν ἡ ΑΒ διῃρημένη εἰς τὰ ὀνόματα κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· τὰ ἄρα ἀπὸ τῶν ΑΓ, ΓΒ ῥητά ἐστι καὶ σύμμετρα ἀλλήλοις· ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ [σύμμετρόν ἐστι τοῖς ἀπὸ τῶν ΑΓ, ΓΒ· ῥητὸν ἄρα ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ]. καί ἐστιν ἴσον τῷ ΔΛ· ῥητὸν ἄρα ἐστὶ τὸ ΔΛ. καὶ παρὰ ῥητὴν τὴν ΔΕ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ ΔΜ καὶ σύμμετρος τῇ ΔΕ μήκει. πάλιν, ἐπεὶ αἱ ΑΓ, ΓΒ ῥηταί εἰσι δυνάμει μόνον σύμμετροι, μέσον ἄρα ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, τουτέστι τὸ ΜΖ. καὶ παρὰ ῥητὴν τὴν ΜΛ παράκειται· ῥητὴ ἄρα καὶ ἡ ΜΗ ἐστι καὶ ἀσύμμετρος τῇ ΜΛ, τουτέστι τῇ ΔΕ, μήκει. ἔστι δὲ καὶ ἡ ΜΔ ῥητὴ καὶ τῇ ΔΕ μήκει σύμμετρος· ἀσύμμετρος ἄρα ἐστὶν ἡ ΔΜ τῇ ΜΗ μήκει. καί εἰσι ῥηταί· αἱ ΔΜ, ΜΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ.

Δεικτέον δή, ὅτι καὶ πρώτη.

Ἐπεὶ τῶν ἀπὸ τῶν ΑΓ, ΓΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΓΒ, καὶ τῶν ΔΘ, ΚΛ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΜΞ. ἔστιν ἄρα ὡς τὸ ΔΘ πρὸς τὸ ΜΞ, οὕτως τὸ ΜΞ πρὸς τὸ ΚΛ, τουτέστιν ὡς ἡ ΔΚ πρὸς τὴν ΜΝ, ἡ ΜΝ πρὸς τὴν ΜΚ· τὸ ἄρα ὑπὸ τῶν ΔΚ, ΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ. καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΓ τῷ ἀπὸ τῆς ΓΒ, σύμμετρόν ἐστι καὶ τὸ ΔΘ τῷ ΚΛ· ὥστε καὶ ἡ ΔΚ τῇ ΚΜ σύμμετρός ἐστιν. καὶ ἐπεὶ μείζονά ἐστι τὰ ἀπὸ τῶν ΑΓ, ΓΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, μεῖζον ἄρα καὶ τὸ ΔΛ τοῦ ΜΖ· ὥστε καὶ ἡ ΔΜ τῆς

ΜΗ μείζων ἐστίν. καί ἐστιν ἴσον τὸ ὑπὸ τῶν ΔΚ, ΚΜ τῷ ἀπὸ τῆς ΜΝ, τουτέστι τῷ τετάρτῳ τοῦ ἀπὸ τῆς ΜΗ, καὶ σύμμετρος ἡ ΔΚ τῇ ΚΜ. ἐὰν δὲ ὦσι δύο εὐθεῖαι ἄνισοι, τῷ δὲ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ καὶ εἰς σύμμετρα αὐτὴν διαιρῇ, ἡ μείζων τῆς ἐλάσσονος μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ· ἡ ΔΜ ἄρα τῆς ΜΗ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καί εἰσι ῥηταὶ αἱ ΔΜ, ΜΗ, καὶ ἡ ΔΜ μεῖζον ὄνομα σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΔΕ μήκει.

Ἡ ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ πρώτη· ὅπερ ἔδει δεῖξαι.

ξα΄. Τὸ ἀπὸ τῆς ἐκ δύο μέσων πρώτης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων δευτέραν.

Ἔστω ἐκ δύο μέσων πρώτη ἡ ΑΒ διῃρημένη εἰς τὰς μέσας κατὰ τὸ Γ, ὧν μείζων ἡ ΑΓ, καὶ ἐκκείσθω ῥητὴ ἡ ΔΕ, καὶ παραβεβλήσθω παρὰ τὴν ΔΕ τῷ ἀπὸ τῆς ΑΒ ἴσον παραλληλόγραμμον τὸ ΔΖ πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶ δευτέρα.

Κατεσκευάσθω γὰρ τὰ αὐτὰ τοῖς πρὸ τούτου. καὶ ἐπεὶ ἡ ΑΒ ἐκ δύο μέσων ἐστὶ πρώτη διῃρημένη κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι· ὥστε καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ μέσα ἐστίν. μέσον ἄρα ἐστὶ τὸ ΔΛ. καὶ παρὰ ῥητὴν τὴν ΔΕ παραβέβληται· ῥητὴ ἄρα ἐστίν ἡ ΜΔ καὶ ἀσύμμετρος τῇ ΔΕ μήκει. πάλιν, ἐπεὶ ῥητόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, ῥητόν ἐστι καὶ τὸ ΜΖ. καὶ παρὰ ῥητὴν τὴν ΜΛ παράκειται· ῥητὴ ἄρα [ἐστὶ] καὶ ἡ ΜΗ καὶ μήκει σύμμετρος τῇ ΜΛ, τουτέστι τῇ ΔΕ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΔΜ τῇ ΜΗ μήκει. καί εἰσι ῥηταί· αἱ ΔΜ, ΜΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ.

Δεικτέον δή, ὅτι καὶ δευτέρα.

Ἐπεὶ γὰρ τὰ ἀπὸ τῶν ΑΓ, ΓΒ μείζονά ἐστι τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, μεῖζον ἄρα καὶ τὸ ΔΛ τοῦ ΜΖ· ὥστε καὶ ἡ ΔΜ τῆς ΜΗ. καὶ ἐπεὶ σύμμετρόν ἐστι τὸ

ἀπὸ τῆς ΑΓ τῷ ἀπὸ τῆς ΓΒ, σύμμετρόν ἐστι καὶ τὸ ΔΘ τῷ ΚΛ· ὥστε καὶ ἡ ΔΚ τῇ ΚΜ σύμμετρός ἐστιν. καί ἐστι τὸ ὑπὸ τῶν ΔΚΜ ἴσον τῷ ἀπὸ τῆς ΜΝ· ἡ ΔΜ ἄρα τῆς ΜΗ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καί ἐστιν ἡ ΜΗ σύμμετρος τῇ ΔΕ μήκει.

Ἡ ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ δευτέρα.

ξβ΄. Τὸ ἀπὸ τῆς ἐκ δύο μέσων δευτέρας παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων τρίτην.

Ἔστω ἐκ δύο μέσων δευτέρα ἡ ΑΒ διῃρημένη εἰς τὰς μέσας κατὰ τὸ Γ, ὥστε τὸ μεῖζον τμῆμα εἶναι τὸ ΑΓ, ῥητὴ δέ τις ἔστω ἡ ΔΕ, καὶ παρὰ τὴν ΔΕ τῷ ἀπὸ τῆς ΑΒ ἴσον παραλληλόγραμμον παραβεβλήσθω τὸ ΔΖ πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶ τρίτη.

Κατεσκευάσθω τὰ αὐτὰ τοῖς προδεδειγμένοις. καὶ ἐπεὶ ἐκ δύο μέσων δευτέρα ἐστὶν ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι· ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ μέσον ἐστίν. καί ἐστιν ἴσον τῷ ΔΛ· μέσον ἄρα καὶ τὸ ΔΛ. καὶ παράκειται παρὰ ῥητὴν τὴν ΔΕ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΜΔ καὶ ἀσύμμετρος τῇ ΔΕ μήκει. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΜΗ ῥητή ἐστι καὶ ἀσύμμετρος τῇ ΜΛ, τουτέστι τῇ ΔΕ, μήκει· ῥητὴ ἄρα ἐστὶν ἑκατέρα τῶν ΔΜ, ΜΗ καὶ ἀσύμμετρος τῇ ΔΕ μήκει. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΓ τῇ ΓΒ μήκει, ὡς δὲ ἡ ΑΓ πρὸς τὴν ΓΒ, οὕτως τὸ ἀπὸ τῆς ΑΓ πρὸς τὸ ὑπὸ τῶν ΑΓΒ, ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΑΓ τῷ ὑπὸ τῶν ΑΓΒ. ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ τῷ δὶς ὑπὸ τῶν ΑΓΒ ἀσύμμετρόν ἐστιν, τουτέστι τὸ ΔΛ τῷ ΜΖ· ὥστε καὶ ἡ ΔΜ τῇ ΜΗ ἀσύμμετρός ἐστιν. καί εἰσι ῥηταί· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ.

Δεικτέον [δή], ὅτι καὶ τρίτη.

Ὁμοίως δὴ τοῖς προτέροις ἐπιλογιούμεθα, ὅτι μείζων ἐστὶν ἡ ΔΜ τῆς ΜΗ, καὶ σύμμετρος ἡ ΔΚ τῇ ΚΜ. καί ἐστι τὸ ὑπὸ τῶν ΔΚΜ ἴσον τῷ ἀπὸ τῆς ΜΝ·

ἡ ΔΜ ἄρα τῆς ΜΗ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καὶ οὐδετέρα τῶν ΔΜ, ΜΗ σύμμετρός ἐστι τῇ ΔΕ μήκει.

Ἡ ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ τρίτη· ὅπερ ἔδει δεῖξαι.

ξγ΄. Τὸ ἀπὸ τῆς μείζονος παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων τετάρτην.

Ἔστω μείζων ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, ὥστε μείζονα εἶναι τὴν ΑΓ τῆς ΓΒ, ῥητὴ δὲ ἡ ΔΕ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΔΕ παραβεβλήσθω τὸ ΔΖ παραλληλόγραμμον πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶ τετάρτη.

Κατεσκευάσθω τὰ αὐτὰ τοῖς προδεδειγμένοις. καὶ ἐπεὶ μείζων ἐστὶν ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δὲ ὑπ᾽ αὐτῶν μέσον. ἐπεὶ οὖν ῥητόν ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, ῥητὸν ἄρα ἐστὶ τὸ ΔΛ· ῥητὴ ἄρα καὶ ἡ ΔΜ καὶ σύμμετρος τῇ ΔΕ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, τουτέστι τὸ ΜΖ, καὶ παρὰ ῥητήν ἐστι τὴν ΜΛ, ῥητὴ ἄρα ἐστὶ καὶ ἡ ΜΗ καὶ ἀσύμμετρος τῇ ΔΕ μήκει· ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΔΜ τῇ ΜΗ μήκει. αἱ ΔΜ, ΜΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ.

Δεικτέον [δή], ὅτι καὶ τετάρτη.

Ὁμοίως δὴ δείξομεν τοῖς πρότερον, ὅτι μείζων ἐστὶν ἡ ΔΜ τῆς ΜΗ, καὶ ὅτι τὸ ὑπὸ ΔΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ. ἐπεὶ οὖν ἀσύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΓ τῷ ἀπὸ τῆς ΓΒ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ΔΘ τῷ ΚΛ· ὥστε ἀσύμμετρος καὶ ἡ ΔΚ τῇ ΚΜ ἐστιν. ἐὰν δὲ ὦσι δύο εὐθεῖαι ἄνισοι, τῷ δὲ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παραλληλόγραμμον παρὰ τὴν μείζονα παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ καὶ εἰς ἀσύμμετρα αὐτὴν διαιρῇ, ἡ μείζων τῆς ἐλάσσονος μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει· ἡ ΔΜ ἄρα τῆς ΜΗ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καί

εἰσιν αἱ ΔΜ, ΜΗ ῥηταὶ δυνάμει μόνον σύμμετροι, καὶ ἡ ΔΜ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΔΕ.

Ἡ ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ τετάρτη· ὅπερ ἔδει δεῖξαι.

ξδ΄. Τὸ ἀπὸ τῆς ῥητὸν καὶ μέσον δυναμένης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων πέμπτην.

Ἔστω ῥητὸν καὶ μέσον δυναμένη ἡ ΑΒ διῃρημένη εἰς τὰς εὐθείας κατὰ τὸ Γ, ὥστε μείζονα εἶναι τὴν ΑΓ, καὶ ἐκκείσθω ῥητὴ ἡ ΔΕ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΔΕ παραβεβλήσθω τὸ ΔΖ πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶ πέμπτη.

Κατεσκευάσθω τὰ αὐτὰ τοῖς πρὸ τούτου. ἐπεὶ οὖν ῥητὸν καὶ μέσον δυναμένη ἐστὶν ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ’ αὐτῶν τετραγώνων μέσον, τὸ δ’ ὑπ’ αὐτῶν ῥητόν. ἐπεὶ οὖν μέσον ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, μέσον ἄρα ἐστὶ τὸ ΔΛ· ὥστε ῥητή ἐστιν ἡ ΔΜ καὶ μήκει ἀσύμμετρος τῇ ΔΕ. πάλιν, ἐπεὶ ῥητόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΓΒ, τουτέστι τὸ ΜΖ, ῥητὴ ἄρα ἡ ΜΗ καὶ σύμμετρος τῇ ΔΕ. ἀσύμμετρος ἄρα ἡ ΔΜ τῇ ΜΗ· αἱ ΔΜ, ΜΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ. Λέγω δή, ὅτι καὶ πέμπτη.

Ὁμοίως γὰρ δειχθήσεται, ὅτι τὸ ὑπὸ τῶν ΔΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ, καὶ ἀσύμμετρος ἡ ΔΚ τῇ ΚΜ μήκει· ἡ ΔΜ ἄρα τῆς ΜΗ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καί εἰσιν αἱ ΔΜ, ΜΗ [ῥηταὶ] δυνάμει μόνον σύμμετροι, καὶ ἡ ἐλάσσων ἡ ΜΗ σύμμετρος τῇ ΔΕ μήκει.

Ἡ ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ πέμπτη· ὅπερ ἔδει δεῖξαι.

ξε΄. Τὸ ἀπὸ τῆς δύο μέσα δυναμένης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων ἕκτην.

Ἔστω δύο μέσα δυναμένη ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, ῥητὴ δὲ ἔστω ἡ ΔΕ. καὶ παρὰ τὴν ΔΕ τῷ ἀπὸ τῆς ΑΒ ἴσον παραβεβλήσθω τὸ ΔΖ πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶν ἕκτη.

Κατεσκευάσθω γὰρ τὰ αὐτὰ τοῖς πρότερον. καὶ ἐπεὶ ἡ ΑΒ δύο μέσα δυναμένη ἐστὶ διῃρημένη κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ᾽ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τὸ ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων συγκείμενον τῷ ὑπ᾽ αὐτῶν· ὥστε κατὰ τὰ προδεδειγμένα μέσον ἐστὶν ἑκάτερον τῶν ΔΛ, ΜΖ. καὶ παρὰ ῥητὴν τὴν ΔΕ παράκειται· ῥητὴ ἄρα ἐστὶν ἑκατέρα τῶν ΔΜ, ΜΗ καὶ ἀσύμμετρος τῇ ΔΕ μήκει. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, ἀσύμμετρον ἄρα ἐστὶ τὸ ΔΛ τῷ ΜΖ. ἀσύμμετρος ἄρα καὶ ἡ ΔΜ τῇ ΜΗ· αἱ ΔΜ, ΜΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ.

Λέγω δή, ὅτι καὶ ἕκτη.

Ὁμοίως δὴ πάλιν δείξομεν, ὅτι τὸ ὑπὸ τῶν ΔΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ, καὶ ὅτι ἡ ΔΚ τῇ ΚΜ μήκει ἐστὶν ἀσύμμετρος· καὶ διὰ τὰ αὐτὰ δὴ ἡ ΔΜ τῆς ΜΗ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει. καὶ οὐδετέρα τῶν ΔΜ, ΜΗ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΔΕ μήκει.

Ἡ ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶν ἕκτη· ὅπερ ἔδει δεῖξαι.

ξς΄. Ἡ τῇ ἐκ δύο ὀνομάτων μήκει σύμμετρος καὶ αὐτὴ ἐκ δύο ὀνομάτων ἐστὶ καὶ τῇ τάξει ἡ αὐτή.

Ἔστω ἐκ δύο ὀνομάτων ἡ ΑΒ, καὶ τῇ ΑΒ μήκει σύμμετρος ἔστω ἡ ΓΔ· λέγω, ὅτι ἡ ΓΔ ἐκ δύο ὀνομάτων ἐστὶ καὶ τῇ τάξει ἡ αὐτὴ τῇ ΑΒ.

Ἐπεὶ γὰρ ἐκ δύο ὀνομάτων ἐστὶν ἡ ΑΒ, διῃρήσθω εἰς τὰ ὀνόματα κατὰ τὸ Ε, καὶ ἔστω μεῖζον ὄνομα τὸ ΑΕ· αἱ ΑΕ, ΕΒ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. γεγονέτω ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΑΕ πρὸς τὴν ΓΖ· καὶ

λοιπὴ ἄρα ἡ ΕΒ πρὸς λοιπὴν τὴν ΖΔ ἐστιν, ὡς ἡ ΑΒ πρὸς τὴν ΓΔ. σύμμετρος δὲ ἡ ΑΒ τῇ ΓΔ μήκει. σύμμετρος ἄρα ἐστὶ καὶ ἡ μὲν ΑΕ τῇ ΓΖ, ἡ δὲ ΕΒ τῇ ΖΔ. καί εἰσι ῥηταὶ αἱ ΑΕ, ΕΒ· ῥηταὶ ἄρα εἰσὶ καὶ αἱ ΓΖ, ΖΔ. καὶ [ἐπεί] ἐστιν ὡς ἡ ΑΕ πρὸς ΓΖ, ἡ ΕΒ πρὸς ΖΔ. ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ ΑΕ πρὸς ΕΒ, ἡ ΓΖ πρὸς ΖΔ. αἱ δὲ ΑΕ, ΕΒ δυνάμει μόνον [εἰσὶ] σύμμετροι· καὶ αἱ ΓΖ, ΖΔ ἄρα δυνάμει μόνον εἰσὶ σύμμετροι. καί εἰσι ῥηταί· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΓΔ.

Λέγω δή, ὅτι τῇ τάξει ἐστὶν ἡ αὐτὴ τῇ ΑΒ.

Ἡ γὰρ ΑΕ τῆς ΕΒ μεῖζον δύναται ἤτοι τῷ ἀπὸ συμμέτρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυμμέτρου. εἰ μὲν οὖν ἡ ΑΕ τῆς ΕΒ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ ΓΖ τῆς ΖΔ μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΑΕ τῇ ἐκκειμένῃ ῥητῇ, καὶ ἡ ΓΖ σύμμετρος αὐτῇ ἔσται, καὶ διὰ τοῦτο ἑκατέρα τῶν ΑΒ, ΓΔ ἐκ δύο ὀνομάτων ἐστὶ πρώτη, τουτέστι τῇ τάξει ἡ αὐτή. εἰ δὲ ἡ ΕΒ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ, καὶ ἡ ΖΔ σύμμετρός ἐστιν αὐτῇ, καὶ διὰ τοῦτο πάλιν τῇ τάξει ἡ αὐτὴ ἔσται τῇ ΑΒ· ἑκατέρα γὰρ αὐτῶν ἔσται ἐκ δύο ὀνομάτων δευτέρα. εἰ δὲ οὐδετέρα τῶν ΑΕ, ΕΒ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ, οὐδετέρα τῶν ΓΖ, ΖΔ σύμμετρος αὐτῇ ἔσται, καί ἐστιν ἑκατέρα τρίτη. εἰ δὲ ἡ ΑΕ τῆς ΕΒ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ ἡ ΓΖ τῆς ΖΔ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ εἰ μὲν ἡ ΑΕ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ, καὶ ἡ ΓΖ σύμμετρός ἐστιν αὐτῇ, καί ἐστιν ἑκατέρα τετάρτη. εἰ δὲ ἡ ΕΒ, καὶ ἡ ΖΔ, καὶ ἔσται ἑκατέρα πέμπτη. εἰ δὲ οὐδετέρα τῶν ΑΕ, ΕΒ, καὶ τῶν ΓΖ, ΖΔ οὐδετέρα σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ, καὶ ἔσται ἑκατέρα ἕκτη.

Ὥστε ἡ τῇ ἐκ δύο ὀνομάτων μήκει σύμμετρος ἐκ δύο ὀνομάτων ἐστὶ καὶ τῇ τάξει ἡ αὐτή· ὅπερ ἔδει δεῖξαι.

ξζ′. Ἡ τῇ ἐκ δύο μέσων μήκει σύμμετρος καὶ αὐτὴ ἐκ δύο μέσων ἐστὶ καὶ τῇ τάξει ἡ αὐτή.

Ἔστω ἐκ δύο μέσων ἡ ΑΒ, καὶ τῇ ΑΒ σύμμετρος ἔστω μήκει ἡ ΓΔ· λέγω, ὅτι ἡ ΓΔ ἐκ δύο μέσων ἐστὶ καὶ τῇ τάξει ἡ αὐτὴ τῇ ΑΒ.

Ἐπεὶ γὰρ ἐκ δύο μέσων ἐστὶν ἡ ΑΒ, διῃρήσθω εἰς τὰς μέσας κατὰ τὸ Ε· αἱ ΑΕ, ΕΒ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. καὶ γεγονέτω ὡς ἡ ΑΒ πρὸς ΓΔ, ἡ ΑΕ πρὸς ΓΖ· καὶ λοιπὴ ἄρα ἡ ΕΒ πρὸς λοιπὴν τὴν ΖΔ ἐστιν, ὡς ἡ ΑΒ πρὸς ΓΔ. σύμμετρος δὲ ἡ ΑΒ τῇ ΓΔ μήκει· σύμμετρος ἄρα καὶ ἑκατέρα τῶν ΑΕ, ΕΒ ἑκατέρᾳ τῶν ΓΖ, ΖΔ. μέσαι δὲ αἱ ΑΕ, ΕΒ· μέσαι ἄρα καὶ αἱ ΓΖ, ΖΔ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΕ πρὸς ΕΒ, ἡ ΓΖ πρὸς ΖΔ, αἱ δὲ ΑΕ, ΕΒ δυνάμει μόνον σύμμετροί εἰσιν, καὶ αἱ ΓΖ, ΖΔ [ἄρα] δυνάμει μόνον σύμμετροί εἰσιν. ἐδείχθησαν δὲ καὶ μέσαι· ἡ ΓΔ ἄρα ἐκ δύο μέσων ἐστίν.

Λέγω δή, ὅτι καὶ τῇ τάξει ἡ αὐτή ἐστι τῇ ΑΒ.

Ἐπεὶ γάρ ἐστιν ὡς ἡ ΑΕ πρὸς ΕΒ, ἡ ΓΖ πρὸς ΖΔ, καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ὑπὸ τῶν ΑΕΒ, οὕτως τὸ ἀπὸ τῆς ΓΖ πρὸς τὸ ὑπὸ τῶν ΓΖΔ· ἐναλλὰξ ὡς τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ἀπὸ τῆς ΓΖ, οὕτως τὸ ὑπὸ τῶν ΑΕΒ πρὸς τὸ ὑπὸ τῶν ΓΖΔ. σύμμετρον δὲ τὸ ἀπὸ τῆς ΑΕ τῷ ἀπὸ τῆς ΓΖ· σύμμετρον ἄρα καὶ τὸ ὑπὸ τῶν ΑΕΒ τῷ ὑπὸ τῶν ΓΖΔ. εἴτε οὖν ῥητόν ἐστι τὸ ὑπὸ τῶν ΑΕΒ, καὶ τὸ ὑπὸ τῶν ΓΖΔ ῥητόν ἐστιν [καὶ διὰ τοῦτό ἐστιν ἐκ δύο μέσων πρώτη]. εἴτε μέσον, μέσον, καί ἐστιν ἑκατέρα δευτέρα .

Καὶ διὰ τοῦτο ἔσται ἡ ΓΔ τῇ ΑΒ τῇ τάξει ἡ αὐτή· ὅπερ ἔδει δεῖξαι.

ξη΄. Ἡ τῇ μείζονι σύμμετρος καὶ αὐτὴ μείζων ἐστίν.

Ἔστω μείζων ἡ ΑΒ, καὶ τῇ ΑΒ σύμμετρος ἔστω ἡ ΓΔ· λέγω, ὅτι ἡ ΓΔ μείζων ἐστίν.

Διῃρήσθω ἡ ΑΒ κατὰ τὸ Ε· αἱ ΑΕ, ΕΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ’ αὐτῶν τετραγώνων ῥητόν, τὸ δ’ ὑπ’ αὐτῶν μέσον· καὶ γεγονέτω τὰ αὐτὰ τοῖς πρότερον. καὶ ἐπεί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἥ τε ΑΕ πρὸς τὴν ΓΖ καὶ ἡ ΕΒ πρὸς τὴν ΖΔ, καὶ ὡς ἄρα ἡ ΑΕ πρὸς τὴν ΓΖ, οὕτως ἡ ΕΒ πρὸς τὴν ΖΔ. σύμμετρος δὲ ἡ ΑΒ τῇ ΓΔ.

σύμμετρος ἄρα καὶ ἑκατέρα τῶν ΑΕ, ΕΒ ἑκατέρᾳ τῶν ΓΖ, ΖΔ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΕ πρὸς τὴν ΓΖ, οὕτως ἡ ΕΒ πρὸς τὴν ΖΔ, καὶ ἐναλλὰξ ὡς ἡ ΑΕ πρὸς ΕΒ, οὕτως ἡ ΓΖ πρὸς ΖΔ, καὶ συνθέντι ἄρα ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΓΔ πρὸς τὴν ΔΖ· καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΕ, οὕτως τὸ ἀπὸ τῆς ΓΔ πρὸς τὸ ἀπὸ τῆς ΔΖ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ὡς τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΑΕ, οὕτως τὸ ἀπὸ τῆς ΓΔ πρὸς τὸ ἀπὸ τῆς ΓΖ. καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΑΒ πρὸς τὰ ἀπὸ τῶν ΑΕ, ΕΒ, οὕτως τὸ ἀπὸ τῆς ΓΔ πρὸς τὰ ἀπὸ τῶν ΓΖ, ΖΔ· καὶ ἐναλλὰξ ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΓΔ, οὕτως τὰ ἀπὸ τῶν ΑΕ, ΕΒ πρὸς τὰ ἀπὸ τῶν ΓΖ, ΖΔ. σύμμετρον δὲ τὸ ἀπὸ τῆς ΑΒ τῷ ἀπὸ τῆς ΓΔ· σύμμετρα ἄρα καὶ τὰ ἀπὸ τῶν ΑΕ, ΕΒ τοῖς ἀπὸ τῶν ΓΖ, ΖΔ. καί ἐστι τὰ ἀπὸ τῶν ΑΕ, ΕΒ ἄμα ῥητόν, καὶ τὰ ἀπὸ τῶν ΓΖ, ΖΔ ἄμα ῥητόν ἐστιν. ὁμοίως δὲ καὶ τὸ δὶς ὑπὸ τῶν ΑΕ, ΕΒ σύμμετρόν ἐστι τῷ δὶς ὑπὸ τῶν ΓΖ, ΖΔ. καί ἐστι μέσον τὸ δὶς ὑπὸ τῶν ΑΕ, ΕΒ· μέσον ἄρα καὶ τὸ δὶς ὑπὸ τῶν ΓΖ, ΖΔ. αἱ ΓΖ, ΖΔ ἄρα δυνάμει ἀσύμμετροί εἰσι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ’ αὐτῶν τετραγώνων ἄμα ῥητόν, τὸ δὲ δὶς ὑπ’ αὐτῶν μέσον· ὅλη ἄρα ἡ ΓΔ ἄλογός ἐστιν ἡ καλουμένη μείζων.

Ἡ ἄρα τῇ μείζονι σύμμετρος μείζων ἐστίν· ὅπερ ἔδει δεῖξαι.

ξθ΄. Ἡ τῇ ῥητὸν καὶ μέσον δυναμένῃ σύμμετρος [καὶ αὐτὴ] ῥητὸν καὶ μέσον δυναμένη ἐστίν.

Ἔστω ῥητὸν καὶ μέσον δυναμένη ἡ ΑΒ, καὶ τῇ ΑΒ σύμμετρος ἔστω ἡ ΓΔ· δεικτέον, ὅτι καὶ ἡ ΓΔ ῥητὸν καὶ μέσον δυναμένη ἐστίν.

Διῃρήσθω ἡ ΑΒ εἰς τὰς εὐθείας κατὰ τὸ Ε· αἱ ΑΕ, ΕΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ’ αὐτῶν τετραγώνων μέσον, τὸ δ’ ὑπ’ αὐτῶν ῥητόν· καὶ τὰ αὐτὰ κατεσκευάσθω τοῖς πρότερον. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ ΓΖ, ΖΔ δυνάμει εἰσὶν ἀσύμμετροι, καὶ σύμμετρον τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ, τὸ δὲ ὑπὸ ΑΕ, ΕΒ τῷ ὑπὸ ΓΖ, ΖΔ· ὥστε καὶ τὸ [μὲν]

συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων ἐστὶ μέσον, τὸ δ᾽ ὑπὸ τῶν ΓΖ, ΖΔ ῥητόν.

Ῥητὸν ἄρα καὶ μέσον δυναμένη ἐστὶν ἡ ΓΔ· ὅπερ ἔδει δεῖξαι.

ο΄. Ἡ τῇ δύο μέσα δυναμένῃ σύμμετρος δύο μέσα δυναμένη ἐστίν.

Ἔστω δύο μέσα δυναμένη ἡ ΑΒ, καὶ τῇ ΑΒ σύμμετρος ἡ ΓΔ· δεικτέον, ὅτι καὶ ἡ ΓΔ δύο μέσα δυναμένη ἐστίν.

Ἐπεὶ γὰρ δύο μέσα δυναμένη ἐστὶν ἡ ΑΒ, διῃρήσθω εἰς τὰς εὐθείας κατὰ τὸ Ε· αἱ ΑΕ, ΕΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν [τετραγώνων] μέσον καὶ τὸ ὑπ᾽ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τετραγώνων τῷ ὑπὸ τῶν ΑΕ, ΕΒ· καὶ κατεσκευάσθω τὰ αὐτὰ τοῖς πρότερον. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ ΓΖ, ΖΔ δυνάμει εἰσὶν ἀσύμμετροι καὶ σύμμετρον τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ, τὸ δὲ ὑπὸ τῶν ΑΕ, ΕΒ τῷ ὑπὸ τῶν ΓΖ, ΖΔ· ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων μέσον ἐστὶ καὶ τὸ ὑπὸ τῶν ΓΖ, ΖΔ μέσον καὶ ἔτι ἀσύμμετρον τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων τῷ ὑπὸ τῶν ΓΖ, ΖΔ.

Ἡ ἄρα ΓΔ δύο μέσα δυναμένη ἐστίν· ὅπερ ἔδει δεῖξαι.

οα΄. Ῥητοῦ καὶ μέσου συντιθεμένου τέσσαρες ἄλογοι γίγνονται ἤτοι ἐκ δύο ὀνομάτων ἢ ἐκ δύο μέσων πρώτη ἢ μείζων ἢ ῥητὸν καὶ μέσον δυναμένη.

Ἔστω ῥητὸν μὲν τὸ ΑΒ, μέσον δὲ τὸ ΓΔ· λέγω, ὅτι ἡ τὸ ΑΔ χωρίον δυναμένη ἤτοι ἐκ δύο ὀνομάτων ἐστὶν ἢ ἐκ δύο μέσων πρώτη ἢ μείζων ἢ ῥητὸν καὶ μέσον δυναμένη.

* Βιβλίον ι΄ Πρότασις 71

Τὸ γὰρ ΑΒ τοῦ ΓΔ ἤτοι μεῖζόν ἐστιν ἢ ἔλασσον. ἔστω πρότερον μεῖζον· καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ παραβεβλήσθω παρὰ τὴν ΕΖ τῷ ΑΒ ἴσον τὸ ΕΗ

πλάτος ποιοῦν τὴν ΕΘ· τῷ δὲ ΔΓ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΘΙ πλάτος ποιοῦν τὴν ΘΚ. καὶ ἐπεὶ ῥητόν ἐστι τὸ ΑΒ καί ἐστιν ῥητόν ἐστι τὸ ΑΒ καί ἐστιν ἴσον τῷ ΕΗ, ῥητὸν ἄρα καὶ τὸ ΕΗ. καὶ παρὰ [ῥητὴν] τὴν ΕΖ παραβέβληται πλάτος ποιοῦν τὴν ΕΘ· ἡ ΕΘ ἄρα ῥητή ἐστι καὶ σύμμετρος τῇ ΕΖ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ ΓΔ καί ἐστιν ἴσον τῷ ΘΙ, μέσον ἄρα ἐστὶ καὶ τὸ ΘΙ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΘΚ· ῥητὴ ἄρα ἐστὶν ἡ ΘΚ καὶ ἀσύμμετρος τῇ ΕΖ μήκει. καὶ ἐπεὶ μέσον ἐστὶ τὸ ΓΔ, ῥητὸν δὲ τὸ ΑΒ, ἀσύμμετρον ἄρα ἐστὶ τὸ ΑΒ τῷ ΓΔ· ὥστε καὶ τὸ ΕΗ ἀσύμμετρόν ἐστι τῷ ΘΙ. ὡς δὲ τὸ ΕΗ πρὸς τὸ ΘΙ, οὕτως ἐστὶν ἡ ΕΘ πρὸς τὴν ΘΚ· ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΕΘ τῇ ΘΚ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΕΘ, ΘΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΕΚ διῃρημένη κατὰ τὸ Θ. καὶ ἐπεὶ μεῖζόν ἐστι τὸ ΑΒ τοῦ ΓΔ, ἴσον δὲ τὸ μὲν ΑΒ τῷ ΕΗ, τὸ δὲ ΓΔ τῷ ΘΙ, μεῖζον ἄρα καὶ τὸ ΕΗ τοῦ ΘΙ· καὶ ἡ ΕΘ ἄρα μείζων ἐστὶ τῆς ΘΚ. ἤτοι οὖν ἡ ΕΘ τῆς ΘΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει ἢ τῷ ἀπὸ ἀσυμμέτρου. δυνάσθω πρότερον τῷ ἀπὸ συμμέτρου ἑαυτῇ. καί ἐστιν ἡ μείζων ἡ ΘΕ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ τῇ ΕΖ· ἡ ἄρα ΕΚ ἐκ δύο ὀνομάτων ἐστὶ πρώτη. ῥητὴ δὲ ἡ ΕΖ· ἐὰν δὲ χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων πρώτης, ἡ τὸ χωρίον δυναμένη ἐκ δύο ὀνομάτων ἐστίν. ἡ ἄρα τὸ ΕΙ δυναμένη ἐκ δύο ὀνομάτων ἐστίν· ὥστε καὶ ἡ τὸ ΑΔ δυναμένη ἐκ δύο ὀνομάτων ἐστίν. ἀλλὰ δὴ δυνάσθω ἡ ΕΘ τῆς ΘΚ μεῖζον τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ· καί ἐστιν ἡ μείζων ἡ ΕΘ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ τῇ ΕΖ μήκει· ἡ ἄρα ΕΚ ἐκ δύο ὀνομάτων ἐστὶ τετάρτη. ῥητὴ δὲ ἡ ΕΖ· ἐὰν δὲ χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων τετάρτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη μείζων. ἡ ἄρα τὸ ΕΙ χωρίον δυναμένη μείζων ἐστίν· ὥστε καὶ ἡ τὸ ΑΔ δυναμένη μείζων ἐστίν.

Ἀλλὰ δὴ ἔστω ἔλασσον τὸ ΑΒ τοῦ ΓΔ· καὶ τὸ ΕΗ ἄρα ἔλασσόν ἐστι τοῦ ΘΙ· ὥστε καὶ ἡ ΕΘ ἐλάσσων ἐστὶ τῆς ΘΚ. ἤτοι δὲ ἡ ΘΚ τῆς ΕΘ μεῖζον

δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυμμέτρου. δυνάσθω πρότερον τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει· καί ἐστιν ἡ ἐλάσσων ἡ ΕΘ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ τῇ ΕΖ μήκει· ἡ ἄρα ΕΚ ἐκ δύο ὀνομάτων ἐστὶ δευτέρα. ῥητὴ δὲ ἡ ΕΖ· ἐὰν δὲ χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων δευτέρας, ἡ τὸ χωρίον δυναμένη ἐκ δύο μέσων ἐστὶ πρώτη. ἡ ἄρα τὸ ΕΙ χωρίον δυναμένη ἐκ δύο μέσων ἐστὶ πρώτη· ὥστε καὶ ἡ τὸ ΑΔ δυναμένη ἐκ δύο μέσων ἐστὶ πρώτη. ἀλλὰ δὴ ἡ ΘΚ τῆς ΘΕ μεῖζον δυνάσθω τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καί ἐστιν ἡ ἐλάσσων ἡ ΕΘ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ τῇ ΕΖ· ἡ ἄρα ΕΚ ἐκ δύο ὀνομάτων ἐστὶ πέμπτη. ῥητὴ δὲ ἡ ΕΖ· ἐὰν δὲ χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων πέμπτης, ἡ τὸ χωρίον δυναμένη ῥητὸν καὶ μέσον δυναμένη ἐστίν. ἡ ἄρα τὸ ΕΙ χωρίον δυναμένη ῥητὸν καὶ μέσον δυναμένη ἐστίν· ὥστε καὶ ἡ τὸ ΑΔ χωρίον δυναμένη ῥητὸν καὶ μέσον δυναμένη ἐστίν.

Ῥητοῦ ἄρα καὶ μέσου συντιθεμένου τέσσαρες ἄλογοι γίγνονται ἤτοι ἐκ δύο ὀνομάτων ἢ ἐκ δύο μέσων πρώτη ἢ μείζων ἢ ῥητὸν καὶ μέσον δυναμένη· ὅπερ ἔδει δεῖξαι.

οβ΄. Δύο μέσων ἀσυμμέτρων ἀλλήλοις συντιθεμένων αἱ λοιπαὶ δύο ἄλογοι γίγνονται ἤτοι ἐκ δύο μέσων δευτέρα ἢ [ἡ] δύο μέσα δυναμένη.

Συγκείσθω γὰρ δύο μέσα ἀσύμμετρα ἀλλήλοις τὰ ΑΒ, ΓΔ· λέγω, ὅτι ἡ τὸ ΑΔ χωρίον δυναμένη ἤτοι ἐκ δύο μέσων ἐστὶ δευτέρα ἢ δύο μέσα δυναμένη.

Τὸ γὰρ ΑΒ τοῦ ΓΔ ἤτοι μεῖζόν ἐστιν ἢ ἔλασσον. ἔστω, εἰ τύχοι, πρότερον μεῖζον τὸ ΑΒ τοῦ ΓΔ· καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ τῷ μὲν ΑΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΕΗ πλάτος ποιοῦν τὴν ΕΘ, τῷ δὲ ΓΔ ἴσον τὸ ΘΙ πλάτος ποιοῦν τὴν ΘΚ. καὶ ἐπεὶ μέσον ἐστὶν ἑκάτερον τῶν ΑΒ, ΓΔ, μέσον ἄρα καὶ ἑκάτερον τῶν ΕΗ, ΘΙ. καὶ παρὰ ῥητὴν τὴν ΖΕ παράκειται πλάτος ποιοῦν τὰς ΕΘ, ΘΚ· ἑκατέρα ἄρα τῶν ΕΘ, ΘΚ ῥητή ἐστι καὶ ἀσύμμετρος τῇ ΕΖ μήκει. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ ΑΒ τῷ ΓΔ, καί ἐστιν ἴσον τὸ μὲν ΑΒ

τῷ ΕΗ, τὸ δὲ ΓΔ τῷ ΘΙ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ΕΗ τῷ ΘΙ. ὡς δὲ τὸ ΕΗ πρὸς τὸ ΘΙ, οὕτως ἐστὶν ἡ ΕΘ πρὸς ΘΚ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΘ τῇ ΘΚ μήκει. αἱ ΕΘ, ΘΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΕΚ. ἤτοι δὲ ἡ ΕΘ τῆς ΘΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυμμέτρου. δυνάσθω πρότερον τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει· καὶ οὐδετέρα τῶν ΕΘ, ΘΚ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΕΖ μήκει· ἡ ΕΚ ἄρα ἐκ δύο ὀνομάτων ἐστὶ τρίτη. ῥητὴ δὲ ἡ ΕΖ· ἐὰν δὲ χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων τρίτης, ἡ τὸ χωρίον δυναμένη ἐκ δύο μέσων ἐστὶ δευτέρα· ἡ ἄρα τὸ ΕΙ, τουτέστι τὸ ΑΔ, δυναμένη ἐκ δύο μέσων ἐστὶ δευτέρα. ἀλλὰ δὴ ἡ ΕΘ τῆς ΘΚ μεῖζον δυνάσθω τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει· καὶ ἀσύμμετρός ἐστιν ἑκατέρα τῶν ΕΘ, ΘΚ τῇ ΕΖ μήκει· ἡ ἄρα ΕΚ ἐκ δύο ὀνομάτων ἐστὶν ἕκτη. ἐὰν δὲ χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων ἕκτης, ἡ τὸ χωρίον δυναμένη ἡ δύο μέσα δυναμένη ἐστίν· ὥστε καὶ ἡ τὸ ΑΔ χωρίον δυναμένη ἡ δύο μέσα δυναμένη ἐστίν.

[Ὁμοίως δὴ δείξομεν, ὅτι κἂν ἔλαττον ᾖ τὸ ΑΒ τοῦ ΓΔ, ἡ τὸ ΑΔ χωρίον δυναμένη ἢ ἐκ δύο μέσων δευτέρα ἐστὶν ἤτοι δύο μέσα δυναμένη].

Δύο ἄρα μέσων ἀσυμμέτρων ἀλλήλοις συντιθεμένων αἱ λοιπαὶ δύο ἄλογοι γίγνονται ἤτοι ἐκ δύο μέσων δευτέρα ἢ δύο μέσα δυναμένη.

Ἡ ἐκ δύο ὀνομάτων καὶ αἱ μετ’ αὐτὴν ἄλογοι οὔτε τῇ μέσῃ οὔτε ἀλλήλαις εἰσὶν αἱ αὐταί. τὸ μὲν γὰρ ἀπὸ μέσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ῥητὴν καὶ ἀσύμμετρον τῇ παρ’ ἣν παράκειται μήκει. τὸ δὲ ἀπὸ τῆς ἐκ δύο ὀνομάτων παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων πρώτην. τὸ δὲ ἀπὸ τῆς ἐκ δύο μέσων πρώτης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων δευτέραν. τὸ δὲ ἀπὸ τῆς ἐκ δύο μέσων δευτέρας παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων τρίτην. τὸ δὲ ἀπὸ τῆς μείζονος παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων τετάρτην. τὸ δὲ ἀπὸ τῆς ῥητὸν καὶ μέσον

δυναμένης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων πέμπτην. τὸ δὲ ἀπὸ τῆς δύο μέσα δυναμένης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων ἕκτην. τὰ δ’ εἰρημένα πλάτη διαφέρει τοῦ τε πρώτου καὶ ἀλλήλων, τοῦ μὲν πρώτου, ὅτι ῥητή ἐστιν, ἀλλήλων δέ, ὅτι τῇ τάξει οὐκ εἰσὶν αἱ αὐταί· ὥστε καὶ αὐταὶ αἱ ἄλογοι διαφέρουσιν ἀλλήλων.

ογ΄. Ἐὰν ἀπὸ ῥητῆς ῥητὴ ἀφαιρεθῇ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ ἀποτομή.

Ἀπὸ γὰρ ῥητῆς τῆς ΑΒ ῥητὴ ἀφῃρήσθω ἡ ΒΓ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ· λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει, καί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ὑπὸ τῶν ΑΒ, ΒΓ, ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τῷ ὑπὸ τῶν ΑΒ, ΒΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΒ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα, τῷ δὲ ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. καὶ ἐπειδήπερ τὰ ἀπὸ τῶν ΑΒ, ΒΓ ἴσα ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΑ, καὶ λοιπῷ ἄρα τῷ ἀπὸ τῆς ΑΓ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ. ῥητὰ δὲ τὰ ἀπὸ τῶν ΑΒ, ΒΓ· ἄλογος ἄρα ἐστὶν ἡ ΑΓ· καλείσθω δὲ ἀποτομή. ὅπερ ἔδει δεῖξαι.

οδ΄. Ἐὰν ἀπὸ μέσης μέση ἀφαιρεθῇ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ῥητὸν περιέχουσα, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ πρώτη.

Ἀπὸ γὰρ μέσης τῆς ΑΒ μέση ἀφῃρήσθω ἡ ΒΓ δυνάμει μόνον σύμμετρος οὖσα τῇ ΑΒ, μετὰ δὲ τῆς ΑΒ ῥητὸν ποιοῦσα τὸ ὑπὸ τῶν ΑΒ, ΒΓ· λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ πρώτη.

Ἐπεὶ γὰρ αἱ ΑΒ, ΒΓ μέσαι εἰσίν, μέσα ἐστὶ καὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ. ῥητὸν δὲ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· ἀσύμμετρα ἄρα τὰ ἀπὸ τῶν ΑΒ, ΒΓ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· καὶ λοιπῷ ἄρα τῷ ἀπὸ τῆς ΑΓ ἀσύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν

ΑΒ, ΒΓ, ἐπεὶ κἂν τὸ ὅλον ἑνὶ αὐτῶν ἀσύμμετρον ᾖ, καὶ τὰ ἐξ ἀρχῆς μεγέθη ἀσύμμετρα ἔσται. ῥητὸν δὲ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· ἄλογον ἄρα τὸ ἀπὸ τῆς ΑΓ· ἄλογος ἄρα ἐστὶν ἡ ΑΓ· καλείσθω δὲ μέσης ἀποτομὴ πρώτη.

οε΄. Ἐὰν ἀπὸ μέσης μέση ἀφαιρεθῇ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλη, μετὰ δὲ τῆς ὅλης μέσον περιέχουσα, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ δευτέρα.

Ἀπὸ γὰρ μέσης τῆς ΑΒ μέση ἀφῃρήσθω ἡ ΓΒ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ τῇ ΑΒ, μετὰ δὲ τῆς ὅλης τῆς ΑΒ μέσον περιέχουσα τὸ ὑπὸ τῶν ΑΒ, ΒΓ· λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ δευτέρα.

Ἐκκείσθω γὰρ ῥητὴ ἡ ΔΙ, καὶ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον παρὰ τὴν ΔΙ παραβεβλήσθω τὸ ΔΕ πλάτος ποιοῦν τὴν ΔΗ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ἴσον παρὰ τὴν ΔΙ παραβεβλήσθω τὸ ΔΘ πλάτος ποιοῦν τὴν ΔΖ· λοιπὸν ἄρα τὸ ΖΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ. καὶ ἐπεὶ μέσα καὶ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ, μέσον ἄρα καὶ τὸ ΔΕ. καὶ παρὰ ῥητὴν τὴν ΔΙ παράκειται πλάτος ποιοῦν τὴν ΔΗ· ῥητὴ ἄρα ἐστὶν ἡ ΔΗ καὶ ἀσύμμετρος τῇ ΔΙ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ, καὶ τὸ δὶς ἄρα ὑπὸ τῶν ΑΒ, ΒΓ μέσον ἐστίν. καί ἐστιν ἴσον τῷ ΔΘ· καὶ τὸ ΔΘ ἄρα μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΔΙ παραβέβληται πλάτος ποιοῦν τὴν ΔΖ· ῥητὴ ἄρα ἐστὶν ἡ ΔΖ καὶ ἀσύμμετρος τῇ ΔΙ μήκει. καὶ ἐπεὶ αἱ ΑΒ, ΒΓ δυνάμει μόνον σύμμετροί εἰσιν, ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΒ τῇ ΒΓ μήκει· ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΑΒ τετράγωνον τῷ ὑπὸ τῶν ΑΒ, ΒΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΒ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ, τῷ δὲ ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· ἀσύμμετρον ἄρα ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ τοῖς ἀπὸ τῶν ΑΒ, ΒΓ. ἴσον δὲ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ τὸ ΔΕ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ τὸ ΔΘ· ἀσύμμετρον ἄρα [ἐστὶ] τὸ ΔΕ τῷ ΔΘ. ὡς δὲ τὸ ΔΕ πρὸς τὸ ΔΘ, οὕτως ἡ ΗΔ πρὸς τὴν ΔΖ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΗΔ τῇ ΔΖ. καί εἰσιν ἀμφότεραι ῥηταί·

αἱ ἄρα ΗΔ, ΔΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΖΗ ἄρα ἀποτομή ἐστιν. ῥητὴ δὲ ἡ ΔΙ· τὸ δὲ ὑπὸ ῥητῆς καὶ ἀλόγου περιεχόμενον ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν. καὶ δύναται τὸ ΖΕ ἡ ΑΓ· ἡ ΑΓ ἄρα ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ δευτέρα. ὅπερ ἔδει δεῖξαι.

ος΄. Ἐὰν ἀπὸ εὐθείας εὐθεῖα ἀφαιρεθῇ δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τὰ μὲν ἀπ᾽ αὐτῶν ἅμα ῥητόν, τὸ δ᾽ ὑπ᾽ αὐτῶν μέσον, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ ἐλάσσων.

Ἀπὸ γὰρ εὐθείας τῆς ΑΒ εὐθεῖα ἀφῃρήσθω ἡ ΒΓ δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ ποιοῦσα τὰ προκείμενα. λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

Ἐπεὶ γὰρ τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τετραγώνων ῥητόν ἐστιν, τὸ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ μέσον, ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· καὶ ἀναστρέψαντι λοιπῷ τῷ ἀπὸ τῆς ΑΓ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ. ῥητὰ δὲ τὰ ἀπὸ τῶν ΑΒ, ΒΓ. ἄλογον ἄρα τὸ ἀπὸ τῆς ΑΓ· ἄλογος ἄρα ἡ ΑΓ· καλείσθω δὲ ἐλάσσων. ὅπερ ἔδει δεῖξαι.

οζ΄. Ἐὰν ἀπὸ εὐθείας εὐθεῖα ἀφαιρεθῇ δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον, τὸ δὲ δὶς ὑπ᾽ αὐτῶν ῥητόν, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα.

Ἀπὸ γὰρ εὐθείας τῆς ΑΒ εὐθεῖα ἀφῃρήσθω ἡ ΒΓ δυνάμει ἀσύμμετρος οὖσα τῇ ΑΒ ποιοῦσα τὰ προκείμενα· λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν ἡ προειρημένη.

Ἐπεὶ γὰρ τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τετραγώνων μέσον ἐστίν, τὸ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ῥητόν, ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· καὶ λοιπὸν ἄρα τὸ ἀπὸ τῆς ΑΓ ἀσύμμετρόν ἐστι

τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. καί ἐστι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ῥητόν· τὸ ἄρα ἀπὸ τῆς ΑΓ ἄλογόν ἐστιν· ἄλογος ἄρα ἐστὶν ἡ ΑΓ· καλείσθω δὲ ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα. ὅπερ ἔδει δεῖξαι.

οη΄. Ἐὰν ἀπὸ εὐθείας εὐθεῖα ἀφαιρεθῇ δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον τό τε δὶς ὑπ᾽ αὐτῶν μέσον καὶ ἔτι τὰ ἀπ᾽ αὐτῶν τετράγωνα ἀσύμμετρα τῷ δὶς ὑπ᾽ αὐτῶν, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα.

Ἀπὸ γὰρ εὐθείας τῆς ΑΒ εὐθεῖα ἀφῃρήσθω ἡ ΒΓ δυνάμει ἀσύμμετρος οὖσα τῇ ΑΒ ποιοῦσα τὰ προκείμενα· λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν ἡ καλουμένη ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα. Ἐκκείσθω γὰρ ῥητὴ ἡ ΔΙ, καὶ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον παρὰ τὴν ΔΙ παραβεβλήσθω τὸ ΔΕ πλάτος ποιοῦν τὴν ΔΗ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ἴσον ἀφῃρήσθω τὸ ΔΘ [πλάτος ποιοῦν τὴν ΔΖ]. λοιπὸν ἄρα τὸ ΖΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ· ὥστε ἡ ΑΓ δύναται τὸ ΖΕ. καὶ ἐπεὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τετραγώνων μέσον ἐστὶ καί ἐστιν ἴσον τῷ ΔΕ, μέσον ἄρα [ἐστὶ] τὸ ΔΕ. καὶ παρὰ ῥητὴν τὴν ΔΙ παράκειται πλάτος ποιοῦν τὴν ΔΗ· ῥητὴ ἄρα ἐστὶν ἡ ΔΗ καὶ ἀσύμμετρος τῇ ΔΙ μήκει. πάλιν, ἐπεὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ μέσον ἐστὶ καί ἐστιν ἴσον τῷ ΔΘ, τὸ ἄρα ΔΘ μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΔΙ παράκειται πλάτος ποιοῦν τὴν ΔΖ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΔΖ καὶ ἀσύμμετρος τῇ ΔΙ μήκει. καὶ ἐπεὶ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, ἀσύμμετρον ἄρα καὶ τὸ ΔΕ τῷ ΔΘ. ὡς δὲ τὸ ΔΕ πρὸς τὸ ΔΘ, οὕτως ἐστὶ καὶ ἡ ΔΗ πρὸς τὴν ΔΖ· ἀσύμμετρος ἄρα ἡ ΔΗ τῇ ΔΖ. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΗΔ, ΔΖ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ἀποτομὴ ἄρα ἐστὶν ἡ ΖΗ· ῥητὴ δὲ ἡ ΖΘ. τὸ δὲ ὑπὸ ῥητῆς καὶ ἀποτομῆς περιεχόμενον [ὀρθογώνιον] ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν. καὶ δύναται

τὸ ΖΕ ἡ ΑΓ· ἡ ΑΓ ἄρα ἄλογός ἐστιν· καλείσθω δὲ ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα. ὅπερ ἔδει δεῖξαι.

οθ΄. Τῇ ἀποτομῇ μία [μόνον] προσαρμόζει εὐθεῖα ῥητὴ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ.

Ἔστω ἀποτομὴ ἡ ΑΒ, προσαρμόζουσα δὲ αὐτῇ ἡ ΒΓ· αἱ ΑΓ, ΓΒ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· λέγω, ὅτι τῇ ΑΒ ἑτέρα οὐ προσαρμόζει ῥητὴ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ.

Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ ΒΔ· καὶ αἱ ΑΔ, ΔΒ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ ἐπεί, ᾧ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΔ, ΔΒ, τούτῳ ὑπερέχει καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ· τῷ γὰρ αὐτῷ τῷ ἀπὸ τῆς ΑΒ ἀμφότερα ὑπερέχει· ἐναλλὰξ ἄρα, ᾧ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, τούτῳ ὑπερέχει [καὶ] τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. τὰ δὲ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ· ῥητὰ γὰρ ἀμφότερα. καὶ τὸ δὶς ἄρα ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ· ὅπερ ἐστὶν ἀδύνατον· μέσα γὰρ ἀμφότερα, μέσον δὲ μέσου οὐχ ὑπερέχει ῥητῷ. τῇ ἄρα ΑΒ ἑτέρα οὐ προσαρμόζει ῥητὴ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ.

Μία ἄρα μόνη τῇ ἀποτομῇ προσαρμόζει ῥητὴ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ· ὅπερ ἔδει δεῖξαι.

π΄. Τῇ μέσης ἀποτομῇ πρώτῃ μία μόνον προσαρμόζει εὐθεῖα μέση δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ῥητὸν περιέχουσα.

Ἔστω γὰρ μέσης ἀποτομὴ πρώτη ἡ ΑΒ, καὶ τῇ ΑΒ προσαρμοζέτω ἡ ΒΓ· αἱ ΑΓ, ΓΒ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι τὸ ὑπὸ τῶν ΑΓ, ΓΒ· λέγω, ὅτι τῇ ΑΒ ἑτέρα οὐ προσαρμόζει μέση δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ῥητὸν περιέχουσα.

Εἰ γὰρ δυνατόν, προσαρμοζέτω καὶ ἡ ΔΒ. αἱ ἄρα ΑΔ, ΔΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι τὸ ὑπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεί, ᾧ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΔ, ΔΒ, τούτῳ ὑπερέχει καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ· τῷ γὰρ αὐτῷ [πάλιν] ὑπερέχουσι τῷ ἀπὸ τῆς ΑΒ· ἐναλλὰξ ἄρα, ᾧ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, τούτῳ ὑπερέχει καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. τὸ δὲ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ· ῥητὰ γὰρ ἀμφότερα. καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ ἄρα τῶν ἀπὸ τῶν ΑΓ, ΓΒ [τετραγώνων] ὑπερέχει ῥητῷ· ὅπερ ἐστὶν ἀδύνατον· μέσα γάρ ἐστιν ἀμφότερα, μέσον δὲ μέσου οὐχ ὑπερέχει ῥητῷ.

Τῇ ἄρα μέσης ἀποτομῇ πρώτῃ μία μόνον προσαρμόζει εὐθεῖα μέση δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ῥητὸν περιέχουσα· ὅπερ ἔδει δεῖξαι.

πα΄. Τῇ μέσης ἀποτομῇ δευτέρᾳ μία μόνον προσαρμόζει εὐθεῖα μέση δυνάμει μόνον σύμμετρος τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης μέσον περιέχουσα.

Ἔστω μέσης ἀποτομὴ δευτέρα ἡ ΑΒ καὶ τῇ ΑΒ προσαρμόζουσα ἡ ΒΓ· αἱ ἄρα ΑΓ, ΓΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι τὸ ὑπὸ τῶν ΑΓ, ΓΒ· λέγω, ὅτι τῇ ΑΒ ἑτέρα οὐ προσαρμόσει εὐθεῖα μέση δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης μέσον περιέχουσα.

Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ ΒΔ· καὶ αἱ ΑΔ, ΔΒ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι τὸ ὑπὸ τῶν ΑΔ, ΔΒ. καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ τοῖς μὲν ἀπὸ τῶν ΑΓ, ΓΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΕΗ πλάτος ποιοῦν τὴν ΕΜ· τῷ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον ἀφῃρήσθω τὸ ΘΗ πλάτος ποιοῦν τὴν ΘΜ· λοιπὸν ἄρα τὸ ΕΛ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ· ὥστε ἡ ΑΒ δύναται τὸ ΕΛ. πάλιν δὴ τοῖς ἀπὸ τῶν ΑΔ, ΔΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΕΙ πλάτος ποιοῦν τὴν ΕΝ· ἔστι δὲ καὶ τὸ ΕΛ ἴσον τῷ ἀπὸ τῆς ΑΒ τετραγώνῳ· λοιπὸν ἄρα τὸ ΘΙ ἴσον ἐστὶ τῷ δὶς

ὑπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεὶ μέσαι εἰσὶν αἱ ΑΓ, ΓΒ, μέσα ἄρα ἐστὶ καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ. καί ἐστιν ἴσα τῷ ΕΗ· μέσον ἄρα καὶ τὸ ΕΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΕΜ· ῥητὴ ἄρα ἐστὶν ἡ ΕΜ καὶ ἀσύμμετρος τῇ ΕΖ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ ὑπὸ τῶν ΑΓ, ΓΒ, καὶ τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ μέσον ἐστίν. καί ἐστιν ἴσον τῷ ΘΗ· καὶ τὸ ΘΗ ἄρα μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΘΜ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΘΜ καὶ ἀσύμμετρος τῇ ΕΖ μήκει. καὶ ἐπεὶ αἱ ΑΓ, ΓΒ δυνάμει μόνον σύμμετροί εἰσιν, ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΓ τῇ ΓΒ μήκει. ὡς δὲ ἡ ΑΓ πρὸς τὴν ΓΒ, οὕτως ἐστὶ τὸ ἀπὸ τῆς ΑΓ πρὸς τὸ ὑπὸ τῶν ΑΓ, ΓΒ· ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΓ τῷ ὑπὸ τῶν ΑΓ, ΓΒ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΓ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΓ, ΓΒ, τῷ δὲ ὑπὸ τῶν ΑΓ, ΓΒ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ· ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. καί ἐστι τοῖς μὲν ἀπὸ τῶν ΑΓ, ΓΒ ἴσον τὸ ΕΗ, τῷ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον τὸ ΗΘ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΕΗ τῷ ΘΗ. ὡς δὲ τὸ ΕΗ πρὸς τὸ ΘΗ, οὕτως ἐστὶν ἡ ΕΜ πρὸς τὴν ΘΜ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΜ τῇ ΜΘ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΕΜ, ΜΘ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΕΘ, προσαρμόζουσα δὲ αὐτῇ ἡ ΘΜ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΘΝ αὐτῇ προσαρμόζει· τῇ ἄρα ἀποτομῇ ἄλλη καὶ ἄλλη προσαρμόζει εὐθεῖα δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ· ὅπερ ἐστὶν ἀδύνατον.

Τῇ ἄρα μέσης ἀποτομῇ δευτέρᾳ μία μόνον προσαρμόζει εὐθεῖα μέση δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης μέσον περιέχουσα· ὅπερ ἔδει δεῖξαι.

πβ΄. Τῇ ἐλάσσονι μία μόνον προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ ποιοῦσα μετὰ τῆς ὅλης τὸ μὲν ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δὲ δὶς ὑπ᾽ αὐτῶν μέσον.

Ἔστω ἡ ἐλάσσων ἡ ΑΒ, καὶ τῇ ΑΒ προσαρμόζουσα ἔστω ἡ ΒΓ· αἱ ἄρα ΑΓ, ΓΒ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δὲ δὶς ὑπ᾽ αὐτῶν μέσον· λέγω, ὅτι τῇ ΑΒ ἑτέρα εὐθεῖα οὐ προσαρμόσει τὰ αὐτὰ ποιοῦσα.

Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ ΒΔ· καὶ αἱ ΑΔ, ΔΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὰ προειρημένα. καὶ ἐπεί, ᾧ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, τούτῳ ὑπερέχει καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, τὰ δὲ ἀπὸ τῶν ΑΔ, ΔΒ τετράγωνα τῶν ἀπὸ τῶν ΑΓ, ΓΒ τετραγώνων ὑπερέχει ῥητῷ· ῥητὰ γάρ ἐστιν ἀμφότερα· καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ ἄρα τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ· ὅπερ ἐστὶν ἀδύνατον· μέσα γάρ ἐστιν ἀμφότερα.

Τῇ ἄρα ἐλάσσονι μία μόνον προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ καὶ ποιοῦσα τὰ μὲν ἀπ᾽ αὐτῶν τετράγωνα ἅμα ῥητόν, τὸ δὲ δὶς ὑπ᾽ αὐτῶν μέσον· ὅπερ ἔδει δεῖξαι.

πγ΄. Τῇ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσῃ μία μόνον προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον, τὸ δὲ δὶς ὑπ᾽ αὐτῶν ῥητόν.

Ἔστω ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα ἡ ΑΒ, καὶ τῇ ΑΒ προσαρμοζέτω ἡ ΒΓ· αἱ ἄρα ΑΓ, ΓΒ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὰ προκείμενα· λέγω, ὅτι τῇ ΑΒ ἑτέρα οὐ προσαρμόσει τὰ αὐτὰ ποιοῦσα.

Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ ΒΔ· καὶ αἱ ΑΔ, ΔΒ ἄρα εὐθεῖαι δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὰ προκείμενα. ἐπεὶ οὖν, ᾧ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, τούτῳ ὑπερέχει καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἀκολούθως τοῖς πρὸ αὐτοῦ, τὸ δὲ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ· ῥητὰ γάρ ἐστιν ἀμφότερα· καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ ἄρα τῶν ἀπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ· ὅπερ ἐστὶν ἀδύνατον· μέσα γάρ ἐστιν ἀμφότερα. οὐκ ἄρα τῇ ΑΒ ἑτέρα προσαρμόσει

εὐθεῖα δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τὰ προειρημένα· μία ἄρα μόνον προσαρμόσει· ὅπερ ἔδει δεῖξαι.

πδ΄. Τῇ μετὰ μέσου μέσον τὸ ὅλον ποιούσῃ μία μόνη προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον τό τε δὶς ὑπ᾽ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῳ ἐκ τῶν ἀπ᾽ αὐτῶν.

Ἔστω ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα ἡ ΑΒ, προσαρμόζουσα δὲ αὐτῇ ἡ ΒΓ· αἱ ἄρα ΑΓ, ΓΒ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὰ προειρημένα. λέγω, ὅτι τῇ ΑΒ ἑτέρα οὐ προσαρμόσει ποιοῦσα τὰ προειρημένα.

Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ ΒΔ, ὥστε καὶ τὰς ΑΔ, ΔΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τά τε ἀπὸ τῶν ΑΔ, ΔΒ τετράγωνα ἅμα μέσον καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ μέσον καὶ ἔτι τὰ ἀπὸ τῶν ΑΔ, ΔΒ ἀσύμμετρα τῷ δὶς ὑπὸ τῶν ΑΔ, ΔΒ· καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ τοῖς μὲν ἀπὸ τῶν ΑΓ, ΓΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΕΗ πλάτος ποιοῦν τὴν ΕΜ, τῷ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΘΗ πλάτος ποιοῦν τὴν ΘΜ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τῷ ΕΛ· ἡ ἄρα ΑΒ δύναται τὸ ΕΛ. πάλιν τοῖς ἀπὸ τῶν ΑΔ, ΔΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΕΙ πλάτος ποιοῦν τὴν ΕΝ. ἔστι δὲ καὶ τὸ ἀπὸ τῆς ΑΒ ἴσον τῷ ΕΛ· λοιπὸν ἄρα τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ ἴσον [ἐστὶ] τῷ ΘΙ. καὶ ἐπεὶ μέσον ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ καί ἐστιν ἴσον τῷ ΕΗ, μέσον ἄρα ἐστὶ καὶ τὸ ΕΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΕΜ· ῥητὴ ἄρα ἐστὶν ἡ ΕΜ καὶ ἀσύμμετρος τῇ ΕΖ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ καί ἐστιν ἴσον τῷ ΘΗ, μέσον ἄρα καὶ τὸ ΘΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΘΜ· ῥητὴ ἄρα ἐστὶν ἡ ΘΜ καὶ ἀσύμμετρος τῇ ΕΖ μήκει. καὶ ἐπεὶ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, ἀσύμμετρόν ἐστι καὶ τὸ ΕΗ τῷ ΘΗ· ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΕΜ τῇ ΜΘ

μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΕΜ, ΜΘ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΕΘ, προσαρμόζουσα δὲ αὐτῇ ἡ ΘΜ. ὁμοίως δὴ δείξομεν, ὅτι ἡ ΕΘ πάλιν ἀποτομή ἐστιν, προσαρμόζουσα δὲ αὐτῇ ἡ ΘΝ. τῇ ἄρα ἀποτομῇ ἄλλη καὶ ἄλλη προσαρμόζει ῥητὴ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ· ὅπερ ἐδείχθη ἀδύνατον. οὐκ ἄρα τῇ ΑΒ ἑτέρα προσαρμόσει εὐθεῖα.

Τῇ ἄρα ΑΒ μία μόνον προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τά τε ἀπ᾽ αὐτῶν τετράγωνα ἅμα μέσον καὶ τὸ δὶς ὑπ᾽ αὐτῶν μέσον καὶ ἔτι τὰ ἀπ᾽ αὐτῶν τετράγωνα ἀσύμμετρα τῷ δὶς ὑπ᾽ αὐτῶν· ὅπερ ἔδει δεῖξαι.

Ὅροι τρίτοι ς΄.

α΄. Ὑποκειμένης ῥητῆς καὶ ἀποτομῆς, ἐὰν μὲν ἡ ὅλη τῆς προσαρμοζούσης μεῖζον δύνηται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει, καὶ ἡ ὅλη σύμμετρος ᾗ τῇ ἐκκειμένῃ ῥητῇ μήκει, καλείσθω ἀποτομὴ πρώτη.

β΄. Ἐὰν δὲ ἡ προσαρμόζουσα σύμμετρος ᾗ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ὅλη τῆς προσαρμοζούσης μεῖζον δύνηται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καλείσθω ἀποτομὴ δευτέρα.

γ΄. Ἐὰν δὲ μηδετέρα σύμμετρος ᾗ τῇ ἐκκειμένῃ ῥητῇ μήκει, ἡ δὲ ὅλη τῆς προσαρμοζούσης μεῖζον δύνηται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καλείσθω ἀποτομὴ τρίτη.

δ΄. Πάλιν, ἐὰν ἡ ὅλη τῆς προσαρμοζούσης μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ [μήκει], ἐὰν μὲν ἡ ὅλη σύμμετρος ᾗ τῇ ἐκκειμένῃ ῥητῇ μήκει, καλείσθω ἀποτομὴ τετάρτη.

ε΄. Ἐὰν δὲ ἡ προσαρμόζουσα, πέμπτη.

ς΄. Ἐὰν δὲ μηδετέρα, ἕκτη.

Προτάσεις ριε΄.(μέρος 3)

πε΄. Εὑρεῖν τὴν πρώτην ἀποτομήν.

Ἐκκείσθω ῥητὴ ἡ Α, καὶ τῇ Α μήκει σύμμετρος ἔστω ἡ ΒΗ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΒΗ. καὶ ἐκκείσθωσαν δύο τετράγωνοι ἀριθμοὶ οἱ ΔΕ, ΕΖ, ὧν ἡ ὑπεροχὴ ὁ ΖΔ μὴ ἔστω τετράγωνος· οὐδ’ ἄρα ὁ ΕΔ πρὸς τὸν ΔΖ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ πεποιήσθω ὡς ὁ ΕΔ πρὸς τὸν ΔΖ, οὕτως τὸ ἀπὸ τῆς ΒΗ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΓ τετράγωνον· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΗ τῷ ἀπὸ τῆς ΗΓ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΒΗ· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΗΓ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΗΓ. καὶ ἐπεὶ ὁ ΕΔ πρὸς τὸν ΔΖ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ’ ἄρα τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΗ τῇ ΗΓ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΒΗ, ΗΓ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ἄρα ΒΓ ἀποτομή ἐστιν.

Λέγω δή, ὅτι καὶ πρώτη.

Ὧι γὰρ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΒΗ τοῦ ἀπὸ τῆς ΗΓ, ἔστω τὸ ἀπὸ τῆς Θ. καὶ ἐπεί ἐστιν ὡς ὁ ΕΔ πρὸς τὸν ΖΔ, οὕτως τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ, καὶ ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΔΕ πρὸς τὸν ΕΖ, οὕτως τὸ ἀπὸ τῆς ΗΒ πρὸς τὸ ἀπὸ τῆς Θ. ὁ δὲ ΔΕ πρὸς τὸν ΕΖ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἑκάτερος γὰρ τετράγωνός ἐστιν· καὶ τὸ ἀπὸ τῆς ΗΒ ἄρα πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· σύμμετρος ἄρα ἐστὶν ἡ ΒΗ τῇ Θ μήκει. καὶ δύναται ἡ ΒΗ τῆς ΗΓ μεῖζον τῷ ἀπὸ τῆς Θ· ἡ ΒΗ ἄρα τῆς ΗΓ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. καί ἐστιν ἡ ὅλη ἡ ΒΗ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ Α. ἡ ΒΓ ἄρα ἀποτομή ἐστι πρώτη.

Εὕρηται ἄρα ἡ πρώτη ἀποτομὴ ἡ ΒΓ· ὅπερ ἔδει εὑρεῖν.

πς΄. Εὑρεῖν τὴν δευτέραν ἀποτομήν.

Ἐκκείσθω ῥητὴ ἡ Α καὶ τῇ Α σύμμετρος μήκει ἡ ΗΓ. ῥητὴ ἄρα ἐστὶν ἡ ΗΓ. καὶ ἐκκείσθωσαν δύο τετράγωνοι ἀριθμοὶ οἱ ΔΕ, ΕΖ, ὧν ἡ ὑπεροχὴ ὁ ΔΖ μὴ ἔστω τετράγωνος. καὶ πεποιήσθω ὡς ὁ ΖΔ πρὸς τὸν ΔΕ, οὕτως τὸ ἀπὸ τῆς ΓΗ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΒ τετράγωνον. σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΓΗ τετράγωνον τῷ ἀπὸ τῆς ΗΒ τετραγώνῳ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΗΓ. ῥητὸν ἄρα [ἐστὶ] καὶ τὸ ἀπὸ τῆς ΗΒ· ῥητὴ ἄρα ἐστὶν ἡ ΒΗ. καὶ ἐπεὶ τὸ ἀπὸ τῆς ΗΓ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΒ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, ἀσύμμετρός ἐστιν ἡ ΓΗ τῇ ΗΒ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΓΗ, ΗΒ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΒΓ ἄρα ἀποτομή ἐστιν.

Λέγω δή, ὅτι καὶ δευτέρα.

Ὧι γὰρ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΒΗ τοῦ ἀπὸ τῆς ΗΓ, ἔστω τὸ ἀπὸ τῆς Θ. ἐπεὶ οὖν ἐστιν ὡς τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ, οὕτως ὁ ΕΔ ἀριθμὸς πρὸς τὸν ΔΖ ἀριθμόν, ἀναστρέψαντι ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς Θ, οὕτως ὁ ΔΕ πρὸς τὸν ΕΖ. καί ἐστιν ἑκάτερος τῶν ΔΕ, ΕΖ τετράγωνος· τὸ ἄρα ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· σύμμετρος ἄρα ἐστὶν ἡ ΒΗ τῇ Θ μήκει. καὶ δύναται ἡ ΒΗ τῆς ΗΓ μεῖζον τῷ ἀπὸ τῆς Θ· ἡ ΒΗ ἄρα τῆς ΗΓ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. καί ἐστιν ἡ προσαρμόζουσα ἡ ΓΗ τῇ ἐκκειμένῃ ῥητῇ σύμμετρος τῇ Α. ἡ ΒΓ ἄρα ἀποτομή ἐστι δευτέρα.

Εὕρηται ἄρα δευτέρα ἀποτομὴ ἡ ΒΓ· ὅπερ ἔδει δεῖξαι.

πζ΄. Εὑρεῖν τὴν τρίτην ἀποτομήν.

Ἐκκείσθω ῥητὴ ἡ Α, καὶ ἐκκείσθωσαν τρεῖς ἀριθμοὶ οἱ Ε, ΒΓ, ΓΔ λόγον μὴ ἔχοντες πρὸς ἀλλήλους, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον

ἀριθμόν, ὁ δὲ ΓΒ πρὸς τὸν ΒΔ λόγον ἐχέτω, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ πεποιήσθω ὡς μὲν ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΗ τετράγωνον, ὡς δὲ ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΘ. ἐπεὶ οὖν ἐστιν ὡς ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΗ τετράγωνον, σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς Α τετράγωνον τῷ ἀπὸ τῆς ΖΗ τετραγώνῳ. ῥητὸν δὲ τὸ ἀπὸ τῆς Α τετράγωνον. ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΖΗ· ῥητὴ ἄρα ἐστὶν ἡ ΖΗ. καὶ ἐπεὶ ὁ Ε πρὸς τὸν ΒΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ’ ἄρα τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΗ [τετράγωνον] λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Α τῇ ΖΗ μήκει. πάλιν, ἐπεί ἐστιν ὡς ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΘ, σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΗΘ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΖΗ· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΗΘ· ῥητὴ ἄρα ἐστὶν ἡ ΗΘ. καὶ ἐπεὶ ὁ ΒΓ πρὸς τὸν ΓΔ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ’ ἄρα τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ ΗΘ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΖΗ, ΗΘ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΖΘ.

Λέγω δή, ὅτι καὶ τρίτη.

Ἐπεὶ γάρ ἐστιν ὡς μὲν ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΗ, ὡς δὲ ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΘΗ, δι’ ἴσου ἄρα ἐστὶν ὡς ὁ Ε πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΘΗ. ὁ δὲ Ε πρὸς τὸν ΓΔ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ’ ἄρα τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἡ Α τῇ ΗΘ μήκει. οὐδετέρα ἄρα τῶν ΖΗ, ΗΘ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ Α μήκει. ᾧ οὖν μεῖζόν ἐστι τὸ ἀπὸ τῆς ΖΗ τοῦ ἀπὸ

τῆς ΗΘ, ἔστω τὸ ἀπὸ τῆς Κ. ἐπεὶ οὖν ἐστιν ὡς ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΒΓ πρὸς τὸν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ τετράγωνον πρὸς τὸ ἀπὸ τῆς Κ. ὁ δὲ ΒΓ πρὸς τὸν ΒΔ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ τὸ ἀπὸ τῆς ΖΗ ἄρα πρὸς τὸ ἀπὸ τῆς Κ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. σύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ Κ μήκει, καὶ δύναται ἡ ΖΗ τῆς ΗΘ μεῖζον τῷ ἀπὸ συμμέτρου ἑαυτῇ. καὶ οὐδετέρα τῶν ΖΗ, ΗΘ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ Α μήκει· ἡ ΖΘ ἄρα ἀποτομή ἐστι τρίτη.

Εὕρηται ἄρα ἡ τρίτη ἀποτομὴ ἡ ΖΘ· ὅπερ ἔδει δεῖξαι.

πη΄. Εὑρεῖν τὴν τετάρτην ἀποτομήν.

Ἐκκείσθω ῥητὴ ἡ Α καὶ τῇ Α μήκει σύμμετρος ἡ ΒΗ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΒΗ. καὶ ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΔΖ, ΖΕ, ὥστε τὸν ΔΕ ὅλον πρὸς ἑκάτερον τῶν ΔΖ, ΕΖ λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ πεποιήσθω ὡς ὁ ΔΕ πρὸς τὸν ΕΖ, οὕτως τὸ ἀπὸ τῆς ΒΗ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΓ. σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΗ τῷ ἀπὸ τῆς ΗΓ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΒΗ· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΗΓ· ῥητὴ ἄρα ἐστὶν ἡ ΗΓ. καὶ ἐπεὶ ὁ ΔΕ πρὸς τὸν ΕΖ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ’ ἄρα τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΗ τῇ ΗΓ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΒΗ, ΗΓ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΒΓ.

[Λέγω δή, ὅτι καὶ τετάρτη].

Ὧι οὖν μεῖζόν ἐστι τὸ ἀπὸ τῆς ΒΗ τοῦ ἀπὸ τῆς ΗΓ, ἔστω τὸ ἀπὸ τῆς Θ. ἐπεὶ οὖν ἐστιν ὡς ὁ ΔΕ πρὸς τὸν ΕΖ, οὕτως τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ, καὶ ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΕΔ πρὸς τὸν ΔΖ, οὕτως τὸ ἀπὸ τῆς ΗΒ πρὸς τὸ ἀπὸ τῆς Θ. ὁ δὲ ΕΔ πρὸς τὸν ΔΖ λόγον οὐκ ἔχει, ὃν τετράγωνος

ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ’ ἄρα τὸ ἀπὸ τῆς ΗΒ πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΗ τῇ Θ μήκει. καὶ δύναται ἡ ΒΗ τῆς ΗΓ μεῖζον τῷ ἀπὸ τῆς Θ· ἡ ἄρα ΒΗ τῆς ΗΓ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καί ἐστιν ὅλη ἡ ΒΗ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ Α. ἡ ἄρα ΒΓ ἀποτομή ἐστι τετάρτη.

Εὕρηται ἄρα ἡ τετάρτη ἀποτομή· ὅπερ ἔδει δεῖξαι.

πθ΄. Εὑρεῖν τὴν πέμπτην ἀποτομήν.

Ἐκκείσθω ῥητὴ ἡ Α, καὶ τῇ Α μήκει σύμμετρος ἔστω ἡ ΓΗ· ῥητὴ ἄρα [ἐστὶν] ἡ ΓΗ. καὶ ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΔΖ, ΖΕ, ὥστε τὸν ΔΕ πρὸς ἑκάτερον τῶν ΔΖ, ΖΕ λόγον πάλιν μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ πεποιήσθω ὡς ὁ ΖΕ πρὸς τὸν ΕΔ, οὕτως τὸ ἀπὸ τῆς ΓΗ πρὸς τὸ ἀπὸ τῆς ΗΒ. ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΗΒ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΒΗ. καὶ ἐπεί ἐστιν ὡς ὁ ΔΕ πρὸς τὸν ΕΖ, οὕτως τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ, ὁ δὲ ΔΕ πρὸς τὸν ΕΖ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ’ ἄρα τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΗ τῇ ΗΓ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΒΗ, ΗΓ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΒΓ ἄρα ἀποτομή ἐστιν.

Λέγω δή, ὅτι καὶ πέμπτη.

Ὧι γὰρ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΒΗ τοῦ ἀπὸ τῆς ΗΓ, ἔστω τὸ ἀπὸ τῆς Θ. ἐπεὶ οὖν ἐστιν ὡς τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ, οὕτως ὁ ΔΕ πρὸς τὸν ΕΖ, ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΕΔ πρὸς τὸν ΔΖ, οὕτως τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς Θ. ὁ δὲ ΕΔ πρὸς τὸν ΔΖ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ’ ἄρα τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΗ τῇ Θ μήκει. καὶ δύναται ἡ ΒΗ τῆς ΗΓ μεῖζον τῷ ἀπὸ τῆς Θ·

ἡ ΗΒ ἄρα τῆς ΗΓ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει. καί ἐστιν ἡ προσαρμόζουσα ἡ ΓΗ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ τῇ Α μήκει· ἡ ἄρα ΒΓ ἀποτομή ἐστι πέμπτη.

Εὕρηται ἄρα ἡ πέμπτη ἀποτομὴ ἡ ΒΓ· ὅπερ ἔδει δεῖξαι.

ϛ΄. Εὑρεῖν τὴν ἕκτην ἀποτομήν.

Ἐκκείσθω ῥητὴ ἡ Α καὶ τρεῖς ἀριθμοὶ οἱ Ε, ΒΓ, ΓΔ λόγον μὴ ἔχοντες πρὸς ἀλλήλους, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἔτι δὲ καὶ ὁ ΓΒ πρὸς τὸν ΒΔ λόγον μὴ ἐχέτω, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ πεποιήσθω ὡς μὲν ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΖΗ, ὡς δὲ ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ.

Ἐπεὶ οὖν ἐστιν ὡς ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΖΗ, σύμμετρον ἄρα τὸ ἀπὸ τῆς Α τῷ ἀπὸ τῆς ΖΗ. ῥητὸν δὲ τὸ ἀπὸ τῆς Α· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΖΗ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΖΗ. καὶ ἐπεὶ ὁ Ε πρὸς τὸν ΒΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ’ ἄρα τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Α τῇ ΖΗ μήκει. πάλιν, ἐπεί ἐστιν ὡς ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, σύμμετρον ἄρα τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΗΘ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΖΗ· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΗΘ· ῥητὴ ἄρα καὶ ἡ ΗΘ. καὶ ἐπεὶ ὁ ΒΓ πρὸς τὸν ΓΔ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ’ ἄρα τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ ΗΘ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΖΗ, ΗΘ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ἄρα ΖΘ ἀποτομή ἐστιν.

Λέγω δή, ὅτι καὶ ἕκτη.

Ἐπεὶ γάρ ἐστιν ὡς μὲν ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΖΗ, ὡς δὲ ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, δι' ἴσου ἄρα ἐστὶν ὡς ὁ Ε πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΗΘ. ὁ δὲ Ε πρὸς τὸν ΓΔ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ' ἄρα τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Α τῇ ΗΘ μήκει· οὐδετέρα ἄρα τῶν ΖΗ, ΗΘ σύμμετρός ἐστι τῇ Α ῥητῇ μήκει. ᾧ οὖν μεῖζόν ἐστι τὸ ἀπὸ τῆς ΖΗ τοῦ ἀπὸ τῆς ΗΘ, ἔστω τὸ ἀπὸ τῆς Κ. ἐπεὶ οὖν ἐστιν ὡς ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΓΒ πρὸς τὸν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς Κ. ὁ δὲ ΓΒ πρὸς τὸν ΒΔ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ' ἄρα τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς Κ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ Κ μήκει. καὶ δύναται ἡ ΖΗ τῆς ΗΘ μεῖζον τῷ ἀπὸ τῆς Κ· ἡ ΖΗ ἄρα τῆς ΗΘ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει. καὶ οὐδετέρα τῶν ΖΗ, ΗΘ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ Α. ἡ ἄρα ΖΘ ἀποτομή ἐστιν ἕκτη.

Εὕρηται ἄρα ἡ ἕκτη ἀποτομὴ ἡ ΖΘ· ὅπερ ἔδει δεῖξαι.

ιαʹ. Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ ἀποτομῆς πρώτης, ἡ τὸ χωρίον δυναμένη ἀποτομή ἐστιν.

Περιεχέσθω γὰρ χωρίον τὸ ΑΒ ὑπὸ ῥητῆς τῆς ΑΓ καὶ ἀποτομῆς πρώτης τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΒ χωρίον δυναμένη ἀποτομή ἐστιν.

Ἐπεὶ γὰρ ἀποτομή ἐστι πρώτη ἡ ΑΔ, ἔστω αὐτῇ προσαρμόζουσα ἡ ΔΗ· αἱ ΑΗ, ΗΔ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ ὅλη ἡ ΑΗ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΑΓ, καὶ ἡ ΑΗ τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει· ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν ΑΗ παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς σύμμετρα αὐτὴν

διαιρεῖ. τετμήσθω ἡ ΔΗ δίχα κατὰ τὸ Ε, καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ· σύμμετρος ἄρα ἐστὶν ἡ ΑΖ τῇ ΖΗ. καὶ διὰ τῶν Ε, Ζ, Η σημείων τῇ ΑΓ παράλληλοι ἤχθωσαν αἱ ΕΘ, ΖΙ, ΗΚ.

Καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΑΖ τῇ ΖΗ μήκει, καὶ ἡ ΑΗ ἄρα ἑκατέρᾳ τῶν ΑΖ, ΖΗ σύμμετρός ἐστι μήκει. ἀλλὰ ἡ ΑΗ σύμμετρός ἐστι τῇ ΑΓ· καὶ ἑκατέρα ἄρα τῶν ΑΖ, ΖΗ σύμμετρός ἐστι τῇ ΑΓ μήκει. καί ἐστι ῥητὴ ἡ ΑΓ· ῥητὴ ἄρα καὶ ἑκατέρα τῶν ΑΖ, ΖΗ· ὥστε καὶ ἑκάτερον τῶν ΑΙ, ΖΚ ῥητόν ἐστιν. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΔΕ τῇ ΕΗ μήκει, καὶ ἡ ΔΗ ἄρα ἑκατέρᾳ τῶν ΔΕ, ΕΗ σύμμετρός ἐστι μήκει. ῥητὴ δὲ ἡ ΔΗ καὶ ἀσύμμετρος τῇ ΑΓ μήκει· ῥητὴ ἄρα καὶ ἑκατέρα τῶν ΔΕ, ΕΗ καὶ ἀσύμμετρος τῇ ΑΓ μήκει· ἑκάτερον ἄρα τῶν ΔΘ, ΕΚ μέσον ἐστίν.

Κείσθω δὴ τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον τετράγωνον ἀφῃρήσθω κοινὴν γωνίαν ἔχον αὐτῷ τὴν ὑπὸ ΛΟΜ τὸ ΝΞ· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὰ ΛΜ, ΝΞ τετράγωνα. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν ἴσον ἐστὶ τὸ ὑπὸ τῶν ΑΖ, ΖΗ περιεχόμενον ὀρθογώνιον τῷ ἀπὸ τῆς ΕΗ τετραγώνῳ, ἔστιν ἄρα ὡς ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἡ ΕΗ πρὸς τὴν ΖΗ. ἀλλ’ ὡς μὲν ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως τὸ ΑΙ πρὸς τὸ ΕΚ, ὡς δὲ ἡ ΕΗ πρὸς τὴν ΖΗ, οὕτως ἐστὶ τὸ ΕΚ πρὸς τὸ ΚΖ· τῶν ἄρα ΑΙ, ΚΖ μέσον ἀνάλογόν ἐστι τὸ ΕΚ. ἔστι δὲ καὶ τῶν ΛΜ, ΝΞ μέσον ἀνάλογον τὸ ΜΝ, ὡς ἐν τοῖς ἔμπροσθεν ἐδείχθη, καί ἐστι τὸ [μὲν] ΑΙ τῷ ΛΜ τετραγώνῳ ἴσον, τὸ δὲ ΚΖ τῷ ΝΞ· καὶ τὸ ΜΝ ἄρα τῷ ΕΚ ἴσον ἐστίν. ἀλλὰ τὸ μὲν ΕΚ τῷ ΔΘ ἐστιν ἴσον, τὸ δὲ ΜΝ τῷ ΛΞ· τὸ ἄρα ΔΚ ἴσον ἐστὶ τῷ ΥΦΧ γνώμονι καὶ τῷ ΝΞ. ἔστι δὲ καὶ τὸ ΑΚ ἴσον τοῖς ΛΜ, ΝΞ τετραγώνοις· λοιπὸν ἄρα τὸ ΑΒ ἴσον ἐστὶ τῷ ΣΤ. τὸ δὲ ΣΤ τὸ ἀπὸ τῆς ΛΝ ἐστι τετράγωνον· τὸ ἄρα ἀπὸ τῆς ΛΝ τετράγωνον ἴσον ἐστὶ τῷ ΑΒ· ἡ ΛΝ ἄρα δύναται τὸ ΑΒ.

Λέγω δή, ὅτι ἡ ΛΝ ἀποτομή ἐστιν.

Ἐπεὶ γὰρ ῥητόν ἐστιν ἑκάτερον τῶν ΑΙ, ΖΚ, καί ἐστιν ἴσον τοῖς ΛΜ, ΝΞ, καὶ ἑκάτερον ἄρα τῶν ΛΜ, ΝΞ ῥητόν ἐστιν, τουτέστι τὸ ἀπὸ ἑκατέρας τῶν ΛΟ, ΟΝ· καὶ ἑκατέρα ἄρα τῶν ΛΟ, ΟΝ ῥητή ἐστιν. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ ΔΘ καί ἐστιν ἴσον τῷ ΛΞ, μέσον ἄρα ἐστὶ καὶ τὸ ΛΞ. ἐπεὶ οὖν τὸ μὲν ΛΞ μέσον ἐστίν, τὸ δὲ ΝΞ ῥητόν, ἀσύμμετρον ἄρα ἐστὶ τὸ ΛΞ τῷ ΝΞ· ὡς δὲ τὸ ΛΞ πρὸς τὸ ΝΞ, οὕτως ἐστὶν ἡ ΛΟ πρὸς τὴν ΟΝ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΛΟ τῇ ΟΝ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΛΟ, ΟΝ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΛΝ. καὶ δύναται τὸ ΑΒ χωρίον· ἡ ἄρα τὸ ΑΒ χωρίον δυναμένη ἀποτομή ἐστιν.

Ἐὰν ἄρα χωρίον περιέχηται ὑπὸ ῥητῆς, καὶ τὰ ἑξῆς.

ϗβ΄. Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ ἀποτομῆς δευτέρας, ἡ τὸ χωρίον δυναμένη μέσης ἀποτομή ἐστι πρώτη.

Χωρίον γὰρ τὸ ΑΒ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΓ καὶ ἀποτομῆς δευτέρας τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΒ χωρίον δυναμένη μέσης ἀποτομή ἐστι πρώτη.

Ἔστω γὰρ τῇ ΑΔ προσαρμόζουσα ἡ ΔΗ· αἱ ἄρα ΑΗ, ΗΔ ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ προσαρμόζουσα ἡ ΔΗ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΑΓ, ἡ δὲ ὅλη ἡ ΑΗ τῆς προσαρμοζούσης τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. ἐπεὶ οὖν ἡ ΑΗ τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΗΔ ἴσον παρὰ τὴν ΑΗ παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς σύμμετρα αὐτὴν διαιρεῖ. τετμήσθω οὖν ἡ ΔΗ δίχα κατὰ τὸ Ε· καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ· σύμμετρος ἄρα ἐστὶν ἡ ΑΖ τῇ ΖΗ μήκει. καὶ ἡ ΑΗ ἄρα ἑκατέρᾳ τῶν ΑΖ, ΖΗ σύμμετρός ἐστι μήκει. ῥητὴ δὲ ἡ ΑΗ καὶ ἀσύμμετρος τῇ ΑΓ μήκει· καὶ ἑκατέρα ἄρα τῶν ΑΖ, ΖΗ ῥητή ἐστι καὶ ἀσύμμετρος τῇ ΑΓ μήκει· ἑκάτερον ἄρα τῶν ΑΙ, ΖΚ μέσον ἐστίν. πάλιν, ἐπεὶ σύμμετρός ἐστιν ἡ ΔΕ τῇ ΕΗ, καὶ ἡ ΔΗ ἄρα ἑκατέρᾳ τῶν ΔΕ, ΕΗ σύμμετρός ἐστιν. ἀλλ᾽ ἡ ΔΗ

σύμμετρός ἐστι τῇ ΑΓ μήκει. [ῥητὴ ἄρα καὶ ἑκατέρα τῶν ΔΕ, ΕΗ καὶ σύμμετρος τῇ ΑΓ μήκει.] ἑκάτερον ἄρα τῶν ΔΘ, ΕΚ ῥητόν ἐστιν.

Συνεστάτω οὖν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον ἀφῃρήσθω τὸ ΝΞ περὶ τὴν αὐτὴν γωνίαν ὃν τῷ ΛΜ τὴν ὑπὸ τῶν ΛΟΜ· περὶ τὴν αὐτὴν ἄρα ἐστὶ διάμετρον τὰ ΛΜ, ΝΞ τετράγωνα. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν τὰ ΑΙ, ΖΚ μέσα ἐστὶ καί ἐστιν ἴσα τοῖς ἀπὸ τῶν ΛΟ, ΟΝ, καὶ τὰ ἀπὸ τῶν ΛΟ, ΟΝ [ἄρα] μέσα ἐστίν· καὶ αἱ ΛΟ, ΟΝ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΖ, ΖΗ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΗ, ἔστιν ἄρα ὡς ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἡ ΕΗ πρὸς τὴν ΖΗ· ἀλλ’ ὡς μὲν ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως τὸ ΑΙ πρὸς τὸ ΕΚ· ὡς δὲ ἡ ΕΗ πρὸς τὴν ΖΗ, οὕτως [ἐστὶ] τὸ ΕΚ πρὸς τὸ ΖΚ· τῶν ἄρα ΑΙ, ΖΚ μέσον ἀνάλογόν ἐστι τὸ ΕΚ. ἔστι δὲ καὶ τῶν ΛΜ, ΝΞ τετραγώνων μέσον ἀνάλογον τὸ ΜΝ· καί ἐστιν ἴσον τὸ μὲν ΑΙ τῷ ΛΜ, τὸ δὲ ΖΚ τῷ ΝΞ· καὶ τὸ ΜΝ ἄρα ἴσον ἐστὶ τῷ ΕΚ. ἀλλὰ τῷ μὲν ΕΚ ἴσον [ἐστὶ] τὸ ΔΘ, τῷ δὲ ΜΝ ἴσον τὸ ΛΞ· ὅλον ἄρα τὸ ΔΚ ἴσον ἐστὶ τῷ ΥΦΧ γνώμονι καὶ τῷ ΝΞ. ἐπεὶ οὖν ὅλον τὸ ΑΚ ἴσον ἐστὶ τοῖς ΛΜ, ΝΞ, ὧν τὸ ΔΚ ἴσον ἐστὶ τῷ ΥΦΧ γνώμονι καὶ τῷ ΝΞ, λοιπὸν ἄρα τὸ ΑΒ ἴσον ἐστὶ τῷ ΤΣ. τὸ δὲ ΤΣ ἐστι τὸ ἀπὸ τῆς ΛΝ· τὸ ἀπὸ τῆς ΛΝ ἄρα ἴσον ἐστὶ τῷ ΑΒ χωρίῳ· ἡ ΛΝ ἄρα δύναται τὸ ΑΒ χωρίον.

Λέγω [δή], ὅτι ἡ ΛΝ μέσης ἀποτομή ἐστι πρώτη.

Ἐπεὶ γὰρ ῥητόν ἐστι τὸ ΕΚ καί ἐστιν ἴσον τῷ ΛΞ, ῥητὸν ἄρα ἐστὶ τὸ ΛΞ, τουτέστι τὸ ὑπὸ τῶν ΛΟ, ΟΝ. μέσον δὲ ἐδείχθη τὸ ΝΞ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΛΞ τῷ ΝΞ· ὡς δὲ τὸ ΛΞ πρὸς τὸ ΝΞ, οὕτως ἐστὶν ἡ ΛΟ πρὸς ΟΝ· αἱ ΛΟ, ΟΝ ἄρα ἀσύμμετροί εἰσι μήκει. αἱ ἄρα ΛΟ, ΟΝ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι· ἡ ΛΝ ἄρα μέσης ἀποτομή ἐστι πρώτη· καὶ δύναται τὸ ΑΒ χωρίον.

Ἡ ἄρα τὸ ΑΒ χωρίον δυναμένη μέσης ἀποτομή ἐστι πρώτη· ὅπερ ἔδει δεῖξαι.

ϗγ΄. Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ ἀποτομῆς τρίτης, ἡ τὸ χωρίον δυναμένη μέσης ἀποτομή ἐστι δευτέρα.

Χωρίον γὰρ τὸ ΑΒ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΓ καὶ ἀποτομῆς τρίτης τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΒ χωρίον δυναμένη μέσης ἀποτομή ἐστι δευτέρα.

Ἔστω γὰρ τῇ ΑΔ προσαρμόζουσα ἡ ΔΗ· αἱ ΑΗ, ΗΔ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ οὐδετέρα τῶν ΑΗ, ΗΔ σύμμετρός ἐστι μήκει τῇ ἐκκειμένῃ ῥητῇ τῇ ΑΓ, ἡ δὲ ὅλη ἡ ΑΗ τῆς προσαρμοζούσης τῆς ΔΗ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. ἐπεὶ οὖν ἡ ΑΗ τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν ΑΗ παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς σύμμετρα αὐτὴν διελεῖ. τετμήσθω οὖν ἡ ΔΗ δίχα κατὰ τὸ Ε, καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ. καὶ ἤχθωσαν διὰ τῶν Ε, Ζ, Η σημείων τῇ ΑΓ παράλληλοι αἱ ΕΘ, ΖΙ, ΗΚ· σύμμετροι ἄρα εἰσὶν αἱ ΑΖ, ΖΗ· σύμμετρον ἄρα καὶ τὸ ΑΙ τῷ ΖΚ. καὶ ἐπεὶ αἱ ΑΖ, ΖΗ σύμμετροί εἰσι μήκει, καὶ ἡ ΑΗ ἄρα ἑκατέρᾳ τῶν ΑΖ, ΖΗ σύμμετρός ἐστι μήκει. ῥητὴ δὲ ἡ ΑΗ καὶ ἀσύμμετρος τῇ ΑΓ μήκει· ὥστε καὶ αἱ ΑΖ, ΖΗ. ἑκάτερον ἄρα τῶν ΑΙ, ΖΚ μέσον ἐστίν. πάλιν, ἐπεὶ σύμμετρός ἐστιν ἡ ΔΕ τῇ ΕΗ μήκει, καὶ ἡ ΔΗ ἄρα ἑκατέρᾳ τῶν ΔΕ, ΕΗ σύμμετρός ἐστι μήκει. ῥητὴ δὲ ἡ ΗΔ καὶ ἀσύμμετρος τῇ ΑΓ μήκει· ῥητὴ ἄρα καὶ ἑκατέρα τῶν ΔΕ, ΕΗ καὶ ἀσύμμετρος τῇ ΑΓ μήκει· ἑκάτερον ἄρα τῶν ΔΘ, ΕΚ μέσον ἐστίν. καὶ ἐπεὶ αἱ ΑΗ, ΗΔ δυνάμει μόνον σύμμετροί εἰσιν, ἀσύμμετρος ἄρα ἐστὶ μήκει ἡ ΑΗ τῇ ΗΔ. ἀλλ᾽ ἡ μὲν ΑΗ τῇ ΑΖ σύμμετρός ἐστι μήκει, ἡ δὲ ΔΗ τῇ ΕΗ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΖ τῇ ΕΗ μήκει. ὡς δὲ ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἐστὶ τὸ ΑΙ πρὸς τὸ ΕΚ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΑΙ τῷ ΕΚ.

Συνεστάτω οὖν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον ἀφῃρήσθω τὸ ΝΞ περὶ τὴν αὐτὴν γωνίαν ὂν τῷ ΛΜ· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὰ ΛΜ, ΝΞ. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν τὸ ὑπὸ τῶν ΑΖ, ΖΗ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΗ, ἔστιν ἄρα

ὡς ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἡ ΕΗ πρὸς τὴν ΖΗ. ἀλλ᾽ ὡς μὲν ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἐστὶ τὸ ΑΙ πρὸς τὸ ΕΚ· ὡς δὲ ἡ ΕΗ πρὸς τὴν ΖΗ, οὕτως ἐστὶ τὸ ΕΚ πρὸς τὸ ΖΚ· καὶ ὡς ἄρα τὸ ΑΙ πρὸς τὸ ΕΚ, οὕτως τὸ ΕΚ πρὸς τὸ ΖΚ· τῶν ἄρα ΑΙ, ΖΚ μέσον ἀνάλογόν ἐστι τὸ ΕΚ. ἔστι δὲ καὶ τῶν ΛΜ, ΝΞ τετραγώνων μέσον ἀνάλογον τὸ ΜΝ· καί ἐστιν ἴσον τὸ μὲν ΑΙ τῷ ΛΜ, τὸ δὲ ΖΚ τῷ ΝΞ· καὶ τὸ ΕΚ ἄρα ἴσον ἐστὶ τῷ ΜΝ. ἀλλὰ τὸ μὲν ΜΝ ἴσον ἐστὶ τῷ ΛΞ, τὸ δὲ ΕΚ ἴσον [ἐστὶ] τῷ ΔΘ· καὶ ὅλον ἄρα τὸ ΔΚ ἴσον ἐστὶ τῷ ΥΦΧ γνώμονι καὶ τῷ ΝΞ. ἔστι δὲ καὶ τὸ ΑΚ ἴσον τοῖς ΛΜ, ΝΞ· λοιπὸν ἄρα τὸ ΑΒ ἴσον ἐστὶ τῷ ΣΤ, τουτέστι τῷ ἀπὸ τῆς ΛΝ τετραγώνῳ· ἡ ΛΝ ἄρα δύναται τὸ ΑΒ χωρίον.

Λέγω, ὅτι ἡ ΛΝ μέσης ἀποτομή ἐστι δευτέρα.

Ἐπεὶ γὰρ μέσα ἐδείχθη τὰ ΑΙ, ΖΚ καί ἐστιν ἴσα τοῖς ἀπὸ τῶν ΛΟ, ΟΝ, μέσον ἄρα καὶ ἑκάτερον τῶν ἀπὸ τῶν ΛΟ, ΟΝ· μέση ἄρα ἑκατέρα τῶν ΛΟ, ΟΝ. καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ΑΙ τῷ ΖΚ, σύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΛΟ τῷ ἀπὸ τῆς ΟΝ. πάλιν, ἐπεὶ ἀσύμμετρον ἐδείχθη τὸ ΑΙ τῷ ΕΚ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ΛΜ τῷ ΜΝ, τουτέστι τὸ ἀπὸ τῆς ΛΟ τῷ ὑπὸ τῶν ΛΟ, ΟΝ· ὥστε καὶ ἡ ΛΟ ἀσύμμετρός ἐστι τῇ ΟΝ· αἱ ΛΟ, ΟΝ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι.

Λέγω δή, ὅτι καὶ μέσον περιέχουσιν.

Ἐπεὶ γὰρ μέσον ἐδείχθη τὸ ΕΚ καί ἐστιν ἴσον τῷ ὑπὸ τῶν ΛΟ, ΟΝ, μέσον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΛΟ, ΟΝ· ὥστε αἱ ΛΟ, ΟΝ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι. ἡ ΛΝ ἄρα μέσης ἀποτομή ἐστι δευτέρα· καὶ δύναται τὸ ΑΒ χωρίον.

Ἡ ἄρα τὸ ΑΒ χωρίον δυναμένη μέσης ἀποτομή ἐστι δευτέρα· ὅπερ ἔδει δεῖξαι.

ϗδ΄. Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ ἀποτομῆς τετάρτης, ἡ τὸ χωρίον δυναμένη ἐλάσσων ἐστίν.

Χωρίον γὰρ τὸ ΑΒ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΓ καὶ ἀποτομῆς τετάρτης τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΒ χωρίον δυναμένη ἐλάσσων ἐστίν.

Ἔστω γὰρ τῇ ΑΔ προσαρμόζουσα ἡ ΔΗ· αἱ ἄρα ΑΗ, ΗΔ ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΗ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΑΓ μήκει, ἡ δὲ ὅλη ἡ ΑΗ τῆς προσαρμοζούσης τῆς ΔΗ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει. ἐπεὶ οὖν ἡ ΑΗ τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει, ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν ΑΗ παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς ἀσύμμετρα αὐτὴν διελεῖ. τετμήσθω οὖν ἡ ΔΗ δίχα κατὰ τὸ Ε, καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ· ἀσύμμετρος ἄρα ἐστὶ μήκει ἡ ΑΖ τῇ ΖΗ. ἤχθωσαν οὖν διὰ τῶν Ε, Ζ, Η παράλληλοι ταῖς ΑΓ, ΒΔ αἱ ΕΘ, ΖΙ, ΗΚ. ἐπεὶ οὖν ῥητή ἐστιν ἡ ΑΗ καὶ σύμμετρος τῇ ΑΓ μήκει, ῥητὸν ἄρα ἐστὶν ὅλον τὸ ΑΚ. πάλιν, ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΔΗ τῇ ΑΓ μήκει, καί εἰσιν ἀμφότεραι ῥηταί, μέσον ἄρα ἐστὶ τὸ ΔΚ. πάλιν, ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΖ τῇ ΖΗ μήκει, ἀσύμμετρον ἄρα καὶ τὸ ΑΙ τῷ ΖΚ. συνεστάτω οὖν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον ἀφῃρήσθω περὶ τὴν αὐτὴν γωνίαν τὴν ὑπὸ τῶν ΛΟΜ τὸ ΝΞ. περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὰ ΛΜ, ΝΞ τετράγωνα. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν τὸ ὑπὸ τῶν ΑΖ, ΖΗ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΗ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἡ ΕΗ πρὸς τὴν ΖΗ. ἀλλ’ ὡς μὲν ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἐστὶ τὸ ΑΙ πρὸς τὸ ΕΚ, ὡς δὲ ἡ ΕΗ πρὸς τὴν ΖΗ, οὕτως ἐστὶ τὸ ΕΚ πρὸς τὸ ΖΚ· τῶν ἄρα ΑΙ, ΖΚ μέσον ἀνάλογόν ἐστι τὸ ΕΚ. ἔστι δὲ καὶ τῶν ΛΜ, ΝΞ τετραγώνων μέσον ἀνάλογον τὸ ΜΝ, καί ἐστιν ἴσον τὸ μὲν ΑΙ τῷ ΛΜ, τὸ δὲ ΖΚ τῷ ΝΞ· καὶ τὸ ΕΚ ἄρα ἴσον ἐστὶ τῷ ΜΝ. ἀλλὰ τῷ μὲν ΕΚ ἴσον ἐστὶ τὸ ΔΘ, τῷ δὲ ΜΝ ἴσον ἐστὶ τὸ ΛΞ· ὅλον ἄρα τὸ ΔΚ ἴσον ἐστὶ τῷ ΥΦΧ γνώμονι καὶ τῷ ΝΞ. ἐπεὶ οὖν ὅλον τὸ ΑΚ ἴσον ἐστὶ τοῖς ΛΜ, ΝΞ τετραγώνοις, ὧν τὸ ΔΚ ἴσον ἐστὶ τῷ ΥΦΧ

γνώμονι καὶ τῷ ΝΞ τετραγώνῳ, λοιπὸν ἄρα τὸ ΑΒ ἴσον ἐστὶ τῷ ΣΤ, τουτέστι τῷ ἀπὸ τῆς ΛΝ τετραγώνῳ· ἡ ΛΝ ἄρα δύναται τὸ ΑΒ χωρίον.

Λέγω, ὅτι ἡ ΛΝ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

Ἐπεὶ γὰρ ῥητόν ἐστι τὸ ΑΚ καί ἐστιν ἴσον τοῖς ἀπὸ τῶν ΛΟ, ΟΝ τετραγώνοις, τὸ ἄρα συγκείμενον ἐκ τῶν ἀπὸ τῶν ΛΟ, ΟΝ ῥητόν ἐστιν. πάλιν, ἐπεὶ τὸ ΔΚ μέσον ἐστίν, καί ἐστιν ἴσον τὸ ΔΚ τῷ δὶς ὑπὸ τῶν ΛΟ, ΟΝ, τὸ ἄρα δὶς ὑπὸ τῶν ΛΟ, ΟΝ μέσον ἐστίν. καὶ ἐπεὶ ἀσύμμετρον ἐδείχθη τὸ ΑΙ τῷ ΖΚ, ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΛΟ τετράγωνον τῷ ἀπὸ τῆς ΟΝ τετραγώνῳ. αἱ ΛΟ, ΟΝ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δὲ δὶς ὑπ᾽ αὐτῶν μέσον. ἡ ΛΝ ἄρα ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων· καὶ δύναται τὸ ΑΒ χωρίον.

Ἡ ἄρα τὸ ΑΒ χωρίον δυναμένη ἐλάσσων ἐστίν· ὅπερ ἔδει δεῖξαι.

ιε΄. Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ ἀποτομῆς πέμπτης, ἡ τὸ χωρίον δυναμένη [ἡ] μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Χωρίον γὰρ τὸ ΑΒ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΓ καὶ ἀποτομῆς πέμπτης τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΒ χωρίον δυναμένη [ἡ] μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Ἔστω γὰρ τῇ ΑΔ προσαρμόζουσα ἡ ΔΗ· αἱ ἄρα ΑΗ, ΗΔ ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ προς αρμόζουσα ἡ ΗΔ σύμμετρός ἐστι μήκει τῇ ἐκκειμένῃ ῥητῇ τῇ ΑΓ, ἡ δὲ ὅλη ἡ ΑΗ τῆς προσαρμοζούσης τῆς ΔΗ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν ΑΗ παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς ἀσύμμετρα αὐτὴν διελεῖ. τετμήσθω οὖν ἡ ΔΗ δίχα κατὰ τὸ Ε σημεῖον, καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΖ τῇ ΖΗ μήκει. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΗ τῇ ΓΑ μήκει, καί εἰσιν ἀμφότεραι ῥηταί, μέσον ἄρα

ἐστὶ τὸ ΑΚ. πάλιν, ἐπεὶ ῥητή ἐστιν ἡ ΔΗ καὶ σύμμετρος τῇ ΑΓ μήκει, ῥητόν ἐστι τὸ ΔΚ. συνεστάτω οὖν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον τετράγωνον ἀφῃρήσθω τὸ ΝΞ περὶ τὴν αὐτὴν γωνίαν τὴν ὑπὸ ΛΟΜ· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὰ ΛΜ, ΝΞ τετράγωνα. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ὁμοίως δὴ δείξομεν, ὅτι ἡ ΛΝ δύναται τὸ ΑΒ χωρίον.

Λέγω, ὅτι ἡ ΛΝ ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Ἐπεὶ γὰρ μέσον ἐδείχθη τὸ ΑΚ καί ἐστιν ἴσον τοῖς ἀπὸ τῶν ΛΟ, ΟΝ, τὸ ἄρα συγκείμενον ἐκ τῶν ἀπὸ τῶν ΛΟ, ΟΝ μέσον ἐστίν. πάλιν, ἐπεὶ ῥητόν ἐστι τὸ ΔΚ καί ἐστιν ἴσον τῷ δὶς ὑπὸ τῶν ΛΟ, ΟΝ, καὶ αὐτὸ ῥητόν ἐστιν. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ ΑΙ τῷ ΖΚ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΛΟ τῷ ἀπὸ τῆς ΟΝ· αἱ ΛΟ, ΟΝ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον, τὸ δὲ δὶς ὑπ᾽ αὐτῶν ῥητόν. ἡ λοιπὴ ἄρα ἡ ΛΝ ἄλογός ἐστιν ἡ καλουμένη μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα· καὶ δύναται τὸ ΑΒ χωρίον.

Ἡ τὸ ΑΒ ἄρα χωρίον δυναμένη μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.

ιϛ΄. Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ ἀποτομῆς ἕκτης, ἡ τὸ χωρίον δυναμένη μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Χωρίον γὰρ τὸ ΑΒ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΓ καὶ ἀποτομῆς ἕκτης τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΒ χωρίον δυναμένη [ἡ] μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Ἔστω γὰρ τῇ ΑΔ προσαρμόζουσα ἡ ΔΗ· αἱ ἄρα ΑΗ, ΗΔ ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ οὐδετέρα αὐτῶν σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΑΓ μήκει, ἡ δὲ ὅλη ἡ ΑΗ τῆς προσαρμοζούσης τῆς ΔΗ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει. ἐπεὶ οὖν ἡ ΑΗ τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει, ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ

τῆς ΔΗ ἴσον παρὰ τὴν ΑΗ παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς ἀσύμμετρα αὐτὴν διελεῖ. τετμήσθω οὖν ἡ ΔΗ δίχα κατὰ τὸ Ε [σημεῖον], καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΖ τῇ ΖΗ μήκει. ὡς δὲ ἡ ΑΖ πρὸς τὴν ΖΗ, οὕτως ἐστὶ τὸ ΑΙ πρὸς τὸ ΖΚ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΑΙ τῷ ΖΚ. καὶ ἐπεὶ αἱ ΑΗ, ΑΓ ῥηταί εἰσι δυνάμει μόνον σύμμετροι, μέσον ἐστὶ τὸ ΑΚ. πάλιν, ἐπεὶ αἱ ΑΓ, ΔΗ ῥηταί εἰσι καὶ ἀσύμμετροι μήκει, μέσον ἐστὶ καὶ τὸ ΔΚ. ἐπεὶ οὖν αἱ ΑΗ, ΗΔ δυνάμει μόνον σύμμετροί εἰσιν, ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΗ τῇ ΗΔ μήκει. ὡς δὲ ἡ ΑΗ πρὸς τὴν ΗΔ, οὕτως ἐστὶ τὸ ΑΚ πρὸς τὸ ΚΔ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΑΚ τῷ ΚΔ. συνεστάτω οὖν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον ἀφῃρήσθω περὶ τὴν αὐτὴν γωνίαν τὸ ΝΞ· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὰ ΛΜ, ΝΞ τετράγωνα. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ὁμοίως δὴ τοῖς ἐπάνω δείξομεν, ὅτι ἡ ΛΝ δύναται τὸ ΑΒ χωρίον.

Λέγω, ὅτι ἡ ΛΝ [ἡ] μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Ἐπεὶ γὰρ μέσον ἐδείχθη τὸ ΑΚ καί ἐστιν ἴσον τοῖς ἀπὸ τῶν ΛΟ, ΟΝ, τὸ ἄρα συγκείμενον ἐκ τῶν ἀπὸ τῶν ΛΟ, ΟΝ μέσον ἐστίν. πάλιν, ἐπεὶ μέσον ἐδείχθη τὸ ΔΚ καί ἐστιν ἴσον τῷ δὶς ὑπὸ τῶν ΛΟ, ΟΝ, καὶ τὸ δὶς ὑπὸ τῶν ΛΟ, ΟΝ μέσον ἐστίν. καὶ ἐπεὶ ἀσύμμετρον ἐδείχθη τὸ ΑΚ τῷ ΔΚ, ἀσύμμετρα [ἄρα] ἐστὶ καὶ τὰ ἀπὸ τῶν ΛΟ, ΟΝ τετράγωνα τῷ δὶς ὑπὸ τῶν ΛΟ, ΟΝ. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ ΑΙ τῷ ΖΚ, ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΛΟ τῷ ἀπὸ τῆς ΟΝ· αἱ ΛΟ, ΟΝ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον καὶ τὸ δὶς ὑπ᾽ αὐτῶν μέσον ἔτι τε τὰ ἀπ᾽ αὐτῶν τετράγωνα ἀσύμμετρα τῷ δὶς ὑπ᾽ αὐτῶν. ἡ ἄρα ΛΝ ἄλογός ἐστιν ἡ καλουμένη μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα· καὶ δύναται τὸ ΑΒ χωρίον.

Ἡ ἄρα τὸ χωρίον δυναμένη μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.

ϗζ΄. Τὸ ἀπὸ ἀποτομῆς παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πρώτην.

Ἔστω ἀποτομὴ ἡ ΑΒ, ῥητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστι πρώτη.

Ἔστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΒΗ τὸ ΚΛ. ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ· ὧν τὸ ΓΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ· λοιπὸν ἄρα τὸ ΖΛ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. τετμήσθω ἡ ΖΜ δίχα κατὰ τὸ Ν σημεῖον, καὶ ἤχθω διὰ τοῦ Ν τῇ ΓΔ παράλληλος ἡ ΝΞ· ἑκάτερον ἄρα τῶν ΖΞ, ΛΝ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. καὶ ἐπεὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ ῥητά ἐστιν, καί ἐστι τοῖς ἀπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΔΜ, ῥητὸν ἄρα ἐστὶ τὸ ΔΜ. καὶ παρὰ ῥητὴν τὴν ΓΔ παραβέβληται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ἡ ΓΜ καὶ σύμμετρος τῇ ΓΔ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ, καὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΖΛ, μέσον ἄρα τὸ ΖΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα ἐστὶν ἡ ΖΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ τὰ μὲν ἀπὸ τῶν ΑΗ, ΗΒ ῥητά ἐστιν, τὸ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ μέσον, ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. καὶ τοῖς μὲν ἀπὸ τῶν ΑΗ, ΗΒ ἴσον ἐστὶ τὸ ΓΛ, τῷ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ τὸ ΖΛ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΔΜ τῷ ΖΛ. ὡς δὲ τὸ ΔΜ πρὸς τὸ ΖΛ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΖΜ. ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΜ τῇ ΖΜ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΓΖ ἄρα ἀποτομή ἐστιν.

Λέγω δή, ὅτι καὶ πρώτη.

Ἐπεὶ γὰρ τῶν ἀπὸ τῶν ΑΗ, ΗΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καί ἐστι τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΒΗ ἴσον τὸ ΚΛ, τῷ δὲ ὑπὸ τῶν ΑΗ, ΗΒ τὸ ΝΛ, καὶ τῶν ΓΘ, ΚΛ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΝΛ·

ἔστιν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως τὸ ΝΛ πρὸς τὸ ΚΛ. ἀλλ᾽ ὡς μὲν τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΝΜ· ὡς δὲ τὸ ΝΛ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΚΜ· τὸ ἄρα ὑπὸ τῶν ΓΚ, ΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΝΜ, τουτέστι τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ. καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ, σύμμετρόν [ἐστι] καὶ τὸ ΓΘ τῷ ΚΛ. ὡς δὲ τὸ ΓΘ πρὸς τὸ ΚΛ, οὕτως ἡ ΓΚ πρὸς τὴν ΚΜ· σύμμετρος ἄρα ἐστὶν ἡ ΓΚ τῇ ΚΜ. ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ ἴσον παρὰ τὴν ΓΜ παραβέβληται ἐλλεῖπον εἴδει τετραγώνῳ τὸ ὑπὸ τῶν ΓΚ, ΚΜ, καί ἐστι σύμμετρος ἡ ΓΚ τῇ ΚΜ, ἡ ἄρα ΓΜ τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. καί ἐστιν ἡ ΓΜ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ τῇ ΓΔ μήκει· ἡ ἄρα ΓΖ ἀποτομή ἐστι πρώτη.

Τὸ ἄρα ἀπὸ ἀποτομῆς παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πρώτην· ὅπερ ἔδει δεῖξαι.

ιη΄. Τὸ ἀπὸ μέσης ἀποτομῆς πρώτης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν δευτέραν.

Ἔστω μέσης ἀποτομὴ πρώτη ἡ ΑΒ, ῥητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστι δευτέρα.

Ἔστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι. καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΘ πλάτος ποιοῦν τὴν ΓΚ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ πλάτος ποιοῦν τὴν ΚΜ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ· μέσον ἄρα καὶ τὸ ΓΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παράκειται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ἡ ΓΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ὧν τὸ ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τῷ ΓΕ, λοιπὸν ἄρα τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ ἴσον ἐστὶ τῷ ΖΛ. ῥητὸν δὲ [ἐστι] τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ· ῥητὸν ἄρα τὸ ΖΛ. καὶ παρὰ ῥητὴν τὴν ΖΕ παράκειται

πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΖΜ καὶ σύμμετρος τῇ ΓΔ μήκει. ἐπεὶ οὖν τὰ μὲν ἀπὸ τῶν ΑΗ, ΗΒ, τουτέστι τὸ ΓΛ, μέσον ἐστίν, τὸ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ, τουτέστι τὸ ΖΛ, ῥητόν, ἀσύμμετρον ἄρα ἐστὶ τὸ ΓΛ τῷ ΖΛ. ὡς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΖΜ· ἀσύμμετρος ἄρα ἡ ΓΜ τῇ ΖΜ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΓΖ ἄρα ἀποτομή ἐστιν.

Λέγω δή, ὅτι καὶ δευτέρα.

Τετμήσθω γὰρ ἡ ΖΜ δίχα κατὰ τὸ Ν, καὶ ἤχθω διὰ τοῦ Ν τῇ ΓΔ παράλληλος ἡ ΝΞ· ἑκάτερον ἄρα τῶν ΖΞ, ΝΛ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. καὶ ἐπεὶ τῶν ἀπὸ τῶν ΑΗ, ΗΒ τετραγώνων μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καί ἐστιν ἴσον τὸ μὲν ἀπὸ τῆς ΑΗ τῷ ΓΘ, τὸ δὲ ὑπὸ τῶν ΑΗ, ΗΒ τῷ ΝΛ, τὸ δὲ ἀπὸ τῆς ΒΗ τῷ ΚΛ, καὶ τῶν ΓΘ, ΚΛ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΝΛ· ἔστιν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως τὸ ΝΛ πρὸς τὸ ΚΛ. ἀλλ’ ὡς μὲν τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΝΜ, ὡς δὲ τὸ ΝΛ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΜΚ· ὡς ἄρα ἡ ΓΚ πρὸς τὴν ΝΜ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΚΜ· τὸ ἄρα ὑπὸ τῶν ΓΚ, ΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΝΜ, τουτέστι τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ. [καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΒΗ, σύμμετρόν ἐστι καὶ τὸ ΓΘ τῷ ΚΛ, τουτέστιν ἡ ΓΚ τῇ ΚΜ.] ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΜΖ ἴσον παρὰ τὴν μείζονα τὴν ΓΜ παραβέβληται ἐλλεῖπον εἴδει τετραγώνῳ τὸ ὑπὸ τῶν ΓΚ, ΚΜ καὶ εἰς σύμμετρα αὐτὴν διαιρεῖ, ἡ ἄρα ΓΜ τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. καί ἐστιν ἡ προσαρμόζουσα ἡ ΖΜ σύμμετρος μήκει τῇ ἐκκειμένῃ ῥητῇ τῇ ΓΔ· ἡ ἄρα ΓΖ ἀποτομή ἐστι δευτέρα.

Τὸ ἄρα ἀπὸ μέσης ἀποτομῆς πρώτης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν δευτέραν· ὅπερ ἔδει δεῖξαι.

ϡθ΄. Τὸ ἀπὸ μέσης ἀποτομῆς δευτέρας παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τρίτην.

Ἔστω μέσης ἀποτομὴ δευτέρα ἡ ΑΒ, ῥητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστι τρίτη.

Ἔστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι. καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΘ πλάτος ποιοῦν τὴν ΓΚ, τῷ δὲ ἀπὸ τῆς ΒΗ ἴσον παρὰ τὴν ΚΘ παραβεβλήσθω τὸ ΚΛ πλάτος ποιοῦν τὴν ΚΜ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ [καί ἐστι μέσα τὰ ἀπὸ τῶν ΑΗ, ΗΒ]· μέσον ἄρα καὶ τὸ ΓΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παραβέβληται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ἡ ΓΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ ὅλον τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ὧν τὸ ΓΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ, λοιπὸν ἄρα τὸ ΛΖ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. τετμήσθω οὖν ἡ ΖΜ δίχα κατὰ τὸ Ν σημεῖον, καὶ τῇ ΓΔ παράλληλος ἤχθω ἡ ΝΞ· ἑκάτερον ἄρα τῶν ΖΞ, ΝΛ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. μέσον δὲ τὸ ὑπὸ τῶν ΑΗ, ΗΒ· μέσον ἄρα ἐστὶ καὶ τὸ ΖΛ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα καὶ ἡ ΖΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ αἱ ΑΗ, ΗΒ δυνάμει μόνον εἰσὶ σύμμετροι, ἀσύμμετρος ἄρα [ἐστὶ] μήκει ἡ ΑΗ τῇ ΗΒ· ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΑΗ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΗ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΗ, ΗΒ, τῷ δὲ ὑπὸ τῶν ΑΗ, ΗΒ τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ· ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΑΗ, ΗΒ ἴσον ἐστὶ τὸ ΓΛ, τῷ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ ἴσον ἐστὶ τὸ ΖΛ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΓΛ τῷ ΖΛ. ὡς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΖΜ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΜ τῇ ΖΜ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΓΖ.

Λέγω δή, ὅτι καὶ τρίτη.

Ἐπεὶ γὰρ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ, σύμμετρον ἄρα καὶ τὸ ΓΘ τῷ ΚΛ· ὥστε καὶ ἡ ΓΚ τῇ ΚΜ. καὶ ἐπεὶ τῶν ἀπὸ τῶν ΑΗ, ΗΒ

μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καί ἐστι τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ, τῷ δὲ ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΝΛ, καὶ τῶν ΓΘ, ΚΛ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΝΛ· ἔστιν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως τὸ ΝΛ πρὸς τὸ ΚΛ. ἀλλ’ ὡς μὲν τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΝΜ, ὡς δὲ τὸ ΝΛ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΚΜ· ὡς ἄρα ἡ ΓΚ πρὸς τὴν ΜΝ, οὕτως ἐστὶν ἡ ΜΝ πρὸς τὴν ΚΜ· τὸ ἄρα ὑπὸ τῶν ΓΚ, ΚΜ ἴσον ἐστὶ τῷ [ἀπὸ τῆς ΜΝ, τουτέστι τῷ] τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ. ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ ἴσον παρὰ τὴν ΓΜ παραβέβληται ἐλλεῖπον εἴδει τετραγώνῳ καὶ εἰς σύμμετρα αὐτὴν διαιρεῖ, ἡ ΓΜ ἄρα τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καὶ οὐδετέρα τῶν ΓΜ, ΜΖ σύμμετρός ἐστι μήκει τῇ ἐκκειμένῃ ῥητῇ τῇ ΓΔ· ἡ ἄρα ΓΖ ἀποτομή ἐστι τρίτη.

Τὸ ἄρα ἀπὸ μέσης ἀποτομῆς δευτέρας παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τρίτην· ὅπερ ἔδει δεῖξαι.

ρ΄. Τὸ ἀπὸ ἐλάσσονος παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τετάρτην.

Ἔστω ἐλάσσων ἡ ΑΒ, ῥητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ ῥητὴν τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστι τετάρτη.

Ἔστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΗ, ΗΒ τετραγώνων ῥητόν, τὸ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ μέσον. καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΘ πλάτος ποιοῦν τὴν ΓΚ, τῷ δὲ ἀπὸ τῆς ΒΗ ἴσον τὸ ΚΛ πλάτος ποιοῦν τὴν ΚΜ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ. καί ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΗ, ΗΒ ῥητόν· ῥητὸν ἄρα ἐστὶ καὶ τὸ ΓΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παράκειται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα καὶ ἡ ΓΜ καὶ σύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ ὅλον τὸ

ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ὧν τὸ ΓΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ, λοιπὸν ἄρα τὸ ΖΛ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. τετμήσθω οὖν ἡ ΖΜ δίχα κατὰ τὸ Ν σημεῖον, καὶ ἤχθω διὰ τοῦ Ν ὁποτέρᾳ τῶν ΓΔ, ΜΛ παράλληλος ἡ ΝΞ· ἑκάτερον ἄρα τῶν ΖΞ, ΝΛ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. καὶ ἐπεὶ τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ μέσον ἐστὶ καί ἐστιν ἴσον τῷ ΖΛ, καὶ τὸ ΖΛ ἄρα μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΖΕ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα ἐστὶν ἡ ΖΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΗ, ΗΒ ῥητόν ἐστιν, τὸ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ μέσον, ἀσύμμετρα [ἄρα] ἐστὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. ἴσον δέ [ἐστι] τὸ ΓΛ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, τῷ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΖΛ· ἀσύμμετρον ἄρα [ἐστὶ] τὸ ΓΛ τῷ ΖΛ. ὡς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΜΖ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΜ τῇ ΜΖ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΓΖ.

Λέγω [δή], ὅτι καὶ τετάρτη.

Ἐπεὶ γὰρ αἱ ΑΗ, ΗΒ δυνάμει εἰσὶν ἀσύμμετροι, ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ. καί ἐστι τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΓΘ τῷ ΚΛ. ὡς δὲ τὸ ΓΘ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΚΜ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΚ τῇ ΚΜ μήκει. καὶ ἐπεὶ τῶν ἀπὸ τῶν ΑΗ, ΗΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καί ἐστιν ἴσον τὸ μὲν ἀπὸ τῆς ΑΗ τῷ ΓΘ, τὸ δὲ ἀπὸ τῆς ΗΒ τῷ ΚΛ, τὸ δὲ ὑπὸ τῶν ΑΗ, ΗΒ τῷ ΝΛ, τῶν ἄρα ΓΘ, ΚΛ μέσον ἀνάλογόν ἐστι τὸ ΝΛ· ἔστιν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως τὸ ΝΛ πρὸς τὸ ΚΛ. ἀλλ’ ὡς μὲν τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΝΜ, ὡς δὲ τὸ ΝΛ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΚΜ· ὡς ἄρα ἡ ΓΚ πρὸς τὴν ΜΝ, οὕτως ἐστὶν ἡ ΜΝ πρὸς τὴν ΚΜ· τὸ ἄρα ὑπὸ τῶν ΓΚ, ΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ, τουτέστι τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ. ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΜΖ ἴσον παρὰ τὴν ΓΜ

παραβέβληται ἐλλεῖπον εἴδει τετραγώνῳ τὸ ὑπὸ τῶν ΓΚ, ΚΜ καὶ εἰς ἀσύμμετρα αὐτὴν διαιρεῖ, ἡ ἄρα ΓΜ τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καί ἐστιν ὅλη ἡ ΓΜ σύμμετρος μήκει τῇ ἐκκειμένῃ ῥητῇ τῇ ΓΔ· ἡ ἄρα ΓΖ ἀποτομή ἐστι τετάρτη.

Τὸ ἄρα ἀπὸ ἐλάσσονος καὶ τὰ ἑξῆς.

ρα΄. Τὸ ἀπὸ τῆς μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πέμπτην.

Ἔστω ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα ἡ ΑΒ, ῥητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστι πέμπτη.

Ἔστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ εὐθεῖαι δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον, τὸ δὲ δὶς ὑπ᾽ αὐτῶν ῥητόν. καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ. τὸ δὲ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΗ, ΗΒ ἅμα μέσον ἐστίν· μέσον ἄρα ἐστὶ τὸ ΓΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παράκειται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ἡ ΓΜ καὶ ἀσύμμετρος τῇ ΓΔ. καὶ ἐπεὶ ὅλον τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ὧν τὸ ΓΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ, λοιπὸν ἄρα τὸ ΖΛ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. τετμήσθω οὖν ἡ ΖΜ δίχα κατὰ τὸ Ν, καὶ ἤχθω διὰ τοῦ Ν ὁποτέρᾳ τῶν ΓΔ, ΜΛ παράλληλος ἡ ΝΞ· ἑκάτερον ἄρα τῶν ΖΞ, ΝΛ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. καὶ ἐπεὶ τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ ῥητόν ἐστι καί [ἐστιν] ἴσον τῷ ΖΛ, ῥητὸν ἄρα ἐστὶ τὸ ΖΛ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα ἐστὶν ἡ ΖΜ καὶ σύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ τὸ μὲν ΓΛ μέσον ἐστίν, τὸ δὲ ΖΛ ῥητόν, ἀσύμμετρον ἄρα ἐστὶ τὸ ΓΛ τῷ ΖΛ. ὡς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ἡ ΓΜ πρὸς τὴν ΜΖ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΜ τῇ

ΜΖ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΓΖ.

Λέγω δή, ὅτι καὶ πέμπτη.

Ὁμοίως γὰρ δείξομεν, ὅτι τὸ ὑπὸ τῶν ΓΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΝΜ, τουτέστι τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ, ἴσον δὲ τὸ μὲν ἀπὸ τῆς ΑΗ τῷ ΓΘ, τὸ δὲ ἀπὸ τῆς ΗΒ τῷ ΚΛ, ἀσύμμετρον ἄρα τὸ ΓΘ τῷ ΚΛ. ὡς δὲ τὸ ΓΘ πρὸς τὸ ΚΛ, οὕτως ἡ ΓΚ πρὸς τὴν ΚΜ· ἀσύμμετρος ἄρα ἡ ΓΚ τῇ ΚΜ μήκει. ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ ἴσον παρὰ τὴν ΓΜ παραβέβληται ἐλλεῖπον εἴδει τετραγώνῳ καὶ εἰς ἀσύμμετρα αὐτὴν διαιρεῖ, ἡ ἄρα ΓΜ τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καί ἐστιν ἡ προσαρμόζουσα ἡ ΖΜ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ τῇ ΓΔ· ἡ ἄρα ΓΖ ἀποτομή ἐστι πέμπτη· ὅπερ ἔδει δεῖξαι.

ρβ΄. Τὸ ἀπὸ τῆς μετὰ μέσου μέσον τὸ ὅλον ποιούσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν ἕκτην.

Ἔστω ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα ἡ ΑΒ, ῥητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστιν ἕκτη.

Ἔστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον καὶ τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ μέσον καὶ ἀσύμμετρον τὰ ἀπὸ τῶν ΑΗ, ΗΒ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. παραβεβλήσθω οὖν παρὰ τὴν ΓΔ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ πλάτος ποιοῦν τὴν ΓΚ, τῷ δὲ ἀπὸ τῆς ΒΗ τὸ ΚΛ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ· μέσον ἄρα [ἐστὶ] καὶ τὸ ΓΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παράκειται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ἡ ΓΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. ἐπεὶ οὖν τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ᾧ δὲ τὸ ΓΕ ἴσον τῷ ἀπὸ τῆς ΑΒ, λοιπὸν ἄρα τὸ ΖΛ ἴσον ἐστὶ δὶς ὑπὸ τῶν ΑΗ,

ΗΒ. καί ἐστι τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ μέσον· καὶ τὸ ΖΛ ἄρα μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΖΕ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα ἐστὶν ἡ ΖΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ ἀσύμμετρά ἐστι τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ, καί ἐστι τοῖς μὲν ἀπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΓΛ, τῷ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΖΛ, ἀσύμμετρον ἄρα [ἐστὶ] τὸ ΓΛ τῷ ΖΛ. ὡς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΜΖ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΜ τῇ ΜΖ μήκει. καί εἰσιν ἀμφότεραι ῥηταί. αἱ ΓΜ, ΜΖ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΓΖ.

Λέγω δή, ὅτι καὶ ἕκτη.

Ἐπεὶ γὰρ τὸ ΖΛ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ, τετμήσθω δίχα ἡ ΖΜ κατὰ τὸ Ν, καὶ ἤχθω διὰ τοῦ Ν τῇ ΓΔ παράλληλος ἡ ΝΞ· ἑκάτερον ἄρα τῶν ΖΞ, ΝΛ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. καὶ ἐπεὶ αἱ ΑΗ, ΗΒ δυνάμει εἰσὶν ἀσύμμετροι, ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον ἐστὶ τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον ἐστὶ τὸ ΚΛ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΓΘ τῷ ΚΛ. ὡς δὲ τὸ ΓΘ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΚΜ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΚ τῇ ΚΜ. καὶ ἐπεὶ τῶν ἀπὸ τῶν ΑΗ, ΗΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καί ἐστι τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ, τῷ δὲ ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΝΛ, καὶ τῶν ἄρα ΓΘ, ΚΛ μέσον ἀνάλογόν ἐστι τὸ ΝΛ· ἔστιν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως τὸ ΝΛ πρὸς τὸ ΚΛ. καὶ διὰ τὰ αὐτὰ ἡ ΓΜ τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ οὐδετέρα αὐτῶν σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΓΔ· ἡ ΓΖ ἄρα ἀποτομή ἐστιν ἕκτη· ὅπερ ἔδει δεῖξαι.

ργ΄. Ἡ τῇ ἀποτομῇ μήκει σύμμετρος ἀποτομή ἐστι καὶ τῇ τάξει ἡ αὐτή.

Ἔστω ἀποτομὴ ἡ ΑΒ, καὶ τῇ ΑΒ μήκει σύμμετρος ἔστω ἡ ΓΔ· λέγω, ὅτι καὶ ἡ ΓΔ ἀποτομή ἐστι καὶ τῇ τάξει ἡ αὐτὴ τῇ ΑΒ.

Ἐπεὶ γὰρ ἀποτομή ἐστιν ἡ ΑΒ, ἔστω αὐτῇ προσαρμόζουσα ἡ ΒΕ· αἱ ΑΕ, ΕΒ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ τῷ τῆς ΑΒ πρὸς τὴν ΓΔ λόγῳ ὁ αὐτὸς γεγονέτω ὁ τῆς ΒΕ πρὸς τὴν ΔΖ· καὶ ὡς ἓν ἄρα πρὸς ἕν, πάντα [ἐστὶ] πρὸς πάντα· ἔστιν ἄρα καὶ ὡς ὅλη ἡ ΑΕ πρὸς ὅλην τὴν ΓΖ, οὕτως ἡ ΑΒ πρὸς τὴν ΓΔ. σύμμετρος δὲ ἡ ΑΒ τῇ ΓΔ μήκει. σύμμετρος ἄρα καὶ ἡ ΑΕ μὲν τῇ ΓΖ, ἡ δὲ ΒΕ τῇ ΔΖ. καὶ αἱ ΑΕ, ΕΒ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· καὶ αἱ ΓΖ, ΖΔ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. [ἀποτομὴ ἄρα ἐστὶν ἡ ΓΔ.

Λέγω δή, ὅτι καὶ τῇ τάξει ἡ αὐτὴ τῇ ΑΒ.]

Ἐπεὶ οὖν ἐστιν ὡς ἡ ΑΕ πρὸς τὴν ΓΖ, οὕτως ἡ ΒΕ πρὸς τὴν ΔΖ, ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ ΑΕ πρὸς τὴν ΕΒ, οὕτως ἡ ΓΖ πρὸς τὴν ΖΔ. ἤτοι δὴ ἡ ΑΕ τῆς ΕΒ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυμμέτρου. εἰ μὲν οὖν ἡ ΑΕ τῆς ΕΒ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ ΓΖ τῆς ΖΔ μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΑΕ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ΓΖ, εἰ δὲ ἡ ΒΕ, καὶ ἡ ΔΖ, εἰ δὲ οὐδετέρα τῶν ΑΕ, ΕΒ, καὶ οὐδετέρα τῶν ΓΖ, ΖΔ. εἰ δὲ ἡ ΑΕ [τῆς ΕΒ] μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ ἡ ΓΖ τῆς ΖΔ μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΑΕ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ΓΖ, εἰ δὲ ἡ ΒΕ, καὶ ἡ ΔΖ, εἰ δὲ οὐδετέρα τῶν ΑΕ, ΕΒ, οὐδετέρα τῶν ΓΖ, ΖΔ.

Ἀποτομὴ ἄρα ἐστὶν ἡ ΓΔ καὶ τῇ τάξει ἡ αὐτὴ τῇ ΑΒ· ὅπερ ἔδει δεῖξαι.

ρδ΄. Ἡ τῇ μέσης ἀποτομῇ σύμμετρος μέσης ἀποτομή ἐστι καὶ τῇ τάξει ἡ αὐτή.

Ἔστω μέσης ἀποτομὴ ἡ ΑΒ, καὶ τῇ ΑΒ μήκει σύμμετρος ἔστω ἡ ΓΔ· λέγω, ὅτι καὶ ἡ ΓΔ μέσης ἀποτομή ἐστι καὶ τῇ τάξει ἡ αὐτὴ τῇ ΑΒ.

Ἐπεὶ γὰρ μέσης ἀποτομή ἐστιν ἡ ΑΒ, ἔστω αὐτῇ προσαρμόζουσα ἡ ΕΒ. αἱ ΑΕ, ΕΒ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. καὶ γεγονέτω ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΒΕ πρὸς τὴν ΔΖ· σύμμετρος ἄρα [ἐστὶ] καὶ ἡ ΑΕ τῇ

ΓΖ, ἡ δὲ ΒΕ τῇ ΔΖ. αἱ δὲ ΑΕ, ΕΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι· καὶ αἱ ΓΖ, ΖΔ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι· μέσης ἄρα ἀποτομή ἐστιν ἡ ΓΔ.

Λέγω δή, ὅτι καὶ τῇ τάξει ἐστὶν ἡ αὐτὴ τῇ ΑΒ.

Ἐπεὶ [γάρ] ἐστιν ὡς ἡ ΑΕ πρὸς τὴν ΕΒ, οὕτως ἡ ΓΖ πρὸς τὴν ΖΔ [ἀλλ' ὡς μὲν ἡ ΑΕ πρὸς τὴν ΕΒ, οὕτως τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ὑπὸ τῶν ΑΕ, ΕΒ, ὡς δὲ ἡ ΓΖ πρὸς τὴν ΖΔ, οὕτως τὸ ἀπὸ τῆς ΓΖ πρὸς τὸ ὑπὸ τῶν ΓΖ, ΖΔ], ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ὑπὸ τῶν ΑΕ, ΕΒ, οὕτως τὸ ἀπὸ τῆς ΓΖ πρὸς τὸ ὑπὸ τῶν ΓΖ, ΖΔ [καὶ ἐναλλὰξ ὡς τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ἀπὸ τῆς ΓΖ, οὕτως τὸ ὑπὸ τῶν ΑΕ, ΕΒ πρὸς τὸ ὑπὸ τῶν ΓΖ, ΖΔ]. σύμμετρον δὲ τὸ ἀπὸ τῆς ΑΕ τῷ ἀπὸ τῆς ΓΖ· σύμμετρον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΑΕ, ΕΒ τῷ ὑπὸ τῶν ΓΖ, ΖΔ. εἴτε οὖν ῥητόν ἐστι τὸ ὑπὸ τῶν ΑΕ, ΕΒ, ῥητὸν ἔσται καὶ τὸ ὑπὸ τῶν ΓΖ, ΖΔ, εἴτε μέσον [ἐστὶ] τὸ ὑπὸ τῶν ΑΕ, ΕΒ, μέσον [ἐστὶ] καὶ τὸ ὑπὸ τῶν ΓΖ, ΖΔ.

Μέσης ἄρα ἀποτομή ἐστιν ἡ ΓΔ καὶ τῇ τάξει ἡ αὐτὴ τῇ ΑΒ· ὅπερ ἔδει δεῖξαι.

ρε΄. Ἡ τῇ ἐλάσσονι σύμμετρος ἐλάσσων ἐστίν.

Ἔστω γὰρ ἐλάσσων ἡ ΑΒ καὶ τῇ ΑΒ σύμμετρος ἡ ΓΔ· λέγω, ὅτι καὶ ἡ ΓΔ ἐλάσσων ἐστίν.

Γεγονέτω γὰρ τὰ αὐτά· καὶ ἐπεὶ αἱ ΑΕ, ΕΒ δυνάμει εἰσὶν ἀσύμμετροι, καὶ αἱ ΓΖ, ΖΔ ἄρα δυνάμει εἰσὶν ἀσύμμετροι. ἐπεὶ οὖν ἐστιν ὡς ἡ ΑΕ πρὸς τὴν ΕΒ, οὕτως ἡ ΓΖ πρὸς τὴν ΖΔ, ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ἀπὸ τῆς ΕΒ, οὕτως τὸ ἀπὸ τῆς ΓΖ πρὸς τὸ ἀπὸ τῆς ΖΔ. συνθέντι ἄρα ἐστὶν ὡς τὰ ἀπὸ τῶν ΑΕ, ΕΒ πρὸς τὸ ἀπὸ τῆς ΕΒ, οὕτως τὰ ἀπὸ τῶν ΓΖ, ΖΔ πρὸς τὸ ἀπὸ τῆς ΖΔ [καὶ ἐναλλάξ]· σύμμετρον δέ ἐστι τὸ ἀπὸ τῆς ΒΕ τῷ ἀπὸ τῆς ΔΖ· σύμμετρον ἄρα καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τετραγώνων τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων. ῥητὸν δέ ἐστι τὸ

συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τετραγώνων· ῥητὸν ἄρα ἐστὶ καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων. πάλιν, ἐπεί ἐστιν ὡς τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ὑπὸ τῶν ΑΕ, ΕΒ, οὕτως τὸ ἀπὸ τῆς ΓΖ πρὸς τὸ ὑπὸ τῶν ΓΖ, ΖΔ, σύμμετρον δὲ τὸ ἀπὸ τῆς ΑΕ τετράγωνον τῷ ἀπὸ τῆς ΓΖ τετραγώνῳ, σύμμετρον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΑΕ, ΕΒ τῷ ὑπὸ τῶν ΓΖ, ΖΔ. μέσον δὲ τὸ ὑπὸ τῶν ΑΕ, ΕΒ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΓΖ, ΖΔ· αἱ ΓΖ, ΖΔ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ’ αὐτῶν τετραγώνων ῥητόν, τὸ δ’ ὑπ’ αὐτῶν μέσον.

Ἐλάσσων ἄρα ἐστὶν ἡ ΓΔ· ὅπερ ἔδει δεῖξαι.

ρς΄. Ἡ τῇ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσῃ σύμμετρος μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Ἔστω μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα ἡ ΑΒ καὶ τῇ ΑΒ σύμμετρος ἡ ΓΔ· λέγω, ὅτι καὶ ἡ ΓΔ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Ἔστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΕ· αἱ ΑΕ, ΕΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τετραγώνων μέσον, τὸ δ’ ὑπ’ αὐτῶν ῥητόν. καὶ τὰ αὐτὰ κατεσκευάσθω. ὁμοίως δὴ δείξομεν τοῖς πρότερον, ὅτι αἱ ΓΖ, ΖΔ ἐν τῷ αὐτῷ λόγῳ εἰσὶ ταῖς ΑΕ, ΕΒ, καὶ σύμμετρόν ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τετραγώνων τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων, τὸ δὲ ὑπὸ τῶν ΑΕ, ΕΒ τῷ ὑπὸ τῶν ΓΖ, ΖΔ· ὥστε καὶ αἱ ΓΖ, ΖΔ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων μέσον, τὸ δ’ ὑπ’ αὐτῶν ῥητόν.

Ἡ ΓΔ ἄρα μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.

ρζ΄. Ἡ τῇ μετὰ μέσου μέσον τὸ ὅλον ποιούσῃ σύμμετρος καὶ αὐτὴ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Ἔστω μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα ἡ ΑΒ, καὶ τῇ ΑΒ ἔστω σύμμετρος ἡ ΓΔ· λέγω, ὅτι καὶ ἡ ΓΔ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Ἔστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΕ, καὶ τὰ αὐτὰ κατεσκευάσθω· αἱ ΑΕ, ΕΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ᾽ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τὸ συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων τῷ ὑπ᾽ αὐτῶν. καί εἰσιν, ὡς ἐδείχθη, αἱ ΑΕ, ΕΒ σύμμετροι ταῖς ΓΖ, ΖΔ, καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τετραγώνων τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ, τὸ δὲ ὑπὸ τῶν ΑΕ, ΕΒ τῷ ὑπὸ τῶν ΓΖ, ΖΔ· καὶ αἱ ΓΖ, ΖΔ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ᾽ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τὸ συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν [τετραγώνων] τῷ ὑπ᾽ αὐτῶν.

Ἡ ΓΔ ἄρα μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.

ρη΄. Ἀπὸ ῥητοῦ μέσου ἀφαιρουμένου ἡ τὸ λοιπὸν χωρίον δυναμένη μία δύο ἀλόγων γίνεται ἤτοι ἀποτομὴ ἢ ἐλάσσων.

Ἀπὸ γὰρ ῥητοῦ τοῦ ΒΓ μέσον ἀφῃρήσθω τὸ ΒΔ· λέγω, ὅτι ἡ τὸ λοιπὸν δυναμένη τὸ ΕΓ μία δύο ἀλόγων γίνεται ἤτοι ἀποτομὴ ἢ ἐλάσσων.

Ἐκκείσθω γὰρ ῥητὴ ἡ ΖΗ, καὶ τῷ μὲν ΒΓ ἴσον παρὰ τὴν ΖΗ παραβεβλήσθω ὀρθογώνιον παραλληλόγραμμον τὸ ΗΘ, τῷ δὲ ΔΒ ἴσον ἀφῃρήσθω τὸ ΗΚ· λοιπὸν ἄρα τὸ ΕΓ ἴσον ἐστὶ τῷ ΛΘ. ἐπεὶ οὖν ῥητὸν μέν ἐστι τὸ ΒΓ, μέσον δὲ τὸ ΒΔ, ἴσον δὲ τὸ μὲν ΒΓ τῷ ΗΘ, τὸ δὲ ΒΔ τῷ ΗΚ, ῥητὸν μὲν ἄρα ἐστὶ τὸ ΗΘ, μέσον δὲ τὸ ΗΚ. καὶ παρὰ ῥητὴν τὴν ΖΗ παράκειται· ῥητὴ μὲν ἄρα ἡ ΖΘ καὶ σύμμετρος τῇ ΖΗ μήκει, ῥητὴ δὲ ἡ ΖΚ καὶ ἀσύμμετρος τῇ ΖΗ μήκει· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΘ τῇ ΖΚ μήκει. αἱ ΖΘ, ΖΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΚΘ,

προσαρμόζουσα δὲ αὐτῇ ἡ ΚΖ. ἤτοι δὴ ἡ ΘΖ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἢ οὔ.

Δυνάσθω πρότερον τῷ ἀπὸ συμμέτρου. καί ἐστιν ὅλη ἡ ΘΖ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ΖΗ· ἀποτομὴ ἄρα πρώτη ἐστὶν ἡ ΚΘ. τὸ δ’ ὑπὸ ῥητῆς καὶ ἀποτομῆς πρώτης περιεχόμενον ἡ δυναμένη ἀποτομή ἐστιν. ἡ ἄρα τὸ ΛΘ, τουτέστι τὸ ΕΓ, δυναμένη ἀποτομή ἐστιν.

Εἰ δὲ ἡ ΘΖ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καί ἐστιν ὅλη ἡ ΖΘ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ΖΗ, ἀποτομὴ τετάρτη ἐστὶν ἡ ΚΘ. τὸ δ’ ὑπὸ ῥητῆς καὶ ἀποτομῆς τετάρτης περιεχόμενον ἡ δυναμένη ἐλάσσων ἐστίν· ὅπερ ἔδει δεῖξαι.

ρθ΄. Ἀπὸ μέσου ῥητοῦ ἀφαιρουμένου ἄλλαι δύο ἄλογοι γίνονται ἤτοι μέσης ἀποτομὴ πρώτη ἢ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα.

Ἀπὸ γὰρ μέσου τοῦ ΒΓ ῥητὸν ἀφῃρήσθω τὸ ΒΔ. λέγω, ὅτι ἡ τὸ λοιπὸν τὸ ΕΓ δυναμένη μία δύο ἀλόγων γίνεται ἤτοι μέσης ἀποτομὴ πρώτη ἢ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα.

Ἐκκείσθω γὰρ ῥητὴ ἡ ΖΗ, καὶ παραβεβλήσθω ὁμοίως τὰ χωρία. ἔστι δὴ ἀκολούθως ῥητὴ μὲν ἡ ΖΘ καὶ ἀσύμμετρος τῇ ΖΗ μήκει, ῥητὴ δὲ ἡ ΚΖ καὶ σύμμετρος τῇ ΖΗ μήκει· αἱ ΖΘ, ΖΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΚΘ, προσαρμόζουσα δὲ ταύτῃ ἡ ΖΚ. ἤτοι δὴ ἡ ΘΖ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυμμέτρου.

Εἰ μὲν οὖν ἡ ΘΖ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καί ἐστιν ἡ προσαρμόζουσα ἡ ΖΚ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ΖΗ, ἀποτομὴ δευτέρα ἐστὶν ἡ ΚΘ. ῥητὴ δὲ ἡ ΖΗ· ὥστε ἡ τὸ ΛΘ, τουτέστι τὸ ΕΓ, δυναμένη μέσης ἀποτομὴ πρώτη ἐστίν.

Εἰ δὲ ἡ ΘΖ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου, καί ἐστιν ἡ προσαρμόζουσα ἡ ΖΚ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ΖΗ, ἀποτομὴ

πέμπτη ἐστὶν ἡ ΚΘ· ὥστε ἡ τὸ ΕΓ δυναμένη μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.

ρι΄. Ἀπὸ μέσου μέσου ἀφαιρουμένου ἀσυμμέτρου τῷ ὅλῳ αἱ λοιπαὶ δύο ἄλογοι γίνονται ἤτοι μέσης ἀποτομὴ δευτέρα ἢ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα.

Ἀφῃρήσθω γὰρ ὡς ἐπὶ τῶν προκειμένων καταγραφῶν ἀπὸ μέσου τοῦ ΒΓ μέσον τὸ ΒΔ ἀσύμμετρον τῷ ὅλῳ· λέγω, ὅτι ἡ τὸ ΕΓ δυναμένη μία ἐστὶ δύο ἀλόγων ἤτοι μέσης ἀποτομὴ δευτέρα ἢ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα.

Ἐπεὶ γὰρ μέσον ἐστὶν ἑκάτερον τῶν ΒΓ, ΒΔ, καὶ ἀσύμμετρον τὸ ΒΓ τῷ ΒΔ, ἔσται ἀκολούθως ῥητὴ ἑκατέρα τῶν ΖΘ, ΖΚ καὶ ἀσύμμετρος τῇ ΖΗ μήκει. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ ΒΓ τῷ ΒΔ, τουτέστι τὸ ΗΘ τῷ ΗΚ, ἀσύμμετρος καὶ ἡ ΘΖ τῇ ΖΚ· αἱ ΖΘ, ΖΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΚΘ [προσαρμόζουσα δὲ ἡ ΖΚ. ἤτοι δὴ ἡ ΖΘ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἢ τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ].

Εἰ μὲν δὴ ἡ ΖΘ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ οὐθετέρα τῶν ΖΘ, ΖΚ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ΖΗ, ἀποτομὴ τρίτη ἐστὶν ἡ ΚΘ. ῥητὴ δὲ ἡ ΚΛ, τὸ δ’ ὑπὸ ῥητῆς καὶ ἀποτομῆς τρίτης περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν, καλεῖται δὲ μέσης ἀποτομὴ δευτέρα· ὥστε ἡ τὸ ΛΘ, τουτέστι τὸ ΕΓ, δυναμένη μέσης ἀποτομή ἐστι δευτέρα.

Εἰ δὲ ἡ ΖΘ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ [μήκει], καὶ οὐθετέρα τῶν ΘΖ, ΖΚ σύμμετρός ἐστι τῇ ΖΗ μήκει, ἀποτομὴ ἕκτη ἐστὶν ἡ ΚΘ. τὸ δ’ ὑπὸ ῥητῆς καὶ ἀποτομῆς ἕκτης ἡ δυναμένη ἐστὶ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα. ἡ τὸ ΛΘ ἄρα, τουτέστι τὸ ΕΓ, δυναμένη μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.

ρια΄. Ἡ ἀποτομὴ οὐκ ἔστιν ἡ αὐτὴ τῇ ἐκ δύο ὀνομάτων.

Ἔστω ἀποτομὴ ἡ ΑΒ· λέγω, ὅτι ἡ ΑΒ οὐκ ἔστιν ἡ αὐτὴ τῇ ἐκ δύο ὀνομάτων.

Εἰ γὰρ δυνατόν, ἔστω· καὶ ἐκκείσθω ῥητὴ ἡ ΔΓ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω ὀρθογώνιον τὸ ΓΕ πλάτος ποιοῦν τὴν ΔΕ. ἐπεὶ οὖν ἀποτομή ἐστιν ἡ ΑΒ, ἀποτομὴ πρώτη ἐστὶν ἡ ΔΕ. ἔστω αὐτῇ προσαρμόζουσα ἡ ΕΖ· αἱ ΔΖ, ΖΕ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΔΖ τῆς ΖΕ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ ΔΖ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ΔΓ. πάλιν, ἐπεὶ ἐκ δύο ὀνομάτων ἐστὶν ἡ ΑΒ, ἐκ δύο ἄρα ὀνομάτων πρώτη ἐστὶν ἡ ΔΕ. διῃρήσθω εἰς τὰ ὀνόματα κατὰ τὸ Η, καὶ ἔστω μεῖζον ὄνομα τὸ ΔΗ· αἱ ΔΗ, ΗΕ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΔΗ τῆς ΗΕ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ τὸ μεῖζον ἡ ΔΗ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ΔΓ. καὶ ἡ ΔΖ ἄρα τῇ ΔΗ σύμμετρός ἐστι μήκει· καὶ λοιπὴ ἄρα ἡ ΗΖ σύμμετρός ἐστι τῇ ΔΖ μήκει. [ἐπεὶ οὖν σύμμετρός ἐστιν ἡ ΔΖ τῇ ΗΖ, ῥητὴ δέ ἐστιν ἡ ΔΖ, ῥητὴ ἄρα ἐστὶ καὶ ἡ ΗΖ. ἐπεὶ οὖν σύμμετρός ἐστιν ἡ ΔΖ τῇ ΗΖ μήκει] ἀσύμμετρος δὲ ἡ ΔΖ τῇ ΕΖ μήκει· ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΖΗ τῇ ΕΖ μήκει. αἱ ΗΖ, ΖΕ ἄρα ῥηταί [εἰσι] δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΕΗ. ἀλλὰ καὶ ῥητή· ὅπερ ἐστὶν ἀδύνατον.

Ἡ ἄρα ἀποτομὴ οὐκ ἔστιν ἡ αὐτὴ τῇ ἐκ δύο ὀνομάτων· ὅπερ ἔδει δεῖξαι.

Πόρισμα

Ἡ ἀποτομὴ καὶ αἱ μετ’ αὐτὴν ἄλογοι οὔτε τῇ μέσῃ οὔτε ἀλλήλαις εἰσὶν αἱ αὐταί.

Τὸ μὲν γὰρ ἀπὸ μέσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ῥητὴν καὶ ἀσύμμετρον τῇ, παρ’ ἣν παράκειται, μήκει, τὸ δὲ ἀπὸ ἀποτομῆς παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πρώτην, τὸ δὲ ἀπὸ μέσης ἀποτομῆς πρώτης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν δευτέραν, τὸ δὲ ἀπὸ μέσης ἀποτομῆς δευτέρας παρὰ ῥητὴν παραβαλλόμενον

πλάτος ποιεῖ ἀποτομὴν τρίτην, τὸ δὲ ἀπὸ ἐλάσσονος παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τετάρτην, τὸ δὲ ἀπὸ τῆς μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πέμπτην, τὸ δὲ ἀπὸ τῆς μετὰ μέσου μέσον τὸ ὅλον ποιούσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν ἕκτην. ἐπεὶ οὖν τὰ εἰρημένα πλάτη διαφέρει τοῦ τε πρώτου καὶ ἀλλήλων, τοῦ μὲν πρώτου, ὅτι ῥητή ἐστιν, ἀλλήλων δέ, ἐπεὶ τῇ τάξει οὐκ εἰσὶν αἱ αὐταί, δῆλον, ὡς καὶ αὐταὶ αἱ ἄλογοι διαφέρουσιν ἀλλήλων. καὶ ἐπεὶ δέδεικται ἡ ἀποτομὴ οὐκ οὖσα ἡ αὐτὴ τῇ ἐκ δύο ὀνομάτων, ποιοῦσι δὲ πλάτη παρὰ ῥητὴν παραβαλλόμεναι αἱ μετὰ τὴν ἀποτομὴν ἀποτομὰς ἀκολούθως ἑκάστη τῇ τάξει τῇ καθ’ αὐτήν, αἱ δὲ μετὰ τὴν ἐκ δύο ὀνομάτων τὰς ἐκ δύο ὀνομάτων καὶ αὐταὶ τῇ τάξει ἀκολούθως, ἕτεραι ἄρα εἰσὶν αἱ μετὰ τὴν ἀποτομὴν καὶ ἕτεραι αἱ μετὰ τὴν ἐκ δύο ὀνομάτων, ὡς εἶναι τῇ τάξει πάσας ἀλόγους ,

Μέσην,
Ἐκ δύο ὀνομάτων,
Ἐκ δύο μέσων πρώτην,
Ἐκ δύο μέσων δευτέραν,
Μείζονα,
Ῥητὸν καὶ μέσον δυναμένην,
Δύο μέσα δυναμένην,
Ἀποτομήν,
Μέσης ἀποτομὴν πρώτην,
Μέσης ἀποτομὴν δευτέραν,
Ἐλάσσονα,
Μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσαν,
Μετὰ μέσου μέσον τὸ ὅλον ποιοῦσαν.

ριβ΄. Τὸ ἀπὸ ῥητῆς παρὰ τὴν ἐκ δύο ὀνομάτων παραβαλλόμενον πλάτος ποιεῖ ἀποτομήν, ἧς τὰ ὀνόματα σύμμετρά ἐστι τοῖς τῆς ἐκ δύο ὀνομάτων ὀνόμασι καὶ

ἔτι ἐν τῷ αὐτῷ λόγῳ, καὶ ἔτι ἡ γινομένη ἀποτομὴ τὴν αὐτὴν ἔξει τάξιν τῇ ἐκ δύο ὀνομάτων.

Ἔστω ῥητὴ μὲν ἡ Α, ἐκ δύο ὀνομάτων δὲ ἡ ΒΓ, ἧς μεῖζον ὄνομα ἔστω ἡ ΔΓ, καὶ τῷ ἀπὸ τῆς Α ἴσον ἔστω τὸ ὑπὸ τῶν ΒΓ, ΕΖ· λέγω, ὅτι ἡ ΕΖ ἀποτομή ἐστιν, ἧς τὰ ὀνόματα σύμμετρά ἐστι τοῖς ΓΔ, ΔΒ, καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ ἔτι ἡ ΕΖ τὴν αὐτὴν ἕξει τάξιν τῇ ΒΓ.

Ἔστω γὰρ πάλιν τῷ ἀπὸ τῆς Α ἴσον τὸ ὑπὸ τῶν ΒΔ, Η. ἐπεὶ οὖν τὸ ὑπὸ τῶν ΒΓ, ΕΖ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΒΔ, Η, ἔστιν ἄρα ὡς ἡ ΓΒ πρὸς τὴν ΒΔ, οὕτως ἡ Η πρὸς τὴν ΕΖ. μείζων δὲ ἡ ΓΒ τῆς ΒΔ· μείζων ἄρα ἐστὶ καὶ ἡ Η τῆς ΕΖ. ἔστω τῇ Η ἴση ἡ ΕΘ· ἔστιν ἄρα ὡς ἡ ΓΒ πρὸς τὴν ΒΔ, οὕτως ἡ ΘΕ πρὸς τὴν ΕΖ· διελόντι ἄρα ἐστὶν ὡς ἡ ΓΔ πρὸς τὴν ΒΔ, οὕτως ἡ ΘΖ πρὸς τὴν ΖΕ. γεγονέτω ὡς ἡ ΘΖ πρὸς τὴν ΖΕ, οὕτως ἡ ΖΚ πρὸς τὴν ΚΕ· καὶ ὅλη ἄρα ἡ ΘΚ πρὸς ὅλην τὴν ΚΖ ἐστιν, ὡς ἡ ΖΚ πρὸς ΚΕ· ὡς γὰρ ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα. ὡς δὲ ἡ ΖΚ πρὸς ΚΕ, οὕτως ἐστὶν ἡ ΓΔ πρὸς τὴν ΔΒ· καὶ ὡς ἄρα ἡ ΘΚ πρὸς ΚΖ, οὕτως ἡ ΓΔ πρὸς τὴν ΔΒ. σύμμετρον δὲ τὸ ἀπὸ τῆς ΓΔ τῷ ἀπὸ τῆς ΔΒ· σύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΘΚ τῷ ἀπὸ τῆς ΚΖ. καί ἐστιν ὡς τὸ ἀπὸ τῆς ΘΚ πρὸς τὸ ἀπὸ τῆς ΚΖ, οὕτως ἡ ΘΚ πρὸς τὴν ΚΕ, ἐπεὶ αἱ τρεῖς αἱ ΘΚ, ΚΖ, ΚΕ ἀνάλογόν εἰσιν. σύμμετρος ἄρα ἡ ΘΚ τῇ ΚΕ μήκει· ὥστε καὶ ἡ ΘΕ τῇ ΕΚ σύμμετρός ἐστι μήκει. καὶ ἐπεὶ τὸ ἀπὸ τῆς Α ἴσον ἐστὶ τῷ ὑπὸ τῶν ΕΘ, ΒΔ, ῥητὸν δέ ἐστι τὸ ἀπὸ τῆς Α, ῥητὸν ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΕΘ, ΒΔ. καὶ παρὰ ῥητὴν τὴν ΒΔ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ ΕΘ καὶ σύμμετρος τῇ ΒΔ μήκει· ὥστε καὶ ἡ σύμμετρος αὐτῇ ἡ ΕΚ ῥητή ἐστι καὶ σύμμετρος τῇ ΒΔ μήκει. ἐπεὶ οὖν ἐστιν ὡς ἡ ΓΔ πρὸς ΔΒ, οὕτως ἡ ΖΚ πρὸς ΚΕ, αἱ δὲ ΓΔ, ΔΒ δυνάμει μόνον εἰσὶ σύμμετροι, καὶ αἱ ΖΚ, ΚΕ δυνάμει μόνον εἰσὶ σύμμετροι. ῥητὴ δέ ἐστιν ἡ ΚΕ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΖΚ. αἱ ΖΚ, ΚΕ ἄρα ῥηταὶ δυνάμει μόνον εἰσὶ σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΕΖ.

Ἤτοι δὲ ἡ ΓΔ τῆς ΔΒ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυμμέτρου.

Εἰ μὲν οὖν ἡ ΓΔ τῆς ΔΒ μεῖζον δύναται τῷ ἀπὸ συμμέτρου [ἑαυτῇ], καὶ ἡ ΖΚ τῆς ΚΕ μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΓΔ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ΖΚ· εἰ δὲ ἡ ΒΔ, καὶ ἡ ΚΕ· εἰ δὲ οὐδετέρα τῶν ΓΔ, ΔΒ, καὶ οὐδετέρα τῶν ΖΚ, ΚΕ.

Εἰ δὲ ἡ ΓΔ τῆς ΔΒ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ ἡ ΖΚ τῆς ΚΕ μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ εἰ μὲν ἡ ΓΔ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ΖΚ· εἰ δὲ ἡ ΒΔ, καὶ ἡ ΚΕ· εἰ δὲ οὐδετέρα τῶν ΓΔ, ΔΒ, καὶ οὐδετέρα τῶν ΖΚ, ΚΕ· ὥστε ἀποτομή ἐστιν ἡ ΖΕ, ἧς τὰ ὀνόματα τὰ ΖΚ, ΚΕ σύμμετρά ἐστι τοῖς τῆς ἐκ δύο ὀνομάτων ὀνόμασι τοῖς ΓΔ, ΔΒ καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ τὴν αὐτὴν τάξιν ἔχει τῇ ΒΓ· ὅπερ ἔδει δεῖξαι.

ριγ΄. Τὸ ἀπὸ ῥητῆς παρὰ ἀποτομὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων, ἧς τὰ ὀνόματα σύμμετρά ἐστι τοῖς τῆς ἀποτομῆς ὀνόμασι καὶ ἐν τῷ αὐτῷ λόγῳ, ἔτι δὲ ἡ γινομένη ἐκ δύο ὀνομάτων τὴν αὐτὴν τάξιν ἔχει τῇ ἀποτομῇ.

Ἔστω ῥητὴ μὲν ἡ Α, ἀποτομὴ δὲ ἡ ΒΔ, καὶ τῷ ἀπὸ τῆς Α ἴσον ἔστω τὸ ὑπὸ τῶν ΒΔ, ΚΘ, ὥστε τὸ ἀπὸ τῆς Α ῥητῆς παρὰ τὴν ΒΔ ἀποτομὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ΚΘ· λέγω, ὅτι ἐκ δύο ὀνομάτων ἐστὶν ἡ ΚΘ, ἧς τὰ ὀνόματα σύμμετρά ἐστι τοῖς τῆς ΒΔ ὀνόμασι καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ ἔτι ἡ ΚΘ τὴν αὐτὴν ἔχει τάξιν τῇ ΒΔ.

Ἔστω γὰρ τῇ ΒΔ προσαρμόζουσα ἡ ΔΓ· αἱ ΒΓ, ΓΔ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ τῷ ἀπὸ τῆς Α ἴσον ἔστω καὶ τὸ ὑπὸ τῶν ΒΓ, Η. ῥητὸν δὲ τὸ ἀπὸ τῆς Α· ῥητὸν ἄρα καὶ τὸ ὑπὸ τῶν ΒΓ, Η. καὶ παρὰ ῥητὴν τὴν ΒΓ παραβέβληται· ῥητὴ ἄρα ἐστὶν ἡ Η καὶ σύμμετρος τῇ ΒΓ μήκει. ἐπεὶ οὖν τὸ ὑπὸ τῶν ΒΓ, Η ἴσον ἐστὶ τῷ ὑπὸ τῶν ΒΔ, ΚΘ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ

ΓΒ πρὸς ΒΔ, οὕτως ἡ ΚΘ πρὸς Η. μείζων δὲ ἡ ΒΓ τῆς ΒΔ· μείζων ἄρα καὶ ἡ ΚΘ τῆς Η. κείσθω τῇ Η ἴση ἡ ΚΕ· σύμμετρος ἄρα ἐστὶν ἡ ΚΕ τῇ ΒΓ μήκει. καὶ ἐπεί ἐστιν ὡς ἡ ΓΒ πρὸς ΒΔ, οὕτως ἡ ΘΚ πρὸς ΚΕ, ἀναστρέψαντι ἄρα ἐστὶν ὡς ἡ ΒΓ πρὸς τὴν ΓΔ, οὕτως ἡ ΚΘ πρὸς ΘΕ. γεγονέτω ὡς ἡ ΚΘ πρὸς ΘΕ, οὕτως ἡ ΘΖ πρὸς ΖΕ· καὶ λοιπὴ ἄρα ἡ ΚΖ πρὸς ΖΘ ἐστιν, ὡς ἡ ΚΘ πρὸς ΘΕ, τουτέστιν [ὡς] ἡ ΒΓ πρὸς ΓΔ. αἱ δὲ ΒΓ, ΓΔ δυνάμει μόνον [εἰσὶ] σύμμετροι· καὶ αἱ ΚΖ, ΖΘ ἄρα δυνάμει μόνον εἰσὶ σύμμετροι. καὶ ἐπεί ἐστιν ὡς ἡ ΚΘ πρὸς ΘΕ, ἡ ΚΖ πρὸς ΖΘ, ἀλλ’ ὡς ἡ ΚΘ πρὸς ΘΕ, ἡ ΘΖ πρὸς ΖΕ, καὶ ὡς ἄρα ἡ ΚΖ πρὸς ΖΘ, ἡ ΘΖ πρὸς ΖΕ· ὥστε καὶ ὡς ἡ πρώτη πρὸς τὴν τρίτην, τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας· καὶ ὡς ἄρα ἡ ΚΖ πρὸς ΖΕ, οὕτως τὸ ἀπὸ τῆς ΚΖ πρὸς τὸ ἀπὸ τῆς ΖΘ. σύμμετρον δέ ἐστι τὸ ἀπὸ τῆς ΚΖ τῷ ἀπὸ τῆς ΖΘ· αἱ γὰρ ΚΖ, ΖΘ δυνάμει εἰσὶ σύμμετροι· σύμμετρος ἄρα ἐστὶ καὶ ἡ ΚΖ τῇ ΖΕ μήκει· ὥστε ἡ ΚΖ καὶ τῇ ΚΕ σύμμετρός [ἐστι] μήκει. ῥητὴ δέ ἐστιν ἡ ΚΕ καὶ σύμμετρος τῇ ΒΓ μήκει· ῥητὴ ἄρα καὶ ἡ ΚΖ καὶ σύμμετρος τῇ ΒΓ μήκει. καὶ ἐπεί ἐστιν ὡς ἡ ΒΓ πρὸς ΓΔ, οὕτως ἡ ΚΖ πρὸς ΖΘ, ἐναλλὰξ ὡς ἡ ΒΓ πρὸς ΚΖ, οὕτως ἡ ΔΓ πρὸς ΖΘ. σύμμετρος δὲ ἡ ΒΓ τῇ ΚΖ· σύμμετρος ἄρα καὶ ἡ ΖΘ τῇ ΓΔ μήκει. αἱ ΒΓ, ΓΔ δὲ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· καὶ αἱ ΚΖ, ΖΘ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ὀνομάτων ἐστὶν ἄρα ἡ ΚΘ.

Εἰ μὲν οὖν ἡ ΒΓ τῆς ΓΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ ΚΖ τῆς ΖΘ μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΒΓ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ΚΖ, εἰ δὲ ἡ ΓΔ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ΖΘ, εἰ δὲ οὐδετέρα τῶν ΒΓ, ΓΔ, οὐδετέρα τῶν ΚΖ, ΖΘ.

Εἰ δὲ ἡ ΒΓ τῆς ΓΔ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ ἡ ΚΖ τῆς ΖΘ μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΒΓ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ΚΖ, εἰ δὲ ἡ ΓΔ, καὶ ἡ ΖΘ, εἰ δὲ οὐδετέρα τῶν ΒΓ, ΓΔ, οὐδετέρα τῶν ΚΖ, ΖΘ.

Ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΚΘ, ἧς τὰ ὀνόματα τὰ ΚΖ, ΖΘ σύμμετρά [ἐστι] τοῖς τῆς ἀποτομῆς ὀνόμασι τοῖς ΒΓ, ΓΔ καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ ἔτι ἡ ΚΘ τῇ ΒΓ τὴν αὐτὴν ἕξει τάξιν· ὅπερ ἔδει δεῖξαι.

ριδ΄. Ἐὰν χωρίον περιέχηται ὑπὸ ἀποτομῆς καὶ τῆς ἐκ δύο ὀνομάτων, ἧς τὰ ὀνόματα σύμμετρά τέ ἐστι τοῖς τῆς ἀποτομῆς ὀνόμασι καὶ ἐν τῷ αὐτῷ λόγῳ, ἡ τὸ χωρίον δυναμένη ῥητή ἐστιν.

Περιεχέσθω γὰρ χωρίον τὸ ὑπὸ τῶν ΑΒ, ΓΔ ὑπὸ ἀποτομῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων τῆς ΓΔ, ἧς μεῖζον ὄνομα ἔστω τὸ ΓΕ, καὶ ἔστω τὰ ὀνόματα τῆς ἐκ δύο ὀνομάτων τὰ ΓΕ, ΕΔ σύμμετρά τε τοῖς τῆς ἀποτομῆς ὀνόμασι τοῖς ΑΖ, ΖΒ καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ ἔστω ἡ τὸ ὑπὸ τῶν ΑΒ, ΓΔ δυναμένη ἡ Η· λέγω, ὅτι ῥητή ἐστιν ἡ Η.

Ἐκκείσθω γὰρ ῥητὴ ἡ Θ, καὶ τῷ ἀπὸ τῆς Θ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω πλάτος ποιοῦν τὴν ΚΛ· ἀποτομὴ ἄρα ἐστὶν ἡ ΚΛ, ἧς τὰ ὀνόματα ἔστω τὰ ΚΜ, ΜΛ σύμμετρα τοῖς τῆς ἐκ δύο ὀνομάτων ὀνόμασι τοῖς ΓΕ, ΕΔ καὶ ἐν τῷ αὐτῷ λόγῳ. ἀλλὰ καὶ αἱ ΓΕ, ΕΔ σύμμετροί τέ εἰσι ταῖς ΑΖ, ΖΒ καὶ ἐν τῷ αὐτῷ λόγῳ· ἔστιν ἄρα ὡς ἡ ΑΖ πρὸς τὴν ΖΒ, οὕτως ἡ ΚΜ πρὸς ΜΛ. ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ ΑΖ πρὸς τὴν ΚΜ, οὕτως ἡ ΒΖ πρὸς τὴν ΛΜ· καὶ λοιπὴ ἄρα ἡ ΑΒ πρὸς λοιπὴν τὴν ΚΛ ἐστιν ὡς ἡ ΑΖ πρὸς ΚΜ. σύμμετρος δὲ ἡ ΑΖ τῇ ΚΜ· σύμμετρος ἄρα ἐστὶ καὶ ἡ ΑΒ τῇ ΚΛ. καί ἐστιν ὡς ἡ ΑΒ πρὸς ΚΛ, οὕτως τὸ ὑπὸ τῶν ΓΔ, ΑΒ πρὸς τὸ ὑπὸ τῶν ΓΔ, ΚΛ· σύμμετρον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΓΔ, ΑΒ τῷ ὑπὸ τῶν ΓΔ, ΚΛ. ἴσον δὲ τὸ ὑπὸ τῶν ΓΔ, ΚΛ τῷ ἀπὸ τῆς Θ· σύμμετρον ἄρα ἐστὶ τὸ ὑπὸ τῶν ΓΔ, ΑΒ τῷ ἀπὸ τῆς Θ. τῷ δὲ ὑπὸ τῶν ΓΔ, ΑΒ ἴσον ἐστὶ τὸ ἀπὸ τῆς Η· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς Η τῷ ἀπὸ τῆς Θ. ῥητὸν δὲ τὸ ἀπὸ τῆς Θ· ῥητὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς Η· ῥητὴ ἄρα ἐστὶν ἡ Η. καὶ δύναται τὸ ὑπὸ τῶν ΓΔ, ΑΒ.

Ἐὰν ἄρα χωρίον περιέχηται ὑπὸ ἀποτομῆς καὶ τῆς ἐκ δύο ὀνομάτων, ἧς τὰ ὀνόματα σύμμετρά ἐστι τοῖς τῆς ἀποτομῆς ὀνόμασι καὶ ἐν τῷ αὐτῷ λόγῳ, ἡ

τὸ χωρίον δυναμένη ῥητή ἐστιν.

Πόρισμα

Καὶ γέγονεν ἡμῖν καὶ διὰ τούτου φανερόν, ὅτι δυνατόν ἐστι ῥητὸν χωρίον ὑπὸ ἀλόγων εὐθειῶν περιέχεσθαι. ὅπερ ἔδει δεῖξαι.

ριε΄. Ἀπὸ μέσης ἄπειροι ἄλογοι γίνονται, καὶ οὐδεμία οὐδεμιᾷ τῶν πρότερον ἡ αὐτή.

Ἔστω μέση ἡ Α· λέγω, ὅτι ἀπὸ τῆς Α ἄπειροι ἄλογοι γίνονται, καὶ οὐδεμία οὐδεμιᾷ τῶν πρότερον ἡ αὐτή.

Ἐκκείσθω ῥητὴ ἡ Β, καὶ τῷ ὑπὸ τῶν Β, Α ἴσον ἔστω τὸ ἀπὸ τῆς Γ· ἄλογος ἄρα ἐστὶν ἡ Γ· τὸ γὰρ ὑπὸ ἀλόγου καὶ ῥητῆς ἄλογόν ἐστιν. καὶ οὐδεμιᾷ τῶν πρότερον ἡ αὐτή· τὸ γὰρ ἀπ᾽ οὐδεμιᾶς τῶν πρότερον παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ μέσην. πάλιν δὴ τῷ ὑπὸ τῶν Β, Γ ἴσον ἔστω τὸ ἀπὸ τῆς Δ· ἄλογον ἄρα ἐστὶ τὸ ἀπὸ τῆς Δ. ἄλογος ἄρα ἐστὶν ἡ Δ· καὶ οὐδεμιᾷ τῶν πρότερον ἡ αὐτή· τὸ γὰρ ἀπ᾽ οὐδεμιᾶς τῶν πρότερον παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν Γ. ὁμοίως δὴ τῆς τοιαύτης τάξεως ἐπ᾽ ἄπειρον προβαινούσης φανερόν, ὅτι ἀπὸ τῆς μέσης ἄπειροι ἄλογοι γίνονται, καὶ οὐδεμία οὐδεμιᾷ τῶν πρότερον ἡ αὐτή· ὅπερ ἔδει δεῖξαι].

Παράρτημα

Ἀσύμμετρος ἡ διάμετρος τῇ πλευρᾷ μήκει.

Προκείσθω ἡμῖν δεῖξαι, ὅτι ἐπὶ τῶν τετραγώνων σχημάτων ἀσύμμετρός ἐστιν ἡ διάμετρος τῇ πλευρᾷ μήκει.

Ἔστω τετράγωνον τὸ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἡ ΑΓ· λέγω, ὅτι ἡ ΓΑ ἀσύμμετρός ἐστι τῇ ΑΒ μήκει.

Εἰ γὰρ δυνατόν, ἔστω σύμμετρος· λέγω, ὅτι συμβήσεται τὸν αὐτὸν ἀριθμὸν ἄρτιον εἶναι καὶ περισσόν. φανερὸν μὲν οὖν, ὅτι τὸ ἀπὸ τῆς ΑΓ διπλάσιον τοῦ ἀπὸ τῆς ΑΒ. καὶ ἐπὶ σύμμετρός ἐστιν ἡ ΓΑ τῇ ΑΒ, ἡ ΓΑ ἄρα πρὸς τὴν ΑΒ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. ἐχέτω, ὃν ὁ ΕΖ πρὸς Η, καὶ ἔστωσαν οἱ ΕΖ, Η ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς· οὐκ ἄρα μονὰς ἐστὶν ὁ ΕΖ. εἰ γὰρ ἔσται μονὰς ὁ ΕΖ, ἔχει δὲ λόγον πρὸς τὸν Η, ὃν ἔχει ἡ ΑΓ πρὸς τὴν ΑΒ, καὶ μείζων ἡ ΑΓ τῆς ΑΒ, μείζων ἄρα καὶ ἡ ΕΖ τοῦ Η ἀριθμοῦ· ὅπερ ἄτοπον· οὐκ ἄρα μονάς ἐστιν ὁ ΕΖ· ἀριθμὸς ἄρα. καὶ ἐπεί ἐστιν ὡς ἡ ΓΑ πρὸς τὴν ΑΒ, οὕτως ὁ ΕΖ πρὸς τὸν Η, καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΓΑ πρὸς τὸ ἀπὸ τῆς ΑΒ, οὕτως ὁ ἀπὸ τοῦ ΕΖ πρὸς τὸν ἀπὸ τοῦ Η.

διπλάσιον δὲ τὸ ἀπὸ τῆς ΓΑ τοῦ ἀπὸ τῆς ΑΒ· διπλάσιον ἄρα καὶ ὁ ἀπὸ τοῦ ΕΖ τοῦ ἀπὸ τοῦ Η· ἄρτιος ἄρα ἐστὶν ὁ ἀπὸ τοῦ ΕΖ· ὥστε καὶ αὐτὸς ἄρτιός ἐστιν. εἰ γὰρ ἦν περισσός, καὶ ὁ ἀπ΄αὐτοῦ τετράγωνος περισσὸς ἦν, ἐπειδήπερ, ἐὰν περισσοὶ ἀριθμοὶ ὁποσοιοῦν συντεθῶσιν, τὸ δὲ πλῆθος αὐτῶν περισσὸν ᾖ, ὁ ὅλος περισσὸς ἐστιν. ὁ ΕΖ ἄρα ἄρτιός ἐστιν. τετμήσθω δίχα κατὰ τὸ Θ. καὶ ἐπεὶ οἱ ΕΖ, Η ἀλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων [αὐτοῖς], πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ὁ ΕΖ ἄρτιος· περισσὸς ἄρα ἐστὶν ὁ Η. εἰ γὰρ ἦν ἄρτιος, τοὺς ΕΖ, Η δυὰς ἐμέτρει· πᾶς γὰρ ἄρτιος ἔχει μέρος ἥμισυ· πρώτους ὄντας πρὸς ἀλλήλους. ὅπερ ἐστὶν ἀδὺνατον· οὐκ ἄρα ἄρτιός ἐστιν ὁ Η· περισσὸς ἄρα, καὶ ἐπὶ διπλάσιος ὁ ΕΖ τοῦ ΕΘ, τετραπλάσιος ἄρα ὁ ἀπὸ ΕΖ τοῦ ἀπὸ ΕΘ. διπλάσιος δὲ ὁ ἀπὸ τοῦ ΕΖ τοῦ ἀπὸ τοῦ Η. διπλάσιος ἄρα ὁ ἀπὸ τοῦ Η τοῦ ἀπὸ ΕΘ· ἄρτιος ἄρα ἐστὶν ὁ ἀπὸ τοῦ Η. ἄρτιος ἄρα διὰ τὰ εἰρημένα ὁ Η· ἀλλὰ καὶ περισσός. ὅσπερ ἐστὶν ἀδύνατον. οὐκ ἄρα σύμμετρός ἐστιν ἡ ΓΑ τῇ ΑΒ μήκει· ὅπερ ἔδει δεῖξαι.

Ἄλλως

[Δεικτέον καὶ ἑτέρως, ὅτι ἀσύμμετρός ἐστιν ἡ τοῦ τετραγώνου διάμετρος τῇ πλευρᾷ].

Ἔστω ἀντὶ μὲν τῆς διαμέτρου ἡ Α, ἀντὶ δὲ τῆς πλευρᾶς ἡ Β. λέγω, ὅτι ἀσύμμετρός ἐστιν ἡ Α τῇ Β μήκει. εἰ γὰρ δυνατόν, ἔστω [σύμμετρος· καὶ γεγονέτω] πάλιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ὁ ΕΖ ἀριθμὸς πρὸς τὸν Η, καὶ ἔστωσαν ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς οἱ ΕΖ, Η· οἱ ΕΖ, Η ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. λέγω πρῶτον, ὅτι ὁ Η οὐκ ἔστι μονάς. εἰ γὰρ δυνατόν, ἔστω μονάς. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ὁ ΕΖ πρὸς τὸν Η, καὶ ὡς ἄρα τὸ ἀπὸ τῆς Α, πρὸς τὸ ἀπὸ τῆς Β, οὕτως ὁ ἀπὸ τοῦ ΕΖ πρὸς τὸν ἀπὸ τοῦ Η. διπλάσιον δὲ τὸ ἀπὸ τῆς Α τοῦ ἀπὸ τῆς Β· διπλάσιος ἄρα καὶ ὁ ἀπὸ τοῦ ΕΖ τοῦ ἀπὸ τοῦ Η. καὶ ἐστι μονὰς ὁ Η· δυὰς ἄρα ὁ ἀπὸ ΕΖ τετράγωνος· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα μονάς ἐστιν ὁ Η· ἀριθμὸς ἄρα. καὶ ἐπεί ἐστιν ὡς τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς Β, οὕτως ὁ ἀπὸ ΕΖ πρὸς τὸν ἀπὸ τοῦ Η, καὶ ἀνάπαλιν ὡς τὸ ἀπὸ τῆς Β πρὸς τὸ ἀπὸ τῆς Α, οὕτως ὁ ἀπὸ τοῦ Η πρὸς τὸν ἀπὸ τοῦ ΕΖ, μετρεῖ δὲ τὸ ἀπὸ τῆς Β τὸ ἀπὸ τῆς Α, μετρεῖ ἄρα καὶ ὁ ἀπὸ τοῦ Η τετράγωνος τὸν ἀπὸ τοῦ ΕΖ· ὥστε καὶ ἡ πλευρὰ αὐτὴ ὁ Η τὸν ΕΖ μετρεῖ. μετρεῖ δὲ καὶ ἑαυτὸν ὁ Η· ὁ Η ἄρα τοὺς ΕΖ, Η μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα σύμμετρός ἐστιν ἡ Α τῇ Β μήκει. ἀσύμμετρος ἄρα ἐστίν· ὅπερ ἔδει δεῖξαι.

Σχόλιον: Ἐπίπεδα, ἀσύμμετρα ἀλληλοις.

Εὑρημένων δὴ τῶν μήκει ἀσυμμέτρων εὐθειῶν, ὡς τῶν Α, Β, εὑρίσκεται καὶ ἄλλα πλεῖστα μεγέθη ἐκ δύο διαστάσεων, λέγω δὴ ἐπίπεδα, ἀσύμμετρα ἀλλήλοις. ἐὰν γὰρ τῶν Α, Β εὐθειῶν μέσην ἀνάλογον λάβωμεν τὴν Γ, ἔσται ὡς ἡ Α πρὸς τὴν Β, οὕτως τὸ ἀπὸ τῆς Α ἐπίπεδον πρὸς τὸ ἀπὸ τῆς Γ τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον, εἴτε τετράγωνα εἴη τὰ ἀναγραφόμενα εἴτε ἕτερα εὐθύγραμμα ὅμοια, εἴτε κύκλοι περὶ διαμέτρους τὰς Α, Γ, ἐπείπερ οἱ κύκλοι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα. εὕρηνται ἄρα καὶ ἐπίπεδα χωρία ἀσύμμετρα ἀλλήλοις· ὅπερ ἔδει δεῖξαι.

Δεδειγμένων δὴ καὶ τῶν ἐκ δύο διαστάσεων διαφόρων ἀσυμμέτρων χωρίων δείξομεν τοῖς ἀπὸ τῆς τῶν στερεῶν θεωρίας, ὡς ἔστι καὶ στερεὰ σύμμετρά τε καὶ ἀσύμμετρα ἀλλήλοις. ἐὰν γὰρ ἐπὶ τῶν Α, Β τετραγώνων ἢ τῶν ἴσων αὐτοῖς εὐθυγράμμων ἀναστήσωμεν ἰσοϋψῆ στερεὰ παραλληλεπίπεδα ἢ πυραμίδας, ἢ πρίσματα, ἔσται τὰ ἀνασταθέντα πρὸς ἄλληλα ὡς αἱ βάσεις. καὶ εἰ μὲν σύμμετροί εἰσιν αἱ βάσεις, σύμμετρα ἔσται καὶ τὰ στερεά, εἰ δὲ ἀσύμμετροι, ἀσύμμετρα. ὅπερ ἔδει δεῖξαι.

Ἀλλὰ μὴν καὶ δύο κύκλων ὄντων τῶν Α, Β, ἐὰν ἀπ᾽ αὐτῶν ἰσοϋψεῖς κώνους ἢ κυλίνδρους ἀναγράψωμεν, ἔσονται πρὸς ἀλλήλους ὡς αἱ βάσεις, τουτέστιν ὡς οἱ Α, Β κύκλοι. καὶ εἰ μὲν σύμμετροί εἰσιν οἱ κύκλοι, σύμμετροι ἔσονται καὶ οἱ τε κῶνοι πρὸς ἀλλήλους καὶ οἱ κύλινδροι, εἰ δὲ ἀσύμμετροί εἰσιν οἱ κύκλοι, ἀσύμμετροι ἔσονται καὶ οἱ κῶνοι καὶ οἱ κύλινδροι. καὶ φανερὸν ἡμῖν γέγονεν, ὅτι οὐ μόνον ἐπὶ γραμμῶν καὶ ἐπιφανειῶν ἐστι συμμετρία τε καὶ ἀσυμμετρία, ἀλλα καὶ ἐπὶ τῶν στερεῶν σχημάτων.

BOOK XI.

Ὅροι κη΄.

α΄. Στερεόν ἐστι τὸ μῆκος καὶ πλάτος καὶ βάθος ἔχον.

β΄. Στερεοῦ δὲ πέρας ἐπιφάνεια.

γ΄. Εὐθεῖα πρὸς ἐπίπεδον ὀρθή ἐστιν, ὅταν πρὸς πάσας τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ [ὑποκειμένῳ] ἐπιπέδῳ ὀρθὰς ποιῇ γωνίας.

δ΄. Ἐπίπεδον πρὸς ἐπίπεδον ὀρθόν ἐστιν, ὅταν αἱ τῇ κοινῇ τομῇ τῶν ἐπιπέδων πρὸς ὀρθὰς ἀγόμεναι εὐθεῖαι ἐν ἑνὶ τῶν ἐπιπέδων τῷ λοιπῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν.

ε΄. Εὐθείας πρὸς ἐπίπεδον κλίσις ἐστίν, ὅταν ἀπὸ τοῦ μετεώρου πέρατος τῆς εὐθείας ἐπὶ τὸ ἐπίπεδον κάθετος ἀχθῇ, καὶ ἀπὸ τοῦ γενομένου σημείου ἐπὶ τὸ ἐν τῷ ἐπιπέδῳ πέρας τῆς εὐθείας εὐθεῖα ἐπιζευχθῇ, ἡ περιεχομένη γωνία ὑπὸ τῆς ἀχθείσης καὶ τῆς ἐφεστώσης.

ς΄. Ἐπιπέδου πρὸς ἐπίπεδον κλίσις ἐστὶν ἡ περιεχομένη ὀξεῖα γωνία ὑπὸ τῶν πρὸς ὀρθὰς τῇ κοινῇ τομῇ ἀγομένων πρὸς τῷ αὐτῷ σημείῳ ἐν ἑκατέρῳ τῶν ἐπιπέδων.

ζ΄. Ἐπίπεδον πρὸς ἐπίπεδον ὁμοίως κεκλίσθαι λέγεται καὶ ἕτερον πρὸς ἕτερον, ὅταν αἱ εἰρημέναι τῶν κλίσεων γωνίαι ἴσαι ἀλλήλαις ὦσιν.

η΄. Παράλληλα ἐπίπεδά ἐστι τὰ ἀσύμπτωτα.

θ΄. Ὅμοια στερεὰ σχήματά ἐστι τὰ ὑπὸ ὁμοίων ἐπιπέδων περιεχόμενα ἴσων τὸ πλῆθος.

ι΄. Ἴσα δὲ καὶ ὅμοια στερεὰ σχήματά ἐστι τὰ ὑπὸ ὁμοίων ἐπιπέδων περιεχόμενα ἴσων τῷ πλήθει καὶ τῷ μεγέθει.

ια΄. Στερεὰ γωνία ἐστὶν ἡ ὑπὸ πλειόνων ἢ δύο γραμμῶν ἁπτομένων ἀλλήλων καὶ μὴ ἐν τῇ αὐτῇ ἐπιφανείᾳ οὐσῶν πρὸς πάσαις ταῖς γραμμαῖς κλίσις. Ἄλλως· στερεὰ γωνία ἐστὶν ἡ ὑπὸ πλειόνων ἢ δύο γωνιῶν ἐπιπέδων περιεχομένη μὴ οὐσῶν ἐν τῷ αὐτῷ ἐπιπέδῳ πρὸς ἑνὶ σημείῳ συνισταμένων.

ιβ΄. Πυραμίς ἐστι σχῆμα στερεὸν ἐπιπέδοις περιεχόμενον ἀπὸ ἑνὸς ἐπιπέδου πρὸς ἑνὶ σημείῳ συνεστώς.

ιγ΄. Πρίσμα ἐστὶ σχῆμα στερεὸν ἐπιπέδοις περιεχόμενον, ὧν δύο τὰ ἀπεναντίον ἴσα τε καὶ ὅμοιά ἐστι καὶ παράλληλα, τὰ δὲ λοιπὰ παραλληλόγραμμα.

ιδ΄. Σφαῖρά ἐστιν, ὅταν ἡμικυκλίου μενούσης τῆς διαμέτρου περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα

ιε΄. Ἄξων δὲ τῆς σφαίρας ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ ἡμικύκλιον στρέφεται.

ις΄. Κέντρον δὲ τῆς σφαίρας ἐστὶ τὸ αὐτό, ὃ καὶ τοῦ ἡμικυκλίου.

ιζ΄. Διάμετρος δὲ τῆς σφαίρας ἐστὶν εὐθεῖά τις διὰ τοῦ κέντρου ἠγμένη καὶ περατουμένη ἐφ’ ἑκάτερα τὰ μέρη ὑπὸ τῆς ἐπιφανείας τῆς σφαίρας.

ιη΄. Κῶνός ἐστιν, ὅταν ὀρθογωνίου τριγώνου μενούσης μιᾶς πλευρᾶς τῶν περὶ τὴν ὀρθὴν γωνίαν περιενεχθὲν τὸ τρίγωνον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα. κἂν μὲν ἡ μένουσα εὐθεῖα ἴση ᾖ τῇ λοιπῇ [τῇ] περὶ τὴν ὀρθὴν περιφερομένῃ, ὀρθογώνιος ἔσται ὁ κῶνος, ἐὰν δὲ ἐλάττων, ἀμβλυγώνιος, ἐὰν δὲ μείζων, ὀξυγώνιος.

ιθ΄. Ἄξων δὲ τοῦ κώνου ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ τρίγωνον στρέφεται.

κ΄. Βάσις δὲ ὁ κύκλος ὁ ὑπὸ τῆς περιφερομένης εὐθείας γραφόμενος.

κα΄. Κύλινδρός ἐστιν, ὅταν ὀρθογωνίου παραλληλογράμμου μενούσης μιᾶς πλευρᾶς τῶν περὶ τὴν ὀρθὴν γωνίαν περιενεχθὲν τὸ παραλληλόγραμμον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα.

κβ΄. Ἄξων δὲ τοῦ κυλίνδρου ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ παραλληλόγραμμον στρέφεται.

κγ΄. Βάσεις δὲ οἱ κύκλοι οἱ ὑπὸ τῶν ἀπεναντίον περιαγομένων δύο πλευρῶν γραφόμενοι.

κδ΄. Ὅμοιοι κῶνοι καὶ κύλινδροί εἰσιν, ὧν οἵ τε ἄξονες καὶ αἱ διάμετροι τῶν βάσεων ἀνάλογόν εἰσιν.

κε΄. Κύβος ἐστὶ σχῆμα στερεὸν ὑπὸ ἓξ τετραγώνων ἴσων περιεχόμενον.

κς΄. Ὀκτάεδρόν ἐστι σχῆμα στερεὸν ὑπὸ ὀκτὼ τριγώνων ἴσων καὶ ἰσοπλεύρων περιεχόμενον.

κζ΄. Εἰκοσάεδρόν ἐστι σχῆμα στερεὸν ὑπὸ εἴκοσι τριγώνων ἴσων καὶ ἰσοπλεύρων περιεχόμενον.

κη΄. Δωδεκάεδρόν ἐστι σχῆμα στερεὸν ὑπὸ δώδεκα πενταγώνων ἴσων καὶ ἰσοπλεύρων καὶ ἰσογωνίων περιεχόμενον.

Προτάσεις λθ΄.

α΄. Εὐθείας γραμμῆς μέρος μέν τι οὐκ ἔστιν ἐν τῷ ὑποκειμένῳ, ἐπιπέδῳ, μέρος δέ τι ἐν μετεωροτέρῳ.

Εἰ γὰρ δυνατόν, εὐθείας γραμμῆς τῆς ΑΒΓ μέρος μέν τι τὸ ΑΒ ἔστω ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, μέρος δέ τι τὸ ΒΓ ἐν μετεωροτέρῳ.

Ἔσται δή τις τῇ ΑΒ συνεχὴς εὐθεῖα ἐπ᾽ εὐθείας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ. ἔστω ἡ ΒΔ· δύο ἄρα εὐθειῶν τῶν ΑΒΓ, ΑΒΔ κοινὸν τμῆμά ἐστιν ἡ ΑΒ· ὅπερ ἐστὶν ἀδύνατον, ἐπειδήπερ ἐὰν κέντρῳ τῷ Β καὶ διαστήματι τῷ ΑΒ κύκλον γράψωμεν, αἱ διάμετροι ἀνίσους ἀπολήψονται τοῦ κύκλου περιφερείας.

Εὐθείας ἄρα γραμμῆς μέρος μέν τι οὐκ ἔστιν ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, τὸ δὲ ἐν μετεωροτέρῳ· ὅπερ ἔδει δεῖξαι.

β΄. Ἐὰν δύο εὐθεῖαι τέμνωσιν ἀλλήλας, ἐν ἑνί εἰσιν ἐπιπέδῳ, καὶ πᾶν τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ.

Δύο γὰρ εὐθεῖαι αἱ ΑΒ, ΓΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε σημεῖον· λέγω, ὅτι αἱ ΑΒ, ΓΔ ἐν ἑνί εἰσιν ἐπιπέδῳ, καὶ πᾶν τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ.

Εἰλήφθω γὰρ ἐπὶ τῶν ΕΓ, ΕΒ τυχόντα σημεῖα τὰ Ζ, Η, καὶ ἐπεζεύχθωσαν αἱ ΓΒ, ΖΗ, καὶ διήχθωσαν αἱ ΖΘ, ΗΚ· λέγω πρῶτον, ὅτι τὸ ΕΓΒ τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ. εἰ γάρ ἐστι τοῦ ΕΓΒ τριγώνου μέρος ἤτοι τὸ ΖΘΓ ἢ τὸ ΗΒΚ ἐν τῷ ὑποκειμένῳ [ἐπιπέδῳ], τὸ δὲ λοιπὸν ἐν ἄλλῳ, ἔσται καὶ μιᾶς τῶν ΕΓ, ΕΒ εὐθειῶν μέρος μέν τι ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, τὸ δὲ ἐν ἄλλῳ. εἰ δὲ τοῦ ΕΓΒ τριγώνου τὸ ΖΓΒΗ μέρος ᾖ ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, τὸ δὲ λοιπὸν ἐν ἄλλῳ, ἔσται καὶ ἀμφοτέρων τῶν ΕΓ, ΕΒ εὐθειῶν μέρος μέν τι ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, τὸ δὲ ἐν ἄλλῳ· ὅπερ ἄτοπον ἐδείχθη. τὸ ἄρα ΕΓΒ

τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ. ἐν ᾧ δέ ἐστι τὸ ΕΓΒ τρίγωνον, ἐν τούτῳ καὶ ἑκατέρα τῶν ΕΓ, ΕΒ, ἐν ᾧ δὲ ἑκατέρα τῶν ΕΓ, ΕΒ, ἐν τούτῳ καὶ αἱ ΑΒ, ΓΔ. αἱ ΑΒ, ΓΔ ἄρα εὐθεῖαι ἐν ἑνί εἰσιν ἐπιπέδῳ, καὶ πᾶν τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ· ὅπερ ἔδει δεῖξαι.

γ΄. Ἐὰν δύο ἐπίπεδα τέμνῃ ἄλληλα, ἡ κοινὴ αὐτῶν τομὴ εὐθεῖά ἐστιν.

Δύο γὰρ ἐπίπεδα τὰ ΑΒ, ΒΓ τεμνέτω ἄλληλα, κοινὴ δὲ αὐτῶν τομὴ ἔστω ἡ ΔΒ γραμμή· λέγω, ὅτι ἡ ΔΒ γραμμὴ εὐθεῖά ἐστιν.

Εἰ γὰρ μή, ἐπεζεύχθω ἀπὸ τοῦ Δ ἐπὶ τὸ Β ἐν μὲν τῷ ΑΒ ἐπιπέδῳ εὐθεῖα ἡ ΔΕΒ, ἐν δὲ τῷ ΒΓ ἐπιπέδῳ εὐθεῖα ἡ ΔΖΒ. ἔσται δὴ δύο εὐθειῶν τῶν ΔΕΒ, ΔΖΒ τὰ αὐτὰ πέρατα, καὶ περιέξουσι δηλαδὴ χωρίον· ὅπερ ἄτοπον. οὐκ ἄρα αἱ ΔΕΒ, ΔΖΒ εὐθεῖαί εἰσιν. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἄλλη τις ἀπὸ τοῦ Δ ἐπὶ τὸ Β ἐπιζευγνυμένη εὐθεῖα ἔσται πλὴν τῆς ΔΒ κοινῆς τομῆς τῶν ΑΒ, ΒΓ ἐπιπέδων.

Ἐὰν ἄρα δύο ἐπίπεδα τέμνῃ ἄλληλα, ἡ κοινὴ αὐτῶν τομὴ εὐθεῖά ἐστιν· ὅπερ ἔδει δεῖξαι.

δ΄. Ἐὰν εὐθεῖα δύο εὐθείαις τεμνούσαις ἀλλήλας πρὸς ὀρθὰς ἐπὶ τῆς κοινῆς τομῆς ἐπισταθῇ, καὶ τῷ δι᾽ αὐτῶν ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.

Εὐθεῖα γάρ τις ἡ ΕΖ δύο εὐθείαις ταῖς ΑΒ, ΓΔ τεμνούσαις ἀλλήλας κατὰ τὸ Ε σημεῖον ἀπὸ τοῦ Ε πρὸς ὀρθὰς ἐφεστάτω· λέγω, ὅτι ἡ ΕΖ καὶ τῷ διὰ τῶν ΑΒ, ΓΔ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

Ἀπειλήφθωσαν γὰρ αἱ ΑΕ, ΕΒ, ΓΕ, ΕΔ ἴσαι ἀλλήλαις, καὶ διήχθω τις διὰ τοῦ Ε, ὡς ἔτυχεν, ἡ ΗΕΘ, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΓΒ, καὶ ἔτι ἀπὸ τυχόντος τοῦ Ζ ἐπεζεύχθωσαν αἱ ΖΑ, ΖΗ, ΖΔ, ΖΓ, ΖΘ, ΖΒ. καὶ ἐπεὶ δύο αἱ ΑΕ, ΕΔ δυσὶ ταῖς ΓΕ, ΕΒ ἴσαι εἰσὶ καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΑΔ βάσει τῇ ΓΒ ἴση ἐστίν, καὶ τὸ ΑΕΔ τρίγωνον τῷ ΓΕΒ τριγώνῳ ἴσον ἔσται· ὥστε καὶ γωνία ἡ ὑπὸ ΔΑΕ γωνίᾳ τῇ ὑπὸ ΕΒΓ ἴση [ἐστίν].

ἔστι δὲ καὶ ἡ ὑπὸ ΑΕΗ γωνία τῇ ὑπὸ ΒΕΘ ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΑΗΕ, ΒΕΘ τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν πρὸς ταῖς ἴσαις γωνίαις τὴν ΑΕ τῇ ΕΒ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξουσιν. ἴση ἄρα ἡ μὲν ΗΕ τῇ ΕΘ, ἡ δὲ ΑΗ τῇ ΒΘ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΕ τῇ ΕΒ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΖΕ, βάσις ἄρα ἡ ΖΑ βάσει τῇ ΖΒ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΖΓ τῇ ΖΔ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΓΒ, ἔστι δὲ καὶ ἡ ΖΑ τῇ ΖΒ ἴση, δύο δὴ αἱ ΖΑ, ΑΔ δυσὶ ταῖς ΖΒ, ΒΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ βάσις ἡ ΖΔ βάσει τῇ ΖΓ ἐδείχθη ἴση· καὶ γωνία ἄρα ἡ ὑπὸ ΖΑΔ γωνίᾳ τῇ ὑπὸ ΖΒΓ ἴση ἐστίν. καὶ ἐπεὶ πάλιν ἐδείχθη ἡ ΑΗ τῇ ΒΘ ἴση, ἀλλὰ μὴν καὶ ἡ ΖΑ τῇ ΖΒ ἴση, δύο δὴ αἱ ΖΑ, ΑΗ δυσὶ ταῖς ΖΒ, ΒΘ ἴσαι εἰσίν. καὶ γωνία ἡ ὑπὸ ΖΑΗ ἐδείχθη ἴση τῇ ὑπὸ ΖΒΘ· βάσις ἄρα ἡ ΖΗ βάσει τῇ ΖΘ ἐστιν ἴση. καὶ ἐπεὶ πάλιν ἴση ἐδείχθη ἡ ΗΕ τῇ ΕΘ, κοινὴ δὲ ἡ ΕΖ, δύο δὴ αἱ ΗΕ, ΕΖ δυσὶ ταῖς ΘΕ, ΕΖ ἴσαι εἰσίν· καὶ βάσις ἡ ΖΗ βάσει τῇ ΖΘ ἴση· γωνία ἄρα ἡ ὑπὸ ΗΕΖ γωνίᾳ τῇ ὑπὸ ΘΕΖ ἴση ἐστίν. ὀρθὴ ἄρα ἑκατέρα τῶν ὑπὸ ΗΕΖ, ΘΕΖ γωνιῶν. ἡ ΖΕ ἄρα πρὸς τὴν ΗΘ τυχόντως διὰ τοῦ Ε ἀχθεῖσαν ὀρθή ἐστιν. ὁμοίως δὴ δείξομεν, ὅτι ἡ ΖΕ καὶ πρὸς πάσας τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. εὐθεῖα δὲ πρὸς ἐπίπεδον ὀρθή ἐστιν, ὅταν πρὸς πάσας τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ αὐτῷ ἐπιπέδῳ ὀρθὰς ποιῇ γωνίας· ἡ ΖΕ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. τὸ δὲ ὑποκείμενον ἐπίπεδόν ἐστι τὸ διὰ τῶν ΑΒ, ΓΔ εὐθειῶν. ἡ ΖΕ ἄρα πρὸς ὀρθάς ἐστι τῷ διὰ τῶν ΑΒ, ΓΔ ἐπιπέδῳ.

Ἐὰν ἄρα εὐθεῖα δύο εὐθείαις τεμνούσαις ἀλλήλας πρὸς ὀρθὰς ἐπὶ τῆς κοινῆς τομῆς ἐπισταθῇ, καὶ τῷ δι' αὐτῶν ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

ε΄. Ἐὰν εὐθεῖα τρισὶν εὐθείαις ἁπτομέναις ἀλλήλων πρὸς ὀρθὰς ἐπὶ τῆς κοινῆς τομῆς ἐπισταθῇ, αἱ τρεῖς εὐθεῖαι ἐν ἑνί εἰσιν ἐπιπέδῳ.

Εὐθεῖα γάρ τις ἡ ΑΒ τρισὶν εὐθείαις ταῖς ΒΓ, ΒΔ, ΒΕ πρὸς ὀρθὰς ἐπὶ τῆς κατὰ τὸ Β ἀφῆς ἐφεστάτω· λέγω, ὅτι αἱ ΒΓ, ΒΔ, ΒΕ ἐν ἑνί εἰσιν ἐπιπέδῳ.

Μὴ γάρ, ἀλλ' εἰ δυνατόν, ἔστωσαν αἱ μὲν ΒΔ, ΒΕ ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, ἡ δὲ ΒΓ ἐν μετεωροτέρῳ, καὶ ἐκβεβλήσθω τὸ διὰ τῶν ΑΒ, ΒΓ ἐπίπεδον· κοινὴν δὴ τομὴν ποιήσει ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ εὐθεῖαν. ποιείτω τὴν ΒΖ. ἐν ἑνὶ ἄρα εἰσὶν ἐπιπέδῳ τῷ διηγμένῳ διὰ τῶν ΑΒ, ΒΓ αἱ τρεῖς εὐθεῖαι αἱ ΑΒ, ΒΓ, ΒΖ. καὶ ἐπεὶ ἡ ΑΒ ὀρθή ἐστι πρὸς ἑκατέραν τῶν ΒΔ, ΒΕ, καὶ τῷ διὰ τῶν ΒΔ, ΒΕ ἄρα ἐπιπέδῳ ὀρθή ἐστιν ἡ ΑΒ. τὸ δὲ διὰ τῶν ΒΔ, ΒΕ ἐπίπεδον τὸ ὑποκείμενόν ἐστιν· ἡ ΑΒ ἄρα ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον. ὥστε καὶ πρὸς πάσας τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας ἡ ΑΒ. ἅπτεται δὲ αὐτῆς ἡ ΒΖ οὖσα ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ· ἡ ἄρα ὑπὸ ΑΒΖ γωνία ὀρθή ἐστιν. ὑπόκειται δὲ καὶ ἡ ὑπὸ ΑΒΓ ὀρθή· ἴση ἄρα ἡ ὑπὸ ΑΒΖ γωνία τῇ ὑπὸ ΑΒΓ. καί εἰσιν ἐν ἑνὶ ἐπιπέδῳ· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ΒΓ εὐθεῖα ἐν μετεωροτέρῳ ἐστὶν ἐπιπέδῳ· αἱ τρεῖς ἄρα εὐθεῖαι αἱ ΒΓ, ΒΔ, ΒΕ ἐν ἑνί εἰσιν ἐπιπέδῳ.

Ἐὰν ἄρα εὐθεῖα τρισὶν εὐθείαις ἁπτομέναις ἀλλήλων ἐπὶ τῆς ἁφῆς πρὸς ὀρθὰς ἐπισταθῇ, αἱ τρεῖς εὐθεῖαι ἐν ἑνί εἰσιν ἐπιπέδῳ· ὅπερ ἔδει δεῖξαι.

ς΄. Ἐὰν δύο εὐθεῖαι τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν, παράλληλοι ἔσονται αἱ εὐθεῖαι.

Δύο γὰρ εὐθεῖαι αἱ ΑΒ, ΓΔ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστωσαν· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ.

Συμβαλλέτωσαν γὰρ τῷ ὑποκειμένῳ ἐπιπέδῳ κατὰ τὰ Β, Δ σημεῖα, καὶ ἐπεζεύχθω ἡ ΒΔ εὐθεῖα, καὶ ἤχθω τῇ ΒΔ πρὸς ὀρθὰς ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ἡ ΔΕ, καὶ κείσθω τῇ ΑΒ ἴση ἡ ΔΕ, καὶ ἐπεζεύχθωσαν αἱ ΒΕ, ΑΕ, ΑΔ.

Καὶ ἐπεὶ ἡ ΑΒ ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ πρὸς πάσας [ἄρα] τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. ἅπτεται δὲ τῆς ΑΒ ἑκατέρα τῶν ΒΔ, ΒΕ οὖσα ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ ΑΒΔ, ΑΒΕ γωνιῶν. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ὑπὸ ΓΔΒ, ΓΔΕ ὀρθή ἐστιν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΔΕ, κοινὴ δὲ ἡ ΒΔ, δύο δὴ αἱ ΑΒ, ΒΔ δυσὶ ταῖς ΕΔ, ΔΒ ἴσαι εἰσίν· καὶ γωνίας ὀρθὰς περιέχουσιν· βάσις ἄρα ἡ ΑΔ βάσει τῇ ΒΕ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΔΕ, ἀλλὰ καὶ ἡ ΑΔ τῇ ΒΕ, δύο δὴ αἱ ΑΒ, ΒΕ δυσὶ ταῖς ΕΔ, ΔΑ ἴσαι εἰσίν· καὶ βάσις αὐτῶν κοινὴ ἡ ΑΕ· γωνία ἄρα ἡ ὑπὸ ΑΒΕ γωνίᾳ τῇ ὑπὸ ΕΔΑ ἐστιν ἴση. ὀρθὴ δὲ ἡ ὑπὸ ΑΒΕ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΕΔΑ· ἡ ΕΔ ἄρα πρὸς τὴν ΔΑ ὀρθή ἐστιν. ἔστι δὲ καὶ πρὸς ἑκατέραν τῶν ΒΔ, ΔΓ ὀρθή. ἡ ΕΔ ἄρα τρισὶν εὐθείαις ταῖς ΒΔ, ΔΑ, ΔΓ πρὸς ὀρθὰς ἐπὶ τῆς ἁφῆς ἐφέστηκεν· αἱ τρεῖς ἄρα εὐθεῖαι αἱ ΒΔ, ΔΑ, ΔΓ ἐν ἑνί εἰσιν ἐπιπέδῳ. ἐν ᾧ δὲ αἱ ΔΒ, ΔΑ, ἐν τούτῳ καὶ ἡ ΑΒ· πᾶν γὰρ τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ· αἱ ἄρα ΑΒ, ΒΔ, ΔΓ εὐθεῖαι ἐν ἑνί εἰσιν ἐπιπέδῳ. καί ἐστιν ὀρθὴ ἑκατέρα τῶν ὑπὸ ΑΒΔ, ΒΔΓ γωνιῶν· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΓΔ.

Ἐὰν ἄρα δύο εὐθεῖαι τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν, παράλληλοι ἔσονται αἱ εὐθεῖαι· ὅπερ ἔδει δεῖξαι.

ζ΄. Ἐὰν ὦσι δύο εὐθεῖαι παράλληλοι, ληφθῇ δὲ ἐφ' ἑκατέρας αὐτῶν τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις.

Ἔστωσαν δύο εὐθεῖαι παράλληλοι αἱ ΑΒ, ΓΔ, καὶ εἰλήφθω ἐφ' ἑκατέρας αὐτῶν τυχόντα σημεῖα τὰ Ε, Ζ· λέγω, ὅτι ἡ ἐπὶ τὰ Ε, Ζ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις.

Μὴ γάρ, ἀλλ' εἰ δυνατόν, ἔστω ἐν μετεωροτέρῳ ὡς ἡ ΕΗΖ, καὶ διήχθω διὰ τῆς ΕΗΖ ἐπίπεδον· τομὴν δὴ ποιήσει ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ εὐθεῖαν. ποιείτω ὡς τὴν ΕΖ· δύο ἄρα εὐθεῖαι αἱ ΕΗΖ, ΕΖ χωρίον περιέξουσιν· ὅπερ

ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ἀπὸ τοῦ Ε ἐπὶ τὸ Ζ ἐπιζευγνυμένη εὐθεῖα ἐν μετεωροτέρῳ ἐστὶν ἐπιπέδῳ· ἐν τῷ διὰ τῶν ΑΒ, ΓΔ ἄρα παραλλήλων ἐστὶν ἐπιπέδῳ ἡ ἀπὸ τοῦ Ε ἐπὶ τὸ Ζ ἐπιζευγνυμένη εὐθεῖα.

Ἐὰν ἄρα ὦσι δύο εὐθεῖαι παράλληλοι, ληφθῇ δὲ ἐφ' ἑκατέρας αὐτῶν τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις· ὅπερ ἔδει δεῖξαι.

η΄. Ἐὰν ὦσι δύο εὐθεῖαι παράλληλοι, ἡ δὲ ἑτέρα αὐτῶν ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ ἡ λοιπὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.

Ἔστωσαν δύο εὐθεῖαι παράλληλοι αἱ ΑΒ, ΓΔ, ἡ δὲ ἑτέρα αὐτῶν ἡ ΑΒ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστω· λέγω, ὅτι καὶ ἡ λοιπὴ ἡ ΓΔ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.

Συμβαλλέτωσαν γὰρ αἱ ΑΒ, ΓΔ τῷ ὑποκειμένῳ ἐπιπέδῳ κατὰ τὰ Β, Δ σημεῖα, καὶ ἐπεζεύχθω ἡ ΒΔ· αἱ ΑΒ, ΓΔ, ΒΔ ἄρα ἐν ἑνί εἰσιν ἐπιπέδῳ. ἤχθω τῇ ΒΔ πρὸς ὀρθὰς ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ἡ ΔΕ, καὶ κείσθω τῇ ΑΒ ἴση ἡ ΔΕ, καὶ ἐπεζεύχθωσαν αἱ ΒΕ, ΑΕ, ΑΔ. καὶ ἐπεὶ ἡ ΑΒ ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν ἡ ΑΒ· ὀρθὴ ἄρα [ἐστὶν] ἑκατέρα τῶν ὑπὸ ΑΒΔ, ΑΒΕ γωνιῶν.

καὶ ἐπεὶ εἰς παραλλήλους τὰς ΑΒ, ΓΔ εὐθεῖα ἐμπέπτωκεν ἡ ΒΔ, αἱ ἄρα ὑπὸ ΑΒΔ, ΓΔΒ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. ὀρθὴ δὲ ἡ ὑπὸ ΑΒΔ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΓΔΒ· ἡ ΓΔ ἄρα πρὸς τὴν ΒΔ ὀρθή ἐστιν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΔΕ, κοινὴ δὲ ἡ ΒΔ, δύο δὴ αἱ ΑΒ, ΒΔ δυσὶ ταῖς ΕΔ, ΔΒ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΑΒΔ γωνίᾳ τῇ ὑπὸ ΕΔΒ ἴση· ὀρθὴ γὰρ ἑκατέρα· βάσις ἄρα ἡ ΑΔ βάσει τῇ ΒΕ ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΑΒ τῇ ΔΕ, ἡ δὲ ΒΕ τῇ ΑΔ, δύο δὴ αἱ ΑΒ, ΒΕ δυσὶ ταῖς ΕΔ, ΔΑ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ. καὶ βάσις αὐτῶν κοινὴ ἡ ΑΕ· γωνία ἄρα ἡ ὑπὸ ΑΒΕ γωνίᾳ τῇ ὑπὸ ΕΔΑ ἐστιν ἴση. ὀρθὴ δὲ ἡ ὑπὸ ΑΒΕ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΕΔΑ· ἡ ΕΔ ἄρα πρὸς τὴν ΑΔ ὀρθή ἐστιν. ἔστι

δὲ καὶ πρὸς τὴν ΔΒ ὀρθή· ἡ ΕΔ ἄρα καὶ τῷ διὰ τῶν ΒΔ, ΔΑ ἐπιπέδῳ ὀρθή ἐστιν. καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ διὰ τῶν ΒΔΑ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας ἡ ΕΔ. ἐν δὲ τῷ διὰ τῶν ΒΔΑ ἐπιπέδῳ ἐστὶν ἡ ΔΓ, ἐπειδήπερ ἐν τῷ διὰ τῶν ΒΔΑ ἐπιπέδῳ εἰσὶν αἱ ΑΒ, ΒΔ, ἐν ᾧ δὲ αἱ ΑΒ, ΒΔ, ἐν τούτῳ ἐστὶ καὶ ἡ ΔΓ. ἡ ΕΔ ἄρα τῇ ΔΓ πρὸς ὀρθάς ἐστιν· ὥστε καὶ ἡ ΓΔ τῇ ΔΕ πρὸς ὀρθάς ἐστιν. ἔστι δὲ καὶ ἡ ΓΔ τῇ ΒΔ πρὸς ὀρθάς. ἡ ΓΔ ἄρα δύο εὐθείαις τεμνούσαις ἀλλήλας ταῖς ΔΕ, ΔΒ ἀπὸ τῆς κατὰ τὸ Δ τομῆς πρὸς ὀρθὰς ἐφέστηκεν· ὥστε ἡ ΓΔ καὶ τῷ διὰ τῶν ΔΕ, ΔΒ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. τὸ δὲ διὰ τῶν ΔΕ, ΔΒ ἐπίπεδον τὸ ὑποκείμενόν ἐστιν· ἡ ΓΔ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

Ἐὰν ἄρα ὦσι δύο εὐθεῖαι παράλληλοι, ἡ δὲ μία αὐτῶν ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ ἡ λοιπὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

θ΄. Αἱ τῇ αὐτῇ εὐθείᾳ παράλληλοι καὶ μὴ οὖσαι αὐτῇ ἐν τῷ αὐτῷ ἐπιπέδῳ καὶ ἀλλήλαις εἰσὶ παράλληλοι.

Ἔστω γὰρ ἑκατέρα τῶν ΑΒ, ΓΔ τῇ ΕΖ παράλληλος μὴ οὖσαι αὐτῇ ἐν τῷ αὐτῷ ἐπιπέδῳ· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ.

Εἰλήφθω γὰρ ἐπὶ τῆς ΕΖ τυχὸν σημεῖον τὸ Η, καὶ ἀπ’ αὐτοῦ τῇ ΕΖ ἐν μὲν τῷ διὰ τῶν ΕΖ, ΑΒ ἐπιπέδῳ πρὸς ὀρθὰς ἤχθω ἡ ΗΘ, ἐν δὲ τῷ διὰ τῶν ΖΕ, ΓΔ τῇ ΕΖ πάλιν πρὸς ὀρθὰς ἤχθω ἡ ΗΚ. καὶ ἐπεὶ ἡ ΕΖ πρὸς ἑκατέραν τῶν ΗΘ, ΗΚ ὀρθή ἐστιν, ἡ ΕΖ ἄρα καὶ τῷ διὰ τῶν ΗΘ, ΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. καί ἐστιν ἡ ΕΖ τῇ ΑΒ παράλληλος· καὶ ἡ ΑΒ ἄρα τῷ διὰ τῶν ΘΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΓΔ τῷ διὰ τῶν ΘΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν· ἑκατέρα ἄρα τῶν ΑΒ, ΓΔ τῷ διὰ τῶν ΘΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. ἐὰν δὲ δύο εὐθεῖαι τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν, παράλληλοί εἰσιν αἱ εὐθεῖαι· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΓΔ· ὅπερ ἔδει δεῖξαι.

ι΄. Ἐὰν δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων ᾖ μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ, ἴσας γωνίας περιέξουσιν.

Δύο γὰρ εὐθεῖαι αἱ ΑΒ, ΒΓ ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας τὰς ΔΕ, ΕΖ ἁπτομένας ἀλλήλων ἔστωσαν μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ· λέγω, ὅτι ἴση ἐστὶν ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΕΖ. Ἀπειλήφθωσαν γὰρ αἱ ΒΑ, ΒΓ, ΕΔ, ΕΖ ἴσαι ἀλλήλαις, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΓΖ, ΒΕ, ΑΓ, ΔΖ. καὶ ἐπεὶ ἡ ΒΑ τῇ ΕΔ ἴση ἐστὶ καὶ παράλληλος, καὶ ἡ ΑΔ ἄρα τῇ ΒΕ ἴση ἐστὶ καὶ παράλληλος.

διὰ τὰ αὐτὰ δὴ καὶ ἡ ΓΖ τῇ ΒΕ ἴση ἐστὶ καὶ παράλληλος· ἑκατέρα ἄρα τῶν ΑΔ, ΓΖ τῇ ΒΕ ἴση ἐστὶ καὶ παράλληλος. αἱ δὲ τῇ αὐτῇ εὐθείᾳ παράλληλοι καὶ μὴ οὖσαι αὐτῇ ἐν τῷ αὐτῷ ἐπιπέδῳ καὶ ἀλλήλαις εἰσὶ παράλληλοι· παράλληλος ἄρα ἐστὶν ἡ ΑΔ τῇ ΓΖ καὶ ἴση. καὶ ἐπιζευγνύουσιν αὐτὰς αἱ ΑΓ, ΔΖ· καὶ ἡ ΑΓ ἄρα τῇ ΔΖ ἴση ἐστὶ καὶ παράλληλος. καὶ ἐπεὶ δύο αἱ ΑΒ, ΒΓ δυσὶ ταῖς ΔΕ, ΕΖ ἴσαι εἰσίν, καὶ βάσις ἡ ΑΓ βάσει τῇ ΔΖ ἴση, γωνία ἄρα ἡ ὑπὸ ΑΒΓ γωνίᾳ τῇ ὑπὸ ΔΕΖ ἐστιν ἴση.

Ἐὰν ἄρα δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων ᾖ μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ, ἴσας γωνίας περιέξουσιν· ὅπερ ἔδει δεῖξαι.

ια΄. Ἀπὸ τοῦ δοθέντος σημείου μετεώρου ἐπὶ τὸ δοθὲν ἐπίπεδον κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Ἔστω τὸ μὲν δοθὲν σημεῖον μετέωρον τὸ Α, τὸ δὲ δοθὲν ἐπίπεδον τὸ ὑποκείμενον· δεῖ δὴ ἀπὸ τοῦ Α σημείου ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Διήχθω γάρ τις ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ εὐθεῖα, ὡς ἔτυχεν, ἡ ΒΓ, καὶ ἤχθω ἀπὸ τοῦ Α σημείου ἐπὶ τὴν ΒΓ κάθετος ἡ ΑΔ. εἰ μὲν οὖν ἡ ΑΔ κάθετός ἐστι καὶ ἐπὶ τὸ ὑποκείμενον ἐπίπεδον, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν. εἰ δὲ οὔ, ἤχθω ἀπὸ τοῦ Δ σημείου τῇ ΒΓ ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΔΕ,

καὶ ἤχθω ἀπὸ τοῦ Α ἐπὶ τὴν ΔΕ κάθετος ἡ ΑΖ, καὶ διὰ τοῦ Ζ σημείου τῇ ΒΓ παράλληλος ἤχθω ἡ ΗΘ.

Καὶ ἐπεὶ ἡ ΒΓ ἑκατέρᾳ τῶν ΔΑ, ΔΕ πρὸς ὀρθάς ἐστιν, ἡ ΒΓ ἄρα καὶ τῷ διὰ τῶν ΕΔΑ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. καί ἐστιν αὐτῇ παράλληλος ἡ ΗΘ· ἐὰν δὲ ὦσι δύο εὐθεῖαι παράλληλοι, ἡ δὲ μία αὐτῶν ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ ἡ λοιπὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· καὶ ἡ ΗΘ ἄρα τῷ διὰ τῶν ΕΔ, ΔΑ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ διὰ τῶν ΕΔ, ΔΑ ἐπιπέδῳ ὀρθή ἐστιν ἡ ΗΘ. ἅπτεται δὲ αὐτῆς ἡ ΑΖ οὖσα ἐν τῷ διὰ τῶν ΕΔ, ΔΑ ἐπιπέδῳ· ἡ ΗΘ ἄρα ὀρθή ἐστι πρὸς τὴν ΖΑ· ὥστε καὶ ἡ ΖΑ ὀρθή ἐστι πρὸς τὴν ΘΗ. ἔστι δὲ ἡ ΑΖ καὶ πρὸς τὴν ΔΕ ὀρθή· ἡ ΑΖ ἄρα πρὸς ἑκατέραν τῶν ΗΘ, ΔΕ ὀρθή ἐστιν. ἐὰν δὲ εὐθεῖα δυσὶν εὐθείαις τεμνούσαις ἀλλήλας ἐπὶ τῆς τομῆς πρὸς ὀρθὰς ἐπισταθῇ, καὶ τῷ δι᾽ αὐτῶν ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ἡ ΖΑ ἄρα τῷ διὰ τῶν ΕΔ, ΗΘ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. τὸ δὲ διὰ τῶν ΕΔ, ΗΘ ἐπίπεδόν ἐστι τὸ ὑποκείμενον· ἡ ΑΖ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

Ἀπὸ τοῦ ἄρα δοθέντος σημείου μετεώρου τοῦ Α ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετος εὐθεῖα γραμμὴ ἦκται ἡ ΑΖ· ὅπερ ἔδει ποιῆσαι.

ιβ΄. Τῷ δοθέντι ἐπιπέδῳ ἀπὸ τοῦ πρὸς αὐτῷ δοθέντος σημείου πρὸς ὀρθὰς εὐθεῖαν γραμμὴν ἀναστῆσαι.

Ἔστω τὸ μὲν δοθὲν ἐπίπεδον τὸ ὑποκείμενον, τὸ δὲ πρὸς αὐτῷ σημεῖον τὸ Α· δεῖ δὴ ἀπὸ τοῦ Α σημείου τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς εὐθεῖαν γραμμὴν ἀναστῆσαι.

Νενοήσθω τι σημεῖον μετέωρον τὸ Β, καὶ ἀπὸ τοῦ Β ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετος ἤχθω ἡ ΒΓ, καὶ διὰ τοῦ Α σημείου τῇ ΒΓ παράλληλος ἤχθω ἡ ΑΔ.

Ἐπεὶ οὖν δύο εὐθεῖαι παράλληλοί εἰσιν αἱ ΑΔ, ΓΒ, ἡ δὲ μία αὐτῶν ἡ ΒΓ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν, καὶ ἡ λοιπὴ ἄρα ἡ ΑΔ τῷ

ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

Τῷ ἄρα δοθέντι ἐπιπέδῳ ἀπὸ τοῦ πρὸς αὐτῷ σημείου τοῦ Α πρὸς ὀρθὰς ἀνέσταται ἡ ΑΔ· ὅπερ ἔδει ποιῆσαι.

ιγ΄. Ἀπὸ τοῦ αὐτοῦ σημείου τῷ αὐτῷ ἐπιπέδῳ δύο εὐθεῖαι πρὸς ὀρθὰς οὐκ ἀναστήσονται ἐπὶ τὰ αὐτὰ μέρη.

Εἰ γὰρ δυνατόν, ἀπὸ τοῦ αὐτοῦ σημείου τοῦ Α τῷ ὑποκειμένῳ ἐπιπέδῳ δύο εὐθεῖαι αἱ ΑΒ, ΑΓ πρὸς ὀρθὰς ἀνεστάτωσαν ἐπὶ τὰ αὐτὰ μέρη, καὶ διήχθω τὸ διὰ τῶν ΒΑ, ΑΓ ἐπίπεδον·

τομὴν δὴ ποιήσει διὰ τοῦ Α ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ εὐθεῖαν. ποιείτω τὴν ΔΑΕ· αἱ ἄρα ΑΒ, ΑΓ, ΔΑΕ εὐθεῖαι ἐν ἑνί εἰσιν ἐπιπέδῳ. καὶ ἐπεὶ ἡ ΓΑ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. ἅπτεται δὲ αὐτῆς ἡ ΔΑΕ οὖσα ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ· ἡ ἄρα ὑπὸ ΓΑΕ γωνία ὀρθή ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΒΑΕ ὀρθή ἐστιν· ἴση ἄρα ἡ ὑπὸ ΓΑΕ τῇ ὑπὸ ΒΑΕ. καί εἰσιν ἐν ἑνὶ ἐπιπέδῳ· ὅπερ ἐστὶν ἀδύνατον.

Οὐκ ἄρα ἀπὸ τοῦ αὐτοῦ σημείου τῷ αὐτῷ ἐπιπέδῳ δύο εὐθεῖαι πρὸς ὀρθὰς ἀνασταθήσονται ἐπὶ τὰ αὐτὰ μέρη· ὅπερ ἔδει δεῖξαι.

ιδ΄. Πρὸς ἃ ἐπίπεδα ἡ αὐτὴ εὐθεῖα ὀρθή ἐστιν, παράλληλα ἔσται τὰ ἐπίπεδα.

Εὐθεῖα γάρ τις ἡ ΑΒ πρὸς ἑκάτερον τῶν ΓΔ, ΕΖ ἐπιπέδων πρὸς ὀρθὰς ἔστω· λέγω, ὅτι παράλληλά ἐστι τὰ ἐπίπεδα.

Εἰ γὰρ μή, ἐκβαλλόμενα συμπεσοῦνται. συμπιπτέτωσαν· ποιήσουσι δὴ κοινὴν τομὴν εὐθεῖαν. ποιείτωσαν τὴν ΗΘ, καὶ εἰλήφθω ἐπὶ τῆς ΗΘ τυχὸν σημεῖον τὸ Κ, καὶ ἐπεζεύχθωσαν αἱ ΑΚ, ΒΚ. καὶ ἐπεὶ ἡ ΑΒ ὀρθή ἐστι πρὸς τὸ ΕΖ ἐπίπεδον, καὶ πρὸς τὴν ΒΚ ἄρα εὐθεῖαν οὖσαν ἐν τῷ ΕΖ ἐκβληθέντι ἐπιπέδῳ ὀρθή ἐστιν ἡ ΑΒ· ἡ ἄρα ὑπὸ ΑΒΚ γωνία ὀρθή ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΒΑΚ ὀρθή ἐστιν. τριγώνου δὴ τοῦ ΑΒΚ αἱ δύο γωνίαι αἱ ὑπὸ

ΑΒΚ, ΒΑΚ δυσὶν ὀρθαῖς εἰσιν ἴσαι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὰ ΓΔ, ΕΖ ἐπίπεδα ἐκβαλλόμενα συμπεσοῦνται· παράλληλα ἄρα ἐστὶ τὰ ΓΔ, ΕΖ ἐπίπεδα.

Πρὸς ἃ ἐπίπεδα ἄρα ἡ αὐτὴ εὐθεῖα ὀρθή ἐστιν, παράλληλά ἐστι τὰ ἐπίπεδα· ὅπερ ἔδει δεῖξαι.

ιε΄. Ἐὰν δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων ὦσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι, παράλληλά ἐστι τὰ δι᾽ αὐτῶν ἐπίπεδα.

Δύο γὰρ εὐθεῖαι ἁπτόμεναι ἀλλήλων αἱ ΑΒ, ΒΓ παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων τὰς ΔΕ, ΕΖ ἔστωσαν μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι· λέγω, ὅτι ἐκβαλλόμενα τὰ διὰ τῶν ΑΒ, ΒΓ, ΔΕ, ΕΖ ἐπίπεδα οὐ συμπεσεῖται ἀλλήλοις.

Ἤχθω γὰρ ἀπὸ τοῦ Β σημείου ἐπὶ τὸ διὰ τῶν ΔΕ, ΕΖ ἐπίπεδον κάθετος ἡ ΒΗ καὶ συμβαλλέτω τῷ ἐπιπέδῳ κατὰ τὸ Η σημεῖον, καὶ διὰ τοῦ Η τῇ μὲν ΕΔ παράλληλος ἤχθω ἡ ΗΘ, τῇ δὲ ΕΖ ἡ ΗΚ. καὶ ἐπεὶ ἡ ΒΗ ὀρθή ἐστι πρὸς τὸ διὰ τῶν ΔΕ, ΕΖ ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ διὰ τῶν ΔΕ, ΕΖ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. ἅπτεται δὲ αὐτῆς ἑκατέρα τῶν ΗΘ, ΗΚ οὖσα ἐν τῷ διὰ τῶν ΔΕ, ΕΖ ἐπιπέδῳ· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ ΒΗΘ, ΒΗΚ γωνιῶν. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΒΑ τῇ ΗΘ, αἱ ἄρα ὑπὸ ΗΒΑ, ΒΗΘ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. ὀρθὴ δὲ ἡ ὑπὸ ΒΗΘ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΗΒΑ· ἡ ΗΒ ἄρα τῇ ΒΑ πρὸς ὀρθάς ἐστιν. διὰ τὰ αὐτὰ δὴ ἡ ΗΒ καὶ τῇ ΒΓ ἐστι πρὸς ὀρθάς. ἐπεὶ οὖν εὐθεῖα ἡ ΗΒ δυσὶν εὐθείαις ταῖς ΒΑ, ΒΓ τεμνούσαις ἀλλήλας πρὸς ὀρθὰς ἐφέστηκεν, ἡ ΗΒ ἄρα καὶ τῷ διὰ τῶν ΒΑ, ΒΓ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. [διὰ τὰ αὐτὰ δὴ ἡ ΒΗ καὶ τῷ διὰ τῶν ΗΘ, ΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. τὸ δὲ διὰ τῶν ΗΘ, ΗΚ ἐπίπεδόν ἐστι τὸ διὰ τῶν ΔΕ, ΕΖ· ἡ ΒΗ ἄρα τῷ διὰ τῶν ΔΕ, ΕΖ ἐπιπέδῳ ἐστὶ πρὸς ὀρθάς. ἐδείχθη δὲ ἡ ΗΒ καὶ τῷ διὰ τῶν ΑΒ, ΒΓ ἐπιπέδῳ πρὸς ὀρθάς].

πρὸς ἃ δὲ ἐπίπεδα ἡ αὐτὴ εὐθεῖα ὀρθή ἐστιν, παράλληλά ἐστι τὰ ἐπίπεδα· παράλληλον ἄρα ἐστὶ τὸ διὰ τῶν ΑΒ, ΒΓ ἐπίπεδον τῷ διὰ τῶν ΔΕ, ΕΖ.

Ἐὰν ἄρα δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων ὦσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ, παράλληλά ἐστι τὰ δι' αὐτῶν ἐπίπεδα· ὅπερ ἔδει δεῖξαι.

ις΄. Ἐὰν δύο ἐπίπεδα παράλληλα ὑπὸ ἐπιπέδου τινὸς τέμνηται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν.

Δύο γὰρ ἐπίπεδα παράλληλα τὰ ΑΒ, ΓΔ ὑπὸ ἐπιπέδου τοῦ ΕΖΗΘ τεμνέσθω, κοιναὶ δὲ αὐτῶν τομαὶ ἔστωσαν αἱ ΕΖ, ΗΘ· λέγω, ὅτι παράλληλός ἐστιν ἡ ΕΖ τῇ ΗΘ.

Εἰ γὰρ μή, ἐκβαλλόμεναι αἱ ΕΖ, ΗΘ ἤτοι ἐπὶ τὰ Ζ, Θ μέρη ἢ ἐπὶ τὰ Ε, Η συμπεσοῦνται. ἐκβεβλήσθωσαν ὡς ἐπὶ τὰ Ζ, Θ μέρη καὶ συμπιπτέτωσαν πρότερον κατὰ τὸ Κ. καὶ ἐπεὶ ἡ ΕΖΚ ἐν τῷ ΑΒ ἐστιν ἐπιπέδῳ, καὶ πάντα ἄρα τὰ ἐπὶ τῆς ΕΖΚ σημεῖα ἐν τῷ ΑΒ ἐστιν ἐπιπέδῳ. ἓν δὲ τῶν ἐπὶ τῆς ΕΖΚ εὐθείας σημείων ἐστὶ τὸ Κ· τὸ Κ ἄρα ἐν τῷ ΑΒ ἐστιν ἐπιπέδῳ. διὰ τὰ αὐτὰ δὴ τὸ Κ καὶ ἐν τῷ ΓΔ ἐστιν ἐπιπέδῳ· τὰ ΑΒ, ΓΔ ἄρα ἐπίπεδα ἐκβαλλόμενα συμπεσοῦνται. οὐ συμπίπτουσι δὲ διὰ τὸ παράλληλα ὑποκεῖσθαι· οὐκ ἄρα αἱ ΕΖ, ΗΘ εὐθεῖαι ἐκβαλλόμεναι ἐπὶ τὰ Ζ, Θ μέρη συμπεσοῦνται. ὁμοίως δὴ δείξομεν, ὅτι αἱ ΕΖ, ΗΘ εὐθεῖαι οὐδὲ ἐπὶ τὰ Ε, Η μέρη ἐκβαλλόμεναι συμπεσοῦνται. αἱ δὲ ἐπὶ μηδέτερα τὰ μέρη συμπίπτουσαι παράλληλοί εἰσιν. παράλληλος ἄρα ἐστὶν ἡ ΕΖ τῇ ΗΘ.

Ἐὰν ἄρα δύο ἐπίπεδα παράλληλα ὑπὸ ἐπιπέδου τινὸς τέμνηται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν· ὅπερ ἔδει δεῖξαι.

ιζ΄. Ἐὰν δύο εὐθεῖαι ὑπὸ παραλλήλων ἐπιπέδων τέμνωνται εἰς τοὺς αὐτοὺς λόγους τμηθήσονται.

Δύο γὰρ εὐθεῖαι αἱ ΑΒ, ΓΔ ὑπὸ παραλλήλων ἐπιπέδων τῶν ΗΘ, ΚΛ, ΜΝ τεμνέσθωσαν κατὰ τὰ Α, Ε, Β, Γ, Ζ, Δ σημεῖα· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΕ εὐθεῖα πρὸς τὴν ΕΒ, οὕτως ἡ ΓΖ πρὸς τὴν ΖΔ.

Ἐπεζεύχθωσαν γὰρ αἱ ΑΓ, ΒΔ, ΑΔ, καὶ συμβαλλέτω ἡ ΑΔ τῷ ΚΛ ἐπιπέδῳ κατὰ τὸ Ξ σημεῖον, καὶ ἐπεζεύχθωσαν αἱ ΕΞ, ΞΖ. καὶ ἐπεὶ δύο ἐπίπεδα παράλληλα τὰ ΚΛ, ΜΝ ὑπὸ ἐπιπέδου τοῦ ΕΒΔΞ τέμνεται, αἱ κοιναὶ αὐτῶν τομαὶ αἱ ΕΞ, ΒΔ παράλληλοί εἰσιν. διὰ τὰ αὐτὰ δὴ ἐπεὶ δύο ἐπίπεδα παράλληλα τὰ ΗΘ, ΚΛ ὑπὸ ἐπιπέδου τοῦ ΑΞΖΓ τέμνεται, αἱ κοιναὶ αὐτῶν τομαὶ αἱ ΑΓ, ΞΖ παράλληλοί εἰσιν. καὶ ἐπεὶ τριγώνου τοῦ ΑΒΔ παρὰ μίαν τῶν πλευρῶν τὴν ΒΔ εὐθεῖα ἦκται ἡ ΕΞ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΑΕ πρὸς ΕΒ, οὕτως ἡ ΑΞ πρὸς ΞΔ. πάλιν ἐπεὶ τριγώνου τοῦ ΑΔΓ παρὰ μίαν τῶν πλευρῶν τὴν ΑΓ εὐθεῖα ἦκται ἡ ΞΖ, ἀνάλογόν ἐστιν ὡς ἡ ΑΞ πρὸς ΞΔ, οὕτως ἡ ΓΖ πρὸς ΖΔ. ἐδείχθη δὲ καὶ ὡς ἡ ΑΞ πρὸς ΞΔ, οὕτως ἡ ΑΕ πρὸς ΕΒ· καὶ ὡς ἄρα ἡ ΑΕ πρὸς ΕΒ, οὕτως ἡ ΓΖ πρὸς ΖΔ.

Ἐὰν ἄρα δύο εὐθεῖαι ὑπὸ παραλλήλων ἐπιπέδων τέμνωνται, εἰς τοὺς αὐτοὺς λόγους τμηθήσονται· ὅπερ ἔδει δεῖξαι.

ιη΄. Ἐὰν εὐθεῖα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ πάντα τὰ δι᾽ αὐτῆς ἐπίπεδα τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.

Εὐθεῖα γάρ τις ἡ ΑΒ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστω· λέγω, ὅτι καὶ πάντα τὰ διὰ τῆς ΑΒ ἐπίπεδα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

Ἐκβεβλήσθω γὰρ διὰ τῆς ΑΒ ἐπίπεδον τὸ ΔΕ, καὶ ἔστω κοινὴ τομὴ τοῦ ΔΕ ἐπιπέδου καὶ τοῦ ὑποκειμένου ἡ ΓΕ, καὶ εἰλήφθω ἐπὶ τῆς ΓΕ τυχὸν σημεῖον τὸ Ζ, καὶ ἀπὸ τοῦ Ζ τῇ ΓΕ πρὸς ὀρθὰς ἤχθω ἐν τῷ ΔΕ ἐπιπέδῳ ἡ ΖΗ. καὶ ἐπεὶ ἡ ΑΒ πρὸς τὸ ὑποκείμενον ἐπίπεδον ὀρθή ἐστιν, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθή ἐστιν ἡ ΑΒ· ὥστε καὶ πρὸς τὴν ΓΕ ὀρθή ἐστιν· ἡ ἄρα ὑπὸ ΑΒΖ γωνία ὀρθή ἐστιν. ἔστι δὲ καὶ ἡ ὑπὸ ΗΖΒ ὀρθή· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΖΗ. ἡ δὲ

ΑΒ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν· καὶ ἡ ΖΗ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. καὶ ἐπίπεδον πρὸς ἐπίπεδον ὀρθόν ἐστιν, ὅταν αἱ τῇ κοινῇ τομῇ τῶν ἐπιπέδων πρὸς ὀρθὰς ἀγόμεναι εὐθεῖαι ἐν ἑνὶ τῶν ἐπιπέδων τῷ λοιπῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν. καὶ τῇ κοινῇ τομῇ τῶν ἐπιπέδων τῇ ΓΕ ἐν ἑνὶ τῶν ἐπιπέδων τῷ ΔΕ πρὸς ὀρθὰς ἀχθεῖσα ἡ ΖΗ ἐδείχθη τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς· τὸ ἄρα ΔΕ ἐπίπεδον ὀρθόν ἐστι πρὸς τὸ ὑποκείμενον. ὁμοίως δὴ δειχθήσεται καὶ πάντα τὰ διὰ τῆς ΑΒ ἐπίπεδα ὀρθὰ τυγχάνοντα πρὸς τὸ ὑποκείμενον ἐπίπεδον.

Ἐὰν ἄρα εὐθεῖα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ πάντα τὰ δι’ αὐτῆς ἐπίπεδα τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

ιθ΄. Ἐὰν δύο ἐπίπεδα τέμνοντα ἄλληλα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ ἡ κοινὴ αὐτῶν τομὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.

Δύο γὰρ ἐπίπεδα τὰ ΑΒ, ΒΓ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστω, κοινὴ δὲ αὐτῶν τομὴ ἔστω ἡ ΒΔ· λέγω, ὅτι ἡ ΒΔ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

Μὴ γάρ, καὶ ἤχθωσαν ἀπὸ τοῦ Δ σημείου ἐν μὲν τῷ ΑΒ ἐπιπέδῳ τῇ ΑΔ εὐθείᾳ πρὸς ὀρθὰς ἡ ΔΕ, ἐν δὲ τῷ ΒΓ ἐπιπέδῳ τῇ ΓΔ πρὸς ὀρθὰς ἡ ΔΖ. καὶ ἐπεὶ τὸ ΑΒ ἐπίπεδον ὀρθόν ἐστι πρὸς τὸ ὑποκείμενον, καὶ τῇ κοινῇ αὐτῶν τομῇ τῇ ΑΔ πρὸς ὀρθὰς ἐν τῷ ΑΒ ἐπιπέδῳ ἦκται ἡ ΔΕ, ἡ ΔΕ ἄρα ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΔΖ ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον. ἀπὸ τοῦ αὐτοῦ ἄρα σημείου τοῦ Δ τῷ ὑποκειμένῳ ἐπιπέδῳ δύο εὐθεῖαι πρὸς ὀρθὰς ἀνεσταμέναι εἰσὶν ἐπὶ τὰ αὐτὰ μέρη· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ ἀπὸ τοῦ Δ σημείου ἀνασταθήσεται πρὸς ὀρθὰς πλὴν τῆς ΔΒ κοινῆς τομῆς τῶν ΑΒ, ΒΓ ἐπιπέδων.

Ἐὰν ἄρα δύο ἐπίπεδα τέμνοντα ἄλληλα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ ἡ κοινὴ αὐτῶν τομὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

κ΄. Ἐὰν στερεὰ γωνία ὑπὸ τριῶν γωνιῶν ἐπιπέδων περιέχηται, δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι.

Στερεὰ γὰρ γωνία ἡ πρὸς τῷ Α ὑπὸ τριῶν γωνιῶν ἐπιπέδων τῶν ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ περιεχέσθω· λέγω, ὅτι τῶν ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ γωνιῶν δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι.

Εἰ μὲν οὖν αἱ ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ γωνίαι ἴσαι ἀλλήλαις εἰσίν, φανερόν, ὅτι δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσιν. εἰ δὲ οὔ, ἔστω μείζων ἡ ὑπὸ ΒΑΓ, καὶ συνεστάτω πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ ὑπὸ ΔΑΒ γωνίᾳ ἐν τῷ διὰ τῶν ΒΑΓ ἐπιπέδῳ ἴση ἡ ὑπὸ ΒΑΕ, καὶ κείσθω τῇ ΑΔ ἴση ἡ ΑΕ, καὶ διὰ τοῦ Ε σημείου διαχθεῖσα ἡ ΒΕΓ τεμνέτω τὰς ΑΒ, ΑΓ εὐθείας κατὰ τὰ Β, Γ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΔΒ, ΔΓ.

καὶ ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΑΕ, κοινὴ δὲ ἡ ΑΒ, δύο δυσὶν ἴσαι· καὶ γωνία ἡ ὑπὸ ΔΑΒ γωνίᾳ τῇ ὑπὸ ΒΑΕ ἴση· βάσις ἄρα ἡ ΔΒ βάσει τῇ ΒΕ ἐστιν ἴση. καὶ ἐπεὶ δύο αἱ ΒΔ, ΔΓ τῆς ΒΓ μείζονές εἰσιν, ὧν ἡ ΔΒ τῇ ΒΕ ἐδείχθη ἴση, λοιπὴ ἄρα ἡ ΔΓ λοιπῆς τῆς ΕΓ μείζων ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΑΕ, κοινὴ δὲ ἡ ΑΓ, καὶ βάσις ἡ ΔΓ βάσεως τῆς ΕΓ μείζων ἐστίν, γωνία ἄρα ἡ ὑπὸ ΔΑΓ γωνίας τῆς ὑπὸ ΕΑΓ μείζων ἐστίν. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΔΑΒ τῇ ὑπὸ ΒΑΕ ἴση· αἱ ἄρα ὑπὸ ΔΑΒ, ΔΑΓ τῆς ὑπὸ ΒΑΓ μείζονές εἰσιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ λοιπαὶ σύνδυο λαμβανόμεναι τῆς λοιπῆς μείζονές εἰσιν.

Ἐὰν ἄρα στερεὰ γωνία ὑπὸ τριῶν γωνιῶν ἐπιπέδων περιέχηται, δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι· ὅπερ ἔδει δεῖξαι .

κα΄. Ἅπασα στερεὰ γωνία ὑπὸ ἐλασσόνων [ἢ] τεσσάρων ὀρθῶν γωνιῶν ἐπιπέδων περιέχεται.

Ἔστω στερεὰ γωνία ἡ πρὸς τῷ Α περιεχομένη ὑπὸ ἐπιπέδων γωνιῶν τῶν ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ· λέγω, ὅτι αἱ ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ τεσσάρων ὀρθῶν

ἐλάσσονές εἰσιν.

Εἰλήφθω γὰρ ἐφ' ἑκάστης τῶν ΑΒ, ΑΓ, ΑΔ τυχόντα σημεῖα τὰ Β, Γ, Δ, καὶ ἐπεζεύχθωσαν αἱ ΒΓ, ΓΔ, ΔΒ. καὶ ἐπεὶ στερεὰ γωνία ἡ πρὸς τῷ Β ὑπὸ τριῶν γωνιῶν ἐπιπέδων περιέχεται τῶν ὑπὸ ΓΒΑ, ΑΒΔ, ΓΒΔ, δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσιν·

αἱ ἄρα ὑπὸ ΓΒΑ, ΑΒΔ τῆς ὑπὸ ΓΒΔ μείζονές εἰσιν. διὰ τὰ αὐτὰ δὴ καὶ αἱ μὲν ὑπὸ ΒΓΑ, ΑΓΔ τῆς ὑπὸ ΒΓΔ μείζονές εἰσιν, αἱ δὲ ὑπὸ ΓΔΑ, ΑΔΒ τῆς ὑπὸ ΓΔΒ μείζονές εἰσιν· αἱ ἓξ ἄρα γωνίαι αἱ ὑπὸ ΓΒΑ, ΑΒΔ, ΒΓΑ, ΑΓΔ, ΓΔΑ, ΑΔΒ τριῶν τῶν ὑπὸ ΓΒΔ, ΒΓΔ, ΓΔΒ μείζονές εἰσιν. ἀλλὰ αἱ τρεῖς αἱ ὑπὸ ΓΒΔ, ΒΔΓ, ΒΓΔ δυσὶν ὀρθαῖς ἴσαι εἰσίν· αἱ ἓξ ἄρα αἱ ὑπὸ ΓΒΑ, ΑΒΔ, ΒΓΑ, ΑΓΔ, ΓΔΑ, ΑΔΒ δύο ὀρθῶν μείζονές εἰσιν. καὶ ἐπεὶ ἑκάστου τῶν ΑΒΓ, ΑΓΔ, ΑΔΒ τριγώνων αἱ τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν, αἱ ἄρα τῶν τριῶν τριγώνων ἐννέα γωνίαι αἱ ὑπὸ ΓΒΑ, ΑΓΒ, ΒΑΓ, ΑΓΔ, ΓΔΑ, ΓΑΔ, ΑΔΒ, ΔΒΑ, ΒΑΔ ἓξ ὀρθαῖς ἴσαι εἰσίν, ὧν αἱ ὑπὸ ΑΒΓ, ΒΓΑ, ΑΓΔ, ΓΔΑ, ΑΔΒ, ΔΒΑ ἓξ γωνίαι δύο ὀρθῶν εἰσι μείζονες· λοιπαὶ ἄρα αἱ ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ τρεῖς [γωνίαι] περιέχουσαι τὴν στερεὰν γωνίαν τεσσάρων ὀρθῶν ἐλάσσονές εἰσιν.

Ἅπασα ἄρα στερεὰ γωνία ὑπὸ ἐλασσόνων [ἢ] τεσσάρων ὀρθῶν γωνιῶν ἐπιπέδων περιέχεται· ὅπερ ἔδει δεῖξαι.

κβ΄. Ἐὰν ὦσι τρεῖς γωνίαι ἐπίπεδοι, ὧν αἱ δύο τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, περιέχωσι δὲ αὐτὰς ἴσαι εὐθεῖαι, δυνατόν ἐστιν ἐκ τῶν ἐπιζευγνυουσῶν τὰς ἴσας εὐθείας τρίγωνον συστήσασθαι.

Ἔστωσαν τρεῖς γωνίαι ἐπίπεδοι αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ, ὧν αἱ δύο τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, αἱ μὲν ὑπὸ ΑΒΓ, ΔΕΖ τῆς ὑπὸ ΗΘΚ, αἱ δὲ ὑπὸ ΔΕΖ, ΗΘΚ τῆς ὑπὸ ΑΒΓ, καὶ ἔτι αἱ ὑπὸ ΗΘΚ, ΑΒΓ τῆς ὑπὸ ΔΕΖ, καὶ ἔστωσαν ἴσαι αἱ ΑΒ, ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ εὐθεῖαι, καὶ ἐπεζεύχθωσαν αἱ ΑΓ, ΔΖ, ΗΚ· λέγω, ὅτι δυνατόν ἐστιν ἐκ τῶν ἴσων ταῖς ΑΓ,

ΔΖ, ΗΚ τρίγωνον συστήσασθαι, τουτέστιν ὅτι τῶν ΑΓ, ΔΖ, ΗΚ δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσιν.

Εἰ μὲν οὖν αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ γωνίαι ἴσαι ἀλλήλαις εἰσίν, φανερόν, ὅτι καὶ τῶν ΑΓ, ΔΖ, ΗΚ ἴσων γινομένων δυνατόν ἐστιν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι. εἰ δὲ οὔ, ἔστωσαν ἄνισοι, καὶ συνεστάτω πρὸς τῇ ΘΚ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Θ τῇ ὑπὸ ΑΒΓ γωνίᾳ ἴση ἡ ὑπὸ ΚΘΛ· καὶ κείσθω μιᾷ τῶν ΑΒ, ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ ἴση ἡ ΘΛ, καὶ ἐπεζεύχθωσαν αἱ ΚΛ, ΗΛ. καὶ ἐπεὶ δύο αἱ ΑΒ, ΒΓ δυσὶ ταῖς ΚΘ, ΘΛ ἴσαι εἰσίν, καὶ γωνία ἡ πρὸς τῷ Β γωνίᾳ τῇ ὑπὸ ΚΘΛ ἴση, βάσις ἄρα ἡ ΑΓ βάσει τῇ ΚΛ ἴση.

καὶ ἐπεὶ αἱ ὑπὸ ΑΒΓ, ΗΘΚ τῆς ὑπὸ ΔΕΖ μείζονές εἰσιν, ἴση δὲ ἡ ὑπὸ ΑΒΓ τῇ ὑπὸ ΚΘΛ, ἡ ἄρα ὑπὸ ΗΘΛ τῆς ὑπὸ ΔΕΖ μείζων ἐστίν. καὶ ἐπεὶ δύο αἱ ΗΘ, ΘΛ δύο ταῖς ΔΕ, ΕΖ ἴσαι εἰσίν, καὶ γωνία ἡ ὑπὸ ΗΘΛ γωνίας τῆς ὑπὸ ΔΕΖ μείζων, βάσις ἄρα ἡ ΗΛ βάσεως τῆς ΔΖ μείζων ἐστίν. ἀλλὰ αἱ ΗΚ, ΚΛ τῆς ΗΛ μείζονές εἰσιν. πολλῷ ἄρα αἱ ΗΚ, ΚΛ τῆς ΔΖ μείζονές εἰσιν. ἴση δὲ ἡ ΚΛ τῇ ΑΓ· αἱ ΑΓ, ΗΚ ἄρα τῆς λοιπῆς τῆς ΔΖ μείζονές εἰσιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ μὲν ΑΓ, ΔΖ τῆς ΗΚ μείζονές εἰσιν, καὶ ἔτι αἱ ΔΖ, ΗΚ τῆς ΑΓ μείζονές εἰσιν. δυνατὸν ἄρα ἐστὶν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι· ὅπερ ἔδει δεῖξαι.

κγ΄. Ἐκ τριῶν γωνιῶν ἐπιπέδων, ὧν αἱ δύο τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, στερεὰν γωνίαν συστήσασθαι· δεῖ δὴ τὰς τρεῖς τεσσάρων ὀρθῶν ἐλάσσονας εἶναι.

Ἔστωσαν αἱ δοθεῖσαι τρεῖς γωνίαι ἐπίπεδοι αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ, ὧν αἱ δύο τῆς λοιπῆς μείζονες ἔστωσαν πάντῃ μεταλαμβανόμεναι, ἔτι δὲ αἱ τρεῖς τεσσάρων ὀρθῶν ἐλάσσονες· δεῖ δὴ ἐκ τῶν ἴσων ταῖς ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ στερεὰν γωνίαν συστήσασθαι.

Ἀπειλήφθωσαν ἴσαι αἱ ΑΒ, ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ, καὶ ἐπεζεύχθωσαν αἱ ΑΓ, ΔΖ, ΗΚ· δυνατὸν ἄρα ἐστὶν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι. συνεστάτω τὸ ΛΜΝ, ὥστε ἴσην εἶναι τὴν μὲν ΑΓ τῇ ΛΜ, τὴν δὲ ΔΖ τῇ ΜΝ, καὶ ἔτι τὴν ΗΚ τῇ ΝΛ, καὶ περιγεγράφθω περὶ τὸ ΛΜΝ τρίγωνον κύκλος ὁ ΛΜΝ καὶ εἰλήφθω αὐτοῦ τὸ κέντρον καὶ ἔστω τὸ Ξ, καὶ ἐπεζεύχθωσαν αἱ ΛΞ, ΜΞ, ΝΞ· λέγω, ὅτι ἡ ΑΒ μείζων ἐστὶ τῆς ΛΞ. εἰ γὰρ μή, ἤτοι ἴση ἐστὶν ἡ ΑΒ τῇ ΛΞ ἢ ἐλάττων. ἔστω πρότερον ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΛΞ, ἀλλὰ ἡ μὲν ΑΒ τῇ ΒΓ ἐστιν ἴση, ἡ δὲ ΞΛ τῇ ΞΜ, δύο δὴ αἱ ΑΒ, ΒΓ δύο ταῖς ΛΞ, ΞΜ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ βάσις ἡ ΑΓ βάσει τῇ ΛΜ ὑπόκειται ἴση· γωνία ἄρα ἡ ὑπὸ ΑΒΓ γωνίᾳ τῇ ὑπὸ ΛΞΜ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ μὲν ὑπὸ ΔΕΖ τῇ ὑπὸ ΜΞΝ ἐστιν ἴση, καὶ ἔτι ἡ ὑπὸ ΗΘΚ τῇ ὑπὸ ΝΞΛ· αἱ ἄρα τρεῖς αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ γωνίαι τρισὶ ταῖς ὑπὸ ΛΞΜ, ΜΞΝ, ΝΞΛ εἰσιν ἴσαι. ἀλλὰ αἱ τρεῖς αἱ ὑπὸ ΛΞΜ, ΜΞΝ, ΝΞΛ τέτταρσιν ὀρθαῖς εἰσιν ἴσαι· καὶ αἱ τρεῖς ἄρα αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ τέτταρσιν ὀρθαῖς ἴσαι εἰσίν. ὑπόκεινται δὲ καὶ τεσσάρων ὀρθῶν ἐλάσσονες· ὅπερ ἄτοπον. οὐκ ἄρα ἡ ΑΒ τῇ ΛΞ ἴση ἐστίν.

λέγω δή, ὅτι οὐδὲ ἐλάττων ἐστὶν ἡ ΑΒ τῆς ΛΞ. εἰ γὰρ δυνατόν, ἔστω· καὶ κείσθω τῇ μὲν ΑΒ ἴση ἡ ΞΟ, τῇ δὲ ΒΓ ἴση ἡ ΞΠ, καὶ ἐπεζεύχθω ἡ ΟΠ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΒΓ, ἴση ἐστὶ καὶ ἡ ΞΟ τῇ ΞΠ· ὥστε καὶ λοιπὴ ἡ ΛΟ τῇ ΠΜ ἐστιν ἴση. παράλληλος ἄρα ἐστὶν ἡ ΛΜ τῇ ΟΠ, καὶ ἰσογώνιον τὸ ΛΜΞ τῷ ΟΠΞ· ἔστιν ἄρα ὡς ἡ ΞΛ πρὸς ΛΜ, οὕτως ἡ ΞΟ πρὸς ΟΠ· ἐναλλὰξ ὡς ἡ ΛΞ πρὸς ΞΟ, οὕτως ἡ ΛΜ πρὸς ΟΠ. μείζων δὲ ἡ ΛΞ τῆς ΞΟ· μείζων ἄρα καὶ ἡ ΛΜ τῆς ΟΠ. ἀλλὰ ἡ ΛΜ κεῖται τῇ ΑΓ ἴση· καὶ ἡ ΑΓ ἄρα τῆς ΟΠ μείζων ἐστίν. ἐπεὶ οὖν δύο αἱ ΑΒ, ΒΓ δυσὶ ταῖς ΟΞ, ΞΠ ἴσαι εἰσίν, καὶ βάσις ἡ ΑΓ βάσεως τῆς ΟΠ μείζων ἐστίν, γωνία ἄρα ἡ ὑπὸ ΑΒΓ γωνίας τῆς ὑπὸ ΟΞΠ μείζων ἐστίν. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ μὲν ὑπὸ ΔΕΖ τῆς ὑπὸ ΜΞΝ μείζων ἐστίν, ἡ δὲ ὑπὸ ΗΘΚ τῆς ὑπὸ ΝΞΛ. αἱ ἄρα τρεῖς γωνίαι αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ τριῶν τῶν ὑπὸ ΛΞΜ, ΜΞΝ, ΝΞΛ μείζονές εἰσιν. ἀλλὰ αἱ ὑπὸ

ΑΒΓ, ΔΕΖ, ΗΘΚ τεσσάρων ὀρθῶν ἐλάσσονες ὑπόκεινται· πολλῷ ἄρα αἱ ὑπὸ ΛΞΜ, ΜΞΝ, ΝΞΛ τεσσάρων ὀρθῶν ἐλάσσονές εἰσιν. ἀλλὰ καὶ ἴσαι· ὅπερ ἐστὶν ἄτοπον. οὐκ ἄρα ἡ ΑΒ ἐλάσσων ἐστὶ τῆς ΛΞ. ἐδείχθη δέ, ὅτι οὐδὲ ἴση· μείζων ἄρα ἡ ΑΒ τῆς ΛΞ. ἀνεστάτω δὴ ἀπὸ τοῦ Ξ σημείου τῷ τοῦ ΛΜΝ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΞΡ, καὶ ᾧ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΑΒ τετράγωνον τοῦ ἀπὸ τῆς ΛΞ, ἐκείνῳ ἴσον ἔστω τὸ ἀπὸ τῆς ΞΡ, καὶ ἐπεζεύχθωσαν αἱ ΡΛ, ΡΜ, ΡΝ. καὶ ἐπεὶ ἡ ΡΞ ὀρθή ἐστι πρὸς τὸ τοῦ ΛΜΝ κύκλου ἐπίπεδον, καὶ πρὸς ἑκάστην ἄρα τῶν ΛΞ, ΜΞ, ΝΞ ὀρθή ἐστιν ἡ ΡΞ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΛΞ τῇ ΞΜ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΞΡ, βάσις ἄρα ἡ ΡΛ βάσει τῇ ΡΜ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΡΝ ἑκατέρᾳ τῶν ΡΛ, ΡΜ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΡΛ, ΡΜ, ΡΝ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ᾧ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΛΞ, ἐκείνῳ ἴσον ὑπόκειται τὸ ἀπὸ τῆς ΞΡ, τὸ ἄρα ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΛΞ, ΞΡ. τοῖς δὲ ἀπὸ τῶν ΛΞ, ΞΡ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΛΡ· ὀρθὴ γὰρ ἡ ὑπὸ ΛΞΡ· τὸ ἄρα ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΡΛ· ἴση ἄρα ἡ ΑΒ τῇ ΡΛ. ἀλλὰ τῇ μὲν ΑΒ ἴση ἐστὶν ἑκάστη τῶν ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ, τῇ δὲ ΡΛ ἴση ἑκατέρα τῶν ΡΜ, ΡΝ· ἑκάστη ἄρα τῶν ΑΒ, ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ ἑκάστῃ τῶν ΡΛ, ΡΜ, ΡΝ ἴση ἐστίν. καὶ ἐπεὶ δύο αἱ ΛΡ, ΡΜ δυσὶ ταῖς ΑΒ, ΒΓ ἴσαι εἰσίν, καὶ βάσις ἡ ΛΜ βάσει τῇ ΑΓ ὑπόκειται ἴση, γωνία ἄρα ἡ ὑπὸ ΛΡΜ γωνίᾳ τῇ ὑπὸ ΑΒΓ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ μὲν ὑπὸ ΜΡΝ τῇ ὑπὸ ΔΕΖ ἐστιν ἴση, ἡ δὲ ὑπὸ ΛΡΝ τῇ ὑπὸ ΗΘΚ.

Ἐκ τριῶν ἄρα γωνιῶν ἐπιπέδων τῶν ὑπὸ ΛΡΜ, ΜΡΝ, ΛΡΝ, αἵ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις ταῖς ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ, στερεὰ γωνία συνέσταται ἡ πρὸς τῷ Ρ περιεχομένη ὑπὸ τῶν ΛΡΜ, ΜΡΝ, ΛΡΝ γωνιῶν· ὅπερ ἔδει ποιῆσαι.

Λῆμμα

Ὃν δὲ τρόπον, ᾧ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΛΞ, ἐκείνῳ ἴσον λαβεῖν ἔστι τὸ ἀπὸ τῆς ΞΡ, δείξομεν οὕτως. ἐκκείσθωσαν αἱ ΑΒ, ΛΞ εὐθεῖαι,

καὶ ἔστω μείζων ἡ ΑΒ, καὶ γεγράφθω ἐπ᾽ αὐτῆς ἡμικύκλιον τὸ ΑΒΓ, καὶ εἰς τὸ ΑΒΓ ἡμικύκλιον ἐνηρμόσθω τῇ ΛΞ εὐθείᾳ μὴ μείζονι οὔσῃ τῆς ΑΒ διαμέτρου ἴση ἡ ΑΓ, καὶ ἐπεζεύχθω ἡ ΓΒ. ἐπεὶ οὖν ἐν ἡμικυκλίῳ τῷ ΑΓΒ γωνία ἐστὶν ἡ ὑπὸ ΑΓΒ, ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΑΓΒ.

τὸ ἄρα ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΓ, ΓΒ. ὥστε τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΑΓ μεῖζόν ἐστι τῷ ἀπὸ τῆς ΓΒ. ἴση δὲ ἡ ΑΓ τῇ ΛΞ. τὸ ἄρα ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΛΞ μεῖζόν ἐστι τῷ ἀπὸ τῆς ΓΒ. ἐὰν οὖν τῇ ΒΓ ἴσην τὴν ΞΡ ἀπολάβωμεν, ἔσται τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΛΞ μεῖζον τῷ ἀπὸ τῆς ΞΡ· ὅπερ προέκειτο ποιῆσαι.

κδ΄. Ἐὰν στερεὸν ὑπὸ παραλλήλων ἐπιπέδων περιέχηται, τὰ ἀπεναντίον αὐτοῦ ἐπίπεδα ἴσα τε καὶ παραλληλόγραμμά ἐστιν.

Στερεὸν γὰρ τὸ ΓΔΘΗ ὑπὸ παραλλήλων ἐπιπέδων περιεχέσθω τῶν ΑΓ, ΗΖ, ΑΘ, ΔΖ, ΒΖ, ΑΕ· λέγω, ὅτι τὰ ἀπεναντίον αὐτοῦ ἐπίπεδα ἴσα τε καὶ παραλληλόγραμμά ἐστιν.

Ἐπεὶ γὰρ δύο ἐπίπεδα παράλληλα τὰ ΒΗ, ΓΕ ὑπὸ ἐπιπέδου τοῦ ΑΓ τέμνεται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν. παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΔΓ. πάλιν, ἐπεὶ δύο ἐπίπεδα παράλληλα τὰ ΒΖ, ΑΕ ὑπὸ ἐπιπέδου τοῦ ΑΓ τέμνεται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν. παράλληλος ἄρα ἐστὶν ἡ ΒΓ τῇ ΑΔ. ἐδείχθη δὲ καὶ ἡ ΑΒ τῇ ΔΓ παράλληλος· παραλληλόγραμμον ἄρα ἐστὶ τὸ ΑΓ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἕκαστον τῶν ΔΖ, ΖΗ, ΗΒ, ΒΖ, ΑΕ παραλληλόγραμμόν ἐστιν.

Ἐπεζεύχθωσαν αἱ ΑΘ, ΔΖ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ μὲν ΑΒ τῇ ΔΓ, ἡ δὲ ΒΘ τῇ ΓΖ, δύο δὴ αἱ ΑΒ, ΒΘ ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας τὰς ΔΓ, ΓΖ ἁπτομένας ἀλλήλων εἰσὶν οὐκ ἐν τῷ αὐτῷ ἐπιπέδῳ· ἴσας ἄρα γωνίας περιέξουσιν· ἴση ἄρα ἡ ὑπὸ ΑΒΘ γωνία τῇ ὑπὸ ΔΓΖ. καὶ ἐπεὶ δύο αἱ ΑΒ, ΒΘ δυσὶ ταῖς ΔΓ, ΓΖ ἴσαι εἰσίν, καὶ γωνία ἡ ὑπὸ ΑΒΘ γωνίᾳ τῇ ὑπὸ ΔΓΖ ἐστιν ἴση, βάσις ἄρα ἡ ΑΘ βάσει τῇ ΔΖ ἐστιν ἴση, καὶ τὸ ΑΒΘ τρίγωνον τῷ ΔΓΖ

τριγώνῳ ἴσον ἐστίν. καί ἐστι τοῦ μὲν ΑΒΘ διπλάσιον τὸ ΒΗ παραλληλόγραμμον, τοῦ δὲ ΔΓΖ διπλάσιον τὸ ΓΕ παραλληλόγραμμον· ἴσον ἄρα τὸ ΒΗ παραλληλόγραμμον τῷ ΓΕ παραλληλογράμμῳ. ὁμοίως δὴ δείξομεν, ὅτι καὶ τὸ μὲν ΑΓ τῷ ΗΖ ἐστιν ἴσον, τὸ δὲ ΑΕ τῷ ΒΖ.

Ἐὰν ἄρα στερεὸν ὑπὸ παραλλήλων ἐπιπέδων περιέχηται, τὰ ἀπεναντίον αὐτοῦ ἐπίπεδα ἴσα τε καὶ παραλληλόγραμμά ἐστιν· ὅπερ ἔδει δεῖξαι.

κε΄. Ἐὰν στερεὸν παραλληλεπίπεδον ἐπιπέδῳ τμηθῇ παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔσται ὡς ἡ βάσις πρὸς τὴν βάσιν, οὕτως τὸ στερεὸν πρὸς τὸ στερεόν.

Στερεὸν γὰρ παραλληλεπίπεδον τὸ ΑΒΓΔ ἐπιπέδῳ τῷ ΖΗ τετμήσθω παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις τοῖς ΡΑ, ΔΘ· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΕΖΦ βάσις πρὸς τὴν ΕΘΓΖ βάσιν, οὕτως τὸ ΑΒΖΥ στερεὸν πρὸς τὸ ΕΗΓΔ στερεόν.

Ἐκβεβλήσθω γὰρ ἡ ΑΘ ἐφ’ ἑκάτερα τὰ μέρη, καὶ κείσθωσαν τῇ μὲν ΑΕ ἴσαι ὁσαιδηποτοῦν αἱ ΑΚ, ΚΛ, τῇ δὲ ΕΘ ἴσαι ὁσαιδηποτοῦν αἱ ΘΜ, ΜΝ, καὶ συμπεπληρώσθω τὰ ΛΟ, ΚΦ, ΘΧ, ΜΣ παραλληλόγραμμα καὶ τὰ ΛΠ, ΚΡ, ΔΜ, ΜΤ στερεά. καὶ ἐπεὶ ἴσαι εἰσὶν αἱ ΛΚ, ΚΑ, ΑΕ εὐθεῖαι ἀλλήλαις, ἴσα ἐστὶ καὶ τὰ μὲν ΛΟ, ΚΦ, ΑΖ παραλληλόγραμμα ἀλλήλοις, τὰ δὲ ΚΞ, ΚΒ, ΑΗ ἀλλήλοις καὶ ἔτι τὰ ΛΨ, ΚΠ, ΑΡ ἀλλήλοις· ἀπεναντίον γάρ. διὰ τὰ αὐτὰ δὴ καὶ τὰ μὲν ΕΓ, ΘΧ, ΜΣ παραλληλόγραμμα ἴσα εἰσὶν ἀλλήλοις, τὰ δὲ ΘΗ, ΘΙ, ΙΝ ἴσα εἰσὶν ἀλλήλοις, καὶ ἔτι τὰ ΔΘ, ΜΩ, ΝΤ· τρία ἄρα ἐπίπεδα τῶν ΛΠ, ΚΡ, ΑΥ στερεῶν τρισὶν ἐπιπέδοις ἐστὶν ἴσα.

ἀλλὰ τὰ τρία τρισὶ τοῖς ἀπεναντίον ἐστὶν ἴσα· τὰ ἄρα τρία στερεὰ τὰ ΛΠ, ΚΡ, ΑΥ ἴσα ἀλλήλοις ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ τὰ τρία στερεὰ τὰ ΕΔ, ΔΜ, ΜΤ ἴσα ἀλλήλοις ἐστίν· ὁσαπλασίων ἄρα ἐστὶν ἡ ΛΖ βάσις τῆς ΑΖ βάσεως, τοσαυταπλάσιόν ἐστι καὶ τὸ ΛΥ στερεὸν τοῦ ΑΥ στερεοῦ. διὰ τὰ αὐτὰ δὴ ὁσαπλασίων ἐστὶν ἡ ΝΖ βάσις τῆς ΖΘ βάσεως, τοσαυταπλάσιόν ἐστι καὶ τὸ

ΝΥ στερεὸν τοῦ ΘΥ στερεοῦ. καὶ εἰ ἴση ἐστὶν ἡ ΛΖ βάσις τῇ ΝΖ βάσει, ἴσον ἐστὶ καὶ τὸ ΛΥ στερεὸν τῷ ΝΥ στερεῷ, καὶ εἰ ὑπερέχει ἡ ΛΖ βάσις τῆς ΝΖ βάσεως, ὑπερέχει καὶ τὸ ΛΥ στερεὸν τοῦ ΝΥ στερεοῦ, καὶ εἰ ἐλλείπει, ἐλλείπει. τεσσάρων δὴ ὄντων μεγεθῶν, δύο μὲν βάσεων τῶν ΑΖ, ΖΘ, δύο δὲ στερεῶν τῶν ΑΥ, ΥΘ, εἴληπται ἰσάκις πολλαπλάσια τῆς μὲν ΑΖ βάσεως καὶ τοῦ ΑΥ στερεοῦ ἥ τε ΛΖ βάσις καὶ τὸ ΛΥ στερεόν, τῆς δὲ ΘΖ βάσεως καὶ τοῦ ΘΥ στερεοῦ ἥ τε ΝΖ βάσις καὶ τὸ ΝΥ στερεόν, καὶ δέδεικται, ὅτι εἰ ὑπερέχει ἡ ΛΖ βάσις τῆς ΖΝ βάσεως, ὑπερέχει καὶ τὸ ΛΥ στερεὸν τοῦ ΝΥ [στερεοῦ], καὶ εἰ ἴση, ἴσον, καὶ εἰ ἐλλείπει, ἐλλείπει. ἔστιν ἄρα ὡς ἡ ΑΖ βάσις πρὸς τὴν ΖΘ βάσιν, οὕτως τὸ ΑΥ στερεὸν πρὸς τὸ ΥΘ στερεόν· ὅπερ ἔδει δεῖξαι.

κς΄. Πρὸς τῇ δοθείσῃ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῇ δοθείσῃ στερεᾷ γωνίᾳ ἴσην στερεὰν γωνίαν συστήσασθαι.

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ πρὸς αὐτῇ δοθὲν σημεῖον τὸ Α, ἡ δὲ δοθεῖσα στερεὰ γωνία ἡ πρὸς τῷ Δ περιεχομένη ὑπὸ τῶν ὑπὸ ΕΔΓ, ΕΔΖ, ΖΔΓ γωνιῶν ἐπιπέδων· δεῖ δὴ πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ πρὸς τῷ Δ στερεᾷ γωνίᾳ ἴσην στερεὰν γωνίαν συστήσασθαι.

Εἰλήφθω γὰρ ἐπὶ τῆς ΔΖ τυχὸν σημεῖον τὸ Ζ, καὶ ἤχθω ἀπὸ τοῦ Ζ ἐπὶ τὸ διὰ τῶν ΕΔ, ΔΓ ἐπίπεδον κάθετος ἡ ΖΗ, καὶ συμβαλλέτω τῷ ἐπιπέδῳ κατὰ τὸ Η, καὶ ἐπεζεύχθω ἡ ΔΗ, καὶ συνεστάτω πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ μὲν ὑπὸ ΕΔΓ γωνίᾳ ἴση ἡ ὑπὸ ΒΑΛ, τῇ δὲ ὑπὸ ΕΔΗ ἴση ἡ ὑπὸ ΒΑΚ, καὶ κείσθω τῇ ΔΗ ἴση ἡ ΑΚ, καὶ ἀνεστάτω ἀπὸ τοῦ Κ σημείου τῷ διὰ τῶν ΒΑΛ ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΚΘ, καὶ κείσθω ἴση τῇ ΗΖ ἡ ΚΘ, καὶ ἐπεζεύχθω ἡ ΘΑ· λέγω, ὅτι ἡ πρὸς τῷ Α στερεὰ γωνία περιεχομένη ὑπὸ τῶν ΒΑΛ, ΒΑΘ, ΘΑΛ γωνιῶν ἴση ἐστὶ τῇ πρὸς τῷ Δ στερεᾷ γωνίᾳ τῇ περιεχομένῃ ὑπὸ τῶν ΕΔΓ, ΕΔΖ, ΖΔΓ γωνιῶν.

Ἀπειλήφθωσαν γὰρ ἴσαι αἱ ΑΒ, ΔΕ, καὶ ἐπεζεύχθωσαν αἱ ΘΒ, ΚΒ, ΖΕ, ΗΕ. καὶ ἐπεὶ ἡ ΖΗ ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ ΖΗΔ, ΖΗΕ γωνιῶν. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ὑπὸ ΘΚΑ, ΘΚΒ γωνιῶν ὀρθή ἐστιν. καὶ ἐπεὶ δύο αἱ ΚΑ, ΑΒ δύο ταῖς ΗΔ, ΔΕ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΚΒ βάσει τῇ ΗΕ ἴση ἐστίν. ἔστι δὲ καὶ ἡ ΚΘ τῇ ΗΖ ἴση· καὶ γωνίας ὀρθὰς περιέχουσιν· ἴση ἄρα καὶ ἡ ΘΒ τῇ ΖΕ. πάλιν ἐπεὶ δύο αἱ ΑΚ, ΚΘ δυσὶ ταῖς ΔΗ, ΗΖ ἴσαι εἰσίν, καὶ γωνίας ὀρθὰς περιέχουσιν, βάσις ἄρα ἡ ΑΘ βάσει τῇ ΖΔ ἴση ἐστίν. ἔστι δὲ καὶ ἡ ΑΒ τῇ ΔΕ ἴση· δύο δὴ αἱ ΘΑ, ΑΒ δύο ταῖς ΔΖ, ΔΕ ἴσαι εἰσίν. καὶ βάσις ἡ ΘΒ βάσει τῇ ΖΕ ἴση· γωνία ἄρα ἡ ὑπὸ ΒΑΘ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΘΑΛ τῇ ὑπὸ ΖΔΓ ἐστιν ἴση [ἐπειδήπερ ἐὰν ἀπολάβωμεν ἴσας τὰς ΑΛ, ΔΓ καὶ ἐπιζεύξωμεν τὰς ΚΛ, ΘΛ, ΗΓ, ΖΓ, ἐπεὶ ὅλη ἡ ὑπὸ ΒΑΛ ὅλῃ τῇ ὑπὸ ΕΔΓ ἐστιν ἴση, ὧν ἡ ὑπὸ ΒΑΚ τῇ ὑπὸ ΕΔΗ ὑπόκειται ἴση, λοιπὴ ἄρα ἡ ὑπὸ ΚΑΛ λοιπῇ τῇ ὑπὸ ΗΔΓ ἐστιν ἴση. καὶ ἐπεὶ δύο αἱ ΚΑ, ΑΛ δυσὶ ταῖς ΗΔ, ΔΓ ἴσαι εἰσίν, καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΚΛ βάσει τῇ ΗΓ ἐστιν ἴση. ἔστι δὲ καὶ ἡ ΚΘ τῇ ΗΖ ἴση· δύο δὴ αἱ ΛΚ, ΚΘ δυσὶ ταῖς ΓΗ, ΗΖ εἰσιν ἴσαι· καὶ γωνίας ὀρθὰς περιέχουσιν· βάσις ἄρα ἡ ΘΛ βάσει τῇ ΖΓ ἐστιν ἴση. καὶ ἐπεὶ δύο αἱ ΘΑ, ΑΛ δυσὶ ταῖς ΖΔ, ΔΓ εἰσιν ἴσαι, καὶ βάσις ἡ ΘΛ βάσει τῇ ΖΓ ἐστιν ἴση, γωνία ἄρα ἡ ὑπὸ ΘΑΛ γωνίᾳ τῇ ὑπὸ ΖΔΓ ἐστιν ἴση]. ἔστι δὲ καὶ ἡ ὑπὸ ΒΑΛ τῇ ὑπὸ ΕΔΓ ἴση.

Πρὸς ἄρα τῇ δοθείσῃ εὐθείᾳ τῇ ΑΒ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ δοθείσῃ στερεᾷ γωνίᾳ τῇ πρὸς τῷ Δ ἴση συνέσταται· ὅπερ ἔδει ποιῆσαι.

κζ΄. Ἀπὸ τῆς δοθείσης εὐθείας τῷ δοθέντι στερεῷ παραλληλεπιπέδῳ ὁμοιόν τε καὶ ὁμοίως κείμενον στερεὸν παραλληλεπίπεδον ἀναγράψαι.

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ δοθὲν στερεὸν παραλληλεπίπεδον τὸ ΓΔ· δεῖ δὴ ἀπὸ τῆς δοθείσης εὐθείας τῆς ΑΒ τῷ δοθέντι στερεῷ παραλληλεπιπέδῳ τῷ ΓΔ ὅμοιόν τε καὶ ὁμοίως κείμενον στερεὸν παραλληλεπίπεδον ἀναγράψαι.

Συνεστάτω γὰρ πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ πρὸς τῷ Γ στερεᾷ γωνίᾳ ἴση ἡ περιεχομένη ὑπὸ τῶν ΒΑΘ, ΘΑΚ, ΚΑΒ, ὥστε ἴσην εἶναι τὴν μὲν ὑπὸ ΒΑΘ γωνίαν τῇ ὑπὸ ΕΓΖ, τὴν δὲ ὑπὸ ΒΑΚ τῇ ὑπὸ ΕΓΗ, τὴν δὲ ὑπὸ ΚΑΘ τῇ ὑπὸ ΗΓΖ· καὶ γεγονέτω ὡς μὲν ἡ ΕΓ πρὸς τὴν ΓΗ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΚ, ὡς δὲ ἡ ΗΓ πρὸς τὴν ΓΖ, οὕτως ἡ ΚΑ πρὸς τὴν ΑΘ. καὶ δι’ ἴσου ἄρα ἐστὶν ὡς ἡ ΕΓ πρὸς τὴν ΓΖ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΘ. καὶ συμπεπληρώσθω τὸ ΘΒ παραλληλόγραμμον καὶ τὸ ΑΛ στερεόν.

Καὶ ἐπεί ἐστιν ὡς ἡ ΕΓ πρὸς τὴν ΓΗ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΚ, καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΕΓΗ, ΒΑΚ αἱ πλευραὶ ἀνάλογόν εἰσιν, ὅμοιον ἄρα ἐστὶ τὸ ΗΕ παραλληλόγραμμον τῷ ΚΒ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν ΚΘ παραλληλόγραμμον τῷ ΗΖ παραλληλογράμμῳ ὅμοιόν ἐστι καὶ ἔτι τὸ ΖΕ τῷ ΘΒ· τρία ἄρα παραλληλόγραμμα τοῦ ΓΔ στερεοῦ τρισὶ παραλληλογράμμοις τοῦ ΑΛ στερεοῦ ὅμοιά ἐστιν. ἀλλὰ τὰ μὲν τρία τρισὶ τοῖς ἀπεναντίον ἴσα τέ ἐστι καὶ ὅμοια, τὰ δὲ τρία τρισὶ τοῖς ἀπεναντίον ἴσα τέ ἐστι καὶ ὅμοια· ὅλον ἄρα τὸ ΓΔ στερεὸν ὅλῳ τῷ ΑΛ στερεῷ ὅμοιόν ἐστιν.

Ἀπὸ τῆς δοθείσης ἄρα εὐθείας τῆς ΑΒ τῷ δοθέντι στερεῷ παραλληλεπιπέδῳ τῷ ΓΔ ὁμοίόν τε καὶ ὁμοίως κείμενον ἀναγέγραπται τὸ ΑΛ· ὅπερ ἔδει ποιῆσαι.

κη΄. Ἐὰν στερεὸν παραλληλεπίπεδον ἐπιπέδῳ τμηθῇ κατὰ τὰς διαγωνίους τῶν ἀπεναντίον ἐπιπέδων, δίχα τμηθήσεται τὸ στερεὸν ὑπὸ τοῦ ἐπιπέδου.

Στερεὸν γὰρ παραλληλεπίπεδον τὸ ΑΒ ἐπιπέδῳ τῷ ΓΔΕΖ τετμήσθω κατὰ τὰς διαγωνίους τῶν ἀπεναντίον ἐπιπέδων τὰς ΓΖ, ΔΕ· λέγω, ὅτι δίχα τμηθήσεται τὸ ΑΒ στερεὸν ὑπὸ τοῦ ΓΔΕΖ ἐπιπέδου.

Ἐπεὶ γὰρ ἴσον ἐστὶ τὸ μὲν ΓΗΖ τρίγωνον τῷ ΓΖΒ τριγώνῳ, τὸ δὲ ΑΔΕ τῷ ΔΕΘ, ἔστι δὲ καὶ τὸ μὲν ΓΑ παραλληλόγραμμον τῷ ΕΒ ἴσον· ἀπεναντίον γάρ· τὸ δὲ ΗΕ τῷ ΓΘ, καὶ τὸ πρίσμα ἄρα τὸ περιεχόμενον ὑπὸ δύο μὲν τριγώνων τῶν ΓΗΖ, ΑΔΕ, τριῶν δὲ παραλληλογράμμων τῶν ΗΕ, ΑΓ, ΓΕ ἴσον ἐστὶ τῷ πρίσματι τῷ περιεχομένῳ ὑπὸ δύο μὲν τριγώνων τῶν ΓΖΒ, ΔΕΘ, τριῶν δὲ παραλληλογράμμων τῶν ΓΘ, ΒΕ, ΓΕ· ὑπὸ γὰρ ἴσων ἐπιπέδων περιέχονται τῷ τε πλήθει καὶ τῷ μεγέθει. ὥστε ὅλον τὸ ΑΒ στερεὸν δίχα τέτμηται ὑπὸ τοῦ ΓΔΕΖ ἐπιπέδου· ὅπερ ἔδει δεῖξαι.

κθ΄. Τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν.

Ἔστω ἐπὶ τῆς αὐτῆς βάσεως τῆς ΑΒ στερεὰ παραλληλεπίπεδα τὰ ΓΜ, ΓΝ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΑΗ, ΑΖ, ΛΜ, ΛΝ, ΓΔ, ΓΕ, ΒΘ, ΒΚ ἐπὶ τῶν αὐτῶν εὐθειῶν ἔστωσαν τῶν ΖΝ, ΔΚ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΓΜ στερεὸν τῷ ΓΝ στερεῷ.

Ἐπεὶ γὰρ παραλληλόγραμμόν ἐστιν ἑκάτερον τῶν ΓΘ, ΓΚ, ἴση ἐστὶν ἡ ΓΒ ἑκατέρᾳ τῶν ΔΘ, ΕΚ· ὥστε καὶ ἡ ΔΘ τῇ ΕΚ ἐστιν ἴση. κοινὴ ἀφῃρήσθω ἡ ΕΘ· λοιπὴ ἄρα ἡ ΔΕ λοιπῇ τῇ ΘΚ ἐστιν ἴση. ὥστε καὶ τὸ μὲν ΔΓΕ τρίγωνον τῷ ΘΒΚ τριγώνῳ ἴσον ἐστίν, τὸ δὲ ΔΗ παραλληλόγραμμον τῷ ΘΝ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΑΖΗ τρίγωνον τῷ ΜΛΝ τριγώνῳ ἴσον ἐστίν. ἔστι δὲ καὶ τὸ μὲν ΓΖ παραλληλόγραμμον τῷ ΒΜ παραλληλογράμμῳ ἴσον, τὸ δὲ ΓΗ τῷ ΒΝ· ἀπεναντίον γάρ· καὶ τὸ πρίσμα ἄρα τὸ περιεχόμενον ὑπὸ δύο μὲν τριγώνων τῶν ΑΖΗ, ΔΓΕ, τριῶν δὲ παραλληλογράμμων τῶν ΑΔ, ΔΗ, ΓΗ ἴσον ἐστὶ τῷ πρίσματι τῷ περιεχομένῳ ὑπὸ δύο μὲν τριγώνων τῶν ΜΛΝ, ΘΒΚ, τριῶν δὲ παραλληλογράμμων τῶν ΒΜ, ΘΝ, ΒΝ. κοινὸν προσκείσθω τὸ στερεόν, οὗ βάσις μὲν τὸ ΑΒ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΗΕΘΜ· ὅλον ἄρα τὸ ΓΜ στερεὸν παραλληλεπίπεδον ὅλῳ τῷ ΓΝ στερεῷ παραλληλεπιπέδῳ ἴσον ἐστίν.

Τὰ ἄρα ἐπὶ τῆς αὐτῆς βάσεως ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

λ΄. Τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι οὐκ εἰσὶν ἐπὶ τῶν αὐτῶν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν.

Ἔστω ἐπὶ τῆς αὐτῆς βάσεως τῆς ΑΒ στερεὰ παραλληλεπίπεδα τὰ ΓΜ, ΓΝ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΑΖ, ΑΗ, ΛΜ, ΛΝ, ΓΔ, ΓΕ, ΒΘ, ΒΚ μὴ ἔστωσαν ἐπὶ τῶν αὐτῶν εὐθειῶν· λέγω, ὅτι ἴσον ἐστὶ τὸ ΓΜ στερεὸν τῷ ΓΝ στερεῷ.

Ἐκβεβλήσθωσαν γὰρ αἱ ΝΚ, ΔΘ καὶ συμπιπτέτωσαν ἀλλήλαις κατὰ τὸ Ρ, καὶ ἔτι ἐκβεβλήσθωσαν αἱ ΖΜ, ΗΕ ἐπὶ τὰ Ο, Π, καὶ ἐπεζεύχθωσαν αἱ ΑΞ, ΛΟ, ΓΠ, ΒΡ. ἴσον δή ἐστι τὸ ΓΜ στερεόν, οὗ βάσις μὲν τὸ ΑΓΒΛ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΖΔΘΜ, τῷ ΓΟ στερεῷ, οὗ βάσις μὲν τὸ ΑΓΒΛ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΞΠΡΟ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΑΓΒΛ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΑΖ, ΑΞ, ΛΜ, ΛΟ, ΓΔ, ΓΠ, ΒΘ, ΒΡ ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν τῶν ΖΟ, ΔΡ. ἀλλὰ τὸ ΓΟ στερεόν, οὗ βάσις μέν ἐστι τὸ ΑΓΒΛ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΞΠΡΟ, ἴσον ἐστὶ τῷ ΓΝ στερεῷ, οὗ βάσις μὲν τὸ ΑΓΒΛ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΗΕΚΝ· ἐπί τε γὰρ πάλιν τῆς αὐτῆς βάσεώς εἰσι τῆς ΑΓΒΛ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΑΗ, ΑΞ, ΓΕ, ΓΠ, ΛΝ, ΛΟ, ΒΚ, ΒΡ ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν τῶν ΗΠ, ΝΡ. ὥστε καὶ τὸ ΓΜ στερεὸν ἴσον ἐστὶ τῷ ΓΝ στερεῷ.

Τὰ ἄρα ἐπὶ τῆς αὐτῆς βάσεως στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι οὐκ εἰσὶν ἐπὶ τῶν αὐτῶν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

λα΄. Τὰ ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν.

Ἔστω ἐπὶ ἴσων βάσεων τῶν ΑΒ, ΓΔ στερεὰ παραλληλεπίπεδα τὰ ΑΕ, ΓΖ ὑπὸ τὸ αὐτὸ ὕψος· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΕ στερεὸν τῷ ΓΖ στερεῷ.

* Βιβλίον ια΄ Πρότασις 31

Ἔστωσαν δὴ πρότερον αἱ ἐφεστηκυῖαι αἱ ΘΚ, ΒΕ, ΑΗ, ΛΜ, ΟΠ, ΔΖ, ΓΞ, ΡΣ πρὸς ὀρθὰς ταῖς ΑΒ, ΓΔ βάσεσιν, καὶ ἐκβεβλήσθω ἐπ᾽ εὐθείας τῇ ΓΡ εὐθεῖα ἡ ΡΤ, καὶ συνεστάτω πρὸς τῇ ΡΤ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Ρ τῇ ὑπὸ ΑΛΒ γωνίᾳ ἴση ἡ ὑπὸ ΤΡΥ, καὶ κείσθω τῇ μὲν ΑΛ ἴση ἡ ΡΤ, τῇ δὲ ΛΒ ἴση ἡ ΡΥ, καὶ συμπεπληρώσθω ἥ τε ΡΧ βάσις καὶ τὸ ΨΥ στερεόν. καὶ ἐπεὶ δύο αἱ ΤΡ, ΡΥ δυσὶ ταῖς ΑΛ, ΛΒ ἴσαι εἰσίν, καὶ γωνίας ἴσας περιέχουσιν, ἴσον ἄρα καὶ ὅμοιον τὸ ΡΧ παραλληλόγραμμον τῷ ΘΛ παραλληλογράμμῳ. καὶ ἐπεὶ πάλιν ἴση μὲν ἡ ΑΛ τῇ ΡΤ, ἡ δὲ ΛΜ τῇ ΡΣ, καὶ γωνίας ὀρθὰς περιέχουσιν, ἴσον ἄρα καὶ ὅμοιόν ἐστι τὸ ΡΨ παραλληλόγραμμον τῷ ΑΜ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΛΕ τῷ ΣΥ ἴσον τέ ἐστι καὶ ὅμοιον· τρία ἄρα παραλληλόγραμμα τοῦ ΑΕ στερεοῦ τρισὶ παραλληλογράμμοις τοῦ ΨΥ στερεοῦ ἴσα τέ ἐστι καὶ ὅμοια. ἀλλὰ τὰ μὲν τρία τρισὶ τοῖς ἀπεναντίον ἴσα τέ ἐστι καὶ ὅμοια, τὰ δὲ τρία τρισὶ τοῖς ἀπεναντίον· ὅλον ἄρα τὸ ΑΕ στερεὸν παραλληλεπίπεδον ὅλῳ τῷ ΨΥ στερεῷ παραλληλεπιπέδῳ ἴσον ἐστίν. διήχθωσαν αἱ ΔΡ, ΧΥ καὶ συμπιπτέτωσαν ἀλλήλαις κατὰ τὸ Ω, καὶ διὰ τοῦ Τ τῇ ΔΩ παράλληλος ἤχθω ἡ αΤϡ, καὶ ἐκβεβλήσθω ἡ ΟΔ κατὰ τὸ α, καὶ συμπεπληρώσθω τὰ ΩΨ, ΡΙ στερεά. ἴσον δή ἐστι τὸ ΨΩ στερεόν, οὗ βάσις μέν ἐστι τὸ ΡΨ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΩϚ, τῷ ΨΥ στερεῷ, οὗ βάσις μὲν τὸ ΡΨ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΥΦ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΡΨ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΡΩ, ΡΥ, Τϡ, ΤΧ, Σς, Σδ, ΨϚ, ΨΦ ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν τῶν ΩΧ, ϛΦ. ἀλλὰ τὸ ΨΥ στερεὸν τῷ ΑΕ ἐστιν ἴσον· καὶ τὸ ΨΩ ἄρα στερεὸν τῷ ΑΕ στερεῷ ἐστιν ἴσον. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΡΥΧΤ παραλληλόγραμμον τῷ ΩΤ

παραλληλογράμμῳ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΡΤ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΡΤ, ΩΧ· ἀλλὰ τὸ ΡΥΧΤ τῷ ΓΔ ἐστιν ἴσον, ἐπεὶ καὶ τῷ ΑΒ, καὶ τὸ ΩΤ ἄρα παραλληλόγραμμον τῷ ΓΔ ἐστιν ἴσον. ἄλλο δὲ τὸ ΔΤ· ἔστιν ἄρα ὡς ἡ ΓΔ βάσις πρὸς τὴν ΔΤ, οὕτως ἡ ΩΤ πρὸς τὴν ΔΤ. καὶ ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ ΓΙ ἐπιπέδῳ τῷ ΡΖ τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ὡς ἡ ΓΔ βάσις πρὸς τὴν ΔΤ βάσιν, οὕτως τὸ ΓΖ στερεὸν πρὸς τὸ ΡΙ στερεόν. διὰ τὰ αὐτὰ δή, ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ ΩΙ ἐπιπέδῳ τῷ ΡΨ τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ὡς ἡ ΩΤ βάσις πρὸς τὴν ΤΔ βάσιν, οὕτως τὸ ΩΨ στερεὸν πρὸς τὸ ΡΙ. ἀλλ’ ὡς ἡ ΓΔ βάσις πρὸς τὴν ΔΤ, οὕτως ἡ ΩΤ πρὸς τὴν ΔΤ· καὶ ὡς ἄρα τὸ ΓΖ στερεὸν πρὸς τὸ ΡΙ στερεόν, οὕτως τὸ ΩΨ στερεὸν πρὸς τὸ ΡΙ. ἑκάτερον ἄρα τῶν ΓΖ, ΩΨ στερεῶν πρὸς τὸ ΡΙ τὸν αὐτὸν ἔχει λόγον· ἴσον ἄρα ἐστὶ τὸ ΓΖ στερεὸν τῷ ΩΨ στερεῷ. ἀλλὰ τὸ ΩΨ τῷ ΑΕ ἐδείχθη ἴσον· καὶ τὸ ΑΕ ἄρα τῷ ΓΖ ἐστιν ἴσον.

Μὴ ἔστωσαν δὴ αἱ ἐφεστηκυῖαι αἱ ΑΗ, ΘΚ, ΒΕ, ΛΜ, ΓΝ, ΟΠ, ΔΖ, ΡΣ πρὸς ὀρθὰς ταῖς ΑΒ, ΓΔ βάσεσιν· λέγω πάλιν, ὅτι ἴσον τὸ ΑΕ στερεὸν τῷ ΓΖ στερεῷ.

ἤχθωσαν γὰρ ἀπὸ τῶν Κ, Ε, Η, Μ, Π, Ζ, Ν, Σ σημείων ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετοι αἱ ΚΞ, ΕΤ, ΗΥ, ΜΦ, ΠΧ, ΖΨ, ΝΩ, ΣΙ, καὶ συμβαλλέτωσαν τῷ ἐπιπέδῳ κατὰ τὰ Ξ, Τ, Υ, Φ, Χ, Ψ, Ω, Ι σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΞΤ, ΞΥ, ΥΦ, ΤΦ, ΧΨ, ΧΩ, ΩΙ, ΙΨ. ἴσον δή ἐστι τὸ ΚΦ στερεὸν τῷ ΠΙ στερεῷ· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν ΚΜ, ΠΣ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι πρὸς ὀρθάς εἰσι ταῖς βάσεσιν. ἀλλὰ τὸ μὲν ΚΦ στερεὸν τῷ ΑΕ στερεῷ ἐστιν ἴσον, τὸ δὲ ΠΙ τῷ ΓΖ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι οὐκ εἰσιν ἐπὶ τῶν αὐτῶν εὐθειῶν. καὶ τὸ ΑΕ ἄρα στερεὸν τῷ ΓΖ στερεῷ ἐστιν ἴσον.

Τὰ ἄρα ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

λβ΄. Τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις.

Ἔστω ὑπὸ τὸ αὐτὸ ὕψος στερεὰ παραλληλεπίπεδα τὰ ΑΒ, ΓΔ· λέγω, ὅτι τὰ ΑΒ, ΓΔ στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις, τουτέστιν ὅτι ἐστὶν ὡς ἡ ΑΕ βάσις πρὸς τὴν ΓΖ βάσιν, οὕτως τὸ ΑΒ στερεὸν πρὸς τὸ ΓΔ στερεόν.

Παραβεβλήσθω γὰρ παρὰ τὴν ΖΗ τῷ ΑΕ ἴσον τὸ ΖΘ, καὶ ἀπὸ βάσεως μὲν τῆς ΖΘ, ὕψους δὲ τοῦ αὐτοῦ τῷ ΓΔ στερεὸν παραλληλεπίπεδον συμπεπληρώσθω τὸ ΗΚ. ἴσον δή ἐστι τὸ ΑΒ στερεὸν τῷ ΗΚ στερεῷ· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν ΑΕ, ΖΘ καὶ ὑπὸ τὸ αὐτὸ ὕψος. καὶ ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ ΓΚ ἐπιπέδῳ τῷ ΔΗ τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ἄρα ὡς ἡ ΓΖ βάσις πρὸς τὴν ΖΘ βάσιν, οὕτως τὸ ΓΔ στερεὸν πρὸς τὸ ΔΘ στερεόν. ἴση δὲ ἡ μὲν ΖΘ βάσις τῇ ΑΕ βάσει, τὸ δὲ ΗΚ στερεὸν τῷ ΑΒ στερεῷ· ἔστιν ἄρα καὶ ὡς ἡ ΑΕ βάσις πρὸς τὴν ΓΖ βάσιν, οὕτως τὸ ΑΒ στερεὸν πρὸς τὸ ΓΔ στερεόν.

Τὰ ἄρα ὑπὸ τὸ αὐτὸ ὕψος ὄντα στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις· ὅπερ ἔδει δεῖξαι.

λγ΄. Τὰ ὅμοια στερεὰ παραλληλεπίπεδα πρὸς ἄλληλα ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν.

Ἔστω ὅμοια στερεὰ παραλληλεπίπεδα τὰ ΑΒ, ΓΔ, ὁμόλογος δὲ ἔστω ἡ ΑΕ τῇ ΓΖ· λέγω, ὅτι τὸ ΑΒ στερεὸν πρὸς τὸ ΓΔ στερεὸν τριπλασίονα λόγον ἔχει, ἤπερ ἡ ΑΕ πρὸς τὴν ΓΖ.

Ἐκβεβλήσθωσαν γὰρ ἐπ᾽ εὐθείας ταῖς ΑΕ, ΗΕ, ΘΕ αἱ ΕΚ, ΕΛ, ΕΜ, καὶ κείσθω τῇ μὲν ΓΖ ἴση ἡ ΕΚ, τῇ δὲ ΖΝ ἴση ἡ ΕΛ, καὶ ἔτι τῇ ΖΡ ἴση ἡ ΕΜ, καὶ συμπεπληρώσθω τὸ ΚΛ παραλληλόγραμμον καὶ τὸ ΚΟ στερεόν.

Καὶ ἐπεὶ δύο αἱ ΚΕ, ΕΛ δυσὶ ταῖς ΓΖ, ΖΝ ἴσαι εἰσίν, ἀλλὰ καὶ γωνία ἡ ὑπὸ ΚΕΛ γωνίᾳ τῇ ὑπὸ ΓΖΝ ἐστιν ἴση, ἐπειδήπερ καὶ ἡ ὑπὸ ΑΕΗ τῇ ὑπὸ ΓΖΝ ἐστιν ἴση διὰ τὴν ὁμοιότητα τῶν ΑΒ, ΓΔ στερεῶν, ἴσον ἄρα ἐστὶ [καὶ ὅμοιον] τὸ ΚΛ παραλληλόγραμμον τῷ ΓΝ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν ΚΜ παραλληλόγραμμον ἴσον ἐστὶ καὶ ὅμοιον τῷ ΓΡ [παραλληλογράμμῳ] καὶ ἔτι τὸ ΕΟ τῷ ΔΖ· τρία ἄρα παραλληλόγραμμα τοῦ ΚΟ στερεοῦ τρισὶ παραλληλογράμμοις τοῦ ΓΔ στερεοῦ ἴσα ἐστὶ καὶ ὅμοια. ἀλλὰ τὰ μὲν τρία τρισὶ τοῖς ἀπεναντίον ἴσα ἐστὶ καὶ ὅμοια, τὰ δὲ τρία τρισὶ τοῖς ἀπεναντίον ἴσα ἐστὶ καὶ ὅμοια· ὅλον ἄρα τὸ ΚΟ στερεὸν ὅλῳ τῷ ΓΔ στερεῷ ἴσον ἐστὶ καὶ ὅμοιον. συμπεπληρώσθω τὸ ΗΚ παραλληλόγραμμον, καὶ ἀπὸ βάσεων μὲν τῶν ΗΚ, ΚΛ παραλληλογράμμων, ὕψους δὲ τοῦ αὐτοῦ τῷ ΑΒ στερεὰ συμπεπληρώσθω τὰ ΕΞ, ΛΠ. καὶ ἐπεὶ διὰ τὴν ὁμοιότητα τῶν ΑΒ, ΓΔ στερεῶν ἐστιν ὡς ἡ ΑΕ πρὸς τὴν ΓΖ, οὕτως ἡ ΕΗ πρὸς τὴν ΖΝ, καὶ ἡ ΕΘ πρὸς τὴν ΖΡ, ἴση δὲ ἡ μέν ΓΖ τῇ ΕΚ, ἡ δὲ ΖΝ τῇ ΕΛ, ἡ δὲ ΖΡ τῇ ΕΜ, ἔστιν ἄρα ὡς ἡ ΑΕ πρὸς τὴν ΕΚ, οὕτως ἡ ΗΕ πρὸς τὴν ΕΛ καὶ ἡ ΘΕ πρὸς τὴν ΕΜ.

ἀλλ’ ὡς μὲν ἡ ΑΕ πρὸς τὴν ΕΚ, οὕτως τὸ ΑΗ [παραλληλόγραμμον] πρὸς τὸ ΗΚ παραλληλόγραμμον, ὡς δὲ ἡ ΗΕ πρὸς τὴν ΕΛ, οὕτως τὸ ΗΚ πρὸς τὸ ΚΛ, ὡς δὲ ἡ ΘΕ πρὸς ΕΜ, οὕτως τὸ ΠΕ πρὸς τὸ ΚΜ· καὶ ὡς ἄρα τὸ ΑΗ παραλληλόγραμμον πρὸς τὸ ΗΚ, οὕτως τὸ ΗΚ πρὸς τὸ ΚΛ καὶ τὸ ΠΕ πρὸς τὸ ΚΜ. ἀλλ’ ὡς μὲν τὸ ΑΗ πρὸς τὸ ΗΚ, οὕτως τὸ ΑΒ στερεὸν πρὸς τὸ ΕΞ στερεόν, ὡς δὲ τὸ ΗΚ πρὸς τὸ ΚΛ, οὕτως τὸ ΞΕ στερεὸν πρὸς τὸ ΠΛ στερεόν, ὡς δὲ τὸ ΠΕ πρὸς τὸ ΚΜ, οὕτως τὸ ΠΛ στερεὸν πρὸς τὸ ΚΟ στερεόν· καὶ ὡς ἄρα τὸ ΑΒ στερεὸν πρὸς τὸ ΕΞ, οὕτως τὸ ΕΞ πρὸς τὸ ΠΛ καὶ τὸ ΠΛ πρὸς τὸ ΚΟ. ἐὰν δὲ τέσσαρα μεγέθη κατὰ τὸ συνεχὲς ἀνάλογον ᾖ, τὸ πρῶτον πρὸς τὸ τέταρτον τριπλασίονα λόγον ἔχει ἤπερ πρὸς τὸ δεύτερον· τὸ ΑΒ ἄρα στερεὸν πρὸς τὸ ΚΟ τριπλασίονα λόγον ἔχει ἤπερ τὸ ΑΒ πρὸς τὸ ΕΞ. ἀλλ’ ὡς τὸ ΑΒ πρὸς τὸ ΕΞ, οὕτως τὸ ΑΗ παραλληλόγραμμον πρὸς τὸ

ΗΚ καὶ ἡ ΑΕ εὐθεῖα πρὸς τὴν ΕΚ· ὥστε καὶ τὸ ΑΒ στερεὸν πρὸς τὸ ΚΟ τριπλασίονα λόγον ἔχει ἥπερ ἡ ΑΕ πρὸς τὴν ΕΚ. ἴσον δὲ τὸ [μὲν] ΚΟ στερεὸν τῷ ΓΔ στερεῷ, ἡ δὲ ΕΚ εὐθεῖα τῇ ΓΖ· καὶ τὸ ΑΒ ἄρα στερεὸν πρὸς τὸ ΓΔ στερεὸν τριπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος αὐτοῦ πλευρὰ ἡ ΑΕ πρὸς τὴν ὁμόλογον πλευρὰν τὴν ΓΖ.

Τὰ ἄρα ὅμοια στερεὰ παραλληλεπίπεδα ἐν τριπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν· ὅπερ ἔδει δεῖξαι.

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, ἔσται ὡς ἡ πρώτη πρὸς τὴν τετάρτην, οὕτω τὸ ἀπὸ τῆς πρώτης στερεὸν παραλληλεπίπεδον πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον, ἐπείπερ καὶ ἡ πρώτη πρὸς τὴν τετάρτην τριπλασίονα λόγον ἔχει ἥπερ πρὸς τὴν δευτέραν.

λδ΄. Τῶν ἴσων στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ὧν στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσα ἐστὶν ἐκεῖνα.

Ἔστω ἴσα στερεὰ παραλληλεπίπεδα τὰ ΑΒ, ΓΔ· λέγω, ὅτι τῶν ΑΒ, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος.

Ἔστωσαν γὰρ πρότερον αἱ ἐφεστηκυῖαι αἱ ΑΗ, ΕΖ, ΛΒ, ΘΚ, ΓΜ, ΝΞ, ΟΔ, ΠΡ πρὸς ὀρθὰς ταῖς βάσεσιν αὐτῶν· λέγω, ὅτι ἐστὶν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΓΜ πρὸς τὴν ΑΗ.

Εἰ μὲν οὖν ἴση ἐστιν ἡ ΕΘ βάσις τῇ ΝΠ βάσει, ἔστι δὲ καὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ ἴσον, ἔσται καὶ ἡ ΓΜ τῇ ΑΗ ἴση. τὰ γὰρ ὑπὸ τὸ αὐτὸ ὕψος στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις [εἰ γὰρ τῶν ΕΘ,

ΝΠ βάσεων ἴσων οὐσῶν μὴ εἴη τὰ ΑΗ, ΓΜ ὕψη ἴσα, οὐδ' ἄρα τὸ ΑΒ στερεὸν ἴσον ἔσται τῷ ΓΔ. ὑπόκειται δὲ ἴσον· οὐκ ἄρα ἄνισόν ἐστι τὸ ΓΜ ὕψος τῷ ΑΗ ὕψει· ἴσον ἄρα]. καὶ ἔσται ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ, οὕτως ἡ ΓΜ πρὸς τὴν ΑΗ, καὶ φανερόν, ὅτι τῶν ΑΒ, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Μὴ ἔστω δὴ ἴση ἡ ΕΘ βάσις τῇ ΝΠ βάσει, ἀλλ' ἔστω μείζων ἡ ΕΘ. ἔστι δὲ καὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ ἴσον· μείζων ἄρα ἐστὶ καὶ ἡ ΓΜ τῆς ΑΗ [εἰ γὰρ μή, οὐδ' ἄρα πάλιν τὰ ΑΒ, ΓΔ στερεὰ ἴσα ἔσται· ὑπόκειται δὲ ἴσα]. κείσθω οὖν τῇ ΑΗ ἴση ἡ ΓΤ, καὶ συμπεπληρώσθω ἀπὸ βάσεως μὲν τῆς ΝΠ, ὕψους δὲ τοῦ ΓΤ, στερεὸν παραλληλεπίπεδον τὸ ΦΓ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ, ἔξωθεν δὲ τὸ ΓΦ, τὰ δὲ ἴσα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ ΑΒ στερεὸν πρὸς τὸ ΓΦ στερεόν, οὕτως τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν. ἀλλ' ὡς μὲν τὸ ΑΒ στερεὸν πρὸς τὸ ΓΦ στερεόν, οὕτως ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν· ἰσοϋψῆ γὰρ τὰ ΑΒ, ΓΦ στερεά· ὡς δὲ τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν, οὕτως ἡ ΜΠ βάσις πρὸς τὴν ΤΠ βάσιν καὶ ἡ ΓΜ πρὸς τὴν ΓΤ· καὶ ὡς ἄρα ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΜΓ πρὸς τὴν ΓΤ. ἴση δὲ ἡ ΓΤ τῇ ΑΗ· καὶ ὡς ἄρα ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΜΓ πρὸς τὴν ΑΗ. τῶν ΑΒ, ΓΔ ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Πάλιν δὴ τῶν ΑΒ, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπονθέτωσαν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ.

Ἔστωσαν [γὰρ] πάλιν αἱ ἐφεστηκυῖαι πρὸς ὀρθὰς ταῖς βάσεσιν, καὶ εἰ μὲν ἴση ἐστὶν ἡ ΕΘ βάσις τῇ ΝΠ βάσει, καί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος, ἴσον ἄρα ἐστὶ καὶ τὸ τοῦ ΓΔ στερεοῦ ὕψος τῷ τοῦ ΑΒ στερεοῦ ὕψει. τὰ δὲ ἐπὶ

ἴσων βάσεων στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ.

Μὴ ἔστω δὴ ἡ ΕΘ βάσις τῇ ΝΠ [βάσει] ἴση, ἀλλ' ἔστω μείζων ἡ ΕΘ· μεῖζον ἄρα ἐστὶ καὶ τὸ τοῦ ΓΔ στερεοῦ ὕψος τοῦ τοῦ ΑΒ στερεοῦ ὕψους, τουτέστιν ἡ ΓΜ τῆς ΑΗ. κείσθω τῇ ΑΗ ἴση πάλιν ἡ ΓΤ, καὶ συμπεπληρώσθω ὁμοίως τὸ ΓΦ στερεόν. ἐπεί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΜΓ πρὸς τὴν ΑΗ, ἴση δὲ ἡ ΑΗ τῇ ΓΤ, ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΓΜ πρὸς τὴν ΓΤ. ἀλλ' ὡς μὲν ἡ ΕΘ [βάσις] πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ ΑΒ στερεὸν πρὸς τὸ ΓΦ στερεόν· ἰσοϋψῆ γάρ ἐστι τὰ ΑΒ, ΓΦ στερεά· ὡς δὲ ἡ ΓΜ πρὸς τὴν ΓΤ, οὕτως ἥ τε ΜΠ βάσις πρὸς τὴν ΠΤ βάσιν καὶ τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν. καὶ ὡς ἄρα τὸ ΑΒ στερεὸν πρὸς τὸ ΓΦ στερεόν, οὕτως τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν· ἑκάτερον ἄρα τῶν ΑΒ, ΓΔ πρὸς τὸ ΓΦ τὸν αὐτὸν ἔχει λόγον. ἴσον ἄρα ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ [ὅπερ ἔδει δεῖξαι].

Μὴ ἔστωσαν δὴ αἱ ἐφεστηκυῖαι αἱ ΖΕ, ΒΛ, ΗΑ, ΘΚ, ΞΝ, ΔΟ, ΜΓ, ΡΠ πρὸς ὀρθὰς ταῖς βάσεσιν αὐτῶν, καὶ ἤχθωσαν ἀπὸ τῶν Ζ, Η, Β, Κ, Ξ, Μ, Δ, Ρ σημείων ἐπὶ τὰ διὰ τῶν ΕΘ, ΝΠ ἐπίπεδα κάθετοι καὶ συμβαλλέτωσαν τοῖς ἐπιπέδοις κατὰ τὰ Σ, Τ, Υ, Φ, Χ, Ψ, Ω, #2, καὶ συμπεπληρώσθω τὰ ΖΦ, ΞΩ στερεά· λέγω, ὅτι καὶ οὕτως ἴσων ὄντων τῶν ΑΒ, ΓΔ στερεῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος.

Ἐπεὶ ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ, ἀλλὰ τὸ μὲν ΑΒ τῷ ΒΤ ἐστιν ἴσον· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΖΚ καὶ ὑπὸ τὸ αὐτὸ ὕψος [ὧν αἱ ἐφεστῶσαι οὐκ εἰσὶν ἐπὶ τῶν αὐτῶν εὐθειῶν]· τὸ δὲ ΓΔ στερεὸν τῷ ΔΨ ἐστιν ἴσον· ἐπί τε γὰρ πάλιν τῆς αὐτῆς βάσεώς εἰσι τῆς ΡΞ καὶ ὑπὸ τὸ αὐτὸ ὕψος [ὧν αἱ ἐφεστῶσαι οὐκ εἰσὶν ἐπὶ τῶν αὐτῶν εὐθειῶν]· καὶ τὸ ΒΤ ἄρα στερεὸν τῷ ΔΨ στερεῷ ἴσον ἐστίν [τῶν δὲ ἴσων στερεῶν παραλληλεπιπέδων, ὧν τὰ ὕψη πρὸς ὀρθάς ἐστι ταῖς βάσεσιν αὐτῶν,

ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν]. ἔστιν ἄρα ὡς ἡ ΖΚ βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΤ στερεοῦ ὕψος. ἴση δὲ ἡ μὲν ΖΚ βάσις τῇ ΕΘ βάσει, ἡ δὲ ΞΡ βάσις τῇ ΝΠ βάσει· ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΤ στερεοῦ ὕψος. τὰ δ’ αὐτὰ ὕψη ἐστὶ τῶν ΔΨ, ΒΤ στερεῶν καὶ τῶν ΔΓ, ΒΑ· ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΔΓ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος. τῶν ΑΒ, ΓΔ ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Πάλιν δὴ τῶν ΑΒ, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπονθέτωσαν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος, ἴση δὲ ἡ μὲν ΕΘ βάσις τῇ ΖΚ βάσει, ἡ δὲ ΝΠ τῇ ΞΡ, ἔστιν ἄρα ὡς ἡ ΖΚ βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος. τὰ δ’ αὐτὰ ὕψη ἐστὶ τῶν ΑΒ, ΓΔ στερεῶν καὶ τῶν ΒΤ, ΔΨ· ἔστιν ἄρα ὡς ἡ ΖΚ βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΤ στερεοῦ ὕψος. τῶν ΒΤ, ΔΨ ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν [ὧν δὲ στερεῶν παραλληλεπιπέδων τὰ ὕψη πρὸς ὀρθάς ἐστι ταῖς βάσεσιν αὐτῶν, ἀντιπεπόνθασι δὲ αἱ βάσεις τοῖς ὕψεσιν, ἴσα ἐστὶν ἐκεῖνα]· ἴσον ἄρα ἐστὶ τὸ ΒΤ στερεὸν τῷ ΔΨ στερεῷ. ἀλλὰ τὸ μὲν ΒΤ τῷ ΒΑ ἴσον ἐστίν· ἐπί τε γὰρ τῆς αὐτῆς βάσεως [εἰσι] τῆς ΖΚ καὶ ὑπὸ τὸ αὐτὸ ὕψος [ὧν αἱ ἐφεστῶσαι οὐκ εἰσὶν ἐπὶ τῶν αὐτῶν εὐθειῶν]. τὸ δὲ ΔΨ στερεὸν τῷ ΔΓ στερεῷ ἴσον ἐστίν [ἐπί τε γὰρ πάλιν τῆς αὐτῆς βάσεώς εἰσι τῆς ΞΡ καὶ ὑπὸ τὸ αὐτὸ ὕψος καὶ οὐκ ἐν ταῖς αὐταῖς εὐθείαις]. καὶ τὸ ΑΒ ἄρα στερεὸν τῷ ΓΔ στερεῷ ἐστιν ἴσον· ὅπερ ἔδει δεῖξαι.

λε΄. Ἐὰν ὦσι δύο γωνίαι ἐπίπεδοι ἴσαι, ἐπὶ δὲ τῶν κορυφῶν αὐτῶν μετέωροι εὐθεῖαι ἐπισταθῶσιν ἴσας γωνίας περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, ἐπὶ δὲ τῶν μετεώρων ληφθῇ τυχόντα σημεῖα, καὶ ἀπ᾽ αὐτῶν ἐπὶ τὰ ἐπίπεδα, ἐν οἷς εἰσιν αἱ ἐξ ἀρχῆς γωνίαι, κάθετοι ἀχθῶσιν, ἀπὸ δὲ τῶν γενομένων σημείων ἐν τοῖς ἐπιπέδοις ἐπὶ τὰς ἐξ ἀρχῆς γωνίας ἐπιζευχθῶσιν εὐθεῖαι, ἴσας γωνίας περιέξουσι μετὰ τῶν μετεώρων.

Ἔστωσαν δύο γωνίαι εὐθύγραμμοι ἴσαι αἱ ὑπὸ ΒΑΓ, ΕΔΖ, ἀπὸ δὲ τῶν Α, Δ σημείων μετέωροι εὐθεῖαι ἐφεστάτωσαν αἱ ΑΗ, ΔΜ ἴσας γωνίας περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, τὴν μὲν ὑπὸ ΜΔΕ τῇ ὑπὸ ΗΑΒ, τὴν δὲ ὑπὸ ΜΔΖ τῇ ὑπὸ ΗΑΓ, καὶ εἰλήφθω ἐπὶ τῶν ΑΗ, ΔΜ τυχόντα σημεῖα τὰ Η, Μ, καὶ ἤχθωσαν ἀπὸ τῶν Η, Μ σημείων ἐπὶ τὰ διὰ τῶν ΒΑΓ, ΕΔΖ ἐπίπεδα κάθετοι αἱ ΗΛ, ΜΝ, καὶ συμβαλλέτωσαν τοῖς ἐπιπέδοις κατὰ τὰ Ν, Λ, καὶ ἐπεζεύχθωσαν αἱ ΛΑ, ΝΔ· λέγω, ὅτι ἴση ἐστὶν ἡ ὑπὸ ΗΑΛ γωνία τῇ ὑπὸ ΜΔΝ γωνίᾳ.

Κείσθω τῇ ΔΜ ἴση ἡ ΑΘ, καὶ ἤχθω διὰ τοῦ Θ σημείου τῇ ΗΛ παράλληλος ἡ ΘΚ. ἡ δὲ ΗΛ κάθετός ἐστιν ἐπὶ τὸ διὰ τῶν ΒΑΓ ἐπίπεδον· καὶ ἡ ΘΚ ἄρα κάθετός ἐστιν ἐπὶ τὸ διὰ τῶν ΒΑΓ ἐπίπεδον. ἤχθωσαν ἀπὸ τῶν Κ, Ν σημείων ἐπὶ τὰς ΑΒ, ΑΓ, ΔΖ, ΔΕ εὐθείας κάθετοι αἱ ΚΓ, ΝΖ, ΚΒ, ΝΕ, καὶ ἐπεζεύχθωσαν αἱ ΘΓ, ΓΒ, ΜΖ, ΖΕ. ἐπεὶ τὸ ἀπὸ τῆς ΘΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΘΚ, ΚΑ, τῷ δὲ ἀπὸ τῆς ΚΑ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΚΓ, ΓΑ, καὶ τὸ ἀπὸ τῆς ΘΑ ἄρα ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΘΚ, ΚΓ, ΓΑ. τοῖς δὲ ἀπὸ τῶν ΘΚ, ΚΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΘΓ· τὸ ἄρα ἀπὸ τῆς ΘΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΘΓ, ΓΑ. ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΘΓΑ γωνία. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΔΖΜ γωνία ὀρθή ἐστιν. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΓΘ γωνία τῇ ὑπὸ ΔΖΜ. ἔστι δὲ καὶ ἡ ὑπὸ ΘΑΓ τῇ ὑπὸ ΜΔΖ ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΜΔΖ, ΘΑΓ δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν τὴν ΘΑ τῇ ΜΔ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει ἑκατέραν ἑκατέρᾳ.

ἴση ἄρα ἐστὶν ἡ ΑΓ τῇ ΔΖ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΑΒ τῇ ΔΕ ἐστιν ἴση [οὕτως· ἐπεζεύχθωσαν αἱ ΘΒ, ΜΕ. καὶ ἐπεὶ τὸ ἀπὸ τῆς ΑΘ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΚ, ΚΘ, τῷ δὲ ἀπὸ τῆς ΑΚ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΑΒ, ΒΚ, τὰ ἄρα ἀπὸ τῶν ΑΒ, ΒΚ, ΚΘ ἴσα ἐστὶ τῷ ἀπὸ ΑΘ. ἀλλὰ τοῖς ἀπὸ τῶν ΒΚ, ΚΘ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΒΘ· ὀρθὴ γὰρ ἡ ὑπὸ ΘΚΒ γωνία διὰ τὸ καὶ τὴν ΘΚ κάθετον εἶναι ἐπὶ τὸ ὑποκείμενον ἐπίπεδον· τὸ ἄρα ἀπὸ τῆς ΑΘ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΒ, ΒΘ· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΑΒΘ γωνία. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΔΕΜ γωνία ὀρθή ἐστιν. ἔστι δὲ καὶ ἡ ὑπὸ ΒΑΘ γωνία τῇ ὑπὸ ΕΔΜ ἴση· ὑπόκεινται γάρ· καὶ ἔστιν ἡ ΑΘ τῇ ΔΜ ἴση· ἴση ἄρα ἐστὶ καὶ ἡ ΑΒ τῇ ΔΕ]. ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν ΑΓ τῇ ΔΖ, ἡ δὲ ΑΒ τῇ ΔΕ, δύο δὴ αἱ ΓΑ, ΑΒ δυσὶ ταῖς ΖΔ, ΔΕ ἴσαι εἰσίν. ἀλλὰ καὶ γωνία ἡ ὑπὸ ΓΑΒ γωνίᾳ τῇ ὑπὸ ΖΔΕ ἐστιν ἴση· βάσις ἄρα ἡ ΒΓ βάσει τῇ ΕΖ ἴση ἐστὶ καὶ τὸ τρίγωνον τῷ τριγώνῳ καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις· ἴση ἄρα ἡ ὑπὸ ΑΓΒ γωνία τῇ ὑπὸ ΔΖΕ. ἔστι δὲ καὶ ὀρθὴ ἡ ὑπὸ ΑΓΚ ὀρθῇ τῇ ὑπὸ ΔΖΝ ἴση· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΒΓΚ λοιπῇ τῇ ὑπὸ ΕΖΝ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΓΒΚ τῇ ὑπὸ ΖΕΝ ἐστιν ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΒΓΚ, ΕΖΝ [τὰς] δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν πρὸς ταῖς ἴσαις γωνίαις τὴν ΒΓ τῇ ΕΖ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξουσιν. ἴση ἄρα ἐστὶν ἡ ΓΚ τῇ ΖΝ. ἔστι δὲ καὶ ἡ ΑΓ τῇ ΔΖ ἴση· δύο δὴ αἱ ΑΓ, ΓΚ δυσὶ ταῖς ΔΖ, ΖΝ ἴσαι εἰσίν· καὶ ὀρθὰς γωνίας περιέχουσιν. βάσις ἄρα ἡ ΑΚ βάσει τῇ ΔΝ ἴση ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΘ τῇ ΔΜ, ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς ΑΘ τῷ ἀπὸ τῆς ΔΜ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΘ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΑΚ, ΚΘ· ὀρθὴ γὰρ ἡ ὑπὸ ΑΚΘ· τῷ δὲ ἀπὸ τῆς ΔΜ ἴσα τὰ ἀπὸ τῶν ΔΝ, ΝΜ· ὀρθὴ γὰρ ἡ ὑπὸ ΔΝΜ· τὰ ἄρα ἀπὸ τῶν ΑΚ, ΚΘ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΔΝ, ΝΜ, ὧν τὸ ἀπὸ τῆς ΑΚ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΝ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΚΘ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΝΜ· ἴση ἄρα ἡ ΘΚ τῇ ΜΝ. καὶ ἐπεὶ δύο αἱ ΘΑ, ΑΚ δυσὶ ταῖς ΜΔ, ΔΝ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ βάσις ἡ ΘΚ βάσει τῇ ΜΝ ἐδείχθη ἴση, γωνία ἄρα ἡ ὑπὸ ΘΑΚ γωνίᾳ τῇ ὑπὸ ΜΔΝ ἐστιν ἴση.

Ἐὰν ἄρα ὦσι δύο γωνίαι ἐπίπεδοι ἴσαι καὶ τὰ ἐξῆς τῆς προτάσεως [ὅπερ ἔδει δεῖξαι].

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι, ἐὰν ὦσι δύο γωνίαι ἐπίπεδοι ἴσαι, ἐπισταθῶσι δὲ ἐπ᾽ αὐτῶν μετέωροι εὐθεῖαι ἴσαι ἴσας γωνίας περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, αἱ ἀπ᾽ αὐτῶν κάθετοι ἀγόμεναι ἐπὶ τὰ ἐπίπεδα, ἐν οἷς εἰσιν αἱ ἐξ ἀρχῆς γωνίαι, ἴσαι ἀλλήλαις εἰσίν. ὅπερ ἔδει δεῖξαι.

λς΄. Ἐὰν τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, τὸ ἐκ τῶν τριῶν στερεὸν παραλληλεπίπεδον ἴσον ἐστὶ τῷ ἀπὸ τῆς μέσης στερεῷ παραλληλεπιπέδῳ ἰσοπλεύρῳ μέν, ἰσογωνίῳ δὲ τῷ προειρημένῳ.

Ἔστωσαν τρεῖς εὐθεῖαι ἀνάλογον αἱ Α, Β, Γ, ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Γ· λέγω, ὅτι τὸ ἐκ τῶν Α, Β, Γ στερεὸν ἴσον ἐστὶ τῷ ἀπὸ τῆς Β στερεῷ ἰσοπλεύρῳ μέν, ἰσογωνίῳ δὲ τῷ προειρημένῳ.

Ἐκκείσθω στερεὰ γωνία ἡ πρὸς τῷ Ε περιεχομένη ὑπὸ τῶν ὑπὸ ΔΕΗ, ΗΕΖ, ΖΕΔ, καὶ κείσθω τῇ μὲν Β ἴση ἑκάστη τῶν ΔΕ, ΗΕ, ΕΖ, καὶ συμπεπληρώσθω τὸ ΕΚ στερεὸν παραλληλεπίπεδον, τῇ δὲ Α ἴση ἡ ΛΜ, καὶ συνεστάτω πρὸς τῇ ΛΜ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Λ τῇ πρὸς τῷ Ε στερεᾷ γωνίᾳ ἴση στερεὰ γωνία ἡ περιεχομένη ὑπὸ τῶν ΝΛΞ, ΞΛΜ, ΜΛΝ, καὶ κείσθω τῇ μὲν Β ἴση ἡ ΛΞ, τῇ δὲ Γ ἴση ἡ ΛΝ. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Γ, ἴση δὲ ἡ μὲν Α τῇ ΛΜ, ἡ δὲ Β ἑκατέρᾳ τῶν ΛΞ, ΕΔ, ἡ δὲ Γ τῇ ΛΝ, ἔστιν ἄρα ὡς ἡ ΛΜ πρὸς τὴν ΕΖ, οὕτως ἡ ΔΕ πρὸς τὴν ΛΝ.

καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΝΛΜ, ΔΕΖ αἱ πλευραὶ ἀντιπεπόνθασιν· ἴσον ἄρα ἐστὶ τὸ ΜΝ παραλληλόγραμμον τῷ ΔΖ παραλληλογράμμῳ. καὶ ἐπεὶ δύο γωνίαι ἐπίπεδοι εὐθύγραμμοι ἴσαι εἰσὶν αἱ ὑπὸ ΔΕΖ, ΝΛΜ, καὶ ἐπ᾽ αὐτῶν μετέωροι εὐθεῖαι ἐφεστᾶσιν αἱ ΛΞ, ΕΗ ἴσαι τε ἀλλήλαις καὶ ἴσας γωνίας

περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, αἱ ἄρα ἀπὸ τῶν Η, Ξ σημείων κάθετοι ἀγόμεναι ἐπὶ τὰ διὰ τῶν ΝΛΜ, ΔΕΖ ἐπίπεδα ἴσαι ἀλλήλαις εἰσίν· ὥστε τὰ ΛΘ, ΕΚ στερεὰ ὑπὸ τὸ αὐτὸ ὕψος ἐστίν. τὰ δὲ ἐπὶ ἴσων βάσεων στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ ΘΛ στερεὸν τῷ ΕΚ στερεῷ. καί ἐστι τὸ μὲν ΛΘ τὸ ἐκ τῶν Α, Β, Γ στερεόν, τὸ δὲ ΕΚ τὸ ἀπὸ τῆς Β στερεόν· τὸ ἄρα ἐκ τῶν Α, Β, Γ στερεὸν παραλληλεπίπεδον ἴσον ἐστὶ τῷ ἀπὸ τῆς Β στερεῷ ἰσοπλεύρῳ μέν, ἰσογωνίῳ δὲ τῷ προειρημένῳ· ὅπερ ἔδει δεῖξαι.

λζ΄. Ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, καὶ τὰ ἀπ᾽ αὐτῶν στερεὰ παραλληλεπίπεδα ὅμοιά τε καὶ ὁμοίως ἀναγραφόμενα ἀνάλογον ἔσται· καὶ ἐὰν τὰ ἀπ᾽ αὐτῶν στερεὰ παραλληλεπίπεδα ὅμοιά τε καὶ ὁμοίως ἀναγραφόμενα ἀνάλογον ᾖ, καὶ αὐταὶ αἱ εὐθεῖαι ἀνάλογον ἔσονται.

Ἔστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἱ ΑΒ, ΓΔ, ΕΖ, ΗΘ, ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ, καὶ ἀναγεγράφθωσαν ἀπὸ τῶν ΑΒ, ΓΔ, ΕΖ, ΗΘ ὅμοιά τε καὶ ὁμοίως κείμενα στερεὰ παραλληλεπίπεδα τὰ ΚΑ, ΛΓ, ΜΕ, ΝΗ· λέγω, ὅτι ἐστὶν ὡς τὸ ΚΑ πρὸς τὸ ΛΓ, οὕτως τὸ ΜΕ πρὸς τὸ ΝΗ.

Ἐπεὶ γὰρ ὅμοιόν ἐστι τὸ ΚΑ στερεὸν παραλληλεπίπεδον τῷ ΛΓ, τὸ ΚΑ ἄρα πρὸς τὸ ΛΓ τριπλασίονα λόγον ἔχει ἥπερ ἡ ΑΒ πρὸς τὴν ΓΔ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΜΕ πρὸς τὸ ΝΗ τριπλασίονα λόγον ἔχει ἥπερ ἡ ΕΖ πρὸς τὴν ΗΘ. καί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ. καὶ ὡς ἄρα τὸ ΑΚ πρὸς τὸ ΛΓ, οὕτως τὸ ΜΕ πρὸς τὸ ΝΗ.

Ἀλλὰ δὴ ἔστω ὡς τὸ ΑΚ στερεὸν πρὸς τὸ ΛΓ στερεόν, οὕτως τὸ ΜΕ στερεὸν πρὸς τὸ ΝΗ· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΒ εὐθεῖα πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ.

Ἐπεὶ γὰρ πάλιν τὸ ΚΑ πρὸς τὸ ΛΓ τριπλασίονα λόγον ἔχει ἥπερ ἡ ΑΒ πρὸς τὴν ΓΔ, ἔχει δὲ καὶ τὸ ΜΕ πρὸς τὸ ΝΗ τριπλασίονα λόγον ἥπερ ἡ ΕΖ

πρὸς τὴν ΗΘ, καί ἐστιν ὡς τὸ ΚΑ πρὸς τὸ ΛΓ, οὕτως τὸ ΜΕ πρὸς τὸ ΝΗ, καὶ ὡς ἄρα ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ.

Ἐὰν ἄρα τέσσαρες εὐθεῖαι ἀνάλογον ὦσι καὶ τὰ ἐξῆς τῆς προτάσεως· ὅπερ ἔδει δεῖξαι.

λη΄. Ἐὰν κύβου τῶν ἀπεναντίον ἐπιπέδων αἱ πλευραὶ δίχα τμηθῶσιν, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβληθῇ, ἡ κοινὴ τομὴ τῶν ἐπιπέδων καὶ ἡ τοῦ κύβου διάμετρος δίχα τέμνουσιν ἀλλήλας.

Κύβου γὰρ τοῦ ΑΖ τῶν ἀπεναντίον ἐπιπέδων τῶν ΓΖ, ΑΘ αἱ πλευραὶ δίχα τετμήσθωσαν κατὰ τὰ Κ, Λ, Μ, Ν, Ξ, Π, Ο, Ρ σημεῖα, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβεβλήσθω τὰ ΚΝ, ΞΡ, κοινὴ δὲ τομὴ τῶν ἐπιπέδων ἔστω ἡ ΥΣ, τοῦ δὲ ΑΖ κύβου διαγώνιος ἡ ΔΗ. λέγω, ὅτι ἴση ἐστὶν ἡ μὲν ΥΤ τῇ ΤΣ, ἡ δὲ ΔΤ τῇ ΤΗ.

Ἐπεζεύχθωσαν γὰρ αἱ ΔΥ, ΥΕ, ΒΣ, ΣΗ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΔΞ τῇ ΟΕ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΔΞΥ, ΥΟΕ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΔΞ τῇ ΟΕ, ἡ δὲ ΞΥ τῇ ΥΟ, καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΔΥ τῇ ΥΕ ἐστιν ἴση, καὶ τὸ ΔΞΥ τρίγωνον τῷ ΟΥΕ τριγώνῳ ἐστὶν ἴσον καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι· ἴση ἄρα ἡ ὑπὸ ΞΥΔ γωνία τῇ ὑπὸ ΟΥΕ γωνίᾳ. διὰ δὴ τοῦτο εὐθεῖά ἐστιν ἡ ΔΥΕ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΒΣΗ εὐθεῖά ἐστιν, καὶ ἴση ἡ ΒΣ τῇ ΣΗ.

καὶ ἐπεὶ ἡ ΓΑ τῇ ΔΒ ἴση ἐστὶ καὶ παράλληλος, ἀλλὰ ἡ ΓΑ καὶ τῇ ΕΗ ἴση τέ ἐστι καὶ παράλληλος, καὶ ἡ ΔΒ ἄρα τῇ ΕΗ ἴση τέ ἐστι καὶ παράλληλος. καὶ ἐπιζευγνύουσιν αὐτὰς εὐθεῖαι αἱ ΔΕ, ΒΗ· παράλληλος ἄρα ἐστὶν ἡ ΔΕ τῇ ΒΗ. ἴση ἄρα ἡ μὲν ὑπὸ ΕΔΤ γωνία τῇ ὑπὸ ΒΗΤ· ἐναλλὰξ γάρ· ἡ δὲ ὑπὸ ΔΤΥ τῇ ὑπὸ ΗΤΣ. δύο δὴ τρίγωνά ἐστι τὰ ΔΤΥ, ΗΤΣ τὰς δύο γωνίας ταῖς δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν τὴν ΔΥ τῇ ΗΣ· ἡμίσειαι γάρ εἰσι τῶν ΔΕ, ΒΗ· καὶ τὰς

λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει. ἴση ἄρα ἡ μὲν ΔΤ τῇ ΤΗ, ἡ δὲ ΥΤ τῇ ΤΣ.

Ἐὰν ἄρα κύβου τῶν ἀπεναντίον ἐπιπέδων αἱ πλευραὶ δίχα τμηθῶσιν, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβληθῇ, ἡ κοινὴ τομὴ τῶν ἐπιπέδων καὶ ἡ τοῦ κύβου διάμετρος δίχα τέμνουσιν ἀλλήλας· ὅπερ ἔδει δεῖξαι.

λθ΄. Ἐὰν ᾖ δύο πρίσματα ἰσοϋψῆ, καὶ τὸ μὲν ἔχῃ βάσιν παραλληλόγραμμον, τὸ δὲ τρίγωνον, διπλάσιον δὲ ᾖ τὸ παραλληλόγραμμον τοῦ τριγώνου, ἴσα ἔσται τὰ πρίσματα.

Ἔστω δύο πρίσματα ἰσοϋψῆ τὰ ΑΒΓΔΕΖ, ΗΘΚΛ ΜΝ, καὶ τὸ μὲν ἐχέτω βάσιν τὸ ΑΖ παραλληλόγραμμον, τὸ δὲ τὸ ΗΘΚ τρίγωνον, διπλάσιον δὲ ἔστω τὸ ΑΖ παραλληλόγραμμον τοῦ ΗΘΚ τριγώνου· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓΔΕΖ πρίσμα τῷ ΗΘΚΛΜΝ πρίσματι.

Συμπεπληρώσθω γὰρ τὰ ΑΞ, ΗΟ στερεά. ἐπεὶ διπλάσιόν ἐστι τὸ ΑΖ παραλληλόγραμμον τοῦ ΗΘΚ τριγώνου, ἔστι δὲ καὶ τὸ ΘΚ παραλληλόγραμμον διπλάσιον τοῦ ΗΘΚ τριγώνου, ἴσον ἄρα ἐστὶ τὸ ΑΖ παραλληλόγραμμον τῷ ΘΚ παραλληλογράμμῳ.

τὰ δὲ ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ ΑΞ στερεὸν τῷ ΗΟ στερεῷ. καί ἐστι τοῦ μὲν ΑΞ στερεοῦ ἥμισυ τὸ ΑΒΓΔΕΖ πρίσμα, τοῦ δὲ ΗΟ στερεοῦ ἥμισυ τὸ ΗΘΚΛΜΝ πρίσμα· ἴσον ἄρα ἐστὶ τὸ ΑΒΓΔΕΖ πρίσμα τῷ ΗΘΚΛΜΝ πρίσματι.

Ἐὰν ἄρα ᾖ δύο πρίσματα ἰσοψῆ, καὶ τὸ μὲν ἔχῃ βάσιν παραλληλόγραμμον, τὸ δὲ τρίγωνον, διπλάσιον δὲ ᾖ τὸ παραλληλόγραμμον τοῦ τριγώνου, ἴσα ἐστὶ τὰ πρίσματα· ὅπερ ἔδει δεῖξαι.

BOOK XII.

Προτάσεις ιη΄.

α΄. Τὰ ἐν τοῖς κύκλοις ὅμοια πολύγωνα πρὸς ἄλληλά ἐστιν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα.

Ἔστωσαν κύκλοι οἱ ΑΒΓ, ΖΗΘ, καὶ ἐν αὐτοῖς ὅμοια πολύγωνα ἔστω τὰ ΑΒΓΔΕ, ΖΗΘΚΛ, διάμετροι δὲ τῶν κύκλων ἔστωσαν αἱ ΒΜ, ΗΝ· λέγω, ὅτι ἐστὶν ὡς τὸ ἀπὸ τῆς ΒΜ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΝ τετράγωνον, οὕτως τὸ ΑΒΓΔΕ πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΕ, ΑΜ, ΗΛ, ΖΝ. καὶ ἐπεὶ ὅμοιον τὸ ΑΒΓΔΕ πολύγωνον τῷ ΖΗΘΚΛ πολυγώνῳ, ἴση ἐστὶ καὶ ἡ ὑπὸ ΒΑΕ γωνία τῇ ὑπὸ ΗΖΛ, καί ἐστιν ὡς ἡ ΒΑ πρὸς τὴν ΑΕ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΛ.

δύο δὴ τρίγωνά ἐστι τὰ ΒΑΕ, ΗΖΛ μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχοντα τὴν ὑπὸ ΒΑΕ τῇ ὑπὸ ΗΖΛ, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΕ τρίγωνον τῷ ΖΗΛ τριγώνῳ. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΕΒ γωνία τῇ ὑπὸ ΖΛΗ. ἀλλ’ ἡ μὲν ὑπὸ ΑΕΒ τῇ ὑπὸ ΑΜΒ ἐστιν ἴση· ἐπὶ γὰρ τῆς αὐτῆς περιφερείας βεβήκασιν· ἡ δὲ ὑπὸ ΖΛΗ τῇ ὑπὸ ΖΝΗ· καὶ ἡ ὑπὸ ΑΜΒ ἄρα τῇ ὑπὸ ΖΝΗ ἐστιν ἴση. ἔστι δὲ καὶ ὀρθὴ ἡ ὑπὸ ΒΑΜ ὀρθῇ τῇ ὑπὸ ΗΖΝ ἴση· καὶ ἡ λοιπὴ ἄρα τῇ λοιπῇ ἐστιν ἴση. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΜ τρίγωνον τῷ ΖΗΝ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΜ πρὸς τὴν ΗΝ, οὕτως ἡ ΒΑ πρὸς τὴν ΗΖ. ἀλλὰ τοῦ μὲν τῆς ΒΜ πρὸς τὴν ΗΝ λόγου διπλασίων ἐστὶν ὁ τοῦ ἀπὸ τῆς ΒΜ τετραγώνου πρὸς τὸ ἀπὸ τῆς ΗΝ τετράγωνον, τοῦ δὲ τῆς ΒΑ πρὸς τὴν ΗΖ διπλασίων ἐστὶν ὁ τοῦ ΑΒΓΔΕ πολυγώνου πρὸς τὸ ΖΗ ΘΚΛ πολύγωνον· καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΒΜ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΝ τετράγωνον, οὕτως τὸ ΑΒΓΔΕ πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον.

Τὰ ἄρα ἐν τοῖς κύκλοις ὅμοια πολύγωνα πρὸς ἄλληλά ἐστιν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα· ὅπερ ἔδει δεῖξαι.

β΄. Οἱ κύκλοι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα.

Ἔστωσαν κύκλοι οἱ ΑΒΓΔ, ΕΖΗΘ, διάμετροι δὲ αὐτῶν [ἔστωσαν] αἱ ΒΔ, ΖΘ· λέγω, ὅτι ἐστὶν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ τετράγωνον.

Εἰ γὰρ μή ἐστιν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ, οὕτως τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, ἔσται ὡς τὸ ἀπὸ τῆς ΒΔ πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος ἤτοι πρὸς ἔλασσόν τι τοῦ ΕΖΗΘ κύκλου χωρίον ἢ πρὸς μεῖζον. ἔστω πρότερον πρὸς ἔλασσον τὸ Σ. καὶ ἐγγεγράφθω εἰς τὸν ΕΖΗΘ κύκλον τετράγωνον τὸ ΕΖΗΘ· τὸ δὴ ἐγγεγραμμένον τετράγωνον μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ ΕΖΗΘ κύκλου, ἐπειδήπερ ἐὰν διὰ τῶν Ε, Ζ, Η, Θ σημείων ἐφαπτομένας [εὐθείας] τοῦ κύκλου ἀγάγωμεν, τοῦ περιγραφομένου περὶ τὸν κύκλον τετραγώνου ἥμισύ ἐστι τὸ ΕΖΗΘ τετράγωνον, τοῦ δὲ περιγραφέντος τετραγώνου ἐλάττων ἐστὶν ὁ κύκλος· ὥστε τὸ ΕΖΗΘ ἐγγεγραμμένον τετράγωνον μεῖζόν ἐστι τοῦ ἡμίσεως τοῦ ΕΖΗΘ κύκλου. τετμήσθωσαν δίχα αἱ ΕΖ, ΖΗ, ΗΘ, ΘΕ περιφέρειαι κατὰ τὰ Κ, Λ, Μ, Ν σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΕΚ, ΚΖ, ΖΛ, ΛΗ, ΗΜ, ΜΘ, ΘΝ, ΝΕ· καὶ ἕκαστον ἄρα τῶν ΕΚΖ, ΖΛΗ, ΗΜΘ, ΘΝΕ τριγώνων μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ καθ’ ἑαυτὸ τμήματος τοῦ κύκλου, ἐπειδήπερ ἐὰν διὰ τῶν Κ, Λ, Μ, Ν σημείων ἐφαπτομένας τοῦ κύκλου ἀγάγωμεν καὶ ἀναπληρώσωμεν τὰ ἐπὶ τῶν ΕΖ, ΖΗ, ΗΘ, ΘΕ εὐθειῶν παραλληλόγραμμα, ἕκαστον τῶν ΕΚΖ, ΖΛΗ, ΗΜΘ, ΘΝΕ τριγώνων ἥμισυ ἔσται τοῦ καθ’ ἑαυτὸ παραλληλογράμμου, ἀλλὰ τὸ καθ’ ἑαυτὸ τμῆμα ἔλαττόν ἐστι τοῦ παραλληλογράμμου· ὥστε ἕκαστον τῶν ΕΚΖ, ΖΛΗ, ΗΜΘ, ΘΝΕ τριγώνων μεῖζόν ἐστι τοῦ ἡμίσεως τοῦ καθ’ ἑαυτὸ τμήματος τοῦ κύκλου. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομέν τινα ἀποτμήματα τοῦ κύκλου, ἃ ἔσται ἐλάσσονα τῆς ὑπεροχῆς, ᾗ ὑπερέχει ὁ ΕΖΗΘ κύκλος τοῦ Σ χωρίου.

ἐδείχθη γὰρ ἐν τῷ πρώτῳ θεωρήματι τοῦ δεκάτου βιβλίου, ὅτι δύο μεγεθῶν ἀνίσων ἐκκειμένων, ἐὰν ἀπὸ τοῦ μείζονος ἀφαιρεθῇ μεῖζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μεῖζον ἢ τὸ ἥμισυ, καὶ τοῦτο ἀεὶ γίγνηται, λειφθήσεταί τι μέγεθος, ὃ ἔσται ἔλασσον τοῦ ἐκκειμένου ἐλάσσονος μεγέθους. λελείφθω οὖν, καὶ ἔστω τὰ ἐπὶ τῶν ΕΚ, ΚΖ, ΖΛ, ΛΗ, ΗΜ, ΜΘ, ΘΝ, ΝΕ τμήματα τοῦ ΕΖΗΘ κύκλου ἐλάττονα τῆς ὑπεροχῆς, ᾗ ὑπερέχει ὁ ΕΖΗΘ κύκλος τοῦ Σ χωρίου. λοιπὸν ἄρα τὸ ΕΚΖΛΗ ΜΘΝ πολύγωνον μεῖζόν ἐστι τοῦ Σ χωρίου. ἐγγεγράφθω καὶ εἰς τὸν ΑΒΓΔ κύκλον τῷ ΕΚΖΛΗΜΘΝ πολυγώνῳ ὅμοιον πολύγωνον τὸ ΑΞΒΟΓΠΔΡ· ἔστιν ἄρα ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ τετράγωνον, οὕτως τὸ ΑΞΒΟΓΠΔΡ πολύγωνον πρὸς τὸ ΕΚΖΛ ΗΜΘΝ πολύγωνον. ἀλλὰ καὶ ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς τὸ Σ χωρίον· καὶ ὡς ἄρα ὁ ΑΒΓΔ κύκλος πρὸς τὸ Σ χωρίον, οὕτως τὸ ΑΞΒΟΓΠΔΡ πολύγωνον πρὸς τὸ ΕΚΖΛΗΜΘΝ πολύγωνον· ἐναλλὰξ ἄρα ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸ ἐν αὐτῷ πολύγωνον, οὕτως τὸ Σ χωρίον πρὸς τὸ ΕΚΖΛΗΜΘΝ πολύγωνον. μείζων δὲ ὁ ΑΒΓΔ κύκλος τοῦ ἐν αὐτῷ πολυγώνου· μεῖζον ἄρα καὶ τὸ Σ χωρίον τοῦ ΕΚΖΛΗΜΘΝ πολυγώνου. ἀλλὰ καὶ ἔλαττον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς ἔλασσόν τι τοῦ ΕΖΗΘ κύκλου χωρίον. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ὡς τὸ ἀπὸ ΖΘ πρὸς τὸ ἀπὸ ΒΔ, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἔλασσόν τι τοῦ ΑΒΓΔ κύκλου χωρίον.

Λέγω δή, ὅτι οὐδὲ ὡς τὸ ἀπὸ τῆς ΒΔ πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς μεῖζόν τι τοῦ ΕΖΗΘ κύκλου χωρίον.

Εἰ γὰρ δυνατόν, ἔστω πρὸς μεῖζον τὸ Σ. ἀνάπαλιν ἄρα [ἐστὶν] ὡς τὸ ἀπὸ τῆς ΖΘ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΔΒ, οὕτως τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον. ἀλλ’ ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔ κύκλου χωρίον· καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΖΘ πρὸς τὸ ἀπὸ τῆς ΒΔ, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἔλασσόν τι τοῦ ΑΒΓΔ κύκλου

χωρίον· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς μεῖζόν τι τοῦ ΕΖΗΘ κύκλου χωρίον. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἔλασσον· ἔστιν ἄρα ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον.

Οἱ ἄρα κύκλοι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα· ὅπερ ἔδει δεῖξαι.

Λῆμμα

Λέγω δή, ὅτι τοῦ Σ χωρίου μείζονος ὄντος τοῦ ΕΖΗΘ κύκλου ἐστὶν ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔ κύκλου χωρίον.

Γεγονέτω γὰρ ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς τὸ Τ χωρίον. λέγω, ὅτι ἔλαττόν ἐστι τὸ Τ χωρίον τοῦ ΑΒΓΔ κύκλου. ἐπεὶ γάρ ἐστιν ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς τὸ Τ χωρίον, ἐναλλάξ ἐστιν ὡς τὸ Σ χωρίον πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς τὸ Τ χωρίον. μεῖζον δὲ τὸ Σ χωρίον τοῦ ΕΖΗΘ κύκλου· μείζων ἄρα καὶ ὁ ΑΒΓΔ κύκλος τοῦ Τ χωρίου. ὥστε ἐστὶν ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔ κύκλου χωρίον· ὅπερ ἔδει δεῖξαι.

γ΄. Πᾶσα πυραμὶς τρίγωνον ἔχουσα βάσιν διαιρεῖται εἰς δύο πυραμίδας ἴσας τε καὶ ὁμοίας ἀλλήλαις καὶ [ὁμοίας] τῇ ὅλῃ τριγώνους ἐχούσας βάσεις καὶ εἰς δύο πρίσματα ἴσα· καὶ τὰ δύο πρίσματα μείζονά ἐστιν ἢ τὸ ἥμισυ τῆς ὅλης πυραμίδος.

Ἔστω πυραμίς, ἧς βάσις μέν ἐστι τὸ ΑΒΓ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον· λέγω, ὅτι ἡ ΑΒΓΔ πυραμὶς διαιρεῖται εἰς δύο πυραμίδας ἴσας

ἀλλήλαις τριγώνους βάσεις ἐχούσας καὶ ὁμοίας τῇ ὅλῃ καὶ εἰς δύο πρίσματα ἴσα· καὶ τὰ δύο πρίσματα μείζονά ἐστιν ἢ τὸ ἥμισυ τῆς ὅλης πυραμίδος.

Τετμήσθωσαν γὰρ αἱ ΑΒ, ΒΓ, ΓΑ, ΑΔ, ΔΒ, ΔΓ δίχα κατὰ τὰ Ε, Ζ, Η, Θ, Κ, Λ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΘΕ, ΕΗ, ΗΘ, ΘΚ, ΚΛ, ΛΘ, ΚΖ, ΖΗ. ἐπεὶ ἴση ἐστὶν ἡ μὲν ΑΕ τῇ ΕΒ, ἡ δὲ ΑΘ τῇ ΔΘ, παράλληλος ἄρα ἐστὶν ἡ ΕΘ τῇ ΔΒ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΘΚ τῇ ΑΒ παράλληλός ἐστιν. παραλληλόγραμμον ἄρα ἐστὶ τὸ ΘΕ ΒΚ· ἴση ἄρα ἐστὶν ἡ ΘΚ τῇ ΕΒ. ἀλλὰ ἡ ΕΒ τῇ ΕΑ ἐστιν ἴση· καὶ ἡ ΑΕ ἄρα τῇ ΘΚ ἐστιν ἴση. ἔστι δὲ καὶ ἡ ΑΘ τῇ ΘΔ ἴση· δύο δὴ αἱ ΕΑ, ΑΘ δυσὶ ταῖς ΚΘ, ΘΔ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΕΑΘ γωνίᾳ τῇ ὑπὸ ΚΘΔ ἴση· βάσις ἄρα ἡ ΕΘ βάσει τῇ ΚΔ ἐστιν ἴση. ἴσον ἄρα καὶ ὅμοιόν ἐστι τὸ ΑΕΘ τρίγωνον τῷ ΘΚΔ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΑΘΗ τρίγωνον τῷ ΘΛΔ τριγώνῳ ἴσον τέ ἐστι καὶ ὅμοιον. καὶ ἐπεὶ δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων αἱ ΕΘ, ΘΗ παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων τὰς ΚΔ, ΔΛ εἰσιν οὐκ ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι, ἴσας γωνίας περιέξουσιν. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΕΘΗ γωνία τῇ ὑπὸ ΚΔΛ γωνίᾳ.

καὶ ἐπεὶ δύο εὐθεῖαι αἱ ΕΘ, ΘΗ δυσὶ ταῖς ΚΔ, ΔΛ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ γωνία ἡ ὑπὸ ΕΘΗ γωνίᾳ τῇ ὑπὸ ΚΔΛ ἐστιν ἴση, βάσις ἄρα ἡ ΕΗ βάσει τῇ ΚΛ [ἐστιν] ἴση· ἴσον ἄρα καὶ ὅμοιόν ἐστι τὸ ΕΘΗ τρίγωνον τῷ ΚΔΛ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΑΕΗ τρίγωνον τῷ ΘΚΛ τριγώνῳ ἴσον τε καὶ ὅμοιόν ἐστιν. ἡ ἄρα πυραμίς, ἧς βάσις μέν ἐστι τὸ ΑΕΗ τρίγωνον, κορυφὴ δὲ τὸ Θ σημεῖον, ἴση καὶ ὁμοία ἐστὶ πυραμίδι, ἧς βάσις μέν ἐστι τὸ ΘΚΛ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον. καὶ ἐπεὶ τριγώνου τοῦ ΑΔΒ παρὰ μίαν τῶν πλευρῶν τὴν ΑΒ ἦκται ἡ ΘΚ, ἰσογώνιόν ἐστι τὸ ΑΔΒ τρίγωνον τῷ ΔΘΚ τριγώνῳ, καὶ τὰς πλευρὰς ἀνάλογον ἔχουσιν· ὅμοιον ἄρα ἐστὶ τὸ ΑΔΒ τρίγωνον τῷ ΔΘΚ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν ΔΒΓ τρίγωνον τῷ ΔΚΛ τριγώνῳ ὅμοιόν ἐστιν, τὸ δὲ ΑΔΓ τῷ ΔΛΘ. καὶ ἐπεὶ δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων αἱ ΒΑ, ΑΓ παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων τὰς ΚΘ, ΘΛ εἰσιν οὐκ ἐν τῷ αὐτῷ ἐπιπέδῳ, ἴσας γωνίας περιέξουσιν. ἴση ἄρα

ἐστὶν ἡ ὑπὸ ΒΑΓ γωνία τῇ ὑπὸ ΚΘΛ. καί ἐστιν ὡς ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΚΘ πρὸς τὴν ΘΛ· ὅμοιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΘΚΛ τριγώνῳ. καὶ πυραμὶς ἄρα, ἧς βάσις μέν ἐστι τὸ ΑΒΓ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, ὁμοία ἐστὶ πυραμίδι, ἧς βάσις μέν ἐστι τὸ ΘΚΛ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον. ἀλλὰ πυραμίς, ἧς βάσις μὲν [ἐστι] τὸ ΘΚΛ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, ὁμοία ἐδείχθη πυραμίδι, ἧς βάσις μέν ἐστι τὸ ΑΕΗ τρίγωνον, κορυφὴ δὲ τὸ Θ σημεῖον [ὥστε καὶ πυραμίς, ἧς βάσις μὲν τὸ ΑΒΓ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, ὁμοία ἐστὶ πυραμίδι, ἧς βάσις μὲν τὸ ΑΕΗ τρίγωνον, κορυφὴ δὲ τὸ Θ σημεῖον]. ἑκατέρα ἄρα τῶν ΑΕΗΘ, ΘΚΛΔ πυραμίδων ὁμοία ἐστὶ τῇ ὅλῃ τῇ ΑΒΓΔ πυραμίδι. ¯Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΖ τῇ ΖΓ, διπλάσιόν ἐστι τὸ ΕΒΖΗ παραλληλόγραμμον τοῦ ΗΖΓ τριγώνου. καὶ ἐπεί, ἐὰν ᾖ δύο πρίσματα ἰσοϋψῆ, καὶ τὸ μὲν ἔχῃ βάσιν παραλληλόγραμμον, τὸ δὲ τρίγωνον, διπλάσιον δὲ ᾖ τὸ παραλληλόγραμμον τοῦ τριγώνου, ἴσα ἐστὶ τὰ πρίσματα, ἴσον ἄρα ἐστὶ τὸ πρίσμα τὸ περιεχόμενον ὑπὸ δύο μὲν τριγώνων τῶν ΒΚΖ, ΕΘΗ, τριῶν δὲ παραλληλογράμμων τῶν ΕΒΖΗ, ΕΒΚΘ, ΘΚΖΗ τῷ πρίσματι τῷ περιεχομένῳ ὑπὸ δύο μὲν τριγώνων τῶν ΗΖΓ, ΘΚΛ, τριῶν δὲ παραλληλογράμμων τῶν ΚΖΓΛ, ΛΓΗΘ, ΘΚΖΗ. καὶ φανερόν, ὅτι ἑκάτερον τῶν πρισμάτων, οὗ τε βάσις τὸ ΕΒΖΗ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΘΚ εὐθεῖα, καὶ οὗ βάσις τὸ ΗΖΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΘΚΛ τρίγωνον, μεῖζόν ἐστιν ἑκατέρας τῶν πυραμίδων, ὧν βάσεις μὲν τὰ ΑΕΗ, ΘΚΛ τρίγωνα, κορυφαὶ δὲ τὰ Θ, Δ σημεῖα, ἐπειδήπερ [καὶ] ἐὰν ἐπιζεύξωμεν τὰς ΕΖ, ΕΚ εὐθείας, τὸ μὲν πρίσμα, οὗ βάσις τὸ ΕΒΖΗ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΘΚ εὐθεῖα, μεῖζόν ἐστι τῆς πυραμίδος, ἧς βάσις τὸ ΕΒΖ τρίγωνον, κορυφὴ δὲ τὸ Κ σημεῖον. ἀλλ’ ἡ πυραμίς, ἧς βάσις τὸ ΕΒΖ τρίγωνον, κορυφὴ δὲ τὸ Κ σημεῖον, ἴση ἐστὶ πυραμίδι, ἧς βάσις τὸ ΑΕΗ τρίγωνον, κορυφὴ δὲ τὸ Θ σημεῖον· ὑπὸ γὰρ ἴσων καὶ ὁμοίων ἐπιπέδων περιέχονται. ὥστε καὶ τὸ πρίσμα, οὗ βάσις μὲν τὸ ΕΒΖΗ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΘΚ εὐθεῖα, μεῖζόν ἐστι πυραμίδος, ἧς βάσις μὲν τὸ ΑΕΗ

τρίγωνον, κορυφὴ δὲ τὸ Θ σημεῖον. ἴσον δὲ τὸ μὲν πρίσμα, οὗ βάσις τὸ ΕΒΖΗ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΘΚ εὐθεῖα, τῷ πρίσματι, οὗ βάσις μὲν τὸ ΗΖΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΘΚΛ τρίγωνον· ἡ δὲ πυραμίς, ἧς βάσις τὸ ΑΕΗ τρίγωνον, κορυφὴ δὲ τὸ Θ σημεῖον, ἴση ἐστὶ πυραμίδι, ἧς βάσις τὸ ΘΚΛ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον. τὰ ἄρα εἰρημένα δύο πρίσματα μείζονά ἐστι τῶν εἰρημένων δύο πυραμίδων, ὧν βάσεις μὲν τὰ ΑΕΗ, ΘΚΛ τρίγωνα, κορυφαὶ δὲ τὰ Θ, Δ σημεῖα.

Ἡ ἄρα ὅλη πυραμίς, ἧς βάσις τὸ ΑΒΓ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, διῄρηται εἵς τε δύο πυραμίδας ἴσας ἀλλήλαις [καὶ ὁμοίας τῇ ὅλῃ] καὶ εἰς δύο πρίσματα ἴσα, καὶ τὰ δύο πρίσματα μείζονά ἐστιν ἢ τὸ ἥμισυ τῆς ὅλης πυραμίδος· ὅπερ ἔδει δεῖξαι.

δ΄. Ἐὰν ὦσι δύο πυραμίδες ὑπὸ τὸ αὐτὸ ὕψος τριγώνους ἔχουσαι βάσεις, διαιρεθῇ δὲ ἑκατέρα αὐτῶν εἵς τε δύο πυραμίδας ἴσας ἀλλήλαις καὶ ὁμοίας τῇ ὅλῃ καὶ εἰς δύο πρίσματα ἴσα, ἔσται ὡς ἡ τῆς μιᾶς πυραμίδος βάσις πρὸς τὴν τῆς ἑτέρας πυραμίδος βάσιν, οὕτως τὰ ἐν τῇ μιᾷ πυραμίδι πρίσματα πάντα πρὸς τὰ ἐν τῇ ἑτέρᾳ πυραμίδι πρίσματα πάντα ἰσοπληθῆ.

Ἔστωσαν δύο πυραμίδες ὑπὸ τὸ αὐτὸ ὕψος τριγώνους ἔχουσαι βάσεις τὰς ΑΒΓ, ΔΕΖ, κορυφὰς δὲ τὰ Η, Θ σημεῖα, καὶ διῃρήσθω ἑκατέρα αὐτῶν εἵς τε δύο πυραμίδας ἴσας ἀλλήλαις καὶ ὁμοίας τῇ ὅλῃ καὶ εἰς δύο πρίσματα ἴσα· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὰ ἐν τῇ ΑΒΓΗ πυραμίδι πρίσματα πάντα πρὸς τὰ ἐν τῇ ΔΕΖΘ πυραμίδι πρίσματα ἰσοπληθῆ.

Ἐπεὶ γὰρ ἴση ἐστὶν ἡ μὲν ΒΞ τῇ ΞΓ, ἡ δὲ ΑΛ τῇ ΛΓ, παράλληλος ἄρα ἐστὶν ἡ ΛΞ τῇ ΑΒ καὶ ὅμοιον τὸ ΑΒΓ τρίγωνον τῷ ΛΞΓ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΔΕΖ τρίγωνον τῷ ΡΦΖ τριγώνῳ ὅμοιόν ἐστιν. καὶ ἐπεὶ διπλασίων ἐστὶν ἡ μὲν ΒΓ τῆς ΓΞ, ἡ δὲ ΕΖ τῆς ΖΦ, ἔστιν ἄρα ὡς ἡ ΒΓ πρὸς τὴν ΓΞ, οὕτως ἡ ΕΖ πρὸς τὴν ΖΦ. καὶ ἀναγέγραπται ἀπὸ μὲν τῶν ΒΓ, ΓΞ ὅμοιά τε καὶ ὁμοίως κείμενα εὐθύγραμμα τὰ ΑΒΓ, ΛΞΓ, ἀπὸ δὲ τῶν ΕΖ, ΖΦ

ὅμοιά τε καὶ ὁμοίως κείμενα [εὐθύγραμμα] τὰ ΔΕΖ, ΡΦΖ. ἔστιν ἄρα ὡς τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΛΞΓ τρίγωνον, οὕτως τὸ ΔΕΖ τρίγωνον πρὸς τὸ ΡΦΖ τρίγωνον· ἐναλλὰξ ἄρα ἐστὶν ὡς τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΔΕΖ [τρίγωνον], οὕτως τὸ ΛΞΓ [τρίγωνον] πρὸς τὸ ΡΦΖ τρίγωνον.

ἀλλ’ ὡς τὸ ΛΞΓ τρίγωνον πρὸς τὸ ΡΦΖ τρίγωνον, οὕτως τὸ πρίσμα, οὗ βάσις μὲν [ἐστι] τὸ ΛΞΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΟΜΝ, πρὸς τὸ πρίσμα, οὗ βάσις μὲν τὸ ΡΦΖ τρίγωνον, ἀπεναντίον δὲ τὸ ΣΤΥ· καὶ ὡς ἄρα τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΔΕΖ τρίγωνον, οὕτως τὸ πρίσμα, οὗ βάσις μὲν τὸ ΛΞΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΟΜΝ, πρὸς τὸ πρίσμα, οὗ βάσις μὲν τὸ ΡΦΖ τρίγωνον, ἀπεναντίον δὲ τὸ ΣΤΥ. ὡς δὲ τὰ εἰρημένα πρίσματα πρὸς ἄλληλα, οὕτως τὸ πρίσμα, οὗ βάσις μὲν τὸ ΚΒΞΛ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΟΜ εὐθεῖα, πρὸς τὸ πρίσμα, οὗ βάσις μὲν τὸ ΠΕΦΡ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΣΤ εὐθεῖα. καὶ τὰ δύο ἄρα πρίσματα, οὗ τε βάσις μὲν τὸ ΚΒΞΛ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΟΜ, καὶ οὗ βάσις μὲν τὸ ΛΞΓ, ἀπεναντίον δὲ τὸ ΟΜΝ, πρὸς τὰ πρίσματα, οὗ τε βάσις μὲν τὸ ΠΕΦΡ, ἀπεναντίον δὲ ἡ ΣΤ εὐθεῖα, καὶ οὗ βάσις μὲν τὸ ΡΦΖ τρίγωνον, ἀπεναντίον δὲ τὸ ΣΤΥ. καὶ ὡς ἄρα ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὰ εἰρημένα δύο πρίσματα πρὸς τὰ εἰρημένα δύο πρίσματα.

Καὶ ὁμοίως, ἐὰν διαιρεθῶσιν αἱ ΟΜΝΗ, ΣΤΥΘ πυραμίδες εἵς τε δύο πρίσματα καὶ δύο πυραμίδας, ἔσται ὡς ἡ ΟΜΝ βάσις πρὸς τὴν ΣΤΥ βάσιν, οὕτως τὰ ἐν τῇ ΟΜ ΝΗ πυραμίδι δύο πρίσματα πρὸς τὰ ἐν τῇ ΣΤΥΘ πυραμίδι δύο πρίσματα. ἀλλ’ ὡς ἡ ΟΜΝ βάσις πρὸς τὴν ΣΤΥ βάσιν, οὕτως ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν· ἴσον γὰρ ἑκάτερον τῶν ΟΜΝ, ΣΤΥ τριγώνων ἑκατέρῳ τῶν ΛΞΓ, ΡΦΖ. καὶ ὡς ἄρα ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὰ τέσσαρα πρίσματα πρὸς τὰ τέσσαρα πρίσματα. ὁμοίως δὲ κἂν τὰς ὑπολειπομένας πυραμίδας διέλωμεν εἵς τε δύο πυραμίδας καὶ εἰς δύο πρίσματα, ἔσται ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὰ ἐν τῇ ΑΒ

ΓΗ πυραμίδι πρίσματα πάντα πρὸς τὰ ἐν τῇ ΔΕΖΘ πυραμίδι πρίσματα πάντα ἰσοπληθῆ· ὅπερ ἔδει δεῖξαι.

Λῆμμα

Ὅτι δέ ἐστιν ὡς τὸ ΛΞΓ τρίγωνον πρὸς τὸ ΡΦΖ τρίγωνον, οὕτως τὸ πρίσμα, οὗ βάσις τὸ ΛΞΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΟΜΝ, πρὸς τὸ πρίσμα, οὗ βάσις μὲν τὸ ΡΦΖ [τρίγωνον], ἀπεναντίον δὲ τὸ ΣΤΥ, οὕτω δεικτέον.

Ἐπὶ γὰρ τῆς αὐτῆς καταγραφῆς νενοήσθωσαν ἀπὸ τῶν Η, Θ κάθετοι ἐπὶ τὰ ΑΒΓ, ΔΕΖ ἐπίπεδα, ἴσαι δηλαδὴ τυγχάνουσαι διὰ τὸ ἰσοϋψεῖς ὑποκεῖσθαι τὰς πυραμίδας. καὶ ἐπεὶ δύο εὐθεῖαι ἥ τε ΗΓ καὶ ἡ ἀπὸ τοῦ Η κάθετος ὑπὸ παραλλήλων ἐπιπέδων τῶν ΑΒΓ, ΟΜΝ τέμνονται, εἰς τοὺς αὐτοὺς λόγους τμηθήσονται. καὶ τέτμηται ἡ ΗΓ δίχα ὑπὸ τοῦ ΟΜΝ ἐπιπέδου κατὰ τὸ Ν· καὶ ἡ ἀπὸ τοῦ Η ἄρα κάθετος ἐπὶ τὸ ΑΒΓ ἐπίπεδον δίχα τμηθήσεται ὑπὸ τοῦ ΟΜΝ ἐπιπέδου. διὰ τὰ αὐτὰ δὴ καὶ ἡ ἀπὸ τοῦ Θ κάθετος ἐπὶ τὸ ΔΕΖ ἐπίπεδον δίχα τμηθήσεται ὑπὸ τοῦ ΣΤΥ ἐπιπέδου. καί εἰσιν ἴσαι αἱ ἀπὸ τῶν Η, Θ κάθετοι ἐπὶ τὰ ΑΒΓ, ΔΕΖ ἐπίπεδα· ἴσαι ἄρα καὶ αἱ ἀπὸ τῶν ΟΜΝ, ΣΤΥ τριγώνων ἐπὶ τὰ ΑΒΓ, ΔΕΖ κάθετοι. ἰσοϋψῆ ἄρα [ἐστὶ] τὰ πρίσματα, ὧν βάσεις μέν εἰσι τὰ ΛΞΓ, ΡΦΖ τρίγωνα, ἀπεναντίον δὲ τὰ ΟΜΝ, ΣΤΥ. ὥστε καὶ τὰ στερεὰ παραλληλεπίπεδα τὰ ἀπὸ τῶν εἰρημένων πρισμάτων ἀναγραφόμενα ἰσοϋψῆ καὶ πρὸς ἄλληλα [εἰσὶν] ὡς αἱ βάσεις· καὶ τὰ ἡμίση ἄρα ἐστὶν ὡς ἡ ΛΞΓ βάσις πρὸς τὴν ΡΦΖ βάσιν, οὕτως τὰ εἰρημένα πρίσματα πρὸς ἄλληλα· ὅπερ ἔδει δεῖξαι.

ε΄. Αἱ ὑπὸ τὸ αὐτὸ ὕψος οὖσαι πυραμίδες καὶ τριγώνους ἔχουσαι βάσεις πρὸς ἀλλήλας εἰσὶν ὡς αἱ βάσεις.

Ἔστωσαν ὑπὸ τὸ αὐτὸ ὕψος πυραμίδες, ὧν βάσεις μὲν τὰ ΑΒΓ, ΔΕΖ τρίγωνα, κορυφαὶ δὲ τὰ Η, Θ σημεῖα· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα.

Εἰ γὰρ μή ἐστιν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα, ἔσται ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς ἤτοι πρὸς ἔλασσόν τι τῆς ΔΕΖΘ πυραμίδος στερεὸν ἢ πρὸς μεῖζον. ἔστω πρότερον πρὸς ἔλασσον τὸ Χ, καὶ διῃρήσθω ἡ ΔΕΖΘ πυραμὶς εἵς τε δύο πυραμίδας ἴσας ἀλλήλαις καὶ ὁμοίας τῇ ὅλῃ καὶ εἰς δύο πρίσματα ἴσα· τὰ δὴ δύο πρίσματα μείζονά ἐστιν ἢ τὸ ἥμισυ τῆς ὅλης πυραμίδος. καὶ πάλιν αἱ ἐκ τῆς διαιρέσεως γινόμεναι πυραμίδες ὁμοίως διῃρήσθωσαν, καὶ τοῦτο ἀεὶ γινέσθω, ἕως οὗ λειφθῶσί τινες πυραμίδες ἀπὸ τῆς ΔΕΖΘ πυραμίδος, αἵ εἰσιν ἐλάττονες τῆς ὑπεροχῆς, ᾗ ὑπερέχει ἡ ΔΕ ΖΘ πυραμὶς τοῦ Χ στερεοῦ. λελείφθωσαν καὶ ἔστωσαν λόγου ἕνεκεν αἱ ΔΠΡΣ, ΣΤΥΘ· λοιπὰ ἄρα τὰ ἐν τῇ ΔΕ ΖΘ πυραμίδι πρίσματα μείζονά ἐστι τοῦ Χ στερεοῦ.

διῃρήσθω καὶ ἡ ΑΒΓΗ πυραμὶς ὁμοίως καὶ ἰσοπληθῶς τῇ ΔΕΖΘ πυραμίδι· ἔστιν ἄρα ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὰ ἐν τῇ ΑΒΓΗ πυραμίδι πρίσματα πρὸς τὰ ἐν τῇ ΔΕΖΘ πυραμίδι πρίσματα. ἀλλὰ καὶ ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς τὸ Χ στερεόν· καὶ ὡς ἄρα ἡ ΑΒΓΗ πυραμὶς πρὸς τὸ Χ στερεόν, οὕτως τὰ ἐν τῇ ΑΒΓΗ πυραμίδι πρίσματα πρὸς τὰ ἐν τῇ ΔΕΖΘ πυραμίδι πρίσματα· ἐναλλὰξ ἄρα ὡς ἡ ΑΒΓΗ πυραμὶς πρὸς τὰ ἐν αὐτῇ πρίσματα, οὕτως τὸ Χ στερεὸν πρὸς τὰ ἐν τῇ ΔΕΖΘ πυραμίδι πρίσματα. μείζων δὲ ἡ ΑΒΓΗ πυραμὶς τῶν ἐν αὐτῇ πρισμάτων· μεῖζον ἄρα καὶ τὸ Χ στερεὸν τῶν ἐν τῇ ΔΕΖΘ πυραμίδι πρισμάτων. ἀλλὰ καὶ ἔλαττον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἐστὶν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς ἔλασσόν τι τῆς ΔΕΖΘ πυραμίδος στερεόν. ὁμοίως δὴ δειχθήσεται, ὅτι οὐδὲ ὡς ἡ ΔΕΖ βάσις πρὸς τὴν ΑΒΓ βάσιν, οὕτως ἡ ΔΕΖΘ πυραμὶς πρὸς ἔλαττόν τι τῆς ΑΒΓΗ πυραμίδος στερεόν.

Λέγω δή, ὅτι οὐκ ἔστιν οὐδὲ ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς μεῖζόν τι τῆς ΔΕΖΘ πυραμίδος στερεόν.

Εἰ γὰρ δυνατόν, ἔστω πρὸς μεῖζον τὸ Χ· ἀνάπαλιν ἄρα ἐστὶν ὡς ἡ ΔΕΖ βάσις πρὸς τὴν ΑΒΓ βάσιν, οὕτως τὸ Χ στερεὸν πρὸς τὴν ΑΒΓΗ πυραμίδα. ὡς δὲ τὸ Χ στερεὸν πρὸς τὴν ΑΒΓΗ πυραμίδα, οὕτως ἡ ΔΕΖΘ πυραμὶς πρὸς ἔλασσόν τι τῆς ΑΒΓΗ πυραμίδος, ὡς ἔμπροσθεν ἐδείχθη· καὶ ὡς ἄρα ἡ ΔΕΖ βάσις πρὸς τὴν ΑΒΓ βάσιν, οὕτως ἡ ΔΕΖΘ πυραμὶς πρὸς ἔλασσόν τι τῆς ΑΒΓΗ πυραμίδος· ὅπερ ἄτοπον ἐδείχθη.

οὐκ ἄρα ἐστὶν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς μεῖζόν τι τῆς ΔΕΖΘ πυραμίδος στερεόν. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἔλασσον. ἔστιν ἄρα ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα· ὅπερ ἔδει δεῖξαι.

ς΄. Αἱ ὑπὸ τὸ αὐτὸ ὕψος οὖσαι πυραμίδες καὶ πολυγώνους ἔχουσαι βάσεις πρὸς ἀλλήλας εἰσὶν ὡς αἱ βάσεις.

Ἔστωσαν ὑπὸ τὸ αὐτὸ ὕψος πυραμίδες, ὧν [αἱ] βάσεις μὲν τὰ ΑΒΓΔΕ, ΖΗΘΚΛ πολύγωνα, κορυφαὶ δὲ τὰ Μ, Ν σημεῖα· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΒΓΔΕ βάσις πρὸς τὴν ΖΗΘΚΛ βάσιν, οὕτως ἡ ΑΒΓΔΕΜ πυραμὶς πρὸς τὴν ΖΗΘΚΛΝ πυραμίδα.

Ἐπεζεύχθωσαν γὰρ αἱ ΑΓ, ΑΔ, ΖΘ, ΖΚ. ἐπεὶ οὖν δύο πυραμίδες εἰσὶν αἱ ΑΒΓΜ, ΑΓΔΜ τριγώνους ἔχουσαι βάσεις καὶ ὕψος ἴσον, πρὸς ἀλλήλας εἰσὶν ὡς αἱ βάσεις· ἔστιν ἄρα ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΑΓΔ βάσιν, οὕτως ἡ ΑΒΓΜ πυραμὶς πρὸς τὴν ΑΓΔΜ πυραμίδα. καὶ συνθέντι ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΑΓΔ βάσιν, οὕτως ἡ ΑΒΓΔΜ πυραμὶς πρὸς τὴν ΑΓΔΜ πυραμίδα. ἀλλὰ καὶ ὡς ἡ ΑΓΔ βάσις πρὸς τὴν ΑΔΕ βάσιν, οὕτως ἡ ΑΓΔΜ πυραμὶς πρὸς τὴν ΑΔΕΜ πυραμίδα.

δι’ ἴσου ἄρα ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΑΔΕ βάσιν, οὕτως ἡ ΑΒΓΔΜ πυραμὶς πρὸς τὴν ΑΔΕΜ πυραμίδα. καὶ συνθέντι πάλιν, ὡς ἡ ΑΒΓΔΕ βάσις πρὸς τὴν ΑΔΕ βάσιν, οὕτως ἡ ΑΒΓΔΕΜ πυραμὶς πρὸς τὴν ΑΔΕΜ πυραμίδα. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ὡς ἡ ΖΗΘΚΛ βάσις πρὸς τὴν ΖΗΘ βάσιν,

οὕτως καὶ ἡ ΖΗΘΚΛΝ πυραμὶς πρὸς τὴν ΖΗΘΝ πυραμίδα. καὶ ἐπεὶ δύο πυραμίδες εἰσὶν αἱ ΑΔ ΕΜ, ΖΗΘΝ τριγώνους ἔχουσαι βάσεις καὶ ὕψος ἴσον, ἔστιν ἄρα ὡς ἡ ΑΔΕ βάσις πρὸς τὴν ΖΗΘ βάσιν, οὕτως ἡ ΑΔΕΜ πυραμὶς πρὸς τὴν ΖΗΘΝ πυραμίδα. ἀλλ' ὡς ἡ ΑΔΕ βάσις πρὸς τὴν ΑΒΓΔΕ βάσιν, οὕτως ἦν ἡ ΑΔ ΕΜ πυραμὶς πρὸς τὴν ΑΒΓΔΕΜ πυραμίδα. καὶ δι' ἴσου ἄρα ὡς ἡ ΑΒΓΔΕ βάσις πρὸς τὴν ΖΗΘ βάσιν, οὕτως ἡ ΑΒΓΔΕΜ πυραμὶς πρὸς τὴν ΖΗΘΝ πυραμίδα. ἀλλὰ μὴν καὶ ὡς ἡ ΖΗΘ βάσις πρὸς τὴν ΖΗΘΚΛ βάσιν, οὕτως ἦν καὶ ἡ ΖΗΘΝ πυραμὶς πρὸς τὴν ΖΗΘΚΛΝ πυραμίδα. καὶ δι' ἴσου ἄρα ὡς ἡ ΑΒΓΔΕ βάσις πρὸς τὴν ΖΗΘΚΛ βάσιν, οὕτως ἡ ΑΒΓΔΕΜ πυραμὶς πρὸς τὴν ΖΗΘΚΛΝ πυραμίδα· ὅπερ ἔδει δεῖξαι.

ζ΄. Πᾶν πρίσμα τρίγωνον ἔχον βάσιν διαιρεῖται εἰς τρεῖς πυραμίδας ἴσας ἀλλήλαις τριγώνους βάσεις ἐχούσας.

Ἔστω πρίσμα, οὗ βάσις μὲν τὸ ΑΒΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΔΕΖ· λέγω, ὅτι τὸ ΑΒΓΔΕΖ πρίσμα διαιρεῖται εἰς τρεῖς πυραμίδας ἴσας ἀλλήλαις τριγώνους ἐχούσας βάσεις.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΔ, ΕΓ, ΓΔ. ἐπεὶ παραλληλόγραμμόν ἐστι τὸ ΑΒΕΔ, διάμετρος δὲ αὐτοῦ ἐστιν ἡ ΒΔ, ἴσον ἄρα ἐστὶ τὸ ΑΒΔ τρίγωνον τῷ ΕΒΔ τριγώνῳ· καὶ ἡ πυραμὶς ἄρα, ἧς βάσις μὲν τὸ ΑΒΔ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, ἴση ἐστὶ πυραμίδι, ἧς βάσις μέν ἐστι τὸ ΔΕΒ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον. ἀλλὰ ἡ πυραμίς, ἧς βάσις μέν ἐστι τὸ ΔΕΒ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, ἡ αὐτή ἐστι πυραμίδι, ἧς βάσις μέν ἐστι τὸ ΕΒΓ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον· ὑπὸ γὰρ τῶν αὐτῶν ἐπιπέδων περιέχεται. καὶ πυραμὶς ἄρα, ἧς βάσις μέν ἐστι τὸ ΑΒΔ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, ἴση ἐστὶ πυραμίδι, ἧς βάσις μέν ἐστι τὸ ΕΒΓ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον.

πάλιν, ἐπεὶ παραλληλόγραμμόν ἐστι τὸ ΖΓΒΕ, διάμετρος δέ ἐστιν αὐτοῦ ἡ ΓΕ, ἴσον ἐστὶ τὸ ΓΕΖ τρίγωνον τῷ ΓΒΕ τριγώνῳ. καὶ πυραμὶς ἄρα, ἧς βάσις

μέν ἐστι τὸ ΒΓΕ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, ἴση ἐστὶ πυραμίδι, ἧς βάσις μέν ἐστι τὸ ΕΓΖ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον. ἡ δὲ πυραμίς, ἧς βάσις μέν ἐστι τὸ ΒΓΕ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, ἴση ἐδείχθη πυραμίδι, ἧς βάσις μέν ἐστι τὸ ΑΒΔ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον· καὶ πυραμὶς ἄρα, ἧς βάσις μέν ἐστι τὸ ΓΕΖ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, ἴση ἐστὶ πυραμίδι, ἧς βάσις μὲν [ἐστι] τὸ ΑΒΔ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον· διῄρηται ἄρα τὸ ΑΒΓΔΕΖ πρίσμα εἰς τρεῖς πυραμίδας ἴσας ἀλλήλαις τριγώνους ἐχούσας βάσεις.

Καὶ ἐπεὶ πυραμίς, ἧς βάσις μέν ἐστι τὸ ΑΒΔ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, ἡ αὐτή ἐστι πυραμίδι, ἧς βάσις τὸ ΓΑΒ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον· ὑπὸ γὰρ τῶν αὐτῶν ἐπιπέδων περιέχονται· ἡ δὲ πυραμίς, ἧς βάσις τὸ ΑΒΔ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, τρίτον ἐδείχθη τοῦ πρίσματος, οὗ βάσις τὸ ΑΒΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΔΕΖ, καὶ ἡ πυραμὶς ἄρα, ἧς βάσις τὸ ΑΒΓ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, τρίτον ἐστὶ τοῦ πρίσματος τοῦ ἔχοντος βάσιν τὴν αὐτὴν τὸ ΑΒΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΔΕΖ.

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι πᾶσα πυραμὶς τρίτον μέρος ἐστὶ τοῦ πρίσματος τοῦ τὴν αὐτὴν βάσιν ἔχοντος αὐτῇ καὶ ὕψος ἴσον [ἐπειδήπερ κἂν ἕτερόν τι σχῆμα εὐθύγραμμον ἔχῃ ἡ βάσις τοῦ πρίσματος, τοιοῦτο καὶ τὸ ἀπεναντίον, καὶ διαιρεῖται εἰς πρίσματα τρίγωνα ἔχοντα τὰς βάσεις καὶ τὰ ἀπεναντίον, καὶ ὡς ἡ ὅλη βάσις πρὸς ἕκαστον]· ὅπερ ἔδει δεῖξαι.

η΄. Αἱ ὅμοιαι πυραμίδες καὶ τριγώνους ἔχουσαι βάσεις ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν.

Ἔστωσαν ὅμοιαι καὶ ὁμοίως κείμεναι πυραμίδες, ὧν βάσεις μέν εἰσι τὰ ΑΒΓ, ΔΕΖ τρίγωνα, κορυφαὶ δὲ τὰ Η, Θ σημεῖα· λέγω, ὅτι ἡ ΑΒΓΗ πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ.

Συμπεπληρώσθω γὰρ τὰ ΒΗΜΛ, ΕΘΠΟ στερεὰ παραλληλεπίπεδα. καὶ ἐπεὶ ὁμοία ἐστὶν ἡ ΑΒΓΗ πυραμὶς τῇ ΔΕΖΘ πυραμίδι, ἴση ἄρα ἐστὶν ἡ μὲν ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΕΖ γωνίᾳ, ἡ δὲ ὑπὸ ΗΒΓ τῇ ὑπὸ ΘΕΖ, ἡ δὲ ὑπὸ ΑΒΗ τῇ ὑπὸ ΔΕΘ, καί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΔΕ, οὕτως ἡ ΒΓ πρὸς τὴν ΕΖ, καὶ ἡ ΒΗ πρὸς τὴν ΕΘ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΔΕ, οὕτως ἡ ΒΓ πρὸς τὴν ΕΖ, καὶ περὶ ἴσας γωνίας αἱ πλευραὶ ἀνάλογόν εἰσιν, ὅμοιον ἄρα ἐστὶ τὸ ΒΜ παραλληλόγραμμον τῷ ΕΠ παραλληλογράμμῳ.

διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν ΒΝ τῷ ΕΡ ὅμοιόν ἐστι, τὸ δὲ ΒΚ τῷ ΕΞ· τὰ τρία ἄρα τὰ ΜΒ, ΒΚ, ΒΝ τρισὶ τοῖς ΕΠ, ΕΞ, ΕΡ ὅμοιά ἐστιν. ἀλλὰ τὰ μὲν τρία τὰ ΜΒ, ΒΚ, ΒΝ τρισὶ τοῖς ἀπεναντίον ἴσα τε καὶ ὅμοιά ἐστιν, τὰ δὲ τρία τὰ ΕΠ, ΕΞ, ΕΡ τρισὶ τοῖς ἀπεναντίον ἴσα τε καὶ ὅμοιά ἐστιν. τὰ ΒΗΜΛ, ΕΘΠΟ ἄρα στερεὰ ὑπὸ ὁμοίων ἐπιπέδων ἴσων τὸ πλῆθος περιέχεται. ὅμοιον ἄρα ἐστὶ τὸ ΒΗΜΛ στερεὸν τῷ ΕΘΠΟ στερεῷ. τὰ δὲ ὅμοια στερεὰ παραλληλεπίπεδα ἐν τριπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν. τὸ ΒΗΜΛ ἄρα στερεὸν πρὸς τὸ ΕΘΠΟ στερεὸν τριπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ ἡ ΒΓ πρὸς τὴν ὁμόλογον πλευρὰν τὴν ΕΖ. ὡς δὲ τὸ ΒΗΜΛ στερεὸν πρὸς τὸ ΕΘΠΟ στερεόν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα, ἐπειδήπερ ἡ πυραμὶς ἕκτον μέρος ἐστὶ τοῦ στερεοῦ διὰ τὸ καὶ τὸ πρίσμα ἥμισυ ὂν τοῦ στερεοῦ παραλληλεπιπέδου τριπλάσιον εἶναι τῆς πυραμίδος. καὶ ἡ ΑΒΓΗ ἄρα πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα τριπλασίονα λόγον ἔχει ἥπερ ἡ ΒΓ πρὸς τὴν ΕΖ· ὅπερ ἔδει δεῖξαι.

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι καὶ αἱ πολυγώνους ἔχουσαι βάσεις ὅμοιαι πυραμίδες πρὸς ἀλλήλας ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν. διαιρεθεισῶν γὰρ αὐτῶν εἰς τὰς ἐν αὐταῖς πυραμίδας τριγώνους βάσεις ἐχούσας τῷ καὶ τὰ ὅμοια πολύγωνα τῶν βάσεων εἰς ὅμοια τρίγωνα διαιρεῖσθαι καὶ ἴσα τῷ πλήθει καὶ ὁμόλογα τοῖς ὅλοις ἔσται ὡς [ἡ] ἐν τῇ

ἑτέρᾳ μία πυραμὶς τρίγωνον ἔχουσα βάσιν πρὸς τὴν ἐν τῇ ἑτέρᾳ μίαν πυραμίδα τρίγωνον ἔχουσαν βάσιν, οὕτως καὶ ἅπασαι αἱ ἐν τῇ ἑτέρᾳ πυραμίδι πυραμίδες τριγώνους ἔχουσαι βάσεις πρὸς τὰς ἐν τῇ ἑτέρᾳ πυραμίδι πυραμίδας τριγώνους βάσεις ἐχούσας, τουτέστιν αὐτὴ ἡ πολύγωνον βάσιν ἔχουσα πυραμὶς πρὸς τὴν πολύγωνον βάσιν ἔχουσαν πυραμίδα. ἡ δὲ τρίγωνον βάσιν ἔχουσα πυραμὶς πρὸς τὴν τρίγωνον βάσιν ἔχουσαν ἐν τριπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν· καὶ ἡ πολύγωνον ἄρα βάσιν ἔχουσα πρὸς τὴν ὁμοίαν βάσιν ἔχουσαν τριπλασίονα λόγον ἔχει ἥπερ ἡ πλευρὰ πρὸς τὴν πλευράν.

θ΄. Τῶν ἴσων πυραμίδων καὶ τριγώνους βάσεις ἐχουσῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ὧν πυραμίδων τριγώνους βάσεις ἐχουσῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσαι εἰσὶν ἐκεῖναι.

Ἔστωσαν γὰρ ἴσαι πυραμίδες τριγώνους βάσεις ἔχουσαι τὰς ΑΒΓ, ΔΕΖ, κορυφὰς δὲ τὰ Η, Θ σημεῖα· λέγω, ὅτι τῶν ΑΒΓΗ, ΔΕΖΘ πυραμίδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καί ἐστιν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὸ τῆς ΔΕΖΘ πυραμίδος ὕψος πρὸς τὸ τῆς ΑΒΓΗ πυραμίδος ὕψος.

Συμπεπληρώσθω γὰρ τὰ ΒΗΜΛ, ΕΘΠΟ στερεὰ παραλληλεπίπεδα. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒΓΗ πυραμὶς τῇ ΔΕΖΘ πυραμίδι, καί ἐστι τῆς μὲν ΑΒΓΗ πυραμίδος ἑξαπλάσιον τὸ ΒΗ ΜΛ στερεόν, τῆς δὲ ΔΕΖΘ πυραμίδος ἑξαπλάσιον τὸ ΕΘΠΟ στερεόν, ἴσον ἄρα ἐστὶ τὸ ΒΗΜΛ στερεὸν τῷ ΕΘΠΟ στερεῷ. τῶν δὲ ἴσων στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· ἔστιν ἄρα ὡς ἡ ΒΜ βάσις πρὸς τὴν ΕΠ βάσιν, οὕτως τὸ τοῦ ΕΘΠΟ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΗΜΛ στερεοῦ ὕψος. ἀλλ’ ὡς ἡ ΒΜ βάσις πρὸς τὴν ΕΠ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΔΕΖ τρίγωνον. καὶ ὡς ἄρα τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΔΕΖ τρίγωνον, οὕτως τὸ τοῦ ΕΘΠΟ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΗΜΛ στερεοῦ ὕψος. ἀλλὰ τὸ μὲν τοῦ ΕΘΠΟ στερεοῦ

ὕψος τὸ αὐτό ἐστι τῷ τῆς ΔΕΖΘ πυραμίδος ὕψει, τὸ δὲ τοῦ ΒΗΜΛ στερεοῦ ὕψος τὸ αὐτό ἐστι τῷ τῆς ΑΒΓΗ πυραμίδος ὕψει· ἔστιν ἄρα ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὸ τῆς ΔΕΖΘ πυραμίδος ὕψος πρὸς τὸ τῆς ΑΒΓΗ πυραμίδος ὕψος. τῶν ΑΒΓΗ, ΔΕΖΘ ἄρα πυραμίδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Ἀλλὰ δὴ τῶν ΑΒΓΗ, ΔΕΖΘ πυραμίδων ἀντιπεπονθέτωσαν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὸ τῆς ΔΕΖΘ πυραμίδος ὕψος πρὸς τὸ τῆς ΑΒΓΗ πυραμίδος ὕψος· λέγω, ὅτι ἴση ἐστὶν ἡ ΑΒΓΗ πυραμὶς τῇ ΔΕΖΘ πυραμίδι.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεί ἐστιν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὸ τῆς ΔΕΖΘ πυραμίδος ὕψος πρὸς τὸ τῆς ΑΒΓΗ πυραμίδος ὕψος, ἀλλ’ ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὸ ΒΜ παραλληλόγραμμον πρὸς τὸ ΕΠ παραλληλόγραμμον, καὶ ὡς ἄρα τὸ ΒΜ παραλληλόγραμμον πρὸς τὸ ΕΠ παραλληλόγραμμον, οὕτως τὸ τῆς ΔΕΖΘ πυραμίδος ὕψος πρὸς τὸ τῆς ΑΒΓΗ πυραμίδος ὕψος. ἀλλὰ τὸ [μὲν] τῆς ΔΕΖΘ πυραμίδος ὕψος τὸ αὐτό ἐστι τῷ τοῦ ΕΘΠΟ παραλληλεπιπέδου ὕψει, τὸ δὲ τῆς ΑΒΓΗ πυραμίδος ὕψος τὸ αὐτό ἐστι τῷ τοῦ ΒΗΜΛ παραλληλεπιπέδου ὕψει· ἔστιν ἄρα ὡς ἡ ΒΜ βάσις πρὸς τὴν ΕΠ βάσιν, οὕτως τὸ τοῦ ΕΘΠΟ παραλληλεπιπέδου ὕψος πρὸς τὸ τοῦ ΒΗΜΛ παραλληλεπιπέδου ὕψος. ὧν δὲ στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσα ἐστὶν ἐκεῖνα· ἴσον ἄρα ἐστὶ τὸ ΒΗΜΛ στερεὸν παραλληλεπίπεδον τῷ ΕΘ ΠΟ στερεῷ παραλληλεπιπέδῳ. καί ἐστι τοῦ μὲν ΒΗΜΛ ἕκτον μέρος ἡ ΑΒΓΗ πυραμίς, τοῦ δὲ ΕΘΠΟ παραλληλεπιπέδου ἕκτον μέρος ἡ ΔΕΖΘ πυραμίς· ἴση ἄρα ἡ ΑΒΓΗ πυραμὶς τῇ ΔΕΖΘ πυραμίδι.

Τῶν ἄρα ἴσων πυραμίδων καὶ τριγώνους βάσεις ἐχουσῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ὧν πυραμίδων τριγώνους βάσεις ἐχουσῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσαι εἰσὶν ἐκεῖναι· ὅπερ ἔδει δεῖξαι.

ι΄. Πᾶς κῶνος κυλίνδρου τρίτον μέρος ἐστὶ τοῦ τὴν αὐτὴν βάσιν ἔχοντος αὐτῷ καὶ ὕψος ἴσον.

Ἐχέτω γὰρ κῶνος κυλίνδρῳ βάσιν τε τὴν αὐτὴν τὸν ΑΒΓΔ κύκλον καὶ ὕψος ἴσον· λέγω, ὅτι ὁ κῶνος τοῦ κυλίνδρου τρίτον ἐστὶ μέρος, τουτέστιν ὅτι ὁ κύλινδρος τοῦ κώνου τριπλασίων ἐστίν.

Εἰ γὰρ μή ἐστιν ὁ κύλινδρος τοῦ κώνου τριπλασίων, ἔσται ὁ κύλινδρος τοῦ κώνου ἤτοι μείζων ἢ τριπλασίων ἢ ἐλάσσων ἢ τριπλασίων. ἔστω πρότερον μείζων ἢ τριπλασίων, καὶ ἐγγεγράφθω εἰς τὸν ΑΒΓΔ κύκλον τετράγωνον τὸ ΑΒΓΔ· τὸ δὴ ΑΒΓΔ τετράγωνον μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ ΑΒΓΔ κύκλου. καὶ ἀνεστάτω ἀπὸ τοῦ ΑΒΓΔ τετραγώνου πρίσμα ἰσουψὲς τῷ κυλίνδρῳ. τὸ δὴ ἀνιστάμενον πρίσμα μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ κυλίνδρου, ἐπειδήπερ κἂν περὶ τὸν ΑΒΓΔ κύκλον τετράγωνον περιγράψωμεν, τὸ ἐγγεγραμμένον εἰς τὸν ΑΒΓΔ κύκλον τετράγωνον ἥμισύ ἐστι τοῦ περιγεγραμμένου· καί ἐστι τὰ ἀπ᾽ αὐτῶν ἀνιστάμενα στερεὰ παραλληλεπίπεδα πρίσματα ἰσοϋψῆ· τὰ δὲ ὑπὸ τὸ αὐτὸ ὕψος ὄντα στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις· καὶ τὸ ἐπὶ τοῦ ΑΒΓΔ ἄρα τετραγώνου ἀνασταθὲν πρίσμα ἥμισύ ἐστι τοῦ ἀνασταθέντος πρίσματος ἀπὸ τοῦ περὶ τὸν ΑΒΓΔ κύκλον περιγραφέντος τετραγώνου· καί ἐστιν ὁ κύλινδρος ἐλάττων τοῦ πρίσματος τοῦ ἀνασταθέντος ἀπὸ τοῦ περὶ τὸν ΑΒΓΔ κύκλον περιγραφέντος τετραγώνου· τὸ ἄρα πρίσμα τὸ ἀνασταθὲν ἀπὸ τοῦ ΑΒΓΔ τετραγώνου ἰσοϋψὲς τῷ κυλίνδρῳ μεῖζόν ἐστι τοῦ ἡμίσεως τοῦ κυλίνδρου. τετμήσθωσαν αἱ ΑΒ, ΒΓ, ΓΔ, ΔΑ περιφέρειαι δίχα κατὰ τὰ Ε, Ζ, Η, Θ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΑΕ, ΕΒ, ΒΖ, ΖΓ, ΓΗ, ΗΔ, ΔΘ, ΘΑ· καὶ ἕκαστον ἄρα τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ καθ᾽ ἑαυτὸ τμήματος τοῦ ΑΒΓΔ κύκλου, ὡς ἔμπροσθεν ἐδείκνυμεν.

ἀνεστάτω ἐφ᾽ ἑκάστου τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων πρίσματα ἰσουψῆ τῷ κυλίνδρῳ· καὶ ἕκαστον ἄρα τῶν ἀνασταθέντων πρισμάτων μεῖζόν ἐστιν ἢ τὸ ἥμισυ μέρος τοῦ καθ᾽ ἑαυτὸ τμήματος τοῦ κυλίνδρου, ἐπειδήπερ

ἐὰν διὰ τῶν Ε, Ζ, Η, Θ σημείων παραλλήλους ταῖς ΑΒ, ΒΓ, ΓΔ, ΔΑ ἀγάγωμεν, καὶ συμπληρώσωμεν τὰ ἐπὶ τῶν ΑΒ, ΒΓ, ΓΔ, ΔΑ παραλληλόγραμμα, καὶ ἀπ᾽ αὐτῶν ἀναστήσωμεν στερεὰ παραλληλεπίπεδα ἰσοϋψῆ τῷ κυλίνδρῳ, ἑκάστου τῶν ἀνασταθέντων ἡμίση ἐστὶ τὰ πρίσματα τὰ ἐπὶ τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων· καί ἐστι τὰ τοῦ κυλίνδρου τμήματα ἐλάττονα τῶν ἀνασταθέντων στερεῶν παραλληλεπιπέδων· ὥστε καὶ τὰ ἐπὶ τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων πρίσματα μείζονά ἐστιν ἢ τὸ ἥμισυ τῶν καθ᾽ ἑαυτὰ τοῦ κυλίνδρου τμημάτων. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ ἀνιστάντες ἐφ᾽ ἑκάστου τῶν τριγώνων πρίσματα ἰσοϋψῆ τῷ κυλίνδρῳ καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομέν τινα ἀποτμήματα τοῦ κυλίνδρου, ἃ ἔσται ἐλάττονα τῆς ὑπεροχῆς, ᾗ ὑπερέχει ὁ κύλινδρος τοῦ τριπλασίου τοῦ κώνου. λελείφθω, καὶ ἔστω τὰ ΑΕ, ΕΒ, ΒΖ, ΖΓ, ΓΗ, ΗΔ, ΔΘ, ΘΑ· λοιπὸν ἄρα τὸ πρίσμα, οὗ βάσις μὲν τὸ ΑΕΒΖ ΓΗΔΘ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ κυλίνδρῳ, μεῖζόν ἐστιν ἢ τριπλάσιον τοῦ κώνου. ἀλλὰ τὸ πρίσμα, οὗ βάσις μέν ἐστι τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ κυλίνδρῳ, τριπλάσιόν ἐστι τῆς πυραμίδος, ἧς βάσις μέν ἐστι τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, κορυφὴ δὲ ἡ αὐτὴ τῷ κώνῳ· καὶ ἡ πυραμὶς ἄρα, ἧς βάσις μὲν [ἐστι] τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, κορυφὴ δὲ ἡ αὐτὴ τῷ κώνῳ, μείζων ἐστὶ τοῦ κώνου τοῦ βάσιν ἔχοντος τὸν ΑΒ ΓΔ κύκλον. ἀλλὰ καὶ ἐλάττων· ἐμπεριέχεται γὰρ ὑπ᾽ αὐτοῦ· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἐστὶν ὁ κύλινδρος τοῦ κώνου μείζων ἢ τριπλάσιος.

Λέγω δή, ὅτι οὐδὲ ἐλάττων ἐστὶν ἢ τριπλάσιος ὁ κύλινδρος τοῦ κώνου.

Εἰ γὰρ δυνατόν, ἔστω ἐλάττων ἢ τριπλάσιος ὁ κύλινδρος τοῦ κώνου· ἀνάπαλιν ἄρα ὁ κῶνος τοῦ κυλίνδρου μείζων ἐστὶν ἢ τρίτον μέρος. ἐγγεγράφθω δὴ εἰς τὸν ΑΒΓΔ κύκλον τετράγωνον τὸ ΑΒΓΔ· τὸ ΑΒΓΔ ἄρα τετράγωνον μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ ΑΒΓΔ κύκλου. καὶ ἀνεστάτω ἀπὸ τοῦ ΑΒΓΔ τετραγώνου πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κώνῳ· ἡ ἄρα

ἀνασταθεῖσα πυραμὶς μείζων ἐστὶν ἢ τὸ ἥμισυ μέρος τοῦ κώνου, ἐπειδήπερ, ὡς ἔμπροσθεν ἐδείκνυμεν, ὅτι ἐὰν περὶ τὸν κύκλον τετράγωνον περιγράψωμεν, ἔσται τὸ ΑΒΓΔ τετράγωνον ἥμισυ τοῦ περὶ τὸν κύκλον περιγεγραμμένου τετραγώνου· καὶ ἐὰν ἀπὸ τῶν τετραγώνων στερεὰ παραλληλεπίπεδα ἀναστήσωμεν ἰσουψῆ τῷ κώνῳ, ἃ καὶ καλεῖται πρίσματα, ἔσται τὸ ἀνασταθὲν ἀπὸ τοῦ ΑΒΓΔ τετραγώνου ἥμισυ τοῦ ἀνασταθέντος ἀπὸ τοῦ περὶ τὸν κύκλον περιγραφέντος τετραγώνου· πρὸς ἄλληλα γάρ εἰσιν ὡς αἱ βάσεις. ὥστε καὶ τὰ τρίτα· καὶ πυραμὶς ἄρα, ἧς βάσις τὸ ΑΒΓΔ τετράγωνον, ἥμισύ ἐστι τῆς πυραμίδος τῆς ἀνασταθείσης ἀπὸ τοῦ περὶ τὸν κύκλον περιγραφέντος τετραγώνου. καί ἐστι μείζων ἡ πυραμὶς ἡ ἀνασταθεῖσα ἀπὸ τοῦ περὶ τὸν κύκλον τετραγώνου τοῦ κώνου· ἐμπεριέχει γὰρ αὐτόν. ἡ ἄρα πυραμὶς, ἧς βάσις τὸ ΑΒΓΔ τετράγωνον, κορυφὴ δὲ ἡ αὐτὴ τῷ κώνῳ, μείζων ἐστὶν ἢ τὸ ἥμισυ τοῦ κώνου. τετμήσθωσαν αἱ ΑΒ, ΒΓ, ΓΔ, ΔΑ περιφέρειαι δίχα κατὰ τὰ Ε, Ζ, Η, Θ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΑΕ, ΕΒ, ΒΖ, ΖΓ, ΓΗ, ΗΔ, ΔΘ, ΘΑ· καὶ ἕκαστον ἄρα τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων μεῖζόν ἐστιν ἢ τὸ ἥμισυ μέρος τοῦ καθ’ ἑαυτὸ τμήματος τοῦ ΑΒΓΔ κύκλου. καὶ ἀνεστάτωσαν ἐφ’ ἑκάστου τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων πυραμίδες τὴν αὐτὴν κορυφὴν ἔχουσαι τῷ κώνῳ· καὶ ἑκάστη ἄρα τῶν ἀνασταθεισῶν πυραμίδων κατὰ τὸν αὐτὸν τρόπον μείζων ἐστὶν ἢ τὸ ἥμισυ μέρος τοῦ καθ’ ἑαυτὴν τμήματος τοῦ κώνου. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ ἀνιστάντες ἐφ’ ἑκάστου τῶν τριγώνων πυραμίδα τὴν αὐτὴν κορυφὴν ἔχουσαν τῷ κώνῳ καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομέν τινα ἀποτμήματα τοῦ κώνου, ἃ ἔσται ἐλάττονα τῆς ὑπεροχῆς, ᾗ ὑπερέχει ὁ κῶνος τοῦ τρίτου μέρους τοῦ κυλίνδρου. λελείφθω, καὶ ἔστω τὰ ἐπὶ τῶν ΑΕ, ΕΒ, ΒΖ, ΖΓ, ΓΗ, ΗΔ, ΔΘ, ΘΑ· λοιπὴ ἄρα ἡ πυραμίς, ἧς βάσις μέν ἐστι τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, κορυφὴ δὲ ἡ αὐτὴ τῷ κώνῳ, μείζων ἐστὶν ἢ τρίτον μέρος τοῦ κυλίνδρου. ἀλλ’ ἡ πυραμίς, ἧς βάσις μέν ἐστι τὸ ΑΕΒΖΓ ΗΔΘ πολύγωνον, κορυφὴ δὲ ἡ αὐτὴ

τῷ κώνῳ, τρίτον ἐστὶ μέρος τοῦ πρίσματος, οὗ βάσις μέν ἐστι τὸ ΑΕΒΖΓ ΗΔΘ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ κυλίνδρῳ· τὸ ἄρα πρίσμα, οὗ βάσις μέν ἐστι τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ κυλίνδρῳ, μεῖζόν ἐστι τοῦ κυλίνδρου, οὗ βάσις ἐστὶν ὁ ΑΒΓΔ κύκλος. ἀλλὰ καὶ ἔλαττον· ἐμπεριέχεται γὰρ ὑπ᾽ αὐτοῦ· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ὁ κύλινδρος τοῦ κώνου ἐλάττων ἐστὶν ἢ τριπλάσιος. ἐδείχθη δέ, ὅτι οὐδὲ μείζων ἢ τριπλάσιος· τριπλάσιος ἄρα ὁ κύλινδρος τοῦ κώνου· ὥστε ὁ κῶνος τρίτον ἐστὶ μέρος τοῦ κυλίνδρου.

Πᾶς ἄρα κῶνος κυλίνδρου τρίτον μέρος ἐστὶ τοῦ τὴν αὐτὴν βάσιν ἔχοντος αὐτῷ καὶ ὕψος ἴσον· ὅπερ ἔδει δεῖξαι.

ια΄. Οἱ ὑπὸ τὸ αὐτὸ ὕψος ὄντες κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις.

Ἔστωσαν ὑπὸ τὸ αὐτὸ ὕψος κῶνοι καὶ κύλινδροι, ὧν βάσεις μὲν [εἰσιν] οἱ ΑΒΓΔ, ΕΖΗΘ κύκλοι, ἄξονες δὲ οἱ ΚΛ, ΜΝ, διάμετροι δὲ τῶν βάσεων αἱ ΑΓ, ΕΗ· λέγω, ὅτι ἐστὶν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς τὸν ΕΝ κῶνον.

Εἰ γὰρ μή, ἔσται ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος ἤτοι πρὸς ἔλασσόν τι τοῦ ΕΝ κώνου στερεὸν ἢ πρὸς μεῖζον. ἔστω πρότερον πρὸς ἔλασσον τὸ Ξ, καὶ ᾧ ἔλασσόν ἐστι τὸ Ξ στερεὸν τοῦ ΕΝ κώνου, ἐκείνῳ ἴσον ἔστω τὸ Ψ στερεόν· ὁ ΕΝ κῶνος ἄρα ἴσος ἐστὶ τοῖς Ξ, Ψ στερεοῖς. ἐγγεγράφθω εἰς τὸν ΕΖΗΘ κύκλον τετράγωνον τὸ ΕΖΗΘ· τὸ ἄρα τετράγωνον μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ κύκλου. ἀνεστάτω ἀπὸ τοῦ ΕΖ ΗΘ τετραγώνου πυραμὶς ἰσοϋψὴς τῷ κώνῳ· ἡ ἄρα ἀνασταθεῖσα πυραμὶς μείζων ἐστὶν ἢ τὸ ἥμισυ τοῦ κώνου, ἐπειδήπερ ἐὰν περιγράψωμεν περὶ τὸν κύκλον τετράγωνον, καὶ ἀπ᾽ αὐτοῦ ἀναστήσωμεν πυραμίδα ἰσουψῆ τῷ κώνῳ, ἡ ἐγγραφεῖσα πυραμὶς ἥμισύ ἐστι τῆς περιγραφείσης· πρὸς ἀλλήλας γάρ εἰσιν ὡς αἱ βάσεις· ἐλάττων δὲ ὁ κῶνος τῆς περιγραφείσης πυραμίδος.

τετμήσθωσαν αἱ ΕΖ, ΖΗ, ΗΘ, ΘΕ περιφέρειαι δίχα κατὰ τὰ Ο, Π, Ρ, Σ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΘΟ, ΟΕ, ΕΠ, ΠΖ, ΖΡ, ΡΗ, ΗΣ, ΣΘ.

ἕκαστον ἄρα τῶν ΘΟΕ, ΕΠΖ, ΖΡΗ, ΗΣΘ τριγώνων μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ καθ' ἑαυτὸ τμήματος τοῦ κύκλου. ἀνεστάτω ἐφ' ἑκάστου τῶν ΘΟΕ, ΕΠΖ, ΖΡΗ, ΗΣΘ τριγώνων πυραμὶς ἰσοϋψὴς τῷ κώνῳ· καὶ ἑκάστη ἄρα τῶν ἀνασταθεισῶν πυραμίδων μείζων ἐστὶν ἢ τὸ ἥμισυ τοῦ καθ' ἑαυτὴν τμήματος τοῦ κώνου. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ ἀνιστάντες ἐπὶ ἑκάστου τῶν τριγώνων πυραμίδας ἰσοϋψεῖς τῷ κώνῳ καὶ ἀεὶ τοῦτο ποιοῦντες καταλείψομέν τινα ἀποτμήματα τοῦ κώνου, ἃ ἔσται ἐλάσσονα τοῦ Ψ στερεοῦ. λελείφθω, καὶ ἔστω τὰ ἐπὶ τῶν ΘΟΕ, ΕΠΖ, ΖΡΗ, ΗΣΘ· λοιπὴ ἄρα ἡ πυραμίς, ἧς βάσις τὸ ΘΟΕΠΖΡΗΣ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ κώνῳ, μείζων ἐστὶ τοῦ Ξ στερεοῦ. ἐγγεγράφθω καὶ εἰς τὸν ΑΒΓΔ κύκλον τῷ ΘΟΕΠΖΡΗΣ πολυγώνῳ ὅμοιόν τε καὶ ὁμοίως κείμενον πολύγωνον τὸ ΔΤΑΥΒ ΦΓΧ, καὶ ἀνεστάτω ἐπ' αὐτοῦ πυραμὶς ἰσοϋψὴς τῷ ΑΛ κώνῳ. ἐπεὶ οὖν ἐστιν ὡς τὸ ἀπὸ τῆς ΑΓ πρὸς τὸ ἀπὸ τῆς ΕΗ, οὕτως τὸ ΔΤΑΥΒΦΓΧ πολύγωνον πρὸς τὸ ΘΟΕ ΠΖΡΗΣ πολύγωνον, ὡς δὲ τὸ ἀπὸ τῆς ΑΓ πρὸς τὸ ἀπὸ τῆς ΕΗ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, καὶ ὡς ἄρα ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως τὸ ΔΤΑΥΒΦΓΧ πολύγωνον πρὸς τὸ ΘΟΕΠΖΡΗΣ πολύγωνον. ὡς δὲ ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς τὸ Ξ στερεόν, ὡς δὲ τὸ ΔΤΑΥΒΦΓΧ πολύγωνον πρὸς τὸ ΘΟΕΠΖ ΡΗΣ πολύγωνον, οὕτως ἡ πυραμίς, ἧς βάσις μὲν τὸ ΔΤΑΥΒΦΓΧ πολύγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὴν πυραμίδα, ἧς βάσις μὲν τὸ ΘΟΕΠΖΡΗΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον. καὶ ὡς ἄρα ὁ ΑΛ κῶνος πρὸς τὸ Ξ στερεόν, οὕτως ἡ πυραμίς, ἧς βάσις μὲν τὸ ΔΤΑΥΒΦΓΧ πολύγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὴν πυραμίδα, ἧς βάσις μὲν τὸ ΘΟΕΠΖΡΗΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον· ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ ΑΛ κῶνος πρὸς τὴν ἐν αὐτῷ πυραμίδα, οὕτως τὸ Ξ στερεὸν πρὸς τὴν ἐν τῷ ΕΝ κώνῳ

πυραμίδα. μείζων δὲ ὁ ΑΛ κῶνος τῆς ἐν αὐτῷ πυραμίδος· μεῖζον ἄρα καὶ τὸ Ξ στερεὸν τῆς ἐν τῷ ΕΝ κώνῳ πυραμίδος. ἀλλὰ καὶ ἔλασσον· ὅπερ ἄτοπον. οὐκ ἄρα ἐστὶν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς ἔλασσόν τι τοῦ ΕΝ κώνου στερεόν. ὁμοίως δὴ δείξομεν, ὅτι οὐδέ ἐστιν ὡς ὁ ΕΖΗΘ κύκλος πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΝ κῶνος πρὸς ἔλασσόν τι τοῦ ΑΛ κώνου στερεόν.

Λέγω δή, ὅτι οὐδέ ἐστιν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς μεῖζόν τι τοῦ ΕΝ κώνου στερεόν.

Εἰ γὰρ δυνατόν, ἔστω πρὸς μεῖζον τὸ Ξ· ἀνάπαλιν ἄρα ἐστὶν ὡς ὁ ΕΖΗΘ κύκλος πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως τὸ Ξ στερεὸν πρὸς τὸν ΑΛ κῶνον. ἀλλ’ ὡς τὸ Ξ στερεὸν πρὸς τὸν ΑΛ κῶνον, οὕτως ὁ ΕΝ κῶνος πρὸς ἔλασσόν τι τοῦ ΑΛ κώνου στερεόν· καὶ ὡς ἄρα ὁ ΕΖΗΘ κύκλος πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΝ κῶνος πρὸς ἔλασσόν τι τοῦ ΑΛ κώνου στερεόν· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ἐστὶν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς μεῖζόν τι τοῦ ΕΝ κώνου στερεόν. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἔλασσον· ἔστιν ἄρα ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς τὸν ΕΝ κῶνον.

Ἀλλ’ ὡς ὁ κῶνος πρὸς τὸν κῶνον, ὁ κύλινδρος πρὸς τὸν κύλινδρον· τριπλασίων γὰρ ἑκάτερος ἑκατέρου. καὶ ὡς ἄρα ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως οἱ ἐπ’ αὐτῶν ἰσοϋψεῖς [τοῖς κώνοις] κύλινδροι.

Οἱ ἄρα ὑπὸ τὸ αὐτὸ ὕψος ὄντες κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις· ὅπερ ἔδει δεῖξαι.

ιβ΄. Οἱ ὅμοιοι κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ἐν ταῖς βάσεσι διαμέτρων.

Ἔστωσαν ὅμοιοι κῶνοι καὶ κύλινδροι, ὧν βάσεις μὲν οἱ ΑΒΓΔ, ΕΖΗΘ κύκλοι, διάμετροι δὲ τῶν βάσεων αἱ ΒΔ, ΖΘ, ἄξονες δὲ τῶν κώνων καὶ κυλίνδρων οἱ ΚΛ, ΜΝ· λέγω, ὅτι ὁ κῶνος, οὗ βάσις μὲν [ἐστιν] ὁ ΑΒΓΔ

κύκλος, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὸν κῶνον, οὗ βάσις μὲν [ἐστιν] ὁ ΕΖΗΘ κύκλος, κορυφὴ δὲ τὸ Ν σημεῖον, τριπλασίονα λόγον ἔχει ἥπερ ἡ ΒΔ πρὸς τὴν ΖΘ.

Εἰ γὰρ μὴ ἔχει ὁ ΑΒΓΔΛ κῶνος πρὸς τὸν ΕΖΗΘΝ κῶνον τριπλασίονα λόγον ἥπερ ἡ ΒΔ πρὸς τὴν ΖΘ, ἕξει ὁ ΑΒΓΔΛ κῶνος ἢ πρὸς ἔλασσόν τι τοῦ ΕΖΗΘΝ κώνου στερεὸν τριπλασίονα λόγον ἢ πρὸς μεῖζον. ἐχέτω πρότερον πρὸς ἔλασσον τὸ Ξ, καὶ ἐγγεγράφθω εἰς τὸν ΕΖΗΘ κύκλον τετράγωνον τὸ ΕΖΗΘ· τὸ ἄρα ΕΖΗΘ τετράγωνον μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ ΕΖΗΘ κύκλου. καὶ ἀνεστάτω ἐπὶ τοῦ ΕΖΗΘ τετραγώνου πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κώνῳ· ἡ ἄρα ἀνασταθεῖσα πυραμὶς μείζων ἐστὶν ἢ τὸ ἥμισυ μέρος τοῦ κώνου. τετμήσθωσαν δὴ αἱ ΕΖ, ΖΗ, ΗΘ, ΘΕ περιφέρειαι δίχα κατὰ τὰ Ο, Π, Ρ, Σ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΕΟ, ΟΖ, ΖΠ, ΠΗ, ΗΡ, ΡΘ, ΘΣ, ΣΕ. καὶ ἕκαστον ἄρα τῶν ΕΟΖ, ΖΠΗ, ΗΡΘ, ΘΣΕ τριγώνων μεῖζόν ἐστιν ἢ τὸ ἥμισυ μέρος τοῦ καθ’ ἑαυτὸ τμήματος τοῦ ΕΖΗΘ κύκλου. καὶ ἀνεστάτω ἐφ’ ἑκάστου τῶν ΕΟΖ, ΖΠΗ, ΗΡΘ, ΘΣΕ τριγώνων πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κώνῳ· καὶ ἑκάστη ἄρα τῶν ἀνασταθεισῶν πυραμίδων μείζων ἐστὶν ἢ τὸ ἥμισυ μέρος τοῦ καθ’ ἑαυτὴν τμήματος τοῦ κώνου. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ ἀνιστάντες ἐφ’ ἑκάστου τῶν τριγώνων πυραμίδας τὴν αὐτὴν κορυφὴν ἐχούσας τῷ κώνῳ καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομέν τινα ἀποτμήματα τοῦ κώνου, ἃ ἔσται ἐλάσσονα τῆς ὑπεροχῆς, ᾗ ὑπερέχει ὁ ΕΖΗΘΝ κῶνος τοῦ Ξ στερεοῦ.

λελείφθω, καὶ ἔστω τὰ ἐπὶ τῶν ΕΟ, ΟΖ, ΖΠ, ΠΗ, ΗΡ, ΡΘ, ΘΣ, ΣΕ· λοιπὴ ἄρα ἡ πυραμίς, ἧς βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον, μείζων ἐστὶ τοῦ Ξ στερεοῦ. ἐγγεγράφθω καὶ εἰς τὸν ΑΒΓΔ κύκλον τῷ ΕΟΖΠΗΡΘΣ πολυγώνῳ ὅμοιόν τε καὶ ὁμοίως κείμενον πολύγωνον τὸ ΑΤΒΥΓΦΔΧ, καὶ ἀνεστάτω ἐπὶ τοῦ ΑΤΒΥΓΦΔΧ πολυγώνου πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κώνῳ, καὶ τῶν μὲν περιεχόντων τὴν

πυραμίδα, ἧς βάσις μέν ἐστι τὸ ΑΤΒΥ ΓΦΔΧ πολύγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, ἓν τρίγωνον ἔστω τὸ ΛΒΤ, τῶν δὲ περιεχόντων τὴν πυραμίδα, ἧς βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον, ἓν τρίγωνον ἔστω τὸ ΝΖΟ, καὶ ἐπεζεύχθωσαν αἱ ΚΤ, ΜΟ. καὶ ἐπεὶ ὅμοιός ἐστιν ὁ ΑΒΓΔΛ κῶνος τῷ ΕΖΗΘΝ κώνῳ, ἔστιν ἄρα ὡς ἡ ΒΔ πρὸς τὴν ΖΘ, οὕτως ὁ ΚΛ ἄξων πρὸς τὸν ΜΝ ἄξονα. ὡς δὲ ἡ ΒΔ πρὸς τὴν ΖΘ, οὕτως ἡ ΒΚ πρὸς τὴν ΖΜ· καὶ ὡς ἄρα ἡ ΒΚ πρὸς τὴν ΖΜ, οὕτως ἡ ΚΛ πρὸς τὴν ΜΝ. καὶ ἐναλλὰξ ὡς ἡ ΒΚ πρὸς τὴν ΚΛ, οὕτως ἡ ΖΜ πρὸς τὴν ΜΝ. καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΒΚΛ, ΖΜΝ αἱ πλευραὶ ἀνάλογόν εἰσιν· ὅμοιον ἄρα ἐστὶ τὸ ΒΚΛ τρίγωνον τῷ ΖΜΝ τριγώνῳ. πάλιν, ἐπεί ἐστιν ὡς ἡ ΒΚ πρὸς τὴν ΚΤ, οὕτως ἡ ΖΜ πρὸς τὴν ΜΟ, καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΒΚΤ, ΖΜΟ, ἐπειδήπερ, ὃ μέρος ἐστὶν ἡ ὑπὸ ΒΚΤ γωνία τῶν πρὸς τῷ Κ κέντρῳ τεσσάρων ὀρθῶν, τὸ αὐτὸ μέρος ἐστὶ καὶ ἡ ὑπὸ ΖΜΟ γωνία τῶν πρὸς τῷ Μ κέντρῳ τεσσάρων ὀρθῶν· ἐπεὶ οὖν περὶ ἴσας γωνίας αἱ πλευραὶ ἀνάλογόν εἰσιν, ὅμοιον ἄρα ἐστὶ τὸ ΒΚΤ τρίγωνον τῷ ΖΜΟ τριγώνῳ. πάλιν, ἐπεὶ ἐδείχθη ὡς ἡ ΒΚ πρὸς τὴν ΚΛ, οὕτως ἡ ΖΜ πρὸς τὴν ΜΝ, ἴση δὲ ἡ μὲν ΒΚ τῇ ΚΤ, ἡ δὲ ΖΜ τῇ ΟΜ, ἔστιν ἄρα ὡς ἡ ΤΚ πρὸς τὴν ΚΛ, οὕτως ἡ ΟΜ πρὸς τὴν ΜΝ. καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΤΚΛ, ΟΜΝ· ὀρθαὶ γάρ· αἱ πλευραὶ ἀνάλογόν εἰσιν· ὅμοιον ἄρα ἐστὶ τὸ ΛΚΤ τρίγωνον τῷ ΝΜΟ τριγώνῳ. καὶ ἐπεὶ διὰ τὴν ὁμοιότητα τῶν ΛΚΒ, ΝΜΖ τριγώνων ἐστὶν ὡς ἡ ΛΒ πρὸς τὴν ΒΚ, οὕτως ἡ ΝΖ πρὸς τὴν ΖΜ, διὰ δὲ τὴν ὁμοιότητα τῶν ΒΚΤ, ΖΜΟ τριγώνων ἐστὶν ὡς ἡ ΚΒ πρὸς τὴν ΒΤ, οὕτως ἡ ΜΖ πρὸς τὴν ΖΟ, δι’ ἴσου ἄρα ὡς ἡ ΛΒ πρὸς τὴν ΒΤ, οὕτως ἡ ΝΖ πρὸς τὴν ΖΟ. πάλιν, ἐπεὶ διὰ τὴν ὁμοιότητα τῶν ΛΤΚ, ΝΟΜ τριγώνων ἐστὶν ὡς ἡ ΛΤ πρὸς τὴν ΤΚ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΜ, διὰ δὲ τὴν ὁμοιότητα τῶν ΤΚΒ, ΟΜΖ τριγώνων ἐστὶν ὡς ἡ ΚΤ πρὸς τὴν ΤΒ, οὕτως ἡ ΜΟ πρὸς τὴν ΟΖ, δι’ ἴσου ἄρα ὡς ἡ ΛΤ πρὸς τὴν ΤΒ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΖ. ἐδείχθη δὲ καὶ ὡς ἡ ΤΒ πρὸς τὴν ΒΛ, οὕτως ἡ ΟΖ πρὸς τὴν ΖΝ. δι’ ἴσου ἄρα ὡς ἡ ΤΛ πρὸς τὴν ΛΒ, οὕτως ἡ ΟΝ πρὸς τὴν ΝΖ. τῶν ΛΤΒ,

ΝΟΖ ἄρα τριγώνων ἀνάλογόν εἰσιν αἱ πλευραί· ἰσογώνια ἄρα ἐστὶ τὰ ΛΤΒ, ΝΟΖ τρίγωνα· ὥστε καὶ ὅμοια. καὶ πυραμὶς ἄρα, ἧς βάσις μὲν τὸ ΒΚΤ τρίγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, ὁμοία ἐστὶ πυραμίδι, ἧς βάσις μὲν τὸ ΖΜΟ τρίγωνον, κορυφὴ δὲ τὸ Ν σημεῖον· ὑπὸ γὰρ ὁμοίων ἐπιπέδων περιέχονται ἴσων τὸ πλῆθος. αἱ δὲ ὅμοιαι πυραμίδες καὶ τριγώνους ἔχουσαι βάσεις ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν. ἡ ἄρα ΒΚΤΛ πυραμὶς πρὸς τὴν ΖΜΟΝ πυραμίδα τριπλασίονα λόγον ἔχει ἥπερ ἡ ΒΚ πρὸς τὴν ΖΜ. ὁμοίως δὴ ἐπιζευγνύντες ἀπὸ τῶν Α, Χ, Δ, Φ, Γ, Υ ἐπὶ τὸ Κ εὐθείας καὶ ἀπὸ τῶν Ε, Σ, Θ, Ρ, Η, Π ἐπὶ τὸ Μ καὶ ἀνιστάντες ἐφ’ ἑκάστου τῶν τριγώνων πυραμίδας τὴν αὐτὴν κορυφὴν ἐχούσας τοῖς κώνοις δείξομεν, ὅτι καὶ ἑκάστη τῶν ὁμοταγῶν πυραμίδων πρὸς ἑκάστην ὁμοταγῆ πυραμίδα τριπλασίονα λόγον ἕξει ἥπερ ἡ ΒΚ ὁμόλογος πλευρὰ πρὸς τὴν ΖΜ ὁμόλογον πλευράν, τουτέστιν ἥπερ ἡ ΒΔ πρὸς τὴν ΖΘ. καὶ ὡς ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα· ἔστιν ἄρα καὶ ὡς ἡ ΒΚΤΛ πυραμὶς πρὸς τὴν ΖΜΟΝ πυραμίδα, οὕτως ἡ ὅλη πυραμίς, ἧς βάσις τὸ ΑΤΒΥΓΦΔΧ πολύγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὴν ὅλην πυραμίδα, ἧς βάσις μὲν τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον· ὥστε καὶ πυραμίς, ἧς βάσις μὲν τὸ ΑΤΒΥΓΦΔΧ, κορυφὴ δὲ τὸ Λ, πρὸς τὴν πυραμίδα, ἧς βάσις [μὲν] τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον, τριπλασίονα λόγον ἔχει ἥπερ ἡ ΒΔ πρὸς τὴν ΖΘ. ὑπόκειται δὲ καὶ ὁ κῶνος, οὗ βάσις [μὲν] ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὸ Ξ στερεὸν τριπλασίονα λόγον ἔχων ἥπερ ἡ ΒΔ πρὸς τὴν ΖΘ· ἔστιν ἄρα ὡς ὁ κῶνος, οὗ βάσις μέν ἐστιν ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ, πρὸς τὸ Ξ στερεόν, οὕτως ἡ πυραμίς, ἧς βάσις μὲν τὸ ΑΤΒΥΓΦΔΧ [πολύγωνον], κορυφὴ δὲ τὸ Λ, πρὸς τὴν πυραμίδα, ἧς βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν· ἐναλλὰξ ἄρα, ὡς ὁ κῶνος, οὗ βάσις μὲν ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ, πρὸς τὴν ἐν αὐτῷ πυραμίδα, ἧς βάσις μὲν τὸ ΑΤΒΥΓΦΔΧ πολύγωνον, κορυφὴ δὲ τὸ Λ, οὕτως τὸ Ξ

[στερεὸν] πρὸς τὴν πυραμίδα, ἧς βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν. μείζων δὲ ὁ εἰρημένος κῶνος τῆς ἐν αὐτῷ πυραμίδος· ἐμπεριέχει γὰρ αὐτήν. μεῖζον ἄρα καὶ τὸ Ξ στερεὸν τῆς πυραμίδος, ἧς βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν. ἀλλὰ καὶ ἔλαττον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ὁ κῶνος, οὗ βάσις ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ [σημεῖον], πρὸς ἔλαττόν τι τοῦ κώνου στερεόν, οὗ βάσις μὲν ὁ ΕΖΗΘ κύκλος, κορυφὴ δὲ τὸ Ν σημεῖον, τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ὁ ΕΖΗΘΝ κῶνος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔΛ κώνου στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΖΘ πρὸς τὴν ΒΔ.

Λέγω δή, ὅτι οὐδὲ ὁ ΑΒΓΔΛ κῶνος πρὸς μεῖζόν τι τοῦ ΕΖΗΘΝ κώνου στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ.

Εἰ γὰρ δυνατόν, ἐχέτω πρὸς μεῖζον τὸ Ξ. ἀνάπαλιν ἄρα τὸ Ξ στερεὸν πρὸς τὸν ΑΒΓΔΛ κῶνον τριπλασίονα λόγον ἔχει ἤπερ ἡ ΖΘ πρὸς τὴν ΒΔ. ὡς δὲ τὸ Ξ στερεὸν πρὸς τὸν ΑΒΓΔΛ κῶνον, οὕτως ὁ ΕΖΗΘΝ κῶνος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔΛ κώνου στερεόν. καὶ ὁ ΕΖΗΘΝ ἄρα κῶνος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔΛ κώνου στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΖΘ πρὸς τὴν ΒΔ· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ὁ ΑΒΓΔΛ κῶνος πρὸς μεῖζόν τι τοῦ ΕΖΗΘΝ κώνου στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἔλαττον. ὁ ΑΒΓΔΛ ἄρα κῶνος πρὸς τὸν ΕΖΗΘΝ κῶνον τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ.

Ὡς δὲ ὁ κῶνος πρὸς τὸν κῶνον, ὁ κύλινδρος πρὸς τὸν κύλινδρον· τριπλάσιος γὰρ ὁ κύλινδρος τοῦ κώνου ὁ ἐπὶ τῆς αὐτῆς βάσεως τῷ κώνῳ καὶ ἰσοϋψὴς αὐτῷ. καὶ ὁ κύλινδρος ἄρα πρὸς τὸν κύλινδρον τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ.

Οἱ ἄρα ὅμοιοι κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ἐν ταῖς βάσεσι διαμέτρων· ὅπερ ἔδει δεῖξαι.

ιγ΄. Ἐὰν κύλινδρος ἐπιπέδῳ τμηθῇ παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔσται ὡς ὁ κύλινδρος πρὸς τὸν κύλινδρον, οὕτως ὁ ἄξων πρὸς τὸν ἄξονα.

Κύλινδρος γὰρ ὁ ΑΔ ἐπιπέδῳ τῷ ΗΘ τετμήσθω παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις τοῖς ΑΒ, ΓΔ, καὶ συμβαλλέτω τῷ ἄξονι τὸ ΗΘ ἐπίπεδον κατὰ τὸ Κ σημεῖον· λέγω, ὅτι ἐστὶν ὡς ὁ ΒΗ κύλινδρος πρὸς τὸν ΗΔ κύλινδρον, οὕτως ὁ ΕΚ ἄξων πρὸς τὸν ΚΖ ἄξονα.

Ἐκβεβλήσθω γὰρ ὁ ΕΖ ἄξων ἐφ' ἑκάτερα τὰ μέρη ἐπὶ τὰ Λ, Μ σημεῖα, καὶ ἐκκείσθωσαν τῷ ΕΚ ἄξονι ἴσοι ὁσοιδηποτοῦν οἱ ΕΝ, ΝΛ, τῷ δὲ ΖΚ ἴσοι ὁσοιδηποτοῦν οἱ ΖΞ, ΞΜ, καὶ νοείσθω ὁ ἐπὶ τοῦ ΛΜ ἄξονος κύλινδρος ὁ ΟΧ, οὗ βάσεις οἱ ΟΠ, ΦΧ κύκλοι. καὶ ἐκβεβλήσθω διὰ τῶν Ν, Ξ σημείων ἐπίπεδα παράλληλα τοῖς ΑΒ, ΓΔ καὶ ταῖς βάσεσι τοῦ ΟΧ κυλίνδρου καὶ ποιείτωσαν τοὺς ΡΣ, ΤΥ κύκλους περὶ τὰ Ν, Ξ κέντρα. καὶ ἐπεὶ οἱ ΛΝ, ΝΕ, ΕΚ ἄξονες ἴσοι εἰσὶν ἀλλήλοις, οἱ ἄρα ΠΡ, ΡΒ, ΒΗ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις. ἴσαι δέ εἰσιν αἱ βάσεις· ἴσοι ἄρα καὶ οἱ ΠΡ, ΡΒ, ΒΗ κύλινδροι ἀλλήλοις.

ἐπεὶ οὖν οἱ ΛΝ, ΝΕ, ΕΚ ἄξονες ἴσοι εἰσὶν ἀλλήλοις, εἰσὶ δὲ καὶ οἱ ΠΡ, ΡΒ, ΒΗ κύλινδροι ἴσοι ἀλλήλοις, καί ἐστιν ἴσον τὸ πλῆθος τῷ πλήθει, ὁσαπλασίων ἄρα ὁ ΚΛ ἄξων τοῦ ΕΚ ἄξονος, τοσαυταπλασίων ἔσται καὶ ὁ ΠΗ κύλινδρος τοῦ ΗΒ κυλίνδρου. διὰ τὰ αὐτὰ δὴ καὶ ὁσαπλασίων ἐστὶν ὁ ΜΚ ἄξων τοῦ ΚΖ ἄξονος, τοσαυταπλασίων ἐστὶ καὶ ὁ ΧΗ κύλινδρος τοῦ ΗΔ κυλίνδρου. καὶ εἰ μὲν ἴσος ἐστὶν ὁ ΚΛ ἄξων τῷ ΚΜ ἄξονι, ἴσος ἔσται καὶ ὁ ΠΗ κύλινδρος τῷ ΗΧ κυλίνδρῳ, εἰ δὲ μείζων ὁ ἄξων τοῦ ἄξονος, μείζων καὶ ὁ κύλινδρος τοῦ κυλίνδρου, καὶ εἰ ἐλάσσων, ἐλάσσων. τεσσάρων δὴ μεγεθῶν ὄντων, ἀξόνων μὲν τῶν ΕΚ, ΚΖ, κυλίνδρων δὲ τῶν ΒΗ, ΗΔ, εἴληπται ἰσάκις πολλαπλάσια, τοῦ μὲν ΕΚ ἄξονος καὶ τοῦ ΒΗ κυλίνδρου ὅ τε ΛΚ ἄξων καὶ ὁ ΠΗ κύλινδρος, τοῦ δὲ ΚΖ ἄξονος καὶ τοῦ ΗΔ κυλίνδρου ὅ τε ΚΜ ἄξων καὶ ὁ ΗΧ κύλινδρος, καὶ δέδεικται, ὅτι εἰ ὑπερέχει ὁ ΚΛ ἄξων τοῦ ΚΜ ἄξονος, ὑπερέχει καὶ ὁ ΠΗ κύλινδρος τοῦ ΗΧ κυλίνδρου, καὶ εἰ ἴσος, ἴσος, καὶ εἰ ἐλάσσων, ἐλάσσων. ἔστιν ἄρα ὡς ὁ ΕΚ ἄξων πρὸς τὸν ΚΖ ἄξονα, οὕτως ὁ ΒΗ κύλινδρος πρὸς τὸν ΗΔ κύλινδρον· ὅπερ ἔδει δεῖξαι.

ιδ΄. Οἱ ἐπὶ ἴσων βάσεων ὄντες κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ὕψη.

Ἔστωσαν γὰρ ἐπὶ ἴσων βάσεων τῶν ΑΒ, ΓΔ κύκλων κύλινδροι οἱ ΕΒ, ΖΔ· λέγω, ὅτι ἐστὶν ὡς ὁ ΕΒ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον, οὕτως ὁ ΗΘ ἄξων πρὸς τὸν ΚΛ ἄξονα.

Ἐκβεβλήσθω γὰρ ὁ ΚΛ ἄξων ἐπὶ τὸ Ν σημεῖον, καὶ κείσθω τῷ ΗΘ ἄξονι ἴσος ὁ ΛΝ, καὶ περὶ ἄξονα τὸν ΛΝ κύλινδρος νενοήσθω ὁ ΓΜ. ἐπεὶ οὖν οἱ ΕΒ, ΓΜ κύλινδροι ὑπὸ τὸ αὐτὸ ὕψος εἰσίν, πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις. ἴσαι δέ εἰσιν αἱ βάσεις ἀλλήλαις· ἴσοι ἄρα εἰσὶ καὶ οἱ ΕΒ, ΓΜ κύλινδροι. καὶ ἐπεὶ κύλινδρος ὁ ΖΜ ἐπιπέδῳ τέτμηται τῷ ΓΔ παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ἄρα ὡς ὁ ΓΜ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον, οὕτως ὁ ΛΝ ἄξων πρὸς τὸν ΚΛ ἄξονα. ἴσος δέ ἐστιν ὁ μὲν ΓΜ κύλινδρος τῷ ΕΒ κυλίνδρῳ, ὁ δὲ ΛΝ ἄξων τῷ ΗΘ ἄξονι· ἔστιν ἄρα ὡς ὁ ΕΒ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον, οὕτως ὁ ΗΘ ἄξων πρὸς τὸν ΚΛ ἄξονα. ὡς δὲ ὁ ΕΒ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον, οὕτως ὁ ΑΒΗ κῶνος πρὸς τὸν ΓΔΚ κῶνον. καὶ ὡς ἄρα ὁ ΗΘ ἄξων πρὸς τὸν ΚΛ ἄξονα, οὕτως ὁ ΑΒΗ κῶνος πρὸς τὸν ΓΔΚ κῶνον καὶ ὁ ΕΒ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον· ὅπερ ἔδει δεῖξαι.

ιε΄. Τῶν ἴσων κώνων καὶ κυλίνδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ὧν κώνων καὶ κυλίνδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσοι εἰσὶν ἐκεῖνοι.

Ἔστωσαν ἴσοι κῶνοι καὶ κύλινδροι, ὧν βάσεις μὲν οἱ ΑΒΓΔ, ΕΖΗΘ κύκλοι, διάμετροι δὲ αὐτῶν αἱ ΑΓ, ΕΗ, ἄξονες δὲ οἱ ΚΛ, ΜΝ, οἴτινες καὶ ὕψη εἰσὶ τῶν κώνων ἢ κυλίνδρων, καὶ συμπεπληρώσθωσαν οἱ ΑΞ, ΕΟ κύλινδροι. λέγω, ὅτι τῶν ΑΞ, ΕΟ κυλίνδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς

ὕψεσιν, καί ἐστιν ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος.

Τὸ γὰρ ΛΚ ὕψος τῷ ΜΝ ὕψει ἤτοι ἴσον ἐστὶν ἢ οὔ. ἔστω πρότερον ἴσον. ἔστι δὲ καὶ ὁ ΑΞ κύλινδρος τῷ ΕΟ κυλίνδρῳ ἴσος. οἱ δὲ ὑπὸ τὸ αὐτὸ ὕψος ὄντες κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις· ἴση ἄρα καὶ ἡ ΑΒΓΔ βάσις τῇ ΕΖΗΘ βάσει. ὥστε καὶ ἀντιπέπονθεν, ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος.

ἀλλὰ δὴ μὴ ἔστω τὸ ΛΚ ὕψος τῷ ΜΝ ἴσον, ἀλλ’ ἔστω μεῖζον τὸ ΜΝ, καὶ ἀφῃρήσθω ἀπὸ τοῦ ΜΝ ὕψους τῷ ΚΛ ἴσον τὸ ΠΝ, καὶ διὰ τοῦ Π σημείου τετμήσθω ὁ ΕΟ κύλινδρος ἐπιπέδῳ τῷ ΤΥΣ παραλλήλῳ τοῖς τῶν ΕΖΗΘ, ΡΟ κύκλων ἐπιπέδοις, καὶ ἀπὸ βάσεως μὲν τοῦ ΕΖΗΘ κύκλου, ὕψους δὲ τοῦ ΝΠ κύλινδρος νενοήσθω ὁ ΕΣ. καὶ ἐπεὶ ἴσος ἐστὶν ὁ ΑΞ κύλινδρος τῷ ΕΟ κυλίνδρῳ, ἔστιν ἄρα ὡς ὁ ΑΞ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον, οὕτως ὁ ΕΟ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον. ἀλλ’ ὡς μὲν ὁ ΑΞ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον, οὕτως ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ· ὑπὸ γὰρ τὸ αὐτὸ ὕψος εἰσὶν οἱ ΑΞ, ΕΣ κύλινδροι· ὡς δὲ ὁ ΕΟ κύλινδρος πρὸς τὸν ΕΣ, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΠΝ ὕψος· ὁ γὰρ ΕΟ κύλινδρος ἐπιπέδῳ τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις. ἔστιν ἄρα καὶ ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΠΝ ὕψος. ἴσον δὲ τὸ ΠΝ ὕψος τῷ ΚΛ ὕψει· ἔστιν ἄρα ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος. τῶν ἄρα ΑΞ, ΕΟ κυλίνδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Ἀλλὰ δὴ τῶν ΑΞ, ΕΟ κυλίνδρων ἀντιπεπονθέτωσαν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος· λέγω, ὅτι ἴσος ἐστὶν ὁ ΑΞ κύλινδρος τῷ ΕΟ κυλίνδρῳ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ἐπεί ἐστιν ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος, ἴσον δὲ τὸ ΚΛ ὕψος τῷ ΠΝ ὕψει, ἔστιν ἄρα ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ

ὕψος πρὸς τὸ ΠΝ ὕψος. ἀλλ’ ὡς μὲν ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως ὁ ΑΞ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον· ὑπὸ γὰρ τὸ αὐτὸ ὕψος εἰσίν· ὡς δὲ τὸ ΜΝ ὕψος πρὸς τὸ ΠΝ [ὕψος], οὕτως ὁ ΕΟ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον· ἔστιν ἄρα ὡς ὁ ΑΞ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον, οὕτως ὁ ΕΟ κύλινδρος πρὸς τὸν ΕΣ. ἴσος ἄρα ὁ ΑΞ κύλινδρος τῷ ΕΟ κυλίνδρῳ. ὡσαύτως δὲ καὶ ἐπὶ τῶν κώνων· ὅπερ ἔδει δεῖξαι.

ις΄. Δύο κύκλων περὶ τὸ αὐτὸ κέντρον ὄντων εἰς τὸν μείζονα κύκλον πολύγωνον ἰσόπλευρόν τε καὶ ἀρτιόπλευρον ἐγγράψαι μὴ ψαῦον τοῦ ἐλάσσονος κύκλου.

Ἔστωσαν οἱ δοθέντες δύο κύκλοι οἱ ΑΒΓΔ, ΕΖΗΘ περὶ τὸ αὐτὸ κέντρον τὸ Κ· δεῖ δὴ εἰς τὸν μείζονα κύκλον τὸν ΑΒΓΔ πολύγωνον ἰσόπλευρόν τε καὶ ἀρτιόπλευρον ἐγγράψαι μὴ ψαῦον τοῦ ΕΖΗΘ κύκλου.

Ἤχθω γὰρ διὰ τοῦ Κ κέντρου εὐθεῖα ἡ ΒΚΔ, καὶ ἀπὸ τοῦ Η σημείου τῇ ΒΔ εὐθείᾳ πρὸς ὀρθὰς ἤχθω ἡ ΗΑ καὶ διήχθω ἐπὶ τὸ Γ· ἡ ΑΓ ἄρα ἐφάπτεται τοῦ ΕΖΗΘ κύκλου. τέμνοντες δὴ τὴν ΒΑΔ περιφέρειαν δίχα καὶ τὴν ἡμίσειαν αὐτῆς δίχα καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομεν περιφέρειαν ἐλάσσονα τῆς ΑΔ. λελείφθω, καὶ ἔστω ἡ ΛΔ, καὶ ἀπὸ τοῦ Λ ἐπὶ τὴν ΒΔ κάθετος ἤχθω ἡ ΛΜ καὶ διήχθω ἐπὶ τὸ Ν, καὶ ἐπεζεύχθωσαν αἱ ΛΔ, ΔΝ· ἴση ἄρα ἐστὶν ἡ ΛΔ τῇ ΔΝ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΛΝ τῇ ΑΓ, ἡ δὲ ΑΓ ἐφάπτεται τοῦ ΕΖΗΘ κύκλου, ἡ ΛΝ ἄρα οὐκ ἐφάπτεται τοῦ ΕΖΗΘ κύκλου· πολλῷ ἄρα αἱ ΛΔ, ΔΝ οὐκ ἐφάπτονται τοῦ ΕΖΗΘ κύκλου. ἐὰν δὴ τῇ ΛΔ εὐθείᾳ ἴσας κατὰ τὸ συνεχὲς ἐναρμόσωμεν εἰς τὸν ΑΒΓΔ κύκλον, ἐγγραφήσεται εἰς τὸν ΑΒΓΔ κύκλον πολύγωνον ἰσόπλευρόν τε καὶ ἀρτιόπλευρον μὴ ψαῦον τοῦ ἐλάσσονος κύκλου τοῦ ΕΖΗΘ· ὅπερ ἔδει ποιῆσαι.

ιζ΄. Δύο σφαιρῶν περὶ τὸ αὐτὸ κέντρον οὐσῶν εἰς τὴν μείζονα σφαῖραν στερεὸν πολύεδρον ἐγγράψαι μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν.

Νενοήσθωσαν δύο σφαῖραι περὶ τὸ αὐτὸ κέντρον τὸ Α· δεῖ δὴ εἰς τὴν μείζονα σφαῖραν στερεὸν πολύεδρον ἐγγράψαι μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν.

Τετμήσθωσαν αἱ σφαῖραι ἐπιπέδῳ τινὶ διὰ τοῦ κέντρου· ἔσονται δὴ αἱ τομαὶ κύκλοι, ἐπειδήπερ μενούσης τῆς διαμέτρου καὶ περιφερομένου τοῦ ἡμικυκλίου ἐγίγνετο ἡ σφαῖρα· ὥστε καὶ καθ' οἵας ἂν θέσεως ἐπινοήσωμεν τὸ ἡμικύκλιον, τὸ δι' αὐτοῦ ἐκβαλλόμενον ἐπίπεδον ποιήσει ἐπὶ τῆς ἐπιφανείας τῆς σφαίρας κύκλον. καὶ φανερόν, ὅτι καὶ μέγιστον, ἐπειδήπερ ἡ διάμετρος τῆς σφαίρας, ἥτις ἐστὶ καὶ τοῦ ἡμικυκλίου διάμετρος δηλαδὴ καὶ τοῦ κύκλου, μείζων ἐστὶ πασῶν τῶν εἰς τὸν κύκλον ἢ τὴν σφαῖραν διαγομένων [εὐθειῶν]. ἔστω οὖν ἐν μὲν τῇ μείζονι σφαίρᾳ κύκλος ὁ ΒΓΔΕ, ἐν δὲ τῇ ἐλάσσονι σφαίρᾳ κύκλος ὁ ΖΗΘ, καὶ ἤχθωσαν αὐτῶν δύο διάμετροι πρὸς ὀρθὰς ἀλλήλαις αἱ ΒΔ, ΓΕ, καὶ δύο κύκλων περὶ τὸ αὐτὸ κέντρον ὄντων τῶν ΒΓΔΕ, ΖΗΘ εἰς τὸν μείζονα κύκλον τὸν ΒΓΔΕ πολύγωνον ἰσόπλευρον καὶ ἀρτιόπλευρον ἐγγεγράφθω μὴ ψαῦον τοῦ ἐλάσσονος κύκλου τοῦ ΖΗΘ, οὗ πλευραὶ ἔστωσαν ἐν τῷ ΒΕ τεταρτημορίῳ αἱ ΒΚ, ΚΛ, ΛΜ, ΜΕ, καὶ ἐπιζευχθεῖσα ἡ ΚΑ διήχθω ἐπὶ τὸ Ν, καὶ ἀνεστάτω ἀπὸ τοῦ Α σημείου τῷ τοῦ ΒΓΔΕ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΑΞ καὶ συμβαλλέτω τῇ ἐπιφανείᾳ τῆς σφαίρας κατὰ τὸ Ξ, καὶ διὰ τῆς ΑΞ καὶ ἑκατέρας τῶν ΒΔ, ΚΝ ἐπίπεδα ἐκβεβλήσθω· ποιήσουσι δὴ διὰ τὰ εἰρημένα ἐπὶ τῆς ἐπιφανείας τῆς σφαίρας μεγίστους κύκλους. ποιείτωσαν, ὧν ἡμικύκλια ἔστω ἐπὶ τῶν ΒΔ, ΚΝ διαμέτρων τὰ ΒΞΔ, ΚΞΝ. καὶ ἐπεὶ ἡ ΞΑ ὀρθή ἐστι πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον, καὶ πάντα ἄρα τὰ διὰ τῆς ΞΑ ἐπίπεδά ἐστιν ὀρθὰ πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον·

ὥστε καὶ τὰ ΒΞΔ, ΚΞΝ ἡμικύκλια ὀρθά ἐστι πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον. καὶ ἐπεὶ ἴσα ἐστὶ τὰ ΒΕΔ, ΒΞΔ, ΚΞΝ ἡμικύκλια· ἐπὶ γὰρ ἴσων εἰσὶ διαμέτρων τῶν ΒΔ, ΚΝ· ἴσα ἐστὶ καὶ τὰ ΒΕ, ΒΞ, ΚΞ τεταρτημόρια ἀλλήλοις. ὅσαι ἄρα εἰσὶν ἐν τῷ ΒΕ τεταρτημορίῳ πλευραὶ τοῦ πολυγώνου, τοσαῦταί

εἰσι καὶ ἐν τοῖς ΒΞ, ΚΞ τεταρτημορίοις ἴσαι ταῖς ΒΚ, ΚΛ, ΛΜ, ΜΕ εὐθείαις. ἐγγεγράφθωσαν καὶ ἔστωσαν αἱ ΒΟ, ΟΠ, ΠΡ, ΡΞ, ΚΣ, ΣΤ, ΤΥ, ΥΞ, καὶ ἐπεζεύχθωσαν αἱ ΣΟ, ΤΠ, ΥΡ, καὶ ἀπὸ τῶν Ο, Σ ἐπὶ τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον κάθετοι ἤχθωσαν· πεσοῦνται δὴ ἐπὶ τὰς κοινὰς τομὰς τῶν ἐπιπέδων τὰς ΒΔ, ΚΝ, ἐπειδήπερ καὶ τὰ τῶν ΒΞΔ, ΚΞΝ ἐπίπεδα ὀρθά ἐστι πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον. πιπτέτωσαν, καὶ ἔστωσαν αἱ ΟΦ, ΣΧ, καὶ ἐπεζεύχθω ἡ ΧΦ. καὶ ἐπεὶ ἐν ἴσοις ἡμικυκλίοις τοῖς ΒΞΔ, ΚΞΝ ἴσαι ἀπειλημμέναι εἰσὶν αἱ ΒΟ, ΚΣ, καὶ κάθετοι ἠγμέναι εἰσὶν αἱ ΟΦ, ΣΧ, ἴση [ἄρα] ἐστὶν ἡ μὲν ΟΦ τῇ ΣΧ, ἡ δὲ ΒΦ τῇ ΚΧ. ἔστι δὲ καὶ ὅλη ἡ ΒΑ ὅλῃ τῇ ΚΑ ἴση· καὶ λοιπὴ ἄρα ἡ ΦΑ λοιπῇ τῇ ΧΑ ἐστιν ἴση· ἔστιν ἄρα ὡς ἡ ΒΦ πρὸς τὴν ΦΑ, οὕτως ἡ ΚΧ πρὸς τὴν ΧΑ· παράλληλος ἄρα ἐστὶν ἡ ΧΦ τῇ ΚΒ. καὶ ἐπεὶ ἑκατέρα τῶν ΟΦ, ΣΧ ὀρθή ἐστι πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον, παράλληλος ἄρα ἐστὶν ἡ ΟΦ τῇ ΣΧ. ἐδείχθη δὲ αὐτῇ καὶ ἴση· καὶ αἱ ΧΦ, ΣΟ ἄρα ἴσαι εἰσὶ καὶ παράλληλοι. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΧΦ τῇ ΣΟ, ἀλλὰ ἡ ΧΦ τῇ ΚΒ ἐστι παράλληλος, καὶ ἡ ΣΟ ἄρα τῇ ΚΒ ἐστι παράλληλος. καὶ ἐπιζευγνύουσιν αὐτὰς αἱ ΒΟ, ΚΣ· τὸ ΚΒΟΣ ἄρα τετράπλευρον ἐν ἑνί ἐστιν ἐπιπέδῳ, ἐπειδήπερ, ἐὰν ὦσι δύο εὐθεῖαι παράλληλοι, καὶ ἐφ’ ἑκατέρας αὐτῶν ληφθῇ τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις. διὰ τὰ αὐτὰ δὴ καὶ ἑκάτερον τῶν ΣΟΠΤ, ΤΠΡΥ τετραπλεύρων ἐν ἑνί ἐστιν ἐπιπέδῳ. ἔστι δὲ καὶ τὸ ΥΡΞ τρίγωνον ἐν ἑνὶ ἐπιπέδῳ. ἐὰν δὴ νοήσωμεν ἀπὸ τῶν Ο, Σ, Π, Τ, Ρ, Υ σημείων ἐπὶ τὸ Α ἐπιζευγνυμένας εὐθείας, συσταθήσεταί τι σχῆμα στερεὸν πολύεδρον μεταξὺ τῶν ΒΞ, ΚΞ περιφερειῶν ἐκ πυραμίδων συγκείμενον, ὧν βάσεις μὲν τὰ ΚΒΟΣ, ΣΟΠΤ, ΤΠΡΥ τετράπλευρα καὶ τὸ ΥΡΞ τρίγωνον, κορυφὴ δὲ τὸ Α σημεῖον. ἐὰν δὲ καὶ ἐπὶ ἑκάστης τῶν ΚΛ, ΛΜ, ΜΕ πλευρῶν καθάπερ ἐπὶ τῆς ΒΚ τὰ αὐτὰ κατασκευάσωμεν καὶ ἔτι ἐπὶ τῶν λοιπῶν τριῶν τεταρτημορίων, συσταθήσεταί τι σχῆμα πολύεδρον ἐγγεγραμμένον εἰς τὴν σφαῖραν πυραμίσι

περιεχόμενον, ὧν βάσεις [μὲν] τὰ εἰρημένα τετράπλευρα καὶ τὸ ΥΡΞ τρίγωνον καὶ τὰ ὁμοταγῆ αὐτοῖς, κορυφὴ δὲ τὸ Α σημεῖον.

Λέγω, ὅτι τὸ εἰρημένον πολύεδρον οὐκ ἐφάψεται τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν, ἐφ' ἧς ἐστιν ὁ ΖΗΘ κύκλος.

Ἤχθω ἀπὸ τοῦ Α σημείου ἐπὶ τὸ τοῦ ΚΒΟΣ τετραπλεύρου ἐπίπεδον κάθετος ἡ ΑΨ καὶ συμβαλλέτω τῷ ἐπιπέδῳ κατὰ τὸ Ψ σημεῖον, καὶ ἐπεζεύχθωσαν αἱ ΨΒ, ΨΚ. καὶ ἐπεὶ ἡ ΑΨ ὀρθή ἐστι πρὸς τὸ τοῦ ΚΒΟΣ τετραπλεύρου ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ τοῦ τετραπλεύρου ἐπιπέδῳ ὀρθή ἐστιν. ἡ ΑΨ ἄρα ὀρθή ἐστι πρὸς ἑκατέραν τῶν ΒΨ, ΨΚ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΑΚ, ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς ΑΒ τῷ ἀπὸ τῆς ΑΚ. καί ἐστι τῷ μὲν ἀπὸ τῆς ΑΒ ἴσα τὰ ἀπὸ τῶν ΑΨ, ΨΒ· ὀρθὴ γὰρ ἡ πρὸς τῷ Ψ· τῷ δὲ ἀπὸ τῆς ΑΚ ἴσα τὰ ἀπὸ τῶν ΑΨ, ΨΚ. τὰ ἄρα ἀπὸ τῶν ΑΨ, ΨΒ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΑΨ, ΨΚ. κοινὸν ἀφῃρήσθω τὸ ἀπὸ τῆς ΑΨ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΒΨ λοιπῷ τῷ ἀπὸ τῆς ΨΚ ἴσον ἐστίν· ἴση ἄρα ἡ ΒΨ τῇ ΨΚ. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ ἀπὸ τοῦ Ψ ἐπὶ τὰ Ο, Σ ἐπιζευγνύμεναι εὐθεῖαι ἴσαι εἰσὶν ἑκατέρᾳ τῶν ΒΨ, ΨΚ. ὁ ἄρα κέντρῳ τῷ Ψ καὶ διαστήματι ἑνὶ τῶν ΨΒ, ΨΚ γραφόμενος κύκλος ἥξει καὶ διὰ τῶν Ο, Σ, καὶ ἔσται ἐν κύκλῳ τὸ ΚΒΟΣ τετράπλευρον.

Καὶ ἐπεὶ μείζων ἐστὶν ἡ ΚΒ τῆς ΧΦ, ἴση δὲ ἡ ΧΦ τῇ ΣΟ, μείζων ἄρα ἡ ΚΒ τῆς ΣΟ. ἴση δὲ ἡ ΚΒ ἑκατέρᾳ τῶν ΚΣ, ΒΟ· καὶ ἑκατέρα ἄρα τῶν ΚΣ, ΒΟ τῆς ΣΟ μείζων ἐστίν. καὶ ἐπεὶ ἐν κύκλῳ τετράπλευρόν ἐστι τὸ ΚΒΟΣ, καὶ ἴσαι αἱ ΚΒ, ΒΟ, ΚΣ, καὶ ἐλάττων ἡ ΟΣ, καὶ ἐκ τοῦ κέντρου τοῦ κύκλου ἐστὶν ἡ ΒΨ, τὸ ἄρα ἀπὸ τῆς ΚΒ τοῦ ἀπὸ τῆς ΒΨ μεῖζόν ἐστιν ἢ διπλάσιον. ἤχθω ἀπὸ τοῦ Κ ἐπὶ τὴν ΒΦ κάθετος ἡ ΚΩ. καὶ ἐπεὶ ἡ ΒΔ τῆς ΔΩ ἐλάττων ἐστὶν ἢ διπλῆ, καί ἐστιν ὡς ἡ ΒΔ πρὸς τὴν ΔΩ, οὕτως τὸ ὑπὸ τῶν ΔΒ, ΒΩ πρὸς τὸ ὑπὸ [τῶν] ΔΩ, ΩΒ, ἀναγραφομένου ἀπὸ τῆς ΒΩ τετραγώνου καὶ συμπληρουμένου τοῦ ἐπὶ τῆς ΩΔ παραλληλογράμμου καὶ τὸ ὑπὸ ΔΒ, ΒΩ ἄρα τοῦ ὑπὸ ΔΩ, ΩΒ ἔλαττόν ἐστιν ἢ διπλάσιον. καί ἐστι τῆς ΚΔ

ἐπιζευγνυμένης τὸ μὲν ὑπὸ ΔΒ, ΒΩ ἴσον τῷ ἀπὸ τῆς ΒΚ, τὸ δὲ ὑπὸ τῶν ΔΩ, ΩΒ ἴσον τῷ ἀπὸ τῆς ΚΩ· τὸ ἄρα ἀπὸ τῆς ΚΒ τοῦ ἀπὸ τῆς ΚΩ ἔλασσόν ἐστιν ἢ διπλάσιον. ἀλλὰ τὸ ἀπὸ τῆς ΚΒ τοῦ ἀπὸ τῆς ΒΨ μεῖζόν ἐστιν ἢ διπλάσιον· μεῖζον ἄρα τὸ ἀπὸ τῆς ΚΩ τοῦ ἀπὸ τῆς ΒΨ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΑ τῇ ΚΑ, ἴσον ἐστὶ τὸ ἀπὸ τῆς ΒΑ τῷ ἀπὸ τῆς ΑΚ. καί ἐστι τῷ μὲν ἀπὸ τῆς ΒΑ ἴσα τὰ ἀπὸ τῶν ΒΨ, ΨΑ, τῷ δὲ ἀπὸ τῆς ΚΑ ἴσα τὰ ἀπὸ τῶν ΚΩ, ΩΑ· τὰ ἄρα ἀπὸ τῶν ΒΨ, ΨΑ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΚΩ, ΩΑ, ὧν τὸ ἀπὸ τῆς ΚΩ μεῖζον τοῦ ἀπὸ τῆς ΒΨ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΩΑ ἔλασσόν ἐστι τοῦ ἀπὸ τῆς ΨΑ. μείζων ἄρα ἡ ΑΨ τῆς ΑΩ· πολλῷ ἄρα ἡ ΑΨ μείζων ἐστὶ τῆς ΑΗ. καί ἐστιν ἡ μὲν ΑΨ ἐπὶ μίαν τοῦ πολυέδρου βάσιν, ἡ δὲ ΑΗ ἐπὶ τὴν τῆς ἐλάσσονος σφαίρας ἐπιφάνειαν· ὥστε τὸ πολύεδρον οὐ ψαύσει τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν.

Δύο ἄρα σφαιρῶν περὶ τὸ αὐτὸ κέντρον οὐσῶν εἰς τὴν μείζονα σφαῖραν στερεὸν πολύεδρον ἐγγέγραπται μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν· ὅπερ ἔδει ποιῆσαι.

Πόρισμα

Ἐὰν δὲ καὶ εἰς ἑτέραν σφαῖραν τῷ ἐν τῇ ΒΓΔΕ σφαίρᾳ στερεῷ πολυέδρῳ ὅμοιον στερεὸν πολύεδρον ἐγγραφῇ, τὸ ἐν τῇ ΒΓΔΕ σφαίρᾳ στερεὸν πολύεδρον πρὸς τὸ ἐν τῇ ἑτέρᾳ σφαίρᾳ στερεὸν πολύεδρον τριπλασίονα λόγον ἔχει, ἥπερ ἡ τῆς ΒΓΔΕ σφαίρας διάμετρος πρὸς τὴν τῆς ἑτέρας σφαίρας διάμετρον. διαιρεθέντων γὰρ τῶν στερεῶν εἰς τὰς ὁμοιοπληθεῖς καὶ ὁμοιοταγεῖς πυραμίδας ἔσονται αἱ πυραμίδες ὅμοιαι. αἱ δὲ ὅμοιαι πυραμίδες πρὸς ἀλλήλας ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν· ἡ ἄρα πυραμίς, ἧς βάσις μέν ἐστι τὸ ΚΒΟΣ τετράπλευρον, κορυφὴ δὲ τὸ Α σημεῖον, πρὸς τὴν ἐν τῇ ἑτέρᾳ σφαίρᾳ ὁμοιοταγῆ πυραμίδα τριπλασίονα λόγον ἔχει, ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἥπερ ἡ ΑΒ ἐκ τοῦ κέντρου τῆς σφαίρας τῆς περὶ κέντρον τὸ Α πρὸς τὴν ἐκ τοῦ κέντρου

τῆς ἑτέρας σφαίρας. ὁμοίως καὶ ἑκάστη πυραμὶς τῶν ἐν τῇ περὶ κέντρον τὸ Α σφαίρᾳ πρὸς ἑκάστην ὁμοταγῆ πυραμίδα τῶν ἐν τῇ ἑτέρᾳ σφαίρᾳ τριπλασίονα λόγον ἕξει, ἥπερ ἡ ΑΒ πρὸς τὴν ἐκ τοῦ κέντρου τῆς ἑτέρας σφαίρας. καὶ ὡς ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα· ὥστε ὅλον τὸ ἐν τῇ περὶ κέντρον τὸ Α σφαίρᾳ στερεὸν πολύεδρον πρὸς ὅλον τὸ ἐν τῇ ἑτέρᾳ [σφαίρᾳ] στερεὸν πολύεδρον τριπλασίονα λόγον ἕξει, ἥπερ ἡ ΑΒ πρὸς τὴν ἐκ τοῦ κέντρου τῆς ἑτέρας σφαίρας, τουτέστιν ἥπερ ἡ ΒΔ διάμετρος πρὸς τὴν τῆς ἑτέρας σφαίρας διάμετρον· ὅπερ ἔδει δεῖξαι.

ιη΄. Αἱ σφαῖραι πρὸς ἀλλήλας ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ἰδίων διαμέτρων.

Νενοήσθωσαν σφαῖραι αἱ ΑΒΓ, ΔΕΖ, διάμετροι δὲ αὐτῶν αἱ ΒΓ, ΕΖ· λέγω, ὅτι ἡ ΑΒΓ σφαῖρα πρὸς τὴν ΔΕΖ σφαῖραν τριπλασίονα λόγον ἔχει ἥπερ ἡ ΒΓ πρὸς τὴν ΕΖ.

Εἰ γὰρ μὴ ἡ ΑΒΓ σφαῖρα πρὸς τὴν ΔΕΖ σφαῖραν τριπλασίονα λόγον ἔχει ἥπερ ἡ ΒΓ πρὸς τὴν ΕΖ, ἕξει ἄρα ἡ ΑΒΓ σφαῖρα πρὸς ἐλάσσονά τινα τῆς ΔΕΖ σφαίρας τριπλασίονα λόγον ἢ πρὸς μείζονα ἥπερ ἡ ΒΓ πρὸς τὴν ΕΖ. ἐχέτω πρότερον πρὸς ἐλάσσονα τὴν ΗΘΚ, καὶ νενοήσθω ἡ ΔΕΖ τῇ ΗΘΚ περὶ τὸ αὐτὸ κέντρον, καὶ ἐγγεγράφθω εἰς τὴν μείζονα σφαῖραν τὴν ΔΕΖ στερεὸν πολύεδρον μὴ ψαῦον τῆς ἐλάσσονος σφαίρας τῆς ΗΘΚ κατὰ τὴν ἐπιφάνειαν, ἐγγεγράφθω δὲ καὶ εἰς τὴν ΑΒΓ σφαῖραν τῷ ἐν τῇ ΔΕΖ σφαίρᾳ στερεῷ πολυέδρῳ ὅμοιον στερεὸν πολύεδρον· τὸ ἄρα ἐν τῇ ΑΒΓ στερεὸν πολύεδρον πρὸς τὸ ἐν τῇ ΔΕΖ στερεὸν πολύεδρον τριπλασίονα λόγον ἔχει ἥπερ ἡ ΒΓ πρὸς τὴν ΕΖ.

ἔχει δὲ καὶ ἡ ΑΒΓ σφαῖρα πρὸς τὴν ΗΘΚ σφαῖραν τριπλασίονα λόγον ἥπερ ἡ ΒΓ πρὸς τὴν ΕΖ· ἔστιν ἄρα ὡς ἡ ΑΒΓ σφαῖρα πρὸς τὴν ΗΘΚ σφαῖραν, οὕτως τὸ ἐν τῇ ΑΒΓ σφαίρᾳ στερεὸν πολύεδρον πρὸς τὸ ἐν τῇ ΔΕΖ σφαίρᾳ στερεὸν πολύεδρον· ἐναλλὰξ [ἄρα] ὡς ἡ ΑΒΓ σφαῖρα πρὸς τὸ ἐν

αὐτῇ πολύεδρον, οὕτως ἡ ΗΘΚ σφαῖρα πρὸς τὸ ἐν τῇ ΔΕΖ σφαίρᾳ στερεὸν πολύεδρον. μείζων δὲ ἡ ΑΒΓ σφαῖρα τοῦ ἐν αὐτῇ πολυέδρου· μείζων ἄρα καὶ ἡ ΗΘΚ σφαῖρα τοῦ ἐν τῇ ΔΕΖ σφαίρᾳ πολυέδρου. ἀλλὰ καὶ ἐλάττων· ἐμπεριέχεται γὰρ ὑπ' αὐτοῦ. οὐκ ἄρα ἡ ΑΒΓ σφαῖρα πρὸς ἐλάσσονα τῆς ΔΕΖ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ διάμετρος πρὸς τὴν ΕΖ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἡ ΔΕΖ σφαῖρα πρὸς ἐλάσσονα τῆς ΑΒΓ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΕΖ πρὸς τὴν ΒΓ.

Λέγω δή, ὅτι οὐδὲ ἡ ΑΒΓ σφαῖρα πρὸς μείζονά τινα τῆς ΔΕΖ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ.

Εἰ γὰρ δυνατόν, ἐχέτω πρὸς μείζονα τὴν ΛΜΝ· ἀνάπαλιν ἄρα ἡ ΛΜΝ σφαῖρα πρὸς τὴν ΑΒΓ σφαῖραν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΕΖ διάμετρος πρὸς τὴν ΒΓ διάμετρον. ὡς δὲ ἡ ΛΜΝ σφαῖρα πρὸς τὴν ΑΒΓ σφαῖραν, οὕτως ἡ ΔΕΖ σφαῖρα πρὸς ἐλάσσονά τινα τῆς ΑΒΓ σφαίρας, ἐπειδήπερ μείζων ἐστὶν ἡ ΛΜΝ τῆς ΔΕΖ, ὡς ἔμπροσθεν ἐδείχθη. καὶ ἡ ΔΕΖ ἄρα σφαῖρα πρὸς ἐλάσσονά τινα τῆς ΑΒΓ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΕΖ πρὸς τὴν ΒΓ· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ἡ ΑΒΓ σφαῖρα πρὸς μείζονά τινα τῆς ΔΕΖ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἐλάσσονα. ἡ ἄρα ΑΒΓ σφαῖρα πρὸς τὴν ΔΕΖ σφαῖραν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ· ὅπερ ἔδει δεῖξαι.

BOOK XIII.

Προτάσεις ιη΄.

α΄. Ἐὰν εὐθεῖα γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ μεῖζον τμῆμα προσλαβὸν τὴν ἡμίσειαν τῆς ὅλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου.

Εὐθεῖα γὰρ γραμμὴ ἡ ΑΒ ἄκρον καὶ μέσον λόγον τετμήσθω κατὰ τὸ Γ σημεῖον, καὶ ἔστω μεῖζον τμῆμα τὸ ΑΓ, καὶ ἐκβεβλήσθω ἐπ’ εὐθείας τῇ ΓΑ εὐθεῖα ἡ ΑΔ, καὶ κείσθω τῆς ΑΒ ἡμίσεια ἡ ΑΔ· λέγω, ὅτι πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΓΔ τοῦ ἀπὸ τῆς ΔΑ.

Ἀναγεγράφθωσαν γὰρ ἀπὸ τῶν ΑΒ, ΔΓ τετράγωνα τὰ ΑΕ, ΔΖ, καὶ καταγεγράφθω ἐν τῷ ΔΖ τὸ σχῆμα, καὶ διήχθω ἡ ΖΓ ἐπὶ τὸ Η. καὶ ἐπεὶ ἡ ΑΒ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Γ, τὸ ἄρα ὑπὸ τῶν ΑΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ. καί ἐστι τὸ μὲν ὑπὸ τῶν ΑΒΓ τὸ ΓΕ, τὸ δὲ ἀπὸ τῆς ΑΓ τὸ ΖΘ· ἴσον ἄρα τὸ ΓΕ τῷ ΖΘ.

καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΒΑ τῆς ΑΔ, ἴση δὲ ἡ μὲν ΒΑ τῇ ΚΑ, ἡ δὲ ΑΔ τῇ ΑΘ, διπλῆ ἄρα καὶ ἡ ΚΑ τῆς ΑΘ. ὡς δὲ ἡ ΚΑ πρὸς τὴν ΑΘ, οὕτως τὸ ΓΚ πρὸς τὸ ΓΘ· διπλάσιον ἄρα τὸ ΓΚ τοῦ ΓΘ. εἰσὶ δὲ καὶ τὰ ΛΘ, ΘΓ διπλάσια τοῦ ΓΘ. ἴσον ἄρα τὸ ΚΓ τοῖς ΛΘ, ΘΓ. ἐδείχθη δὲ καὶ τὸ ΓΕ τῷ ΘΖ ἴσον· ὅλον ἄρα τὸ ΑΕ τετράγωνον ἴσον ἐστὶ τῷ ΜΝΞ γνώμονι. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΒΑ τῆς ΑΔ, τετραπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΑ τοῦ ἀπὸ τῆς ΑΔ, τουτέστι τὸ ΑΕ τοῦ ΔΘ. ἴσον δὲ τὸ ΑΕ τῷ ΜΝΞ γνώμονι· καὶ ὁ ΜΝΞ ἄρα γνώμων τετραπλάσιός ἐστι τοῦ ΑΟ· ὅλον ἄρα τὸ ΔΖ πενταπλάσιόν ἐστι τοῦ ΑΟ. καί ἐστι τὸ μὲν ΔΖ τὸ ἀπὸ τῆς ΔΓ, τὸ δὲ ΑΟ τὸ ἀπὸ τῆς ΔΑ· τὸ ἄρα ἀπὸ τῆς ΓΔ πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΔΑ.

Ἐὰν ἄρα εὐθεῖα ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ μεῖζον τμῆμα προσλαβὸν τὴν ἡμίσειαν τῆς ὅλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου· ὅπερ ἔδει δεῖξαι.

β΄. Ἐὰν εὐθεῖα γραμμὴ τμήματος ἑαυτῆς πενταπλάσιον δύνηται, τῆς διπλασίας τοῦ εἰρημένου τμήματος ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμα τὸ λοιπὸν μέρος ἐστὶ τῆς ἐξ ἀρχῆς εὐθείας.

Εὐθεῖα γὰρ γραμμὴ ἡ ΑΒ τμήματος ἑαυτῆς τοῦ ΑΓ πενταπλάσιον δυνάσθω, τῆς δὲ ΑΓ διπλῆ ἔστω ἡ ΓΔ· λέγω, ὅτι τῆς ΓΔ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΓΒ.

Ἀναγεγράφθω γὰρ ἀφ’ ἑκατέρας τῶν ΑΒ, ΓΔ τετράγωνα τὰ ΑΖ, ΓΗ, καὶ καταγεγράφθω ἐν τῷ ΑΖ τὸ σχῆμα, καὶ διήχθω ἡ ΒΕ. καὶ ἐπεὶ πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΑ τοῦ ἀπὸ τῆς ΑΓ, πενταπλάσιόν ἐστι τὸ ΑΖ τοῦ ΑΘ. τετραπλάσιος ἄρα ὁ ΜΝΞ γνώμων τοῦ ΑΘ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΔΓ τῆς ΓΑ, τετραπλάσιον ἄρα ἐστὶ τὸ ἀπὸ ΔΓ τοῦ ἀπὸ ΓΑ, τουτέστι τὸ ΓΗ τοῦ ΑΘ. ἐδείχθη δὲ καὶ ὁ ΜΝΞ γνώμων τετραπλάσιος τοῦ ΑΘ· ἴσος ἄρα ὁ ΜΝΞ γνώμων τῷ ΓΗ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΔΓ τῆς ΓΑ, ἴση δὲ ἡ μὲν ΔΓ τῇ ΓΚ, ἡ δὲ ΑΓ τῇ ΓΘ [διπλῆ ἄρα καὶ ἡ ΚΓ τῆς ΓΘ], διπλάσιον ἄρα καὶ τὸ ΚΒ τοῦ ΒΘ.

εἰσὶ δὲ καὶ τὰ ΛΘ, ΘΒ τοῦ ΘΒ διπλάσια· ἴσον ἄρα τὸ ΚΒ τοῖς ΛΘ, ΘΒ. ἐδείχθη δὲ καὶ ὅλος ὁ ΜΝΞ γνώμων ὅλῳ τῷ ΓΗ ἴσος· καὶ λοιπὸν ἄρα τὸ ΘΖ τῷ ΒΗ ἐστιν ἴσον. καί ἐστι τὸ μὲν ΒΗ τὸ ὑπὸ τῶν ΓΔΒ· ἴση γὰρ ἡ ΓΔ τῇ ΔΗ· τὸ δὲ ΘΖ τὸ ἀπὸ τῆς ΓΒ· τὸ ἄρα ὑπὸ τῶν ΓΔΒ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΒ. ἔστιν ἄρα ὡς ἡ ΔΓ πρὸς τὴν ΓΒ, οὕτως ἡ ΓΒ πρὸς τὴν ΒΔ. μείζων δὲ ἡ ΔΓ τῆς ΓΒ· μείζων ἄρα καὶ ἡ ΓΒ τῆς ΒΔ. τῆς ΓΔ ἄρα εὐθείας ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΓΒ.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμήματος ἑαυτῆς πενταπλάσιον δύνηται, τῆς διπλασίας τοῦ εἰρημένου τμήματος ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμα τὸ λοιπὸν μέρος ἐστὶ τῆς ἐξ ἀρχῆς εὐθείας· ὅπερ ἔδει δεῖξαι.

Λῆμμα

Ὅτι δὲ ἡ διπλῆ τῆς ΑΓ μείζων ἐστὶ τῆς ΒΓ, οὕτως δεικτέον.

Εἰ γὰρ μή, ἔστω, εἰ δυνατόν, ἡ ΒΓ διπλῆ τῆς ΓΑ. τετραπλάσιον ἄρα τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΑ· πενταπλάσια ἄρα τὰ ἀπὸ τῶν ΒΓ, ΓΑ τοῦ ἀπὸ τῆς ΓΑ. ὑπόκειται δὲ καὶ τὸ ἀπὸ τῆς ΒΑ πενταπλάσιον τοῦ ἀπὸ τῆς ΓΑ· τὸ ἄρα ἀπὸ τῆς ΒΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΓ, ΓΑ· ὅπερ ἀδύνατον. οὐκ ἄρα ἡ ΓΒ διπλασία ἐστὶ τῆς ΑΓ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἡ ἐλάττων τῆς ΓΒ διπλασίων ἐστὶ τῆς ΓΑ· πολλῷ γὰρ [μεῖζον] τὸ ἄτοπον.

Ἡ ἄρα τῆς ΑΓ διπλῆ μείζων ἐστὶ τῆς ΓΒ· ὅπερ ἔδει δεῖξαι.

γ΄. Ἐὰν εὐθεῖα γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ ἔλασσον τμῆμα προσλαβὸν τὴν ἡμίσειαν τοῦ μείζονος τμήματος πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τοῦ μείζονος τμήματος τετραγώνου.

Εὐθεῖα γάρ τις ἡ ΑΒ ἄκρον καὶ μέσον λόγον τετμήσθω κατὰ τὸ Γ σημεῖον, καὶ ἔστω μεῖζον τμῆμα τὸ ΑΓ, καὶ τετμήσθω ἡ ΑΓ δίχα κατὰ τὸ Δ· λέγω, ὅτι πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΔ τοῦ ἀπὸ τῆς ΔΓ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΕ, καὶ καταγεγράφθω διπλοῦν τὸ σχῆμα. ἐπεὶ διπλῆ ἐστιν ἡ ΑΓ τῆς ΔΓ, τετραπλάσιον ἄρα τὸ ἀπὸ τῆς ΑΓ τοῦ ἀπὸ τῆς ΔΓ, τουτέστι τὸ ΡΣ τοῦ ΖΗ. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ, καί ἐστι τὸ ὑπὸ τῶν ΑΒΓ τὸ ΓΕ, τὸ ἄρα ΓΕ ἴσον ἐστὶ τῷ ΡΣ.

τετραπλάσιον δὲ τὸ ΡΣ τοῦ ΖΗ· τετραπλάσιον ἄρα καὶ τὸ ΓΕ τοῦ ΖΗ. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΔΓ, ἴση ἐστὶ καὶ ἡ ΘΚ τῇ ΚΖ. ὥστε καὶ τὸ ΗΖ τετράγωνον ἴσον ἐστὶ τῷ ΘΛ τετραγώνῳ. ἴση ἄρα ἡ ΗΚ τῇ ΚΛ, τουτέστιν ἡ ΜΝ τῇ ΝΕ· ὥστε καὶ τὸ ΜΖ τῷ ΖΕ ἐστιν ἴσον. ἀλλὰ τὸ ΜΖ τῷ ΓΗ ἐστιν ἴσον· καὶ τὸ ΓΗ ἄρα τῷ ΖΕ ἐστιν ἴσον. κοινὸν προσκείσθω τὸ ΓΝ· ὁ ἄρα ΞΟΠ γνώμων ἴσος ἐστὶ τῷ ΓΕ. ἀλλὰ τὸ ΓΕ τετραπλάσιον ἐδείχθη τοῦ ΗΖ· καὶ ὁ ΞΟΠ ἄρα γνώμων τετραπλάσιός ἐστι τοῦ ΖΗ τετραγώνου. ὁ ΞΟΠ ἄρα γνώμων καὶ τὸ ΖΗ τετράγωνον πενταπλάσιός ἐστι τοῦ ΖΗ. ἀλλὰ ὁ ΞΟΠ

γνώμων καὶ τὸ ΖΗ τετράγωνόν ἐστι τὸ ΔΝ. καί ἐστι τὸ μὲν ΔΝ τὸ ἀπὸ τῆς ΔΒ, τὸ δὲ ΗΖ τὸ ἀπὸ τῆς ΔΓ. τὸ ἄρα ἀπὸ τῆς ΔΒ πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΔΓ· ὅπερ ἔδει δεῖξαι.

δ΄. Ἐὰν εὐθεῖα γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ ἀπὸ τῆς ὅλης καὶ τοῦ ἐλάσσονος τμήματος, τὰ συναμφότερα τετράγωνα, τριπλάσιά ἐστι τοῦ ἀπὸ τοῦ μείζονος τμήματος τετραγώνου.

Ἔστω εὐθεῖα ἡ ΑΒ, καὶ τετμήσθω ἄκρον καὶ μέσον λόγον κατὰ τὸ Γ, καὶ ἔστω μεῖζον τμῆμα τὸ ΑΓ· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΒ, ΒΓ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΓΑ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔΕΒ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν ἡ ΑΒ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Γ, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΑΓ, τὸ ἄρα ὑπὸ τῶν ΑΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ.

καί ἐστι τὸ μὲν ὑπὸ τῶν ΑΒΓ τὸ ΑΚ, τὸ δὲ ἀπὸ τῆς ΑΓ τὸ ΘΗ· ἴσον ἄρα ἐστὶ τὸ ΑΚ τῷ ΘΗ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΑΖ τῷ ΖΕ, κοινὸν προσκείσθω τὸ ΓΚ· ὅλον ἄρα τὸ ΑΚ ὅλῳ τῷ ΓΕ ἐστιν ἴσον· τὰ ἄρα ΑΚ, ΓΕ τοῦ ΑΚ ἐστι διπλάσια. ἀλλὰ τὰ ΑΚ, ΓΕ ὁ ΛΜΝ γνώμων ἐστὶ καὶ τὸ ΓΚ τετράγωνον· ὁ ἄρα ΛΜΝ γνώμων καὶ τὸ ΓΚ τετράγωνον διπλάσιά ἐστι τοῦ ΑΚ. ἀλλὰ μὴν καὶ τὸ ΑΚ τῷ ΘΗ ἐδείχθη ἴσον· ὁ ἄρα ΛΜΝ γνώμων καὶ [τὸ ΓΚ τετράγωνον διπλάσιά ἐστι τοῦ ΘΗ· ὥστε ὁ ΛΜΝ γνώμων καὶ] τὰ ΓΚ, ΘΗ τετράγωνα τριπλάσιά ἐστι τοῦ ΘΗ τετραγώνου. καί ἐστιν ὁ [μὲν] ΛΜΝ γνώμων καὶ τὰ ΓΚ, ΘΗ τετράγωνα ὅλον τὸ ΑΕ καὶ τὸ ΓΚ, ἅπερ ἐστὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα, τὸ δὲ ΗΘ τὸ ἀπὸ τῆς ΑΓ τετράγωνον. τὰ ἄρα ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΑΓ τετραγώνου· ὅπερ ἔδει δεῖξαι.

ε΄. Ἐὰν εὐθεῖα γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, καὶ προστεθῇ αὐτῇ ἴση τῷ μείζονι τμήματι, ἡ ὅλη εὐθεῖα ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ἐξ ἀρχῆς εὐθεῖα.

Εὐθεῖα γὰρ γραμμὴ ἡ ΑΒ ἄκρον καὶ μέσον λόγον τετμήσθω κατὰ τὸ Γ σημεῖον, καὶ ἔστω μεῖζον τμῆμα ἡ ΑΓ, καὶ τῇ ΑΓ ἴση [κείσθω] ἡ ΑΔ. λέγω, ὅτι ἡ ΔΒ εὐθεῖα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Α, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ἐξ ἀρχῆς εὐθεῖα ἡ ΑΒ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΕ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ ἡ ΑΒ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Γ, τὸ ἄρα ὑπὸ ΑΒΓ ἴσον ἐστὶ τῷ ἀπὸ ΑΓ. καί ἐστι τὸ μὲν ὑπὸ ΑΒΓ τὸ ΓΕ, τὸ δὲ ἀπὸ τῆς ΑΓ τὸ ΓΘ· ἴσον ἄρα τὸ ΓΕ τῷ ΘΓ. ἀλλὰ τῷ μὲν ΓΕ ἴσον ἐστὶ τὸ ΘΕ, τῷ δὲ ΘΓ ἴσον τὸ ΔΘ· καὶ τὸ ΔΘ ἄρα ἴσον ἐστὶ τῷ ΘΕ [κοινὸν προσκείσθω τὸ ΘΒ]. ὅλον ἄρα τὸ ΔΚ ὅλῳ τῷ ΑΕ ἐστιν ἴσον. καί ἐστι τὸ μὲν ΔΚ τὸ ὑπὸ τῶν ΒΔ, ΔΑ· ἴση γὰρ ἡ ΑΔ τῇ ΔΛ· τὸ δὲ ΑΕ τὸ ἀπὸ τῆς ΑΒ· τὸ ἄρα ὑπὸ τῶν ΒΔΑ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ. ἔστιν ἄρα ὡς ἡ ΔΒ πρὸς τὴν ΒΑ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΔ. μείζων δὲ ἡ ΔΒ τῆς ΒΑ· μείζων ἄρα καὶ ἡ ΒΑ τῆς ΑΔ.

Ἡ ἄρα ΔΒ ἄκρον καὶ μέσον λόγον τέμηται κατὰ τὸ Α, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΑΒ· ὅπερ ἔδει δεῖξαι.

ς΄. Ἐὰν εὐθεῖα ῥητὴ ἄκρον καὶ μέσον λόγον τμηθῇ, ἑκάτερον τῶν τμημάτων ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἔστω εὐθεῖα ῥητὴ ἡ ΑΒ καὶ τετμήσθω ἄκρον καὶ μέσον λόγον κατὰ τὸ Γ, καὶ ἔστω μεῖζον τμῆμα ἡ ΑΓ· λέγω, ὅτι ἑκατέρα τῶν ΑΓ, ΓΒ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐκβεβλήσθω γὰρ ἡ ΒΑ, καὶ κείσθω τῆς ΒΑ ἡμίσεια ἡ ΑΔ. ἐπεὶ οὖν εὐθεῖα ἡ ΑΒ τέτμηται ἄκρον καὶ μέσον λόγον κατὰ τὸ Γ, καὶ τῷ μείζονι τμήματι τῷ ΑΓ πρόσκειται ἡ ΑΔ ἡμίσεια οὖσα τῆς ΑΒ, τὸ ἄρα ἀπὸ ΓΔ τοῦ ἀπὸ ΔΑ πενταπλάσιόν ἐστιν.

τὸ ἄρα ἀπὸ ΓΔ πρὸς τὸ ἀπὸ ΔΑ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· σύμμετρον ἄρα τὸ ἀπὸ ΓΔ τῷ ἀπὸ ΔΑ. ῥητὸν δὲ τὸ ἀπὸ ΔΑ· ῥητὴ γὰρ [ἐστιν] ἡ ΔΑ ἡμίσεια οὖσα τῆς ΑΒ ῥητῆς οὔσης· ῥητὸν ἄρα καὶ τὸ ἀπὸ ΓΔ· ῥητὴ

ἄρα ἐστὶ καὶ ἡ ΓΔ. καὶ ἐπεὶ τὸ ἀπὸ ΓΔ πρὸς τὸ ἀπὸ ΔΑ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, ἀσύμμετρος ἄρα μήκει ἡ ΓΔ τῇ ΔΑ· αἱ ΓΔ, ΔΑ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΑΓ. πάλιν, ἐπεὶ ἡ ΑΒ ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΑΓ, τὸ ἄρα ὑπὸ ΑΒ, ΒΓ τῷ ἀπὸ ΑΓ ἴσον ἐστίν. τὸ ἄρα ἀπὸ τῆς ΑΓ ἀποτομῆς παρὰ τὴν ΑΒ ῥητὴν παραβληθὲν πλάτος ποιεῖ τὴν ΒΓ. τὸ δὲ ἀπὸ ἀποτομῆς παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πρώτην· ἀποτομὴ ἄρα πρώτη ἐστὶν ἡ ΓΒ. ἐδείχθη δὲ καὶ ἡ ΓΑ ἀποτομή.

Ἐὰν ἄρα εὐθεῖα ῥητὴ ἄκρον καὶ μέσον λόγον τμηθῇ, ἑκάτερον τῶν τμημάτων ἄλογός ἐστιν ἡ καλουμένη ἀποτομή· ὅπερ ἔδει δεῖξαι.

ζ΄. Ἐὰν πενταγώνου ἰσοπλεύρου αἱ τρεῖς γωνίαι ἤτοι αἱ κατὰ τὸ ἑξῆς ἢ αἱ μὴ κατὰ τὸ ἑξῆς ἴσαι ὦσιν, ἰσογώνιον ἔσται τὸ πεντάγωνον.

Πενταγώνου γὰρ ἰσοπλεύρου τοῦ ΑΒΓΔΕ αἱ τρεῖς γωνίαι πρότερον αἱ κατὰ τὸ ἑξῆς αἱ πρὸς τοῖς Α, Β, Γ ἴσαι ἀλλήλαις ἔστωσαν· λέγω, ὅτι ἰσογώνιόν ἐστι τὸ ΑΒΓΔΕ πεντάγωνον.

Ἐπεζεύχθωσαν γὰρ αἱ ΑΓ, ΒΕ, ΖΔ. καὶ ἐπεὶ δύο αἱ ΓΒ, ΒΑ δυσὶ ταῖς ΒΑ, ΑΕ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ γωνία ἡ ὑπὸ ΓΒΑ γωνίᾳ τῇ ὑπὸ ΒΑΕ ἐστιν ἴση, βάσις ἄρα ἡ ΑΓ βάσει τῇ ΒΕ ἐστιν ἴση, καὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΒΕ τριγώνῳ ἴσον, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὑφ’ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν, ἡ μὲν ὑπὸ ΒΓΑ τῇ ὑπὸ ΒΕΑ, ἡ δὲ ὑπὸ ΑΒΕ τῇ ὑπὸ ΓΑΒ· ὥστε καὶ πλευρὰ ἡ ΑΖ πλευρᾷ τῇ ΒΖ ἐστιν ἴση.

ἐδείχθη δὲ καὶ ὅλη ἡ ΑΓ ὅλῃ τῇ ΒΕ ἴση· καὶ λοιπὴ ἄρα ἡ ΖΓ λοιπῇ τῇ ΖΕ ἐστιν ἴση. ἔστι δὲ καὶ ἡ ΓΔ τῇ ΔΕ ἴση. δύο δὴ αἱ ΖΓ, ΓΔ δυσὶ ταῖς ΖΕ, ΕΔ ἴσαι εἰσίν· καὶ βάσις αὐτῶν κοινὴ ἡ ΖΔ· γωνία ἄρα ἡ ὑπὸ ΖΓΔ γωνίᾳ τῇ ὑπὸ ΖΕΔ ἐστιν ἴση. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΒΓΑ τῇ ὑπὸ ΑΕΒ ἴση· καὶ ὅλη ἄρα ἡ ὑπὸ ΒΓΔ ὅλῃ τῇ ὑπὸ ΑΕΔ ἴση. ἀλλ’ ἡ ὑπὸ ΒΓΔ ἴση ὑπόκειται ταῖς πρὸς τοῖς Α, Β γωνίαις· καὶ ἡ ὑπὸ ΑΕΔ ἄρα ταῖς πρὸς τοῖς Α, Β γωνίαις ἴση ἐστίν.

ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ὑπὸ ΓΔΕ γωνία ἴση ἐστὶ ταῖς πρὸς τοῖς Α, Β, Γ γωνίαις· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓΔΕ πεντάγωνον.

Ἀλλὰ δὴ μὴ ἔστωσαν ἴσαι αἱ κατὰ τὸ ἑξῆς γωνίαι, ἀλλ᾽ ἔστωσαν ἴσαι αἱ πρὸς τοῖς Α, Γ, Δ σημείοις· λέγω, ὅτι καὶ οὕτως ἰσογώνιόν ἐστι τὸ ΑΒΓΔΕ πεντάγωνον.

Ἐπεζεύχθω γὰρ ἡ ΒΔ. καὶ ἐπεὶ δύο αἱ ΒΑ, ΑΕ δυσὶ ταῖς ΒΓ, ΓΔ ἴσαι εἰσὶ καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΒΕ βάσει τῇ ΒΔ ἴση ἐστίν, καὶ τὸ ΑΒΕ τρίγωνον τῷ ΒΓΔ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΕΒ γωνία τῇ ὑπὸ ΓΔΒ. ἔστι δὲ καὶ ἡ ὑπὸ ΒΕΔ γωνία τῇ ὑπὸ ΒΔΕ ἴση, ἐπεὶ καὶ πλευρὰ ἡ ΒΕ πλευρᾷ τῇ ΒΔ ἐστιν ἴση. καὶ ὅλη ἄρα ἡ ὑπὸ ΑΕΔ γωνία ὅλῃ τῇ ὑπὸ ΓΔΕ ἐστιν ἴση. ἀλλὰ ἡ ὑπὸ ΓΔΕ ταῖς πρὸς τοῖς Α, Γ γωνίαις ὑπόκειται ἴση· καὶ ἡ ὑπὸ ΑΕΔ ἄρα γωνία ταῖς πρὸς τοῖς Α, Γ ἴση ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΑΒΓ ἴση ἐστὶ ταῖς πρὸς τοῖς Α, Γ, Δ γωνίαις. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓΔΕ πεντάγωνον· ὅπερ ἔδει δεῖξαι.

η΄. Ἐὰν πενταγώνου ἰσοπλεύρου καὶ ἰσογωνίου τὰς κατὰ τὸ ἑξῆς δύο γωνίας ὑποτείνωσιν εὐθεῖαι, ἄκρον καὶ μέσον λόγον τέμνουσιν ἀλλήλας, καὶ τὰ μείζονα αὐτῶν τμήματα ἴσα ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ.

Πενταγώνου γὰρ ἰσοπλεύρου καὶ ἰσογωνίου τοῦ ΑΒΓ ΔΕ δύο γωνίας τὰς κατὰ τὸ ἑξῆς τὰς πρὸς τοῖς Α, Β ὑποτεινέτωσαν εὐθεῖαι αἱ ΑΓ, ΒΕ τέμνουσαι ἀλλήλας κατὰ τὸ Θ σημεῖον· λέγω, ὅτι ἑκατέρα αὐτῶν ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Θ σημεῖον, καὶ τὰ μείζονα αὐτῶν τμήματα ἴσα ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ.

Περιγεγράφθω γὰρ περὶ τὸ ΑΒΓΔΕ πεντάγωνον κύκλος ὁ ΑΒΓΔΕ. καὶ ἐπεὶ δύο εὐθεῖαι αἱ ΕΑ, ΑΒ δυσὶ ταῖς ΑΒ, ΒΓ ἴσαι εἰσὶ καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΒΕ βάσει τῇ ΑΓ ἴση ἐστίν, καὶ τὸ ΑΒΕ τρίγωνον τῷ ΑΒΓ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι

ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΒΑΓ γωνία τῇ ὑπὸ ΑΒΕ· διπλῆ ἄρα ἡ ὑπὸ ΑΘΕ τῆς ὑπὸ ΒΑΘ. ἔστι δὲ καὶ ἡ ὑπὸ ΕΑΓ τῆς ὑπὸ ΒΑΓ διπλῆ, ἐπειδήπερ καὶ περιφέρεια ἡ ΕΔΓ περιφερείας τῆς ΓΒ ἐστι διπλῆ· ἴση ἄρα ἡ ὑπὸ ΘΑΕ γωνία τῇ ὑπὸ ΑΘΕ· ὥστε καὶ ἡ ΘΕ εὐθεῖα τῇ ΕΑ, τουτέστι τῇ ΑΒ ἐστιν ἴση.

καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΑ εὐθεῖα τῇ ΑΕ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΑΒΕ τῇ ὑπὸ ΑΕΒ. ἀλλὰ ἡ ὑπὸ ΑΒΕ τῇ ὑπὸ ΒΑΘ ἐδείχθη ἴση· καὶ ἡ ὑπὸ ΒΕΑ ἄρα τῇ ὑπὸ ΒΑΘ ἐστιν ἴση. καὶ κοινὴ τῶν δύο τριγώνων τοῦ τε ΑΒΕ καὶ τοῦ ΑΒΘ ἐστιν ἡ ὑπὸ ΑΒΕ· λοιπὴ ἄρα ἡ ὑπὸ ΒΑΕ γωνία λοιπῇ τῇ ὑπὸ ΑΘΒ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΕ τρίγωνον τῷ ΑΒΘ τριγώνῳ· ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΕΒ πρὸς τὴν ΒΑ, οὕτως ἡ ΑΒ πρὸς τὴν ΒΘ. ἴση δὲ ἡ ΒΑ τῇ ΕΘ· ὡς ἄρα ἡ ΒΕ πρὸς τὴν ΕΘ, οὕτως ἡ ΕΘ πρὸς τὴν ΘΒ. μείζων δὲ ἡ ΒΕ τῆς ΕΘ· μείζων ἄρα καὶ ἡ ΕΘ τῆς ΘΒ. ἡ ΒΕ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Θ, καὶ τὸ μεῖζον τμῆμα τὸ ΘΕ ἴσον ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΑΓ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Θ, καὶ τὸ μεῖζον αὐτῆς τμῆμα ἡ ΓΘ ἴσον ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ· ὅπερ ἔδει δεῖξαι.

θ΄. Ἐὰν ἡ τοῦ ἑξαγώνου πλευρὰ καὶ ἡ τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων συντεθῶσιν, ἡ ὅλη εὐθεῖα ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ τοῦ ἑξαγώνου πλευρά.

Ἔστω κύκλος ὁ ΑΒΓ, καὶ τῶν εἰς τὸν ΑΒΓ κύκλον ἐγγραφομένων σχημάτων, δεκαγώνου μὲν ἔστω πλευρὰ ἡ ΒΓ, ἑξαγώνου δὲ ἡ ΓΔ, καὶ ἔστωσαν ἐπ᾽ εὐθείας· λέγω, ὅτι ἡ ὅλη εὐθεῖα ἡ ΒΔ ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΓΔ.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ κύκλου τὸ Ε σημεῖον, καὶ ἐπεζεύχθωσαν αἱ ΕΒ, ΕΓ, ΕΔ, καὶ διήχθω ἡ ΒΕ ἐπὶ τὸ Α. ἐπεὶ δεκαγώνου ἰσοπλεύρου πλευρά ἐστιν ἡ ΒΓ, πενταπλασίων ἄρα ἡ ΑΓΒ περιφέρεια τῆς ΒΓ περιφερείας·

τετραπλασίων ἄρα ἡ ΑΓ περιφέρεια τῆς ΓΒ. ὡς δὲ ἡ ΑΓ περιφέρεια πρὸς τὴν ΓΒ, οὕτως ἡ ὑπὸ ΑΕΓ γωνία πρὸς τὴν ὑπὸ ΓΕΒ· τετραπλασίων ἄρα ἡ ὑπὸ ΑΕΓ τῆς ὑπὸ ΓΕΒ.

καὶ ἐπεὶ ἴση ἡ ὑπὸ ΕΒΓ γωνία τῇ ὑπὸ ΕΓΒ, ἡ ἄρα ὑπὸ ΑΕΓ γωνία διπλασία ἐστὶ τῆς ὑπὸ ΕΓΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΕΓ εὐθεῖα τῇ ΓΔ· ἑκατέρα γὰρ αὐτῶν ἴση ἐστὶ τῇ τοῦ ἑξαγώνου πλευρᾷ τοῦ εἰς τὸν ΑΒΓ κύκλον [ἐγγραφομένου]· ἴση ἐστὶ καὶ ἡ ὑπὸ ΓΕΔ γωνία τῇ ὑπὸ ΓΔΕ γωνίᾳ· διπλασία ἄρα ἡ ὑπὸ ΕΓΒ γωνία τῆς ὑπὸ ΕΔΓ. ἀλλὰ τῆς ὑπὸ ΕΓΒ διπλασία ἐδείχθη ἡ ὑπὸ ΑΕΓ· τετραπλασία ἄρα ἡ ὑπὸ ΑΕΓ τῆς ὑπὸ ΕΔΓ. ἐδείχθη δὲ καὶ τῆς ὑπὸ ΒΕΓ τετραπλασία ἡ ὑπὸ ΑΕΓ· ἴση ἄρα ἡ ὑπὸ ΕΔΓ τῇ ὑπὸ ΒΕΓ. κοινὴ δὲ τῶν δύο τριγώνων, τοῦ τε ΒΕΓ καὶ τοῦ ΒΕΔ, ἡ ὑπὸ ΕΒΔ γωνία· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΒΕΔ τῇ ὑπὸ ΕΓΒ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΕΒΔ τρίγωνον τῷ ΕΒΓ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΕΒ πρὸς τὴν ΒΓ. ἴση δὲ ἡ ΕΒ τῇ ΓΔ. ἔστιν ἄρα ὡς ἡ ΒΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΔΓ πρὸς τὴν ΓΒ. μείζων δὲ ἡ ΒΔ τῆς ΔΓ· μείζων ἄρα καὶ ἡ ΔΓ τῆς ΓΒ. ἡ ΒΔ ἄρα εὐθεῖα ἄκρον καὶ μέσον λόγον τέτμηται [κατὰ τὸ Γ], καὶ τὸ μεῖζον τμῆμα αὐτῆς ἐστιν ἡ ΔΓ· ὅπερ ἔδει δεῖξαι.

ι΄. Ἐὰν εἰς κύκλον πεντάγωνον ἰσόπλευρον ἐγγραφῇ, ἡ τοῦ πενταγώνου πλευρὰ δύναται τήν τε τοῦ ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων.

Ἔστω κύκλος ὁ ΑΒΓΔΕ, καὶ εἰς τὸν ΑΒΓΔΕ κύκλον πεντάγωνον ἰσόπλευρον ἐγγεγράφθω τὸ ΑΒΓΔΕ. λέγω, ὅτι ἡ τοῦ ΑΒΓΔΕ πενταγώνου πλευρὰ δύναται τήν τε τοῦ ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου πλευρὰν τῶν εἰς τὸν ΑΒΓΔΕ κύκλον ἐγγραφομένων.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ κύκλου τὸ Ζ σημεῖον, καὶ ἐπιζευχθεῖσα ἡ ΑΖ διήχθω ἐπὶ τὸ Η σημεῖον, καὶ ἐπεζεύχθω ἡ ΖΒ, καὶ ἀπὸ τοῦ Ζ ἐπὶ τὴν ΑΒ κάθετος ἤχθω ἡ ΖΘ, καὶ διήχθω ἐπὶ τὸ Κ, καὶ ἐπεζεύχθωσαν αἱ ΑΚ, ΚΒ, καὶ

πάλιν ἀπὸ τοῦ Ζ ἐπὶ τὴν ΑΚ κάθετος ἤχθω ἡ ΖΛ, καὶ διήχθω ἐπὶ τὸ Μ, καὶ ἐπεζεύχθω ἡ ΚΝ. ἐπεὶ ἴση ἐστὶν ἡ ΑΒΓΗ περιφέρεια τῇ ΑΕΔΗ περιφερείᾳ, ὧν ἡ ΑΒΓ τῇ ΑΕΔ ἐστιν ἴση, λοιπὴ ἄρα ἡ ΓΗ περιφέρεια λοιπῇ τῇ ΗΔ ἐστιν ἴση. πενταγώνου δὲ ἡ ΓΔ· δεκαγώνου ἄρα ἡ ΓΗ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΖΑ τῇ ΖΒ, καὶ κάθετος ἡ ΖΘ, ἴση ἄρα καὶ ἡ ὑπὸ ΑΖΚ γωνία τῇ ὑπὸ ΚΖΒ. ὥστε καὶ περιφέρεια ἡ ΑΚ τῇ ΚΒ ἐστιν ἴση· διπλῆ ἄρα ἡ ΑΒ περιφέρεια τῆς ΒΚ περιφερείας· δεκαγώνου ἄρα πλευρά ἐστιν ἡ ΑΚ εὐθεῖα. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΑΚ τῆς ΚΜ ἐστι διπλῆ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΒ περιφέρεια τῆς ΒΚ περιφερείας, ἴση δὲ ἡ ΓΔ περιφέρεια τῇ ΑΒ περιφερείᾳ, διπλῆ ἄρα καὶ ἡ ΓΔ περιφέρεια τῆς ΒΚ περιφερείας.

ἔστι δὲ ἡ ΓΔ περιφέρεια καὶ τῆς ΓΗ διπλῆ· ἴση ἄρα ἡ ΓΗ περιφέρεια τῇ ΒΚ περιφερείᾳ. ἀλλὰ ἡ ΒΚ τῆς ΚΜ ἐστι διπλῆ, ἐπεὶ καὶ ἡ ΚΑ· καὶ ἡ ΓΗ ἄρα τῆς ΚΜ ἐστι διπλῆ. ἀλλὰ μὴν καὶ ἡ ΓΒ περιφέρεια τῆς ΒΚ περιφερείας ἐστὶ διπλῆ· ἴση γὰρ ἡ ΓΒ περιφέρεια τῇ ΒΑ. καὶ ὅλη ἄρα ἡ ΗΒ περιφέρεια τῆς ΒΜ ἐστι διπλῆ· ὥστε καὶ γωνία ἡ ὑπὸ ΗΖΒ γωνίας τῆς ὑπὸ ΒΖΜ [ἐστι] διπλῆ. ἔστι δὲ ἡ ὑπὸ ΗΖΒ καὶ τῆς ὑπὸ ΖΑΒ διπλῆ· ἴση γὰρ ἡ ὑπὸ ΖΑΒ τῇ ὑπὸ ΑΒΖ. καὶ ἡ ὑπὸ ΒΖΝ ἄρα τῇ ὑπὸ ΖΑΒ ἐστιν ἴση. κοινὴ δὲ τῶν δύο τριγώνων, τοῦ τε ΑΒΖ καὶ τοῦ ΒΖΝ, ἡ ὑπὸ ΑΒΖ γωνία· λοιπὴ ἄρα ἡ ὑπὸ ΑΖΒ λοιπῇ τῇ ὑπὸ ΒΝΖ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΖ τρίγωνον τῷ ΒΖΝ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΑΒ εὐθεῖα πρὸς τὴν ΒΖ, οὕτως ἡ ΖΒ πρὸς τὴν ΒΝ· τὸ ἄρα ὑπὸ τῶν ΑΒΝ ἴσον ἐστὶ τῷ ἀπὸ ΒΖ. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ΑΛ τῇ ΛΚ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΛΝ, βάσις ἄρα ἡ ΚΝ βάσει τῇ ΑΝ ἐστιν ἴση· καὶ γωνία ἄρα ἡ ὑπὸ ΛΚΝ γωνίᾳ τῇ ὑπὸ ΛΑΝ ἐστιν ἴση. ἀλλὰ ἡ ὑπὸ ΛΑΝ τῇ ὑπὸ ΚΒΝ ἐστιν ἴση· καὶ ἡ ὑπὸ ΛΚΝ ἄρα τῇ ὑπὸ ΚΒΝ ἐστιν ἴση. καὶ κοινὴ τῶν δύο τριγώνων τοῦ τε ΑΚΒ καὶ τοῦ ΑΚΝ ἡ πρὸς τῷ Α. λοιπὴ ἄρα ἡ ὑπὸ ΑΚΒ λοιπῇ τῇ ὑπὸ ΚΝΑ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΚΒΑ τρίγωνον τῷ ΚΝΑ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΑ εὐθεῖα πρὸς τὴν ΑΚ, οὕτως ἡ ΚΑ πρὸς τὴν ΑΝ· τὸ ἄρα ὑπὸ τῶν ΒΑΝ ἴσον ἐστὶ τῷ ἀπὸ

τῆς ΑΚ. ἐδείχθη δὲ καὶ τὸ ὑπὸ τῶν ΑΒΝ ἴσον τῷ ἀπὸ τῆς ΒΖ· τὸ ἄρα ὑπὸ τῶν ΑΒΝ μετὰ τοῦ ὑπὸ ΒΑΝ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΒΑ, ἴσον ἐστὶ τῷ ἀπὸ τῆς ΒΖ μετὰ τοῦ ἀπὸ τῆς ΑΚ. καί ἐστιν ἡ μὲν ΒΑ πενταγώνου πλευρά, ἡ δὲ ΒΖ ἑξαγώνου, ἡ δὲ ΑΚ δεκαγώνου.

Ἡ ἄρα τοῦ πενταγώνου πλευρὰ δύναται τήν τε τοῦ ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων· ὅπερ ἔδει δεῖξαι.

ια΄. Ἐὰν εἰς κύκλον ῥητὴν ἔχοντα τὴν διάμετρον πεντάγωνον ἰσόπλευρον ἐγγραφῇ, ἡ τοῦ πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

Εἰς γὰρ κύκλον τὸν ΑΒΓΔΕ ῥητὴν ἔχοντα τὴν διάμετρον πεντάγωνον ἰσόπλευρον ἐγγεγράφθω τὸ ΑΒΓΔΕ· λέγω, ὅτι ἡ τοῦ [ΑΒΓΔΕ] πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ κύκλου τὸ Ζ σημεῖον, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΖΒ καὶ διήχθωσαν ἐπὶ τὰ Η, Θ σημεῖα, καὶ ἐπεζεύχθω ἡ ΑΓ, καὶ κείσθω τῆς ΑΖ τέταρτον μέρος ἡ ΖΚ. ῥητὴ δὲ ἡ ΑΖ· ῥητὴ ἄρα καὶ ἡ ΖΚ. ἔστι δὲ καὶ ἡ ΒΖ ῥητή· ὅλη ἄρα ἡ ΒΚ ῥητή ἐστιν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓΗ περιφέρεια τῇ ΑΔΗ περιφερείᾳ, ὧν ἡ ΑΒΓ τῇ ΑΕΔ ἐστιν ἴση, λοιπὴ ἄρα ἡ ΓΗ λοιπῇ τῇ ΗΔ ἐστιν ἴση. καὶ ἐὰν ἐπιζεύξωμεν τὴν ΑΔ, συνάγονται ὀρθαὶ αἱ πρὸς τῷ Λ γωνίαι, καὶ διπλῆ ἡ ΓΔ τῆς ΓΛ. διὰ τὰ αὐτὰ δὴ καὶ αἱ πρὸς τῷ Μ ὀρθαί εἰσιν, καὶ διπλῆ ἡ ΑΓ τῆς ΓΜ. ἐπεὶ οὖν ἴση ἐστὶν ἡ ὑπὸ ΑΛΓ γωνία τῇ ὑπὸ ΑΜΖ, κοινὴ δὲ τῶν δύο τριγώνων τοῦ τε ΑΓΛ καὶ τοῦ ΑΜΖ ἡ ὑπὸ ΛΑΓ, λοιπὴ ἄρα ἡ ὑπὸ ΑΓΛ λοιπῇ τῇ ὑπὸ ΜΖΑ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΓΛ τρίγωνον τῷ ΑΜΖ τριγώνῳ· ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΛΓ πρὸς ΓΑ, οὕτως ἡ ΜΖ πρὸς ΖΑ· καὶ τῶν ἡγουμένων τὰ διπλάσια· ὡς ἄρα ἡ τῆς ΛΓ διπλῆ πρὸς τὴν ΓΑ, οὕτως ἡ τῆς ΜΖ διπλῆ πρὸς τὴν ΖΑ. ὡς δὲ ἡ τῆς ΜΖ διπλῆ πρὸς τὴν ΖΑ, οὕτως ἡ ΜΖ πρὸς τὴν ἡμίσειαν τῆς ΖΑ· καὶ ὡς ἄρα ἡ τῆς ΛΓ διπλῆ πρὸς τὴν ΓΑ, οὕτως ἡ ΜΖ πρὸς τὴν ἡμίσειαν τῆς ΖΑ. καὶ τῶν ἑπομένων τὰ

ἡμίσεα· ὡς ἄρα ἡ τῆς ΛΓ διπλῆ πρὸς τὴν ἡμίσειαν τῆς ΓΑ, οὕτως ἡ ΜΖ πρὸς τὸ τέταρτον τῆς ΖΑ.

καί ἐστι τῆς μὲν ΛΓ διπλῆ ἡ ΔΓ, τῆς δὲ ΓΑ ἡμίσεια ἡ ΓΜ, τῆς δὲ ΖΑ τέταρτον μέρος ἡ ΖΚ· ἔστιν ἄρα ὡς ἡ ΔΓ πρὸς τὴν ΓΜ, οὕτως ἡ ΜΖ πρὸς τὴν ΖΚ. συνθέντι καὶ ὡς συναμφότερος ἡ ΔΓΜ πρὸς τὴν ΓΜ, οὕτως ἡ ΜΚ πρὸς ΚΖ· καὶ ὡς ἄρα τὸ ἀπὸ συναμφοτέρου τῆς ΔΓΜ πρὸς τὸ ἀπὸ ΓΜ, οὕτως τὸ ἀπὸ ΜΚ πρὸς τὸ ἀπὸ ΚΖ. καὶ ἐπεὶ τῆς ὑπὸ δύο πλευρὰς τοῦ πενταγώνου ὑποτεινούσης, οἷον τῆς ΑΓ, ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμα ἴσον ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ, τουτέστι τῇ ΔΓ, τὸ δὲ μεῖζον τμῆμα προσλαβὸν τὴν ἡμίσειαν τῆς ὅλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τῆς ὅλης, καί ἐστιν ὅλης τῆς ΑΓ ἡμίσεια ἡ ΓΜ, τὸ ἄρα ἀπὸ τῆς ΔΓΜ ὡς μιᾶς πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΓΜ. ὡς δὲ τὸ ἀπὸ τῆς ΔΓΜ ὡς μιᾶς πρὸς τὸ ἀπὸ τῆς ΓΜ, οὕτως ἐδείχθη τὸ ἀπὸ τῆς ΜΚ πρὸς τὸ ἀπὸ τῆς ΚΖ· πενταπλάσιον ἄρα τὸ ἀπὸ τῆς ΜΚ τοῦ ἀπὸ τῆς ΚΖ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΚΖ· ῥητὴ γὰρ ἡ διάμετρος· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΜΚ· ῥητὴ ἄρα ἐστὶν ἡ ΜΚ [δυνάμει μόνον]. καὶ ἐπεὶ τετραπλασία ἐστὶν ἡ ΒΖ τῆς ΖΚ, πενταπλασία ἄρα ἐστὶν ἡ ΒΚ τῆς ΚΖ· εἰκοσιπενταπλάσιον ἄρα τὸ ἀπὸ τῆς ΒΚ τοῦ ἀπὸ τῆς ΚΖ. πενταπλάσιον δὲ τὸ ἀπὸ τῆς ΜΚ τοῦ ἀπὸ τῆς ΚΖ· πενταπλάσιον ἄρα τὸ ἀπὸ τῆς ΒΚ τοῦ ἀπὸ τῆς ΚΜ· τὸ ἄρα ἀπὸ τῆς ΒΚ πρὸς τὸ ἀπὸ ΚΜ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΚ τῇ ΚΜ μήκει. καί ἐστι ῥητὴ ἑκατέρα αὐτῶν. αἱ ΒΚ, ΚΜ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ἐὰν δὲ ἀπὸ ῥητῆς ῥητὴ ἀφαιρεθῇ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, ἡ λοιπὴ ἄλογός ἐστιν ἀποτομή· ἀποτομὴ ἄρα ἐστὶν ἡ ΜΒ, προσαρμόζουσα δὲ αὐτῇ ἡ ΜΚ. λέγω δή, ὅτι καὶ τετάρτη. ᾧ δὴ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΒΚ τοῦ ἀπὸ τῆς ΚΜ, ἐκείνῳ ἴσον ἔστω τὸ ἀπὸ τῆς Ν· ἡ ΒΚ ἄρα τῆς ΚΜ μεῖζον δύναται τῇ Ν. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΚΖ τῇ ΖΒ, καὶ συνθέντι σύμμετρός ἐστιν ἡ ΚΒ τῇ ΖΒ. ἀλλὰ ἡ ΒΖ τῇ ΒΘ σύμμετρός ἐστιν· καὶ ἡ ΒΚ ἄρα τῇ ΒΘ σύμμετρός ἐστιν. καὶ ἐπεὶ

πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΚ τοῦ ἀπὸ τῆς ΚΜ, τὸ ἄρα ἀπὸ τῆς ΒΚ πρὸς τὸ ἀπὸ τῆς ΚΜ λόγον ἔχει, ὃν ε πρὸς ἕν.ἀναστρέψαντι ἄρα τὸ ἀπὸ τῆς ΒΚ πρὸςυὸ ἀπὸ τῆς Ν λόγον ἔχει, ὅν ε πρὸς δ. οὐχ ὃν τετράγωνος πρὸς τετράγωνον· ἀσύμετρος ἄρα ἐστὶν ἡ ΒΚ τῇ Ν· ἡ ΒΚ ἄρα τῆς ΚΜ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. ἐπεὶ οὖν ὅλη ἡ ΒΚ τῆς προσαρμοζούσης τῆς ΚΜ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ ὅλη ἡ ΒΚ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΒΘ, ἀποτομὴ ἄρα τετάρτη ἐστὶν ἡ ΜΒ. τὸ δὲ ὑπὸ ῥητῆς καὶ ἀποτομῆς τετάρτης περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν, καλεῖται δὲ ἐλάττων. δύναται δὲ τὸ ὑπὸ τῶν ΘΒΜ ἡ ΑΒ διὰ τὸ ἐπιζευγνυμένης τῆς ΑΘ ἰσογώνιον γίνεσθαι τὸ ΑΒΘ τρίγωνον τῷ ΑΒΜ τριγώνῳ καὶ εἶναι ὡς τὴν ΘΒ πρὸς τὴν ΒΑ, οὕτως τὴν ΑΒ πρὸς τὴν ΒΜ.

Ἡ ἄρα ΑΒ τοῦ πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων· ὅπερ ἔδει δεῖξαι.

ιβ΄. Ἐὰν εἰς κύκλον τρίγωνον ἰσόπλευρον ἐγγραφῇ, ἡ τοῦ τριγώνου πλευρὰ δυνάμει τριπλασίων ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ κύκλου.

Ἔστω κύκλος ὁ ΑΒΓ, καὶ εἰς αὐτὸν τρίγωνον ἰσόπλευρον ἐγγεγράφθω τὸ ΑΒΓ· λέγω, ὅτι τοῦ ΑΒΓ τριγώνου μία πλευρὰ δυνάμει τριπλασίων ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ ΑΒΓ κύκλου.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ ΑΒΓ κύκλου τὸ Δ, καὶ ἐπιζευχθεῖσα ἡ ΑΔ διήχθω ἐπὶ τὸ Ε, καὶ ἐπεζεύχθω ἡ ΒΕ. καὶ ἐπεὶ ἰσόπλευρόν ἐστι τὸ ΑΒΓ τρίγωνον, ἡ ΒΕΓ ἄρα περιφέρεια τρίτον μέρος ἐστὶ τῆς τοῦ ΑΒΓ κύκλου περιφερείας.

ἡ ἄρα ΒΕ περιφέρεια ἕκτον ἐστὶ μέρος τῆς τοῦ κύκλου περιφερείας· ἐξαγώνου ἄρα ἐστὶν ἡ ΒΕ εὐθεῖα· ἴση ἄρα ἐστὶ τῇ ἐκ τοῦ κέντρου τῇ ΔΕ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΕ τῆς ΔΕ, τετραπλάσιόν ἐστι τὸ ἀπὸ τῆς ΑΕ τοῦ ἀπὸ τῆς ΕΔ, τουτέστι τοῦ ἀπὸ τῆς ΒΕ. ἴσον δὲ τὸ ἀπὸ τῆς ΑΕ τοῖς ἀπὸ τῶν ΑΒ, ΒΕ·

τὰ ἄρα ἀπὸ τῶν ΑΒ, ΒΕ τετραπλάσιά ἐστι τοῦ ἀπὸ τῆς ΒΕ. διελόντι ἄρα τὸ ἀπὸ τῆς ΑΒ τριπλάσιόν ἐστι τοῦ ἀπὸ ΒΕ. ἴση δὲ ἡ ΒΕ τῇ ΔΕ· τὸ ἄρα ἀπὸ τῆς ΑΒ τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΔΕ.

Ἡ ἄρα τοῦ τριγώνου πλευρὰ δυνάμει τριπλασία ἐστὶ τῆς ἐκ τοῦ κέντρου [τοῦ κύκλου]· ὅπερ ἔδει δεῖξαι.

ιγ΄. Πυραμίδα συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει ἡμιολία ἐστὶ τῆς πλευρᾶς τῆς πυραμίδος.

Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ, καὶ τετμήσθω κατὰ τὸ Γ σημεῖον, ὥστε διπλασίαν εἶναι τὴν ΑΓ τῆς ΓΒ· καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ ἤχθω ἀπὸ τοῦ Γ σημείου τῇ ΑΒ πρὸς ὀρθὰς ἡ ΓΔ, καὶ ἐπεζεύχθω ἡ ΔΑ· καὶ ἐκκείσθω κύκλος ὁ ΕΖΗ ἴσην ἔχων τὴν ἐκ τοῦ κέντρου τῇ ΔΓ, καὶ ἐγγεγράφθω εἰς τὸν ΕΖΗ κύκλον τρίγωνον ἰσόπλευρον τὸ ΕΖΗ· καὶ εἰλήφθω τὸ κέντρον τοῦ κύκλου τὸ Θ σημεῖον, καὶ ἐπεζεύχθωσαν αἱ ΕΘ, ΘΖ, ΘΗ· καὶ ἀνεστάτω ἀπὸ τοῦ Θ σημείου τῷ τοῦ ΕΖΗ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΘΚ, καὶ ἀφῃρήσθω ἀπὸ τῆς ΘΚ τῇ ΑΓ εὐθείᾳ ἴση ἡ ΘΚ, καὶ ἐπεζεύχθωσαν αἱ ΚΕ, ΚΖ, ΚΗ. καὶ ἐπεὶ ἡ ΚΘ ὀρθή ἐστι πρὸς τὸ τοῦ ΕΖΗ κύκλου ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ τοῦ ΕΖΗ κύκλου ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. ἅπτεται δὲ αὐτῆς ἑκάστη τῶν ΘΕ, ΘΖ, ΘΗ· ἡ ΘΚ ἄρα πρὸς ἑκάστην τῶν ΘΕ, ΘΖ, ΘΗ ὀρθή ἐστιν. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΑΓ τῇ ΘΚ, ἡ δὲ ΓΔ τῇ ΘΕ, καὶ ὀρθὰς γωνίας περιέχουσιν, βάσις ἄρα ἡ ΔΑ βάσει τῇ ΚΕ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ΚΖ, ΚΗ τῇ ΔΑ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΚΕ, ΚΖ, ΚΗ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΓ τῆς ΓΒ, τριπλῆ ἄρα ἡ ΑΒ τῆς ΒΓ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΔ πρὸς τὸ ἀπὸ τῆς ΔΓ, ὡς ἐξῆς δειχθήσεται. τριπλάσιον ἄρα τὸ ἀπὸ τῆς ΑΔ τοῦ ἀπὸ τῆς ΔΓ. ἔστι δὲ καὶ τὸ ἀπὸ τῆς ΖΕ τοῦ ἀπὸ τῆς ΕΘ τριπλάσιον, καί ἐστιν ἴση ἡ ΔΓ τῇ ΕΘ· ἴση ἄρα καὶ ἡ ΔΑ τῇ ΕΖ. ἀλλὰ ἡ ΔΑ ἑκάστῃ τῶν ΚΕ, ΚΖ, ΚΗ ἐδείχθη ἴση· καὶ

ἑκάστη ἄρα τῶν ΕΖ, ΖΗ, ΗΕ ἑκάστῃ τῶν ΚΕ, ΚΖ, ΚΗ ἐστιν ἴση· ἰσόπλευρα ἄρα ἐστὶ τὰ τέσσαρα τρίγωνα τὰ ΕΖΗ, ΚΕΖ, ΚΖΗ, ΚΕΗ. πυραμὶς ἄρα συνέσταται ἐκ τεσσάρων τριγώνων ἰσοπλεύρων, ἧς βάσις μέν ἐστι τὸ ΕΖΗ τρίγωνον, κορυφὴ δὲ τὸ Κ σημεῖον.

Δεῖ δὴ αὐτὴν καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ δυνάμει τῆς πλευρᾶς τῆς πυραμίδος.

Ἐκβεβλήσθω γὰρ ἐπ᾽ εὐθείας τῇ ΚΘ εὐθεῖα ἡ ΘΛ, καὶ κείσθω τῇ ΓΒ ἴση ἡ ΘΛ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΓ πρὸς τὴν ΓΔ, οὕτως ἡ ΓΔ πρὸς τὴν ΓΒ, ἴση δὲ ἡ μὲν ΑΓ τῇ ΚΘ, ἡ δὲ ΓΔ τῇ ΘΕ, ἡ δὲ ΓΒ τῇ ΘΛ, ἔστιν ἄρα ὡς ἡ ΚΘ πρὸς τὴν ΘΕ, οὕτως ἡ ΕΘ πρὸς τὴν ΘΛ· τὸ ἄρα ὑπὸ τῶν ΚΘ, ΘΛ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ. καί ἐστιν ὀρθὴ ἑκατέρα τῶν ὑπὸ ΚΘΕ, ΕΘΛ γωνιῶν· τὸ ἄρα ἐπὶ τῆς ΚΛ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Ε [ἐπειδήπερ ἐὰν ἐπιζεύξωμεν τὴν ΕΛ, ὀρθὴ γίνεται ἡ ὑπὸ ΛΕΚ γωνία διὰ τὸ ἰσογώνιον γίνεσθαι τὸ ΕΛΚ τρίγωνον ἑκατέρῳ τῶν ΕΛΘ, ΕΘΚ τριγώνων]. ἐὰν δὴ μενούσης τῆς ΚΛ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἥξει καὶ διὰ τῶν Ζ, Η σημείων ἐπιζευγνυμένων τῶν ΖΛ, ΛΗ καὶ ὀρθῶν ὁμοίως γινομένων τῶν πρὸς τοῖς Ζ, Η γωνιῶν· καὶ ἔσται ἡ πυραμὶς σφαίρᾳ περιειλημμένη τῇ δοθείσῃ. ἡ γὰρ ΚΛ τῆς σφαίρας διάμετρος ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ τῇ ΑΒ, ἐπειδήπερ τῇ μὲν ΑΓ ἴση κεῖται ἡ ΚΘ, τῇ δὲ ΓΒ ἡ ΘΛ.

Λέγω δή, ὅτι ἡ τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ δυνάμει τῆς πλευρᾶς τῆς πυραμίδος.

Ἐπεὶ γὰρ διπλῆ ἐστιν ἡ ΑΓ τῆς ΓΒ, τριπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΓ· ἀναστρέψαντι ἡμιολία ἄρα ἐστὶν ἡ ΒΑ τῆς ΑΓ. ὡς δὲ ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΔ [ἐπειδήπερ ἐπιζευγνυμένης τῆς ΔΒ ἐστιν ὡς ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΔΑ πρὸς τὴν ΑΓ διὰ τὴν ὁμοιότητα τῶν ΔΑΒ, ΔΑΓ τριγώνων, καὶ εἶναι ὡς τὴν πρώτην πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας]. ἡμιόλιον ἄρα καὶ τὸ ἀπὸ τῆς ΒΑ τοῦ

ἀπὸ τῆς ΑΔ. καί ἐστιν ἡ μὲν ΒΑ ἡ τῆς δοθείσης σφαίρας διάμετρος, ἡ δὲ ΑΔ ἴση τῇ πλευρᾷ τῆς πυραμίδος.

Ἡ ἄρα τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ τῆς πλευρᾶς τῆς πυραμίδος· ὅπερ ἔδει δεῖξαι.

Λῆμμα

Δεικτέον, ὅτι ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΔ πρὸς τὸ ἀπὸ τῆς ΔΓ.

Ἐκκείσθω γὰρ ἡ τοῦ ἡμικυκλίου καταγραφή, καὶ ἐπεζεύχθω ἡ ΔΒ, καὶ ἀναγεγράφθω ἀπὸ τῆς ΑΓ τετράγωνον τὸ ΕΓ, καὶ συμπεπληρώσθω τὸ ΖΒ παραλληλόγραμμον.

ἐπεὶ οὖν διὰ τὸ ἰσογώνιον εἶναι τὸ ΔΑΒ τρίγωνον τῷ ΔΑΓ τριγώνῳ ἐστὶν ὡς ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΔΑ πρὸς τὴν ΑΓ, τὸ ἄρα ὑπὸ τῶν ΒΑ, ΑΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΔ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ΕΒ πρὸς τὸ ΒΖ, καί ἐστι τὸ μὲν ΕΒ τὸ ὑπὸ τῶν ΒΑ, ΑΓ· ἴση γὰρ ἡ ΕΑ τῇ ΑΓ· τὸ δὲ ΒΖ τὸ ὑπὸ τῶν ΑΓ, ΓΒ, ὡς ἄρα ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ὑπὸ τῶν ΒΑ, ΑΓ πρὸς τὸ ὑπὸ τῶν ΑΓ, ΓΒ. καί ἐστι τὸ μὲν ὑπὸ τῶν ΒΑ, ΑΓ ἴσον τῷ ἀπὸ τῆς ΑΔ, τὸ δὲ ὑπὸ τῶν ΑΓΒ ἴσον τῷ ἀπὸ τῆς ΔΓ· ἡ γὰρ ΔΓ κάθετος τῶν τῆς βάσεως τμημάτων τῶν ΑΓ, ΓΒ μέση ἀνάλογόν ἐστι διὰ τὸ ὀρθὴν εἶναι τὴν ὑπὸ ΑΔΒ. ὡς ἄρα ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΔ πρὸς τὸ ἀπὸ τῆς ΔΓ· ὅπερ ἔδει δεῖξαι.

ιδ΄. Ὀκτάεδρον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν, ᾗ καὶ τὰ πρότερα, καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασία ἐστὶ τῆς πλευρᾶς τοῦ ὀκταέδρου.

Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ, καὶ τετμήσθω δίχα κατὰ τὸ Γ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ ἤχθω ἀπὸ τοῦ Γ τῇ ΑΒ πρὸς ὀρθὰς ἡ ΓΔ, καὶ ἐπεζεύχθω ἡ ΔΒ, καὶ ἐκκείσθω τετράγωνον τὸ ΕΖΗΘ ἴσην ἔχον ἑκάστην τῶν πλευρῶν τῇ ΔΒ, καὶ ἐπεζεύχθωσαν αἱ ΘΖ, ΕΗ,

καὶ ἀνεστάτω ἀπὸ τοῦ Κ σημείου τῷ τοῦ ΕΖΗΘ τετραγώνου ἐπιπέδῳ πρὸς ὀρθὰς εὐθεῖα ἡ ΚΛ καὶ διήχθω ἐπὶ τὰ ἕτερα μέρη τοῦ ἐπιπέδου ὡς ἡ ΚΜ, καὶ ἀφῃρήσθω ἀφ' ἑκατέρας τῶν ΚΛ, ΚΜ μιᾷ τῶν ΕΚ, ΖΚ, ΗΚ, ΘΚ ἴση ἑκατέρα τῶν ΚΛ, ΚΜ, καὶ ἐπεζεύχθωσαν αἱ ΛΕ, ΛΖ, ΛΗ, ΛΘ, ΜΕ, ΜΖ, ΜΗ, ΜΘ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΚΕ τῇ ΚΘ, καί ἐστιν ὀρθὴ ἡ ὑπὸ ΕΚΘ γωνία, τὸ ἄρα ἀπὸ τῆς ΘΕ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΚ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΛΚ τῇ ΚΕ, καί ἐστιν ὀρθὴ ἡ ὑπὸ ΛΚΕ γωνία, τὸ ἄρα ἀπὸ τῆς ΕΛ διπλάσιόν ἐστι τοῦ ἀπὸ ΕΚ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΘΕ διπλάσιον τοῦ ἀπὸ τῆς ΕΚ· τὸ ἄρα ἀπὸ τῆς ΛΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ· ἴση ἄρα ἐστὶν ἡ ΛΕ τῇ ΕΘ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΛΘ τῇ ΘΕ ἐστιν ἴση· ἰσόπλευρον ἄρα ἐστὶ τὸ ΛΕΘ τρίγωνον. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ὧν βάσεις μέν εἰσιν αἱ τοῦ ΕΖΗΘ τετραγώνου πλευραί, κορυφαὶ δὲ τὰ Λ, Μ σημεῖα, ἰσόπλευρόν ἐστιν· ὀκτάεδρον ἄρα συνέσταται ὑπὸ ὀκτὼ τριγώνων ἰσοπλεύρων περιεχόμενον.

Δεῖ δὴ αὐτὸ καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίων ἐστὶ τῆς τοῦ ὀκταέδρου πλευρᾶς.

Ἐπεὶ γὰρ αἱ τρεῖς αἱ ΛΚ, ΚΜ, ΚΕ ἴσαι ἀλλήλαις εἰσίν, τὸ ἄρα ἐπὶ τῆς ΛΜ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Ε. καὶ διὰ τὰ αὐτά, ἐὰν μενούσης τῆς ΛΜ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἥξει καὶ διὰ τῶν Ζ, Η, Θ σημείων, καὶ ἔσται σφαίρᾳ περιειλημμένον τὸ ὀκτάεδρον. λέγω δή, ὅτι καὶ τῇ δοθείσῃ. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΛΚ τῇ ΚΜ, κοινὴ δὲ ἡ ΚΕ, καὶ γωνίας ὀρθὰς περιέχουσιν, βάσις ἄρα ἡ ΛΕ βάσει τῇ ΕΜ ἐστιν ἴση. καὶ ἐπεὶ ὀρθή ἐστιν ἡ ὑπὸ ΛΕΜ γωνία· ἐν ἡμικυκλίῳ γάρ· τὸ ἄρα ἀπὸ τῆς ΛΜ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΛΕ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΒ, διπλασία ἐστὶν ἡ ΑΒ τῆς ΒΓ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΔ· διπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΛΜ διπλάσιον τοῦ ἀπὸ τῆς ΛΕ. καί ἐστιν ἴσον τὸ ἀπὸ τῆς ΔΒ τῷ ἀπὸ τῆς ΛΕ· ἴση γὰρ κεῖται ἡ

ΕΘ τῇ ΔΒ. ἴσον ἄρα καὶ τὸ ἀπὸ τῆς ΑΒ τῷ ἀπὸ τῆς ΛΜ· ἴση ἄρα ἡ ΑΒ τῇ ΛΜ. καί ἐστιν ἡ ΑΒ ἡ τῆς δοθείσης σφαίρας διάμετρος· ἡ ΛΜ ἄρα ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ.

Περιείληπται ἄρα τὸ ὀκτάεδρον τῇ δοθείσῃ σφαίρᾳ. καὶ συναποδέδεικται, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίων ἐστὶ τῆς τοῦ ὀκταέδρου πλευρᾶς· ὅπερ ἔδει δεῖξαι.

ιε΄. Κύβον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν, ᾗ καὶ τὴν πυραμίδα, καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων ἐστὶ τῆς τοῦ κύβου πλευρᾶς.

Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ καὶ τετμήσθω κατὰ τὸ Γ ὥστε διπλῆν εἶναι τὴν ΑΓ τῆς ΓΒ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ ἀπὸ τοῦ Γ τῇ ΑΒ πρὸς ὀρθὰς ἤχθω ἡ ΓΔ, καὶ ἐπεζεύχθω ἡ ΔΒ, καὶ ἐκκείσθω τετράγωνον τὸ ΕΖΗΘ ἴσην ἔχον τὴν πλευρὰν τῇ ΔΒ, καὶ ἀπὸ τῶν Ε, Ζ, Η, Θ τῷ τοῦ ΕΖΗΘ τετραγώνου ἐπιπέδῳ πρὸς ὀρθὰς ἤχθωσαν αἱ ΕΚ, ΖΛ, ΗΜ, ΘΝ, καὶ ἀφῃρήσθω ἀπὸ ἑκάστης τῶν ΕΚ, ΖΛ, ΗΜ, ΘΝ μιᾷ τῶν ΕΖ, ΖΗ, ΗΘ, ΘΕ ἴση ἑκάστη τῶν ΕΚ, ΖΛ, ΗΜ, ΘΝ, καὶ ἐπεζεύχθωσαν αἱ ΚΛ, ΛΜ, ΜΝ, ΝΚ· κύβος ἄρα συνέσταται ὁ ΖΝ ὑπὸ ἓξ τετραγώνων ἴσων περιεχόμενος. δεῖ δὴ αὐτὸν καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασία ἐστὶ τῆς πλευρᾶς τοῦ κύβου.

Ἐπεζεύχθωσαν γὰρ αἱ ΚΗ, ΕΗ. καὶ ἐπεὶ ὀρθή ἐστιν ἡ ὑπὸ ΚΕΗ γωνία διὰ τὸ καὶ τὴν ΚΕ ὀρθὴν εἶναι πρὸς τὸ ΕΗ ἐπίπεδον δηλαδὴ καὶ πρὸς τὴν ΕΗ εὐθεῖαν, τὸ ἄρα ἐπὶ τῆς ΚΗ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Ε σημείου. πάλιν, ἐπεὶ ἡ ΗΖ ὀρθή ἐστι πρὸς ἑκατέραν τῶν ΖΛ, ΖΕ, καὶ πρὸς τὸ ΖΚ ἄρα ἐπίπεδον ὀρθή ἐστιν ἡ ΗΖ· ὥστε καὶ ἐὰν ἐπιζεύξωμεν τὴν ΖΚ, ἡ ΗΖ ὀρθὴ ἔσται καὶ πρὸς τὴν ΖΚ· καὶ διὰ τοῦτο πάλιν τὸ ἐπὶ τῆς ΗΚ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Ζ. ὁμοίως καὶ διὰ τῶν λοιπῶν τοῦ κύβου σημείων ἥξει. ἐὰν δὴ μενούσης τῆς ΚΗ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ

αὐτὸ ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἔσται σφαίρᾳ περιειλημμένος ὁ κύβος. λέγω δή, ὅτι καὶ τῇ δοθείσῃ. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΗΖ τῇ ΖΕ, καί ἐστιν ὀρθὴ ἡ πρὸς τῷ Ζ γωνία, τὸ ἄρα ἀπὸ τῆς ΕΗ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΖ. ἴση δὲ ἡ ΕΖ τῇ ΕΚ· τὸ ἄρα ἀπὸ τῆς ΕΗ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΚ· ὥστε τὰ ἀπὸ τῶν ΗΕ, ΕΚ, τουτέστι τὸ ἀπὸ τῆς ΗΚ, τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΚ. καὶ ἐπεὶ τριπλασίων ἐστὶν ἡ ΑΒ τῆς ΒΓ, ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΔ, τριπλάσιον ἄρα τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΗΚ τοῦ ἀπὸ τῆς ΚΕ τριπλάσιον. καὶ κεῖται ἴση ἡ ΚΕ τῇ ΔΒ· ἴση ἄρα καὶ ἡ ΚΗ τῇ ΑΒ. καί ἐστιν ἡ ΑΒ τῆς δοθείσης σφαίρας διάμετρος· καὶ ἡ ΚΗ ἄρα ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ.

Τῇ δοθείσῃ ἄρα σφαίρᾳ περιείληπται ὁ κύβος· καὶ συναποδέδεικται, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων ἐστὶ τῆς τοῦ κύβου πλευρᾶς· ὅπερ ἔδει δεῖξαι.

ις΄. Εἰκοσάεδρον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν, ᾗ καὶ τὰ προειρημένα σχήματα, καὶ δεῖξαι, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων.

Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ καὶ τετμήσθω κατὰ τὸ Γ ὥστε τετραπλῆν εἶναι τὴν ΑΓ τῆς ΓΒ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ ἤχθω ἀπὸ τοῦ Γ τῇ ΑΒ πρὸς ὀρθὰς γωνίας εὐθεῖα γραμμὴ ἡ ΓΔ, καὶ ἐπεζεύχθω ἡ ΔΒ, καὶ ἐκκείσθω κύκλος ὁ ΕΖΗΘΚ, οὗ ἡ ἐκ τοῦ κέντρου ἴση ἔστω τῇ ΔΒ, καὶ ἐγγεγράφθω εἰς τὸν ΕΖΗΘΚ κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον τὸ ΕΖΗΘΚ, καὶ τετμήσθωσαν αἱ ΕΖ, ΖΗ, ΗΘ, ΘΚ, ΚΕ περιφέρειαι δίχα κατὰ τὰ Λ, Μ, Ν, Ξ, Ο σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΛΜ, ΜΝ, ΝΞ, ΞΟ, ΟΛ, ΕΟ. ἰσόπλευρον ἄρα ἐστὶ καὶ τὸ ΛΜΝΞΟ πεντάγωνον, καὶ δεκαγώνου ἡ ΕΟ εὐθεῖα. καὶ ἀνεστάτωσαν ἀπὸ τῶν Ε, Ζ, Η, Θ, Κ σημείων τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς γωνίας

εὐθεῖαι αἱ ΕΠ, ΖΡ, ΗΣ, ΘΤ, ΚΥ ἴσαι οὖσαι τῇ ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου, καὶ ἐπεζεύχθωσαν αἱ ΠΡ, ΡΣ, ΣΤ, ΤΥ, ΥΠ, ΠΛ, ΛΡ, ΡΜ, ΜΣ, ΣΝ, ΝΤ, ΤΞ, ΞΥ, ΥΟ, ΟΠ. καὶ ἐπεὶ ἑκατέρα τῶν ΕΠ, ΚΥ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν, παράλληλος ἄρα ἐστὶν ἡ ΕΠ τῇ ΚΥ. ἔστι δὲ αὐτῇ καὶ ἴση· αἱ δὲ τὰς ἴσας τε καὶ παραλλήλους ἐπιζευγνύουσαι ἐπὶ τὰ αὐτὰ μέρη εὐθεῖαι ἴσαι τε καὶ παράλληλοί εἰσιν. ἡ ΠΥ ἄρα τῇ ΕΚ ἴση τε καὶ παράλληλός ἐστιν. πενταγώνου δὲ ἰσοπλεύρου ἡ ΕΚ· πενταγώνου ἄρα ἰσοπλεύρου καὶ ἡ ΠΥ τοῦ εἰς τὸν ΕΖΗΘΚ κύκλον ἐγγραφομένου. διὰ τὰ αὐτὰ δὴ καὶ ἑκάστη τῶν ΠΡ, ΡΣ, ΣΤ, ΤΥ πενταγώνου ἐστὶν ἰσοπλεύρου τοῦ εἰς τὸν ΕΖΗΘΚ κύκλον ἐγγραφομένου· ἰσόπλευρον ἄρα τὸ ΠΡΣΤΥ πεντάγωνον. καὶ ἐπεὶ ἑξαγώνου μέν ἐστιν ἡ ΠΕ, δεκαγώνου δὲ ἡ ΕΟ, καί ἐστιν ὀρθὴ ἡ ὑπὸ ΠΕΟ, πενταγώνου ἄρα ἐστὶν ἡ ΠΟ· ἡ γὰρ τοῦ πενταγώνου πλευρὰ δύναται τήν τε τοῦ ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΟΥ πενταγώνου ἐστὶ πλευρά. ἔστι δὲ καὶ ἡ ΠΥ πενταγώνου· ἰσόπλευρον ἄρα ἐστὶ τὸ ΠΟΥ τρίγωνον.

διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν ΠΛΡ, ΡΜΣ, ΣΝΤ, ΤΞΥ ἰσόπλευρόν ἐστιν. καὶ ἐπεὶ πενταγώνου ἐδείχθη ἑκατέρα τῶν ΠΛ, ΠΟ, ἔστι δὲ καὶ ἡ ΛΟ πενταγώνου, ἰσόπλευρον ἄρα ἐστὶ τὸ ΠΛΟ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν ΛΡΜ, ΜΣΝ, ΝΤΞ, ΞΥΟ τριγώνων ἰσόπλευρόν ἐστιν. εἰλήφθω τὸ κέντρον τοῦ ΕΖΗ ΘΚ κύκλου τὸ Φ σημεῖον· καὶ ἀπὸ τοῦ Φ τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἀνεστάτω ἡ ΦΩ, καὶ ἐκβεβλήσθω ἐπὶ τὰ ἕτερα μέρη ὡς ἡ ΦΨ, καὶ ἀφῃρήσθω ἑξαγώνου μὲν ἡ ΦΧ, δεκαγώνου δὲ ἑκατέρα τῶν ΦΨ, ΧΩ, καὶ ἐπεζεύχθωσαν αἱ ΠΩ, ΠΧ, ΥΩ, ΕΦ, ΛΦ, ΛΨ, ΨΜ. καὶ ἐπεὶ ἑκατέρα τῶν ΦΧ, ΠΕ τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθάς ἐστιν, παράλληλος ἄρα ἐστὶν ἡ ΦΧ τῇ ΠΕ. εἰσὶ δὲ καὶ ἴσαι· καὶ αἱ ΕΦ, ΠΧ ἄρα ἴσαι τε καὶ παράλληλοί εἰσιν. ἑξαγώνου δὲ ἡ ΕΦ· ἑξαγώνου ἄρα καὶ ἡ ΠΧ. καὶ ἐπεὶ ἑξαγώνου μέν ἐστιν ἡ ΠΧ, δεκαγώνου δὲ ἡ ΧΩ, καὶ ὀρθή ἐστιν ἡ ὑπὸ ΠΧΩ γωνία, πενταγώνου ἄρα ἐστὶν ἡ ΠΩ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΥΩ

πενταγώνου ἐστίν, ἐπειδήπερ, ἐὰν ἐπιζεύξωμεν τὰς ΦΚ, ΧΥ, ἴσαι καὶ ἀπεναντίον ἔσονται, καί ἐστιν ἡ ΦΚ ἐκ τοῦ κέντρου οὖσα ἑξαγώνου· ἑξαγώνου ἄρα καὶ ἡ ΧΥ. δεκαγώνου δὲ ἡ ΧΩ, καὶ ὀρθὴ ἡ ὑπὸ ΥΧΩ· πενταγώνου ἄρα ἡ ΥΩ. ἔστι δὲ καὶ ἡ ΠΥ πενταγώνου· ἰσόπλευρον ἄρα ἐστὶ τὸ ΠΥΩ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ὧν βάσεις μέν εἰσιν αἱ ΠΡ, ΡΣ, ΣΤ, ΤΥ εὐθεῖαι, κορυφὴ δὲ τὸ Ω σημεῖον, ἰσόπλευρόν ἐστιν. πάλιν, ἐπεὶ ἑξαγώνου μὲν ἡ ΦΛ, δεκαγώνου δὲ ἡ ΦΨ, καὶ ὀρθή ἐστιν ἡ ὑπὸ ΛΦΨ γωνία, πενταγώνου ἄρα ἐστὶν ἡ ΛΨ. διὰ τὰ αὐτὰ δὴ ἐὰν ἐπιζεύξωμεν τὴν ΜΦ οὖσαν ἑξαγώνου, συνάγεται καὶ ἡ ΜΨ πενταγώνου. ἔστι δὲ καὶ ἡ ΛΜ πενταγώνου· ἰσόπλευρον ἄρα ἐστὶ τὸ ΛΜΨ τρίγωνον. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ὧν βάσεις μέν εἰσιν αἱ ΜΝ, ΝΞ, ΞΟ, ΟΛ, κορυφὴ δὲ τὸ Ψ σημεῖον, ἰσόπλευρόν ἐστιν. συνέσταται ἄρα εἰκοσάεδρον ὑπὸ εἴκοσι τριγώνων ἰσοπλεύρων περιεχόμενον.

Δεῖ δὴ αὐτὸ καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

Ἐπεὶ γὰρ ἑξαγώνου ἐστὶν ἡ ΦΧ, δεκαγώνου δὲ ἡ ΧΩ, ἡ ΦΩ ἄρα ἄκρον καὶ μέσον λόγον τέμηται κατὰ τὸ Χ, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΦΧ· ἔστιν ἄρα ὡς ἡ ΩΦ πρὸς τὴν ΦΧ, οὕτως ἡ ΦΧ πρὸς τὴν ΧΩ. ἴση δὲ ἡ μὲν ΦΧ τῇ ΦΕ, ἡ δὲ ΧΩ τῇ ΦΨ· ἔστιν ἄρα ὡς ἡ ΩΦ πρὸς τὴν ΦΕ, οὕτως ἡ ΕΦ πρὸς τὴν ΦΨ. καί εἰσιν ὀρθαὶ αἱ ὑπὸ ΩΦΕ, ΕΦΨ γωνίαι· ἐὰν ἄρα ἐπιζεύξωμεν τὴν ΕΩ εὐθεῖαν, ὀρθὴ ἔσται ἡ ὑπὸ ΨΕΩ γωνία διὰ τὴν ὁμοιότητα τῶν ΨΕΩ, ΦΕΩ τριγώνων. διὰ τὰ αὐτὰ δὴ ἐπεί ἐστιν ὡς ἡ ΩΦ πρὸς τὴν ΦΧ, οὕτως ἡ ΦΧ πρὸς τὴν ΧΩ, ἴση δὲ ἡ μὲν ΩΦ τῇ ΨΧ, ἡ δὲ ΦΧ τῇ ΧΠ, ἔστιν ἄρα ὡς ἡ ΨΧ πρὸς τὴν ΧΠ, οὕτως ἡ ΠΧ πρὸς τὴν ΧΩ. καὶ διὰ τοῦτο πάλιν ἐὰν ἐπιζεύξωμεν τὴν ΠΨ, ὀρθὴ ἔσται ἡ πρὸς τῷ Π γωνία· τὸ ἄρα ἐπὶ τῆς ΨΩ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Π. καὶ ἐὰν μενούσης τῆς ΨΩ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο

φέρεσθαι, ἥξει καὶ διὰ τοῦ Π καὶ τῶν λοιπῶν σημείων τοῦ εἰκοσαέδρου, καὶ ἔσται σφαίρᾳ περιειλημμένον τὸ εἰκοσάεδρον. λέγω δή, ὅτι καὶ τῇ δοθείσῃ. τετμήσθω γὰρ ἡ ΦΧ δίχα κατὰ τὸ Α#. καὶ ἐπεὶ εὐθεῖα γραμμὴ ἡ ΦΩ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Χ, καὶ τὸ ἔλασσον αὐτῆς τμῆμά ἐστιν ἡ ΩΧ, ἡ ἄρα ΩΧ προσλαβοῦσα τὴν ἡμίσειαν τοῦ μείζονος τμήματος τὴν ΧΑ# πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τοῦ μείζονος τμήματος· πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΩΑ# τοῦ ἀπὸ τῆς Α#Χ. καί ἐστι τῆς μὲν ΩΑ# διπλῆ ἡ ΩΨ, τῆς δὲ Α#Χ διπλῆ ἡ ΦΧ· πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΩΨ τοῦ ἀπὸ τῆς ΧΦ. καὶ ἐπεὶ τετραπλῆ ἐστιν ἡ ΑΓ τῆς ΓΒ, πενταπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΓ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΔ· πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΩΨ πενταπλάσιον τοῦ ἀπὸ τῆς ΦΧ. καί ἐστιν ἴση ἡ ΔΒ τῇ ΦΧ· ἑκατέρα γὰρ αὐτῶν ἴση ἐστὶ τῇ ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου· ἴση ἄρα καὶ ἡ ΑΒ τῇ ΨΩ. καί ἐστιν ἡ ΑΒ ἡ τῆς δοθείσης σφαίρας διάμετρος· καὶ ἡ ΨΩ ἄρα ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ. τῇ ἄρα δοθείσῃ σφαίρᾳ περιείληπται τὸ εἰκοσάεδρον.

Λέγω δή, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων. ἐπεὶ γὰρ ῥητή ἐστιν ἡ τῆς σφαίρας διάμετρος, καί ἐστι δυνάμει πενταπλασίων τῆς ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου, ῥητὴ ἄρα ἐστὶ καὶ ἡ ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου· ὥστε καὶ ἡ διάμετρος αὐτοῦ ῥητή ἐστιν. ἐὰν δὲ εἰς κύκλον ῥητὴν ἔχοντα τὴν διάμετρον πεντάγωνον ἰσόπλευρον ἐγγραφῇ, ἡ τοῦ πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων. ἡ δὲ τοῦ ΕΖΗΘΚ πενταγώνου πλευρὰ ἡ τοῦ εἰκοσαέδρου ἐστίν. ἡ ἄρα τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων.

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει πενταπλασίων ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ κύκλου, ἀφ’ οὗ τὸ εἰκοσάεδρον

ἀναγέγραπται, καὶ ὅτι ἡ τῆς σφαίρας διάμετρος σύγκειται ἔκ τε τῆς τοῦ ἑξαγώνου καὶ δύο τῶν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων. ὅπερ ἔδει δεῖξαι.

ιζ΄. Δωδεκάεδρον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν, ᾗ καὶ τὰ προειρημένα σχήματα, καὶ δεῖξαι, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐκκείσθωσαν τοῦ προειρημένου κύβου δύο ἐπίπεδα πρὸς ὀρθὰς ἀλλήλοις τὰ ΑΒΓΔ, ΓΒΕΖ, καὶ τετμήσθω ἑκάστη τῶν ΑΒ, ΒΓ, ΓΔ, ΔΑ, ΕΖ, ΕΒ, ΖΓ πλευρῶν δίχα κατὰ τὰ Η, Θ, Κ, Λ, Μ, Ν, Ξ, καὶ ἐπεζεύχθωσαν αἱ ΗΚ, ΘΛ, ΜΘ, ΝΞ, καὶ τετμήσθω ἑκάστη τῶν ΝΟ, ΟΞ, ΘΠ ἄκρον καὶ μέσον λόγον κατὰ τὰ Ρ, Σ, Τ σημεῖα, καὶ ἔστω αὐτῶν μείζονα τμήματα τὰ ΡΟ, ΟΣ, ΤΠ, καὶ ἀνεστάτωσαν ἀπὸ τῶν Ρ, Σ, Τ σημείων τοῖς τοῦ κύβου ἐπιπέδοις πρὸς ὀρθὰς ἐπὶ τὰ ἐκτὸς μέρη τοῦ κύβου αἱ ΡΥ, ΣΦ, ΤΧ, καὶ κείσθωσαν ἴσαι ταῖς ΡΟ, ΟΣ, ΤΠ, καὶ ἐπεζεύχθωσαν αἱ ΥΒ, ΒΧ, ΧΓ, ΓΦ, ΦΥ. λέγω, ὅτι τὸ ΥΒΧΓΦ πεντάγωνον ἰσόπλευρόν τε καὶ ἐν ἑνὶ ἐπιπέδῳ καὶ ἔτι ἰσογώνιόν ἐστιν. ἐπεζεύχθωσαν γὰρ αἱ ΡΒ, ΣΒ, ΦΒ. καὶ ἐπεὶ εὐθεῖα ἡ ΝΟ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ρ, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΡΟ, τὰ ἄρα ἀπὸ τῶν ΟΝ, ΝΡ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΡΟ. ἴση δὲ ἡ μὲν ΟΝ τῇ ΝΒ, ἡ δὲ ΟΡ τῇ ΡΥ· τὰ ἄρα ἀπὸ τῶν ΒΝ, ΝΡ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΡΥ. τοῖς δὲ ἀπὸ τῶν ΒΝ, ΝΡ τὸ ἀπὸ τῆς ΒΡ ἐστιν ἴσον· τὸ ἄρα ἀπὸ τῆς ΒΡ τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΡΥ· ὥστε τὰ ἀπὸ τῶν ΒΡ, ΡΥ τετραπλάσιά ἐστι τοῦ ἀπὸ τῆς ΡΥ. τοῖς δὲ ἀπὸ τῶν ΒΡ, ΡΥ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΒΥ· τὸ ἄρα ἀπὸ τῆς ΒΥ τετραπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΥΡ· διπλῆ ἄρα ἐστὶν ἡ ΒΥ τῆς ΡΥ. ἔστι δὲ καὶ ἡ ΦΥ τῆς ΥΡ διπλῆ, ἐπειδήπερ καὶ ἡ ΣΡ τῆς ΟΡ, τουτέστι τῆς ΡΥ, ἐστι διπλῆ· ἴση ἄρα ἡ ΒΥ τῇ ΥΦ. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἑκάστη τῶν ΒΧ, ΧΓ, ΓΦ ἑκατέρᾳ τῶν ΒΥ, ΥΦ ἐστιν ἴση. ἰσόπλευρον ἄρα ἐστὶ τὸ ΒΥΦΓΧ πεντάγωνον. λέγω δή, ὅτι καὶ ἐν ἑνί ἐστιν ἐπιπέδῳ. ἤχθω γὰρ ἀπὸ τοῦ Ο

ἑκατέρᾳ τῶν ΡΥ, ΣΦ παράλληλος ἐπὶ τὰ ἐκτὸς τοῦ κύβου μέρη ἡ ΟΨ, καὶ ἐπεζεύχθωσαν αἱ ΨΘ, ΘΧ· λέγω, ὅτι ἡ ΨΘΧ εὐθεῖά ἐστιν. ἐπεὶ γὰρ ἡ ΘΠ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Τ, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΠΤ, ἔστιν ἄρα ὡς ἡ ΘΠ πρὸς τὴν ΠΤ, οὕτως ἡ ΠΤ πρὸς τὴν ΤΘ. ἴση δὲ ἡ μὲν ΘΠ τῇ ΘΟ, ἡ δὲ ΠΤ ἑκατέρᾳ τῶν ΤΧ, ΟΨ· ἔστιν ἄρα ὡς ἡ ΘΟ πρὸς τὴν ΟΨ, οὕτως ἡ ΧΤ πρὸς τὴν ΤΘ. καί ἐστι παράλληλος ἡ μὲν ΘΟ τῇ ΤΧ· ἑκατέρα γὰρ αὐτῶν τῷ ΒΔ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν· ἡ δὲ ΤΘ τῇ ΟΨ· ἑκατέρα γὰρ αὐτῶν τῷ ΒΖ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. ἐὰν δὲ δύο τρίγωνα συντεθῇ κατὰ μίαν γωνίαν, ὡς τὰ ΨΟΘ, ΘΤΧ, τὰς δύο πλευρὰς ταῖς δυσὶν ἀνάλογον ἔχοντα, ὥστε τὰς ὁμολόγους αὐτῶν πλευρὰς καὶ παραλλήλους εἶναι, αἱ λοιπαὶ εὐθεῖαι ἐπ’ εὐθείας ἔσονται· ἐπ’ εὐθείας ἄρα ἐστὶν ἡ ΨΘ τῇ ΘΧ. πᾶσα δὲ εὐθεῖα ἐν ἑνί ἐστιν ἐπιπέδῳ· ἐν ἑνὶ ἄρα ἐπιπέδῳ ἐστὶ τὸ ΥΒΧΓΦ πεντάγωνον.

Λέγω δή, ὅτι καὶ ἰσογώνιόν ἐστιν.

Ἐπεὶ γὰρ εὐθεῖα γραμμὴ ἡ ΝΟ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ρ, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΟΡ [ἔστιν ἄρα ὡς συναμφότερος ἡ ΝΟ, ΟΡ πρὸς τὴν ΟΝ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΡ], ἴση δὲ ἡ ΟΡ τῇ ΟΣ [ἔστιν ἄρα ὡς ἡ ΣΝ πρὸς τὴν ΝΟ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΣ], ἡ ΝΣ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ο, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΝΟ· τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΟ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΟ. ἴση δὲ ἡ μὲν ΝΟ τῇ ΝΒ, ἡ δὲ ΟΣ τῇ ΣΦ· τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΦ τετράγωνα τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΒ· ὥστε τὰ ἀπὸ τῶν ΦΣ, ΣΝ, ΝΒ τετραπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΒ. τοῖς δὲ ἀπὸ τῶν ΣΝ, ΝΒ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΣΒ· τὰ ἄρα ἀπὸ τῶν ΒΣ, ΣΦ, τουτέστι τὸ ἀπὸ τῆς ΒΦ [ὀρθὴ γὰρ ἡ ὑπὸ ΦΣΒ γωνία], τετραπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΝΒ· διπλῆ ἄρα ἐστὶν ἡ ΦΒ τῆς ΒΝ. ἔστι δὲ καὶ ἡ ΒΓ τῆς ΒΝ διπλῆ· ἴση ἄρα ἐστὶν ἡ ΒΦ τῇ ΒΓ. καὶ ἐπεὶ δύο αἱ ΒΥ, ΥΦ δυσὶ ταῖς ΒΧ, ΧΓ ἴσαι εἰσίν, καὶ βάσις ἡ ΒΦ βάσει τῇ ΒΓ ἴση, γωνία ἄρα ἡ ὑπὸ ΒΥΦ γωνίᾳ τῇ ὑπὸ ΒΧΓ ἐστιν ἴση. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ὑπὸ ΥΦΓ γωνία ἴση ἐστὶ τῇ ὑπὸ ΒΧΓ· αἱ ἄρα ὑπὸ

ΒΧΓ, ΒΥΦ, ΥΦΓ τρεῖς γωνίαι ἴσαι ἀλλήλαις εἰσίν. ἐὰν δὲ πενταγώνου ἰσοπλεύρου αἱ τρεῖς γωνίαι ἴσαι ἀλλήλαις ὦσιν, ἰσογώνιον ἔσται τὸ πεντάγωνον· ἰσογώνιον ἄρα ἐστὶ τὸ ΒΥΦΓΧ πεντάγωνον. ἐδείχθη δὲ καὶ ἰσόπλευρον· τὸ ἄρα ΒΥΦΓΧ πεντάγωνον ἰσόπλευρόν ἐστι καὶ ἰσογώνιον, καί ἐστιν ἐπὶ μιᾶς τοῦ κύβου πλευρᾶς τῆς ΒΓ. ἐὰν ἄρα ἐφ' ἑκάστης τῶν τοῦ κύβου δώδεκα πλευρῶν τὰ αὐτὰ κατασκευάσωμεν, συσταθήσεταί τι σχῆμα στερεὸν ὑπὸ δώδεκα πενταγώνων ἰσοπλεύρων τε καὶ ἰσογωνίων περιεχόμενον, ὃ καλεῖται δωδεκάεδρον.

Δεῖ δὴ αὐτὸ καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐκβεβλήσθω γὰρ ἡ ΨΟ, καὶ ἔστω ἡ ΨΩ· συμβάλλει ἄρα ἡ ΟΩ τῇ τοῦ κύβου διαμέτρῳ, καὶ δίχα τέμνουσιν ἀλλήλας· τοῦτο γὰρ δέδεικται ἐν τῷ παρατελεύτῳ θεωρήματι τοῦ ἑνδεκάτου βιβλίου. τεμνέτωσαν κατὰ τὸ Ω· τὸ Ω ἄρα κέντρον ἐστὶ τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον, καὶ ἡ ΩΟ ἡμίσεια τῆς πλευρᾶς τοῦ κύβου. ἐπεζεύχθω δὴ ἡ ΥΩ. καὶ ἐπεὶ εὐθεῖα γραμμὴ ἡ ΝΣ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ο, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΝΟ, τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΟ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΟ. ἴση δὲ ἡ μὲν ΝΣ τῇ ΨΩ, ἐπειδήπερ καὶ ἡ μὲν ΝΟ τῇ ΟΩ ἐστιν ἴση, ἡ δὲ ΨΟ τῇ ΟΣ. ἀλλὰ μὴν καὶ ἡ ΟΣ τῇ ΨΥ, ἐπεὶ καὶ τῇ ΡΟ· τὰ ἄρα ἀπὸ τῶν ΩΨ, ΨΥ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΟ. τοῖς δὲ ἀπὸ τῶν ΩΨ, ΨΥ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΥΩ· τὸ ἄρα ἀπὸ τῆς ΥΩ τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΝΟ. ἔστι δὲ καὶ ἡ ἐκ τοῦ κέντρου τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον δυνάμει τριπλασίων τῆς ἡμισείας τῆς τοῦ κύβου πλευρᾶς· προδέδεικται γὰρ κύβον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων ἐστὶ τῆς πλευρᾶς τοῦ κύβου. εἰ δὲ ὅλη τῆς ὅλης, καὶ [ἡ] ἡμίσεια τῆς ἡμισείας· καί ἐστιν ἡ ΝΟ ἡμίσεια τῆς τοῦ κύβου πλευρᾶς· ἡ ἄρα ΥΩ ἴση ἐστὶ τῇ ἐκ τοῦ κέντρου τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον. καί ἐστι τὸ Ω κέντρον τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον· τὸ Υ

ἄρα σημεῖον πρὸς τῇ ἐπιφανείᾳ ἐστὶ τῆς σφαίρας. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἑκάστη τῶν λοιπῶν γωνιῶν τοῦ δωδεκαέδρου πρὸς τῇ ἐπιφανείᾳ ἐστὶ τῆς σφαίρας· περιείληπται ἄρα τὸ δωδεκάεδρον τῇ δοθείσῃ σφαίρᾳ.

Λέγω δή, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐπεὶ γὰρ τῆς ΝΟ ἄκρον καὶ μέσον λόγον τετμημένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΡΟ, τῆς δὲ ΟΞ ἄκρον καὶ μέσον λόγον τετμημένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΟΣ, ὅλης ἄρα τῆς ΝΞ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΡΣ. οἷον ἐπεί ἐστιν ὡς ἡ ΝΟ πρὸς τὴν ΟΡ, ἡ ΟΡ πρὸς τὴν ΡΝ, καὶ τὰ διπλάσια· τὰ γὰρ μέρη τοῖς ἰσάκις πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον· ὡς ἄρα ἡ ΝΞ πρὸς τὴν ΡΣ, οὕτως ἡ ΡΣ πρὸς συναμφότερον τὴν ΝΡ, ΣΞ. μείζων δὲ ἡ ΝΞ τῆς ΡΣ· μείζων ἄρα καὶ ἡ ΡΣ συναμφοτέρου τῆς ΝΡ, ΣΞ· ἡ ΝΞ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΡΣ. ἴση δὲ ἡ ΡΣ τῇ ΥΦ· τῆς ἄρα ΝΞ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΥΦ. καὶ ἐπεὶ ῥητή ἐστιν ἡ τῆς σφαίρας διάμετρος καί ἐστι δυνάμει τριπλασίων τῆς τοῦ κύβου πλευρᾶς, ῥητὴ ἄρα ἐστὶν ἡ ΝΞ πλευρὰ οὖσα τοῦ κύβου. ἐὰν δὲ ῥητὴ γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, ἑκάτερον τῶν τμημάτων ἄλογός ἐστιν ἀποτομή.

Ἡ ΥΦ ἄρα πλευρὰ οὖσα τοῦ δωδεκαέδρου ἄλογός ἐστιν ἀποτομή.

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι τῆς τοῦ κύβου πλευρᾶς ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ τοῦ δωδεκαέδρου πλευρά. ὅπερ ἔδει δεῖξαι.

ιη΄. Τὰς πλευρὰς τῶν πέντε σχημάτων ἐκθέσθαι καὶ συγκρῖναι πρὸς ἀλλήλας.

Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ, καὶ τετμήσθω κατὰ τὸ Γ ὥστε ἴσην εἶναι τὴν ΑΓ τῇ ΓΒ, κατὰ δὲ τὸ Δ ὥστε διπλασίονα εἶναι τὴν ΑΔ

τῆς ΔΒ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΕΒ, καὶ ἀπὸ τῶν Γ, Δ τῇ ΑΒ πρὸς ὀρθὰς ἤχθωσαν αἱ ΓΕ, ΔΖ, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΖΒ, ΕΒ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΔ τῆς ΔΒ, τριπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΔ. ἀναστρέψαντι ἡμιολία ἄρα ἐστὶν ἡ ΒΑ τῆς ΑΔ. ὡς δὲ ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ· ἰσογώνιον γάρ ἐστι τὸ ΑΖΒ τρίγωνον τῷ ΑΖΔ τριγώνῳ· ἡμιόλιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΑ τοῦ ἀπὸ τῆς ΑΖ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει ἡμιολία τῆς πλευρᾶς τῆς πυραμίδος. καί ἐστιν ἡ ΑΒ ἡ τῆς σφαίρας διάμετρος· ἡ ΑΖ ἄρα ἴση ἐστὶ τῇ πλευρᾷ τῆς πυραμίδος.

Πάλιν, ἐπεὶ διπλασίων ἐστὶν ἡ ΑΔ τῆς ΔΒ, τριπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΔ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΖ· τριπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΖ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων τῆς τοῦ κύβου πλευρᾶς. καί ἐστιν ἡ ΑΒ ἡ τῆς σφαίρας διάμετρος· ἡ ΒΖ ἄρα τοῦ κύβου ἐστὶ πλευρά.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΒ, διπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΓ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΕ· διπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΕ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίων τῆς τοῦ ὀκταέδρου πλευρᾶς. καί ἐστιν ἡ ΑΒ ἡ τῆς δοθείσης σφαίρας διάμετρος· ἡ ΒΕ ἄρα τοῦ ὀκταέδρου ἐστὶ πλευρά.

Ἤχθω δὴ ἀπὸ τοῦ Α σημείου τῇ ΑΒ εὐθείᾳ πρὸς ὀρθὰς ἡ ΑΗ, καὶ κείσθω ἡ ΑΗ ἴση τῇ ΑΒ, καὶ ἐπεζεύχθω ἡ ΗΓ, καὶ ἀπὸ τοῦ Θ ἐπὶ τὴν ΑΒ κάθετος ἤχθω ἡ ΘΚ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΗΑ τῆς ΑΓ· ἴση γὰρ ἡ ΗΑ τῇ ΑΒ· ὡς δὲ ἡ ΗΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΘΚ πρὸς τὴν ΚΓ, διπλῆ ἄρα καὶ ἡ ΘΚ τῆς ΚΓ. τετραπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΘΚ τοῦ ἀπὸ τῆς ΚΓ· τὰ ἄρα ἀπὸ τῶν ΘΚ, ΚΓ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΘΓ, πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΚΓ. ἴση δὲ ἡ ΘΓ τῇ ΓΒ· πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΚ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΒ τῆς ΓΒ, ὧν ἡ ΑΔ τῆς ΔΒ ἐστι διπλῆ, λοιπὴ ἄρα ἡ ΒΔ λοιπῆς τῆς ΔΓ ἐστι διπλῆ. τριπλῆ ἄρα ἡ ΒΓ τῆς ΓΔ· ἐνναπλάσιον ἄρα τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΔ. πενταπλάσιον δὲ τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΚ·

μεῖζον ἄρα τὸ ἀπὸ τῆς ΓΚ τοῦ ἀπὸ τῆς ΓΔ. μείζων ἄρα ἐστὶν ἡ ΓΚ τῆς ΓΔ. κείσθω τῇ ΓΚ ἴση ἡ ΓΛ, καὶ ἀπὸ τοῦ Λ τῇ ΑΒ πρὸς ὀρθὰς ἤχθω ἡ ΛΜ, καὶ ἐπεζεύχθω ἡ ΜΒ. καὶ ἐπεὶ πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΚ, καί ἐστι τῆς μὲν ΒΓ διπλῆ ἡ ΑΒ, τῆς δὲ ΓΚ διπλῆ ἡ ΚΛ, πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΚΛ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει πενταπλασίων τῆς ἐκ τοῦ κέντρου τοῦ κύκλου, ἀφ’ οὗ τὸ εἰκοσάεδρον ἀναγέγραπται. καί ἐστιν ἡ ΑΒ ἡ τῆς σφαίρας διάμετρος· ἡ ΚΛ ἄρα ἐκ τοῦ κέντρου ἐστὶ τοῦ κύκλου, ἀφ’ οὗ τὸ εἰκοσάεδρον ἀναγέγραπται· ἡ ΚΛ ἄρα ἑξαγώνου ἐστὶ πλευρὰ τοῦ εἰρημένου κύκλου. καὶ ἐπεὶ ἡ τῆς σφαίρας διάμετρος σύγκειται ἔκ τε τῆς τοῦ ἑξαγώνου καὶ δύο τῶν τοῦ δεκαγώνου τῶν εἰς τὸν εἰρημένον κύκλον ἐγγραφομένων, καί ἐστιν ἡ μὲν ΑΒ ἡ τῆς σφαίρας διάμετρος, ἡ δὲ ΚΛ ἑξαγώνου πλευρά, καὶ ἴση ἡ ΑΚ τῇ ΛΒ, ἑκατέρα ἄρα τῶν ΑΚ, ΛΒ δεκαγώνου ἐστὶ πλευρὰ τοῦ ἐγγραφομένου εἰς τὸν κύκλον, ἀφ’ οὗ τὸ εἰκοσάεδρον ἀναγέγραπται. καὶ ἐπεὶ δεκαγώνου μὲν ἡ ΛΒ, ἑξαγώνου δὲ ἡ ΜΛ· ἴση γάρ ἐστι τῇ ΚΛ, ἐπεὶ καὶ τῇ ΘΚ· ἴσον γὰρ ἀπέχουσιν ἀπὸ τοῦ κέντρου· καί ἐστιν ἑκατέρα τῶν ΘΚ, ΚΛ διπλασίων τῆς ΚΓ· πενταγώνου ἄρα ἐστὶν ἡ ΜΒ. ἡ δὲ τοῦ πενταγώνου ἐστὶν ἡ τοῦ εἰκοσαέδρου· εἰκοσαέδρου ἄρα ἐστὶν ἡ ΜΒ.

Καὶ ἐπεὶ ἡ ΖΒ κύβου ἐστὶ πλευρά, τετμήσθω ἄκρον καὶ μέσον λόγον κατὰ τὸ Ν, καὶ ἔστω μεῖζον τμῆμα τὸ ΝΒ· ἡ ΝΒ ἄρα δωδεκαέδρου ἐστὶ πλευρά.

Καὶ ἐπεὶ ἡ τῆς σφαίρας διάμετρος ἐδείχθη τῆς μὲν ΑΖ πλευρᾶς τῆς πυραμίδος δυνάμει ἡμιολία, τῆς δὲ τοῦ ὀκταέδρου τῆς ΒΕ δυνάμει διπλασίων, τῆς δὲ τοῦ κύβου τῆς ΖΒ δυνάμει τριπλασίων, οἵων ἄρα ἡ τῆς σφαίρας διάμετρος δυνάμει ἕξ, τοιούτων ἡ μὲν τῆς πυραμίδος τεσσάρων, ἡ δὲ τοῦ ὀκταέδρου τριῶν, ἡ δὲ τοῦ κύβου δύο. ἡ μὲν ἄρα τῆς πυραμίδος πλευρὰ τῆς μὲν τοῦ ὀκταέδρου πλευρᾶς δυνάμει ἐστὶν ἐπίτριτος, τῆς δὲ τοῦ κύβου δυνάμει διπλῆ, ἡ δὲ τοῦ ὀκταέδρου τῆς τοῦ κύβου δυνάμει ἡμιολία. αἱ μὲν οὖν εἰρημέναι τῶν τριῶν σχημάτων πλευραί, λέγω δὴ πυραμίδος καὶ

ὀκταέδρου καὶ κύβου, πρὸς ἀλλήλας εἰσὶν ἐν λόγοις ῥητοῖς. αἱ δὲ λοιπαὶ δύο, λέγω δὴ ἥ τε τοῦ εἰκοσαέδρου καὶ ἡ τοῦ δωδεκαέδρου, οὔτε πρὸς ἀλλήλας οὔτε πρὸς τὰς προειρημένας εἰσὶν ἐν λόγοις ῥητοῖς· ἄλογοι γάρ εἰσιν, ἡ μὲν ἐλάττων, ἡ δὲ ἀποτομή.

Ὅτι μείζων ἐστὶν ἡ τοῦ εἰκοσαέδρου πλευρὰ ἡ ΜΒ τῆς τοῦ δωδεκαέδρου τῆς ΝΒ, δείξομεν οὕτως.

Ἐπεὶ γὰρ ἰσογώνιόν ἐστι τὸ ΖΔΒ τρίγωνον τῷ ΖΑΒ τριγώνῳ, ἀνάλογόν ἐστιν ὡς ἡ ΔΒ πρὸς τὴν ΒΖ, οὕτως ἡ ΒΖ πρὸς τὴν ΒΑ. καὶ ἐπεὶ τρεῖς εὐθεῖαι ἀνάλογόν εἰσιν, ἔστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας· ἔστιν ἄρα ὡς ἡ ΔΒ πρὸς τὴν ΒΑ, οὕτως τὸ ἀπὸ τῆς ΔΒ πρὸς τὸ ἀπὸ τῆς ΒΖ· ἀνάπαλιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΖΒ πρὸς τὸ ἀπὸ τῆς ΒΔ. τριπλῆ δὲ ἡ ΑΒ τῆς ΒΔ· τριπλάσιον ἄρα τὸ ἀπὸ τῆς ΖΒ τοῦ ἀπὸ τῆς ΒΔ. ἔστι δὲ καὶ τὸ ἀπὸ τῆς ΑΔ τοῦ ἀπὸ τῆς ΔΒ τετραπλάσιον· διπλῆ γὰρ ἡ ΑΔ τῆς ΔΒ· μεῖζον ἄρα τὸ ἀπὸ τῆς ΑΔ τοῦ ἀπὸ τῆς ΖΒ· μείζων ἄρα ἡ ΑΔ τῆς ΖΒ· πολλῷ ἄρα ἡ ΑΛ τῆς ΖΒ μείζων ἐστίν. καὶ τῆς μὲν ΑΛ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΚΛ, ἐπειδήπερ ἡ μὲν ΛΚ ἑξαγώνου ἐστίν, ἡ δὲ ΚΑ δεκαγώνου· τῆς δὲ ΖΒ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΝΒ· μείζων ἄρα ἡ ΚΛ τῆς ΝΒ. ἴση δὲ ἡ ΚΛ τῇ ΛΜ·μείζων ἄρα ἡ ΛΜ τῆς ΝΒ [τῆς δὲ ΛΜ μείζων ἐστὶν ἡ ΜΒ]. πολλῷ ἄρα ἡ ΜΒ πλευρὰ οὖσα τοῦ εἰκοσαέδρου μείζων ἐστὶ τῆς ΝΒ πλευρᾶς οὔσης τοῦ δωδεκαέδρου· ὅπερ ἔδει δεῖξαι.

Λέγω δή, ὅτι παρὰ τὰ εἰρημένα πέντε σχήματα οὐ συσταθήσεται ἕτερον σχῆμα περιεχόμενον ὑπὸ ἰσοπλεύρων τε καὶ ἰσογωνίων ἴσων ἀλλήλοις.

Ὑπὸ μὲν γὰρ δύο τριγώνων ἢ ὅλως ἐπιπέδων στερεὰ γωνία οὐ συνίσταται. ὑπὸ δὲ τριῶν τριγώνων ἡ τῆς πυραμίδος, ὑπὸ δὲ τεσσάρων ἡ τοῦ ὀκταέδρου, ὑπὸ δὲ πέντε ἡ τοῦ εἰκοσαέδρου· ὑπὸ δὲ ἓξ τριγώνων ἰσοπλεύρων τε καὶ ἰσογωνίων πρὸς ἑνὶ σημείῳ συνισταμένων οὐκ ἔσται στερεὰ γωνία· οὔσης γὰρ τῆς τοῦ ἰσοπλεύρου τριγώνου γωνίας διμοίρου ὀρθῆς ἔσονται αἱ ἓξ

τέσσαρσιν ὀρθαῖς ἴσαι· ὅπερ ἀδύνατον· ἅπασα γὰρ στερεὰ γωνία ὑπὸ ἐλασσόνων ἢ τεσσάρων ὀρθῶν περιέχεται. διὰ τὰ αὐτὰ δὴ οὐδὲ ὑπὸ πλειόνων ἢ ἓξ γωνιῶν ἐπιπέδων στερεὰ γωνία συνίσταται. ὑπὸ δὲ τετραγώνων τριῶν ἡ τοῦ κύβου γωνία περιέχεται· ὑπὸ δὲ τεσσάρων ἀδύνατον· ἔσονται γὰρ πάλιν τέσσαρες ὀρθαί. ὑπὸ δὲ πενταγώνων ἰσοπλεύρων καὶ ἰσογωνίων, ὑπὸ μὲν τριῶν ἡ τοῦ δωδεκαέδρου· ὑπὸ δὲ τεσσάρων ἀδύνατον· οὔσης γὰρ τῆς τοῦ πενταγώνου ἰσοπλεύρου γωνίας ὀρθῆς καὶ πέμπτου, ἔσονται αἱ τέσσαρες γωνίαι τεσσάρων ὀρθῶν μείζους· ὅπερ ἀδύνατον. οὐδὲ μὴν ὑπὸ πολυγώνων ἑτέρων σχημάτων περισχεθήσεται στερεὰ γωνία διὰ τὸ αὐτὸ ἄτοπον.

Οὐκ ἄρα παρὰ τὰ εἰρημένα πέντε σχήματα ἕτερον σχῆμα στερεὸν συσταθήσεται ὑπὸ ἰσοπλεύρων τε καὶ ἰσογωνίων περιεχόμενον· ὅπερ ἔδει δεῖξαι.

Λῆμμα

Ὅτι δὲ ἡ τοῦ ἰσοπλεύρου καὶ ἰσογωνίου πενταγώνου γωνία ὀρθή ἐστι καὶ πέμπτου, οὕτω δεικτέον.

Ἔστω γὰρ πεντάγωνον ἰσόπλευρον καὶ ἰσογώνιον τὸ ΑΒΓΔΕ, καὶ περιγεγράφθω περὶ αὐτὸ κύκλος ὁ ΑΒΓ ΔΕ, καὶ εἰλήφθω αὐτοῦ τὸ κέντρον τὸ Ζ, καὶ ἐπεζεύχθωσαν αἱ ΖΑ, ΖΒ, ΖΓ, ΖΔ, ΖΕ.

δίχα ἄρα τέμνουσι τὰς πρὸς τοῖς Α, Β, Γ, Δ, Ε τοῦ πενταγώνου γωνίας. καὶ ἐπεὶ αἱ πρὸς τῷ Ζ πέντε γωνίαι τέσσαρσιν ὀρθαῖς ἴσαι εἰσὶ καί εἰσιν ἴσαι, μία ἄρα αὐτῶν, ὡς ἡ ὑπὸ ΑΖΒ, μιᾶς ὀρθῆς ἐστι παρὰ πέμπτον· λοιπαὶ ἄρα αἱ ὑπὸ ΖΑΒ, ΑΒΖ μιᾶς εἰσιν ὀρθῆς καὶ πέμπτου. ἴση δὲ ἡ ὑπὸ ΖΑΒ τῇ ὑπὸ ΖΒΓ· καὶ ὅλη ἄρα ἡ ὑπὸ ΑΒΓ τοῦ πενταγώνου γωνία μιᾶς ἐστιν ὀρθῆς καὶ πέμπτου· ὅπερ ἔδει δεῖξαι.

The Biography

Ptolemy I, who reigned Egypt from 323 to 285 BC — Proclus states in his commentary on the 'Elements' that Euclid lived in the time of Ptolemy I., king of Egypt, who reigned from 323 to 285 BC.

Euclid by John Sturgeon Mackay

From '1911 Encyclopædia Britannica, Volume 9'

EUCLID**,** Greek mathematician of the 3rd century B.C.; we are ignorant not only of the dates of his birth and death, but also of his parentage, his teachers, and the residence of his early years. In some of the editions of his works he is called *Megarensis*, as if he had been born at Megara in Greece, a mistake which arose from confounding him with another Euclid, a disciple of Socrates. Proclus (A.D. 412–485), the authority for most of our information regarding Euclid, states in his commentary on the first book of the *Elements* that Euclid lived in the time of Ptolemy I., king of Egypt, who reigned from 323 to 285 B.C., that he was younger than the associates of Plato, but older than Eratosthenes (276–196 B.C.) and Archimedes (287–212 B.C.). Euclid is said to have founded the mathematical school of Alexandria, which was at that time becoming a centre, not only of commerce, but of learning and research, and for this service to the cause of exact science he would have deserved commemoration, even if his writings had not secured him a worthier title to fame. Proclus preserves a reply made by Euclid to King Ptolemy, who asked whether he could not learn geometry more easily than by studying the *Elements*— "There is no royal road to geometry." Pappus of Alexandria, in his *Mathematical Collection*, says that Euclid was a man of mild and inoffensive temperament, unpretending, and kind to all genuine students of mathematics. This being all that is known of the life and character of Euclid, it only remains therefore to speak of his works.

Among those which have come down to us the most remarkable is the *Elements* (Στοιχεῖα) (see Geometry). They consist of thirteen books; two more are frequently added, but there is reason to believe that they are the work of a later mathematician, Hypsicles of Alexandria.

The question has often been mooted, to what extent Euclid, in his *Elements*, is a discoverer or a compiler. To this question no entirely satisfactory answer can be given, for scarcely any of the writings of earlier geometers have come down to our times. We are mainly dependent on Pappus and Proclus for the scanty notices we have of Euclid's predecessors, and of the problems which engaged their attention; for the solution of problems, and not the discovery of theorems, would seem to have been their principal object. From these authors we learn that the property of the right-angled triangle had been found out, the principles of geometrical analysis laid down, the restriction of constructions in plane geometry to the straight line and the circle agreed upon, the doctrine of proportion, for both commensurables and incommensurables, as well as loci, plane and solid, and some of the properties of the conic sections investigated, the five regular solids (often called the Platonic bodies) and the relation between the volume of a cone or pyramid and that of its circumscribed cylinder or prism discovered. Elementary works had been written, and the famous problem of the duplication of the cube reduced to the determination of two mean proportionals between two given straight lines. Notwithstanding this amount of discovery, and all that it implied, Euclid must have made a great advance beyond his predecessors (we are told that "he arranged the discoveries of Eudoxus, perfected those of Theaetetus, and reduced to invincible demonstration many things that had previously been more loosely proved"), for his *Elements* supplanted all similar treatises, and, as Apollonius received

the title of “the great geometer,” so Euclid has come down to later ages as “the elementator.”

For the past twenty centuries parts of the *Elements*, notably the first six books, have been used as an introduction to geometry. Though they are now to some extent superseded in most countries, their long retention is a proof that they were, at any rate, not unsuitable for such a purpose. They are, speaking generally, not too difficult for novices in the science; the demonstrations are rigorous, ingenious and often elegant; the mixture of problems and theorems gives perhaps some variety, and makes their study less monotonous; and, if regard be had merely to the metrical properties of space as distinguished from the graphical, hardly any cardinal geometrical truths are omitted. With these excellences are combined a good many defects, some of them inevitable to a system based on a very few axioms and postulates. Thus the arrangement of the propositions seems arbitrary; associated theorems and problems are not grouped together; the classification, in short, is imperfect. Other objections, not to mention minor blemishes, are the prolixity of the style, arising partly from a defective nomenclature, the treatment of parallels depending on an axiom which is not axiomatic, and the sparing use of superposition as a method of proof.

Of the thirty-three ancient books subservient to geometrical analysis, Pappus enumerates first the *Data* (Δεδομένα) of Euclid. He says it contained 90 propositions, the scope of which he describes; it now consists of 95. It is not easy to explain this discrepancy, unless we suppose that some of the propositions, as they existed in the time of Pappus, have since been split into two, or that what were once scholia have since been erected into propositions. The object of the *Data* is to show that when certain things — lines, angles, spaces, ratios, &c. — are given by hypothesis, certain other things are given, that is, are determinable. The book, as we are expressly told, and as we may

gather from its contents, was intended for the investigation of problems; and it has been conjectured that Euclid must have extended the method of the *Data* to the investigation of theorems. What prompts this conjecture is the similarity between the analysis of a theorem and the method, common enough in the *Elements*, of *reductio ad absurdum* — the one setting out from the supposition that the theorem is true, the other from the supposition that it is false, thence in both cases deducing a chain of consequences which ends in a conclusion previously known to be true or false.

The *Introduction to Harmony* (Εἰσαγωγὴ ἁρμονική), and the *Section of the Scale* (Κατατομὴ κανόνος), treat of music. There is good reason for believing that one at any rate, and probably both, of these books are not by Euclid. No mention is made of them by any writer previous to Ptolemy (A.D. 140), or by Ptolemy himself, and in no ancient codex are they ascribed to Euclid.

The *Phaenomena* (Φαινόμενα) contains an exposition of the appearances produced by the motion attributed to the celestial sphere. Pappus, in the few remarks prefatory to his sixth book, complains of the faults, both of omission and commission, of writers on astronomy, and cites as an example of the former the second theorem of Euclid's *Phaenomena*, whence, and from the interpolation of other proofs, David Gregory infers that this treatise is corrupt.

The *Optics* and *Catoptrics* (Ὀπτικά, Κατοπτρικά) are ascribed to Euclid by Proclus, and by Marinus in his preface to the *Data*, but no mention is made of them by Pappus. This latter circumstance, taken in connexion with the fact that two of the propositions in the sixth book of the *Mathematical Collection* prove the same things as three in the *Optics*, is one of the reasons given by Gregory for deeming that work spurious. Several other reasons will be found in Gregory's preface to his edition of Euclid's works.

In some editions of Euclid's works there is given a book on the *Divisions of Superficies*, which consists of a few propositions, showing how a straight line may be drawn to divide in a given ratio triangles, quadrilaterals and pentagons. This was supposed by John Dee of London, who transcribed or translated it, and entrusted it for publication to his friend Federico Commandino of Urbino, to be the treatise of Euclid referred to by Proclus as τὸ περὶ διαιρέσεων βιβλίον. Dee mentions that, in the copy from which he wrote, the book was ascribed to Machomet of Bagdad, and adduces two or three reasons for thinking it to be Euclid's. This opinion, however, he does not seem to have held very strongly, nor does it appear that it was adopted by Commandino. The book does not exist in Greek.

The fragment, in Latin, *De levi et ponderoso*, which is of no value, and was printed at the end of Gregory's edition only in order that nothing might be left out, is mentioned neither by Pappus nor Proclus, and occurs first in Bartholomew Zamberti's edition of 1537. There is no reason for supposing it to be genuine.

The following works attributed to Euclid are not now extant: —

1. Three books on *Porisms* (Περὶ τῶν πορισμάτων) are mentioned both by Pappus and Proclus, and the former gives an abstract of them, with the lemmas assumed. (See Porism.)

2. Two books are mentioned, named Τόπων πρὸς ἐπιφανείᾳ, which is rendered *Locorum ad superficiem* by Commandino and subsequent geometers. These books were subservient to the analysis of loci, but the four lemmas which refer to them and which occur at the end of the seventh book of the *Mathematical Collection*, throw very little light on their contents. R. Simson's opinion was that they treated of curves of double curvature, and he intended at one time to write a treatise on the subject. (See Trail's *Life of Dr Simson*).

3. Pappus says that Euclid wrote four books on the *Conic Sections* (βιβλία τέσσαρα Κωνικῶν), which Apollonius amplified, and to which he added four more. It is known that, in the time of Euclid, the parabola was considered as the section of a right-angled cone, the ellipse that of an acute-angled cone, the hyperbola that of an obtuse-angled cone, and that Apollonius was the first who showed that the three sections could be obtained from any cone. There is good ground therefore for supposing that the first four books of Apollonius's *Conics*, which are still extant, resemble Euclid's *Conics* even less than Euclid's *Elements* do those of Eudoxus and Theaetetus.

4. A book on *Fallacies* (Περὶ ψευδαρίων) is mentioned by Proclus, who says that Euclid wrote it for the purpose of exercising beginners in the detection of errors in reasoning.

This notice of Euclid would be incomplete without some account of the earliest and the most important editions of his works. Passing over the commentators of the Alexandrian school, the first European translator of any part of Euclid is Boëtius (500), author of the *De consolatione philosophiae*. His *Euclidis Megarensis geometriae libri duo* contain nearly all the definitions of the first three books of the *Elements*, the postulates, and most of the axioms. The enunciations, with diagrams but no proofs, are given of most of the propositions in the first, second and fourth books, and a few from the third. Some centuries afterwards, Euclid was translated into Arabic, but the only printed version in that language is the one made of the thirteen books of the *Elements* by Nasir Al-Dīn Al-Tūsī (13th century), which appeared at Rome in 1594.

The first printed edition of Euclid was a translation of the fifteen books of the *Elements* from the Arabic, made, it is supposed, by Adelard of Bath (12th century), with the comments of Campanus of Novara. It appeared at Venice in 1482, printed by Erhardus Ratdolt, and dedicated to the doge Giovanni

Mocenigo. This edition represents Euclid very inadequately; the comments are often foolish, propositions are sometimes omitted, sometimes joined together, useless cases are interpolated, and now and then Euclid's order changed.

The first printed translation from the Greek is that of Bartholomew Zamberti, which appeared at Venice in 1505. Its contents will be seen from the title: *Euclidis megarēsis philosophi platonici MathematicaruF disciplinarū Janitoris: Habent in hoc volumine quicūqF ad mathematicā substantiā aspirāt: elemētorum libros xiii cū expositione Theonis insignis mathematici ... Quibus ... adjuncta. Deputatum scilicet Euclidi volumē xiiii cū expositiōe Hypsi. Alex. ItidēqF Phaeno. Specu. Perspe. cum expositione Theonis ac mirandus ille liber Datorum cum expostiōe Pappi Mechanici una cū Marini dialectici protheoria. Bar. Zāber. Vene. Interpte.*

The first printed Greek text was published at Basel, in 1533, with the title Εὐκλείδου Στοιχεῖων βιβλ. ιέ ἐκ τῶν Θέωνος συνουσιῶν. It was edited by Simon Grynaeus from two MSS. sent to him, the one from Venice by Lazarus Bayfius, and the other from Paris by John Ruellius. The four books of Proclus's commentary are given at the end from an Oxford MS. supplied by John Claymundus.

The English edition, the only one which contains all the extant works attributed to Euclid, is that of Dr David Gregory, published at Oxford in 1703, with the title, Εὐκλείδου τὰ σωζόμενα. *Euclidis quae supersunt omnia*. The text is that of the Basel edition, corrected from the MSS. bequeathed by Sir Henry Savile, and from Savile's annotations on his own copy. The Latin translation, which accompanies the Greek on the same page, is for the most part that of Commandino. The French edition has the title, *Les Œuvres d'Euclide, traduites en Latin et en Français, d'après un manuscrit très-ancien qui était resté inconnu jusqu'à nos jours. Par F. Peyrard, Traducteur*

des œuvres d'Archimède. It was published at Paris in three volumes, the first of which appeared in 1814, the second in 1816 and the third in 1818. It contains the *Elements* and the *Data*, which are, says the editor, certainly the only works which remain to us of this ever-celebrated geometer. The texts of the Basel and Oxford editions were collated with 23 MSS., one of which belonged to the library of the Vatican, but had been sent to Paris by the comte de Peluse (Monge). The Vatican MS. was supposed to date from the 9th century; and to its readings Peyrard gave the greatest weight. What may be called the German edition has the title Εὐκλείδου Στοιχεῖα. *Euclidis Elementa ex optimis libris in usum Tironum Graece edita ab Ernesto Ferdinando August*. It was published at Berlin in two parts, the first of which appeared in 1826 and the second in 1829. The above mentioned texts were collated with three other MSS. Modern standard editions are by Dr Heiberg of Copenhagen, *Euclidis Elementa, edidit et Latine interpretatus est J. L. Heiberg*. vols. i.-v. (Lipsiae, 1883–1888), and by T. L. Heath, *The Thirteen Books of Euclid's Elements*, vols. i.-iii. (Cambridge, 1908).

Of translations of the *Elements* into modern languages the number is very large. The first English translation, published at London in 1570, has the title, *The Elements of Geometrie of the most auncient Philosopher Euclide of Megara. Faithfully* (*now first*) *translated into the Englishe toung, by H. Billingsley, Citizen of London. Whereunto are annexed certaine Scholies, Annotations and Inventions, of the best Mathematiciens, both of time past and in this our age*. The first French translation of the whole of the *Elements* has the title, *Les Quinze Livres des Elements d'Euclide. Traduicts de Latin en François. Par D. Henrion, Mathematicien*. The first edition of it was published at Paris in 1615, and a second, corrected and augmented, in 1623. Pierre Forcadel de Beziés had published at Paris in 1564 a translation of the first six books of the *Elements*, and in 1565 of the seventh, eighth and ninth

books. An Italian translation, with the title, *Euclide Megarense acutissimo philosopho solo introduttore delle Scientie Mathematice. Diligentemente rassettato, et alla integrità ridotto, per il degno professore di tal Scientie Nicolò Tartalea Brisciano*, was published at Venice in 1569, and Federico Commandino's translation appeared at Urbino in 1575; a Spanish version, *Los Seis Libros primeros de la geometria de Euclides. Traduzidos en lēgua Española por Rodrigo Camorano, Astrologo y Mathematico*, at Seville in 1576; and a Turkish one, translated from the edition of J. Bonnycastle by Husaīn Rifkī, at Bulak in 1825. Dr Robert Simson's editions of the first six and the eleventh and twelfth books of the *Elements*, and of the *Data*.

Authorities. — The authors and editions above referred to; Fabricius, *Bibliotheca Graeca*, vol. iv.; Murhard's *Litteratur der mathematischen Wissenschaften*; Heilbronner's *Historia matheseos universae*; De Morgan's article "Eucleides" in Smith's *Dictionary of Biography and Mythology*; Moritz Cantor's *Geschichte der Mathematik*, vol. i. (J. S. M.)

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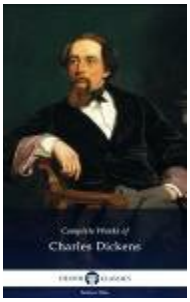

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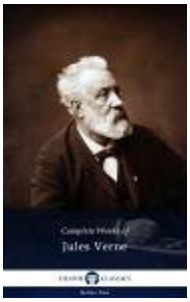

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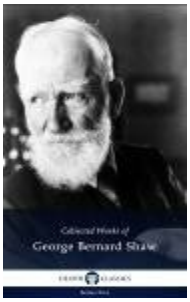

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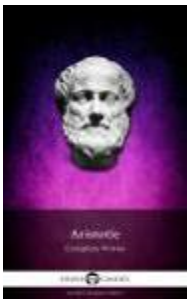

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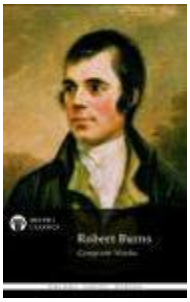
Athenaeus
Augustine
Aulus Gellius
Bede
Callimachus
Cassius Dio
Cato
Catullus
Cicero
Claudian
Clement of Alexandria
Cornelius Nepos
Demosthenes
Dio Chrysostom
Diodorus Siculus
Dionysius of Halicarnassus
Diogenes Laërtius
Ennius

Epictetus
Euclid
Euripides
Eusebius
Eutropius
Florus
Frontius
Fronto
Gregory I
Herodotus
Hesiod
Hippocrates
Homer
Horace
Isocrates
Josephus
Julian
Julius Caesar
Juvenal
Livy
Longus
Lucan
Lucian
Lucretius
Marcus Aurelius
Manetho
Martial
Nonnus
Origen
Ovid
Pausanias
Petronius
Pindar
Plato
Plautus
Pliny the Elder
Pliny the Younger
Plotinus
Plutarch

Polybius
Procopius
Propertius
Quintus Curtius Rufus
Quintus Smyrnaeus
Sallust
Sappho
Seneca the Younger
Septuagint
Sidonius
Sophocles
Statius
Strabo
Suetonius
Tacitus
Terence
Tertullian
Theocritus
Theophrastus
Thucydides
Tibullus
Varro
Virgil
Vitruvius
Xenophon



Delphi Poets Series

Adam Mickiewicz
A. E. Housman
Alexander Pope

Alfred, Lord Tennyson
Algernon Charles Swinburne
Andrew Marvell
Beowulf
Charlotte Smith
Christina Rossetti
Confederation Poets
D. H Lawrence (poetry)
Dante Alighieri (English)
Dante Gabriel Rossetti
Delphi Poetry Anthology
Edgar Allan Poe (poetry)
Edmund Spenser
Edmund Waller
Edward Lear
Edward Thomas
Edwin Arlington Robinson
Ella Wheeler Wilcox
Elizabeth Barrett Browning
Emily Dickinson
Epic of Gilgamesh
Ezra Pound
Friedrich Schiller (English)
George Chapman
George Herbert
Gerard Manley Hopkins
Giacomo Leopardi
Hafez
Heinrich Heine
Henry Howard, Earl of Surrey
Henry Lawson
Henry Wadsworth Longfellow
Isaac Rosenberg
James Russell Lowell
Johan Ludvig Runeberg
John Clare
John Donne
John Dryden
John Gay

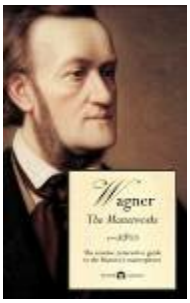
John Greenleaf Whittier
John Gower
John Keats
John Milton
John Wilmot, Earl of Rochester
Joseph Addison
Kahlil Gibran
Leigh Hunt
Lord Byron
Ludovico Ariosto
Luís de Camões
Mary Robinson
Matthew Arnold
Matthew Prior
Michael Drayton
Nikolai Nekrasov
Paul Laurence Dunbar
Percy Bysshe Shelley
Petrarch
Rabindranath Tagore
Ralph Waldo Emerson
Robert Browning
Robert Burns
Robert Frost
Robert Southey
Rumi
Rupert Brooke
Saadi
Sanskrit Epics
Sara Teasdale
Samuel Taylor Coleridge
Sir Philip Sidney
Sir Thomas Wyatt
Sir Walter Raleigh
Thomas Chatterton
Thomas Gray
Thomas Hardy (poetry)
Thomas Hood
Thomas Moore

Torquato Tasso
T. S. Eliot
W. B. Yeats
Walter Savage Landor
Walt Whitman
Wilfred Owen
William Blake
William Congreve
William Cowper
William Wordsworth

Masters of Art

Albrecht Dürer
Amedeo Modigliani
Anthony van Dyck
Artemisia Gentileschi
Berthe Morisot
Camille Pissarro
Canaletto
Caravaggio
Caspar David Friedrich
Claude Lorrain
Claude Monet
Correggio
Dante Gabriel Rossetti
Diego Velázquez
Donatello
Edgar Degas
Édouard Manet
Edvard Munch

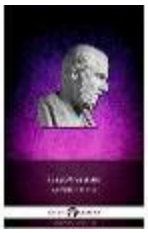
El Greco
Eugène Delacroix
Francisco Goya
Georges Seurat
Giotto
Giovanni Bellini
Gustave Courbet
Gustav Klimt
Hieronymus Bosch
Hokusai
Jacques-Louis David
James Abbott McNeill Whistler
Jan van Eyck
Jean-Auguste-Dominique Ingres
J. M. W. Turner
Johannes Vermeer
John Constable
Leonardo da Vinci
Masaccio
Michelangelo
Paul Cézanne
Paul Gauguin
Paul Klee
Peter Paul Rubens
Pierre-Auguste Renoir
Pieter Bruegel the Elder
Piet Mondrian
Sandro Botticelli
Raphael
Rembrandt van Rijn
Thomas Gainsborough
Tintoretto
Titian
Vincent van Gogh
Wassily Kandinsky
William Hogarth



Great Composers

Antonio Vivaldi
Antonín Dvořák
Franz Schubert
Frédéric Chopin
Giuseppe Verdi
Johann Sebastian Bach
Joseph Haydn
Ludwig van Beethoven
Piotr Illitch Tchaïkovsky
Richard Wagner
Wolfgang Amadeus Mozart

Alphabetical List of Titles



A. E. Housman
A. E. W. Mason
Abraham Lincoln
Achilles Tatius
Adam Mickiewicz
Adam Smith
Aeschines
Aeschylus
Albrecht Dürer

Aldous Huxley
Alexander Pope
Alexander Pushkin
Alexandre Dumas (English)
Alfred, Lord Tennyson
Algernon Blackwood
Algernon Charles Swinburne
Ambrose Bierce
Amedeo Modigliani
Ammianus Marcellinus
Anatole France
Andrew Lang
Andrew Marvell
Ann Radcliffe
Anna Katharine Green
Anthony Hope
Anthony Trollope
Anthony van Dyck
Anton Chekhov
Antonín Dvořák
Antonio Vivaldi
Aphra Behn
Apollodorus
Apollonius of Rhodes
Appian
Apuleius
Aristophanes
Aristotle
Arnold Bennett
Arrian
Artemisia Gentileschi
Arthur Machen
Arthur Morrison
Arthur Schopenhauer
Athenaeus
August Strindberg
Augustine
Aulus Gellius
Baroness Emma Orczy

Baruch Spinoza
Beatrix Potter
Beaumont and Fletcher
Bede
Ben Jonson
Benjamin Disraeli
Beowulf
Berthe Morisot
Booth Tarkington
Bram Stoker
Bret Harte
C. S. Forester
C. S. Lewis
Callimachus
Camille Pissarro
Canaletto
Captain Frederick Marryat
Captain Mayne Reid
Caravaggio
Carolyn Wells
Caspar David Friedrich
Cassius Dio
Cato
Catullus
Charles and Mary Lamb
Charles Brockden Brown
Charles Darwin
Charles Dickens
Charles Kingsley
Charles Lever
Charles Reade
Charlotte M. Yonge
Charlotte Perkins Gilman
Charlotte Smith
Christina Rossetti
Christopher Marlowe
Cicero
Claude Lorrain
Claude Monet

Claudian
Clement of Alexandria
Confederation Poets
Confucius
Cornelius Nepos
Correggio
D. H Lawrence (poetry)
D.H. Lawrence
Daniel Defoe
Dante Alighieri (English)
Dante Gabriel Rossetti
Dante Gabriel Rossetti
David Hume
Delphi Poetry Anthology
Demosthenes
Dickensiana Volume I
Diego Velázquez
Dio Chrysostom
Diodorus Siculus
Diogenes Laërtius
Dionysius of Halicarnassus
Donatello
E. F. Benson
E. M. Delafield
E. M. Forster
E. Nesbit
E. Phillips Oppenheim
E. W. Hornung
Earl Derr Biggers
Edgar Allan Poe
Edgar Allan Poe (poetry)
Edgar Degas
Edgar Rice Burroughs
Edgar Wallace
Edith Wharton
Edmund Burke
Edmund Spenser
Edmund Waller
Édouard Manet

Edvard Munch
Edward Bulwer-Lytton
Edward Gibbon
Edward Lear
Edward Thomas
Edwin Arlington Robinson
El Greco
Elizabeth Barrett Browning
Elizabeth Gaskell
Elizabeth von Arnim
Ella Wheeler Wilcox
Ellen Wood
Émile Zola
Emily Dickinson
Ennius
Epic of Gilgamesh
Epictetus
Erasmus
Ernest Bramah
Ernest Hemingway
Euclid
Eugène Delacroix
Eugene Sue
Euripides
Eusebius
Eutropius
Evelyn Waugh
Ezra Pound
F. Marion Crawford
F. Scott Fitzgerald
Fergus Hume
Florus
Ford Madox Ford
Frances Burney
Frances Hodgson Burnett
Frances Trollope
Francis Bacon
Francisco Goya
Frank Norris

Frank R. Stockton
Franz Schubert
Fred M. White
Frédéric Chopin
Frederick Douglass
Friedrich Nietzsche
Friedrich Schiller (English)
Frontius
Fronto
Fyodor Dostoyevsky
G. A. Henty
G. K. Chesterton
Galileo Galilei
Gaston Leroux
Geoffrey Chaucer
Georg Wilhelm Friedrich Hegel
George Berkeley
George Bernard Shaw
George Chapman
George Eliot
George Gissing
George Herbert
George MacDonald
George Meredith
George Moore
George Orwell
Georges Seurat
Gerard Manley Hopkins
Gertrude Stein
Giacomo Leopardi
Gilbert and Sullivan
Giotto
Giovanni Bellini
Giovanni Boccaccio
Giuseppe Verdi
Grant Allen
Gregory I
Gustav Klimt
Gustave Courbet

Gustave Flaubert (English)
Guy Boothby
Guy de Maupassant
H. G. Wells
H. P. Lovecraft
H. Rider Haggard
Hafez
Hall Caine
Hans Christian Andersen
Harriet Beecher Stowe
Heinrich Heine
Henrik Ibsen
Henry David Thoreau
Henry Fielding
Henry Howard, Earl of Surrey
Henry James
Henry Lawson
Henry Wadsworth Longfellow
Henryk Sienkiewicz
Herman Melville
Herodotus
Hesiod
Hieronymus Bosch
Hilaire Belloc
Hippocrates
Hokusai
Homer
Honoré de Balzac (English)
Horace
Horace Walpole
Howard Pyle
Hugh Walpole
Ian Fleming
Immanuel Kant
Isaac Rosenberg
Isocrates
Ivan Turgenev
Izaak Walton
J. M. Barrie

J. M. Synge
J. M. W. Turner
J. S. Fletcher
J. W. von Goethe (English)
Jack London
Jacques-Louis David
James Abbott McNeill Whistler
James Fenimore Cooper
James Joyce
James Russell Lowell
Jan van Eyck
Jane Austen
Jean-Auguste-Dominique Ingres
Jean-Jacques Rousseau
Jerome K. Jerome
Johan Ludvig Runeberg
Johann Sebastian Bach
Johanna Spyri
Johannes Vermeer
John Buchan
John Bunyan
John Clare
John Constable
John Donne
John Dryden
John Galsworthy
John Galt
John Gay
John Gower
John Greenleaf Whittier
John Keats
John Kendrick Bangs
John Locke
John Milton
John Muir
John Ruskin
John Steinbeck
John Stuart Mill
John Webster

John Wilmot, Earl of Rochester
Jonathan Swift
Joseph Addison
Joseph Addison
Joseph Conrad
Joseph Haydn
Josephus
Jules Verne
Julian
Julius Caesar
Juvenal
Kahlil Gibran
Karl Marx
Kate Chopin
Katherine Mansfield
Kenneth Grahame
L. Frank Baum
L. M. Montgomery
Lafcadio Hearn
Laurence Sterne
Leigh Hunt
Leo Tolstoy
Leonardo da Vinci
Lewis Carroll
Livy
Longus
Lord Byron
Lord Dunsany
Louisa May Alcott
Lucan
Lucian
Lucretius
Ludovico Ariosto
Ludwig van Beethoven
Luís de Camões
Lytton Strachey
M. E. Braddon
M. R. James
Manetho

Marcel Proust (English)
Marcus Aurelius
Margaret Oliphant
Maria Edgeworth
Marie Corelli
Mark Twain
Martial
Martin Luther
Mary Robinson
Mary Shelley
Mary Wollstonecraft
Masaccio
Matthew Arnold
Matthew Prior
Maurice Leblanc
Max Brand
Maxim Gorky
Michael Drayton
Michel de Montaigne
Michelangelo
Miguel de Cervantes
Molière
Nathaniel Hawthorne
Niccolò Machiavelli
Nikolai Gogol
Nikolai Nekrasov
Nonnus
Norse Sagas
O. Henry
Oliver Goldsmith
One Thousand and One Nights
Origen
Oscar Wilde
Ouida
Ovid
Paul Cézanne
Paul Gauguin
Paul Klee
Paul Laurence Dunbar

Pausanias
Percy Bysshe Shelley
Peter Paul Rubens
Petrarch
Petronius
Pierre-Auguste Renoir
Piet Mondrian
Pieter Bruegel the Elder
Pindar
Piotr Illitch Tchaïkovsky
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Pliny the Younger
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Plutarch
Polybius
Procopius
Propertius
Quintus Curtius Rufus
Quintus Smyrnaeus
R. Austin Freeman
R. D. Blackmore
R. M. Ballantyne
R. S. Surtees
Rabindranath Tagore
Radclyffe Hall
Rafael Sabatini
Ralph Waldo Emerson
Raphael
Rembrandt van Rijn
René Descartes
Richard Brinsley Sheridan
Richard Marsh
Richard Wagner
Robert Browning
Robert Burns
Robert E. Howard
Robert Frost

Robert Louis Stevenson
Robert Southey
Robert W. Chambers
Rudyard Kipling
Rumi
Rupert Brooke
Saadi
Saki
Sallust
Samuel Butler
Samuel Johnson
Samuel Pepys
Samuel Richardson
Samuel Taylor Coleridge
Sandro Botticelli
Sanskrit Epics
Sappho
Sara Teasdale
Sax Rohmer
Seneca the Younger
Septuagint
Sheridan Le Fanu
Sherwood Anderson
Sidonius
Sigmund Freud
Sir Arthur Conan Doyle
Sir Issac Newton
Sir Philip Sidney
Sir Richard Burton
Sir Thomas Malory
Sir Thomas More
Sir Thomas Wyatt
Sir Walter Raleigh
Sir Walter Scott
Sophocles
Stanley J. Weyman
Statius
Stendhal
Stephen Crane

Stephen Leacock
Strabo
Suetonius
T. S. Eliot
Tacitus
Talbot Mundy
Terence
Tertullian
The Brontës
The Brothers Grimm
The Harvard Classics
Theocritus
Theodore Dreiser
Theophrastus
Thomas Babington Macaulay
Thomas Carlyle
Thomas Chatterton
Thomas De Quincey
Thomas Dekker
Thomas Gainsborough
Thomas Gray
Thomas Hardy
Thomas Hardy (poetry)
Thomas Hobbes
Thomas Hood
Thomas Jefferson
Thomas Love Peacock
Thomas Middleton
Thomas Moore
Thomas Paine
Thucydides
Tibullus
Tintoretto
Titian
Tobias Smollett
Torquato Tasso
Varro
Victor Hugo
Vincent van Gogh

Virgil
Virginia Woolf
Vitruvius
Voltaire
W. B. Yeats
W. Somerset Maugham
W. W. Jacobs
Walt Whitman
Walter Pater
Walter Savage Landor
Washington Irving
Wassily Kandinsky
Wilfred Owen
Wilkie Collins
Willa Cather
William Blake
William Congreve
William Cowper
William Dean Howells
William Faulkner
William Harrison Ainsworth
William Hazlitt
William Hogarth
William Hope Hodgson
William James
William Le Queux
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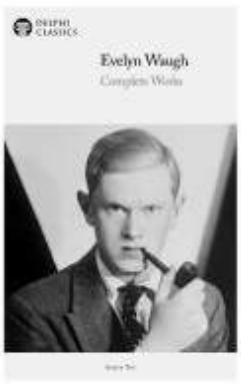

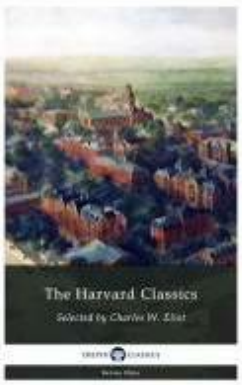

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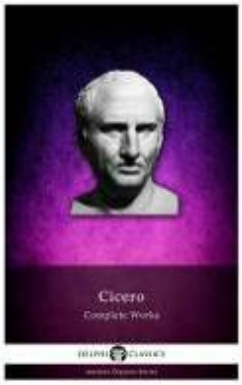

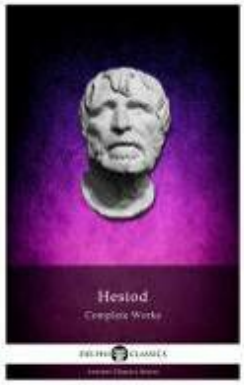

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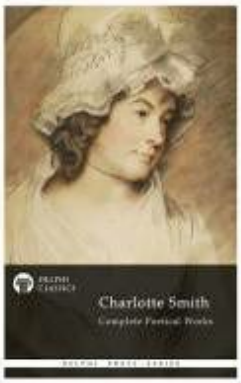

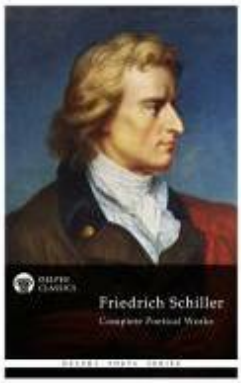

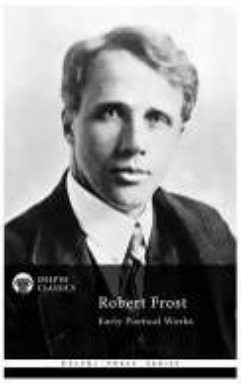

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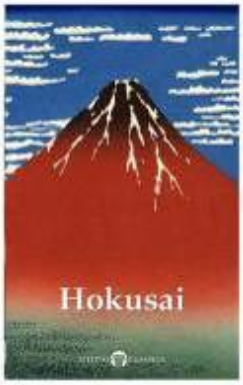

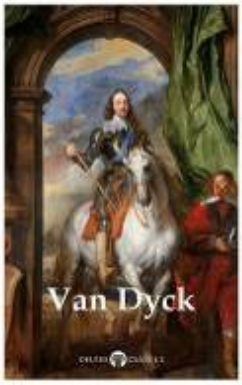

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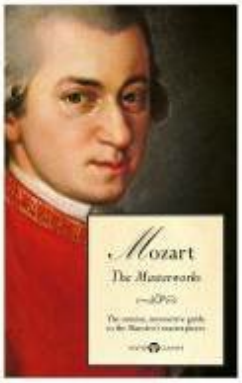

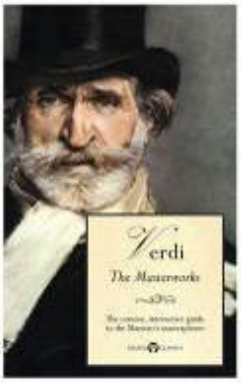

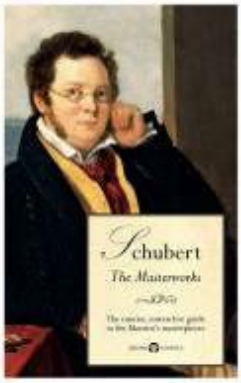

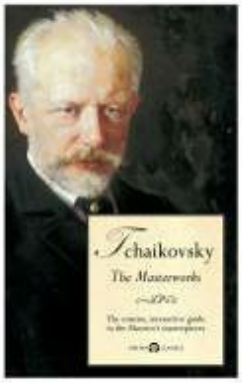

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